Semistable Reduction of overconvergent F-isocrystals

I: Isocrystals and Rigid Cohomology

The aim of this talk is to give a necessary background of isocrystals and rigid geometry in order to understand the semistable reduction conjecture. I will also do a brief literature review of this field.

1.1 Tubes and Strict Neighborhoods

This subsection is just a short version of [2, Section 1]. For details and proofs, one can consult the original paper.

Notation 1.1.1. Let K be a complete non-archimedean field (not necessarily discrete valued) of characteristic 0. Let $\mathcal{O}_K$ and k denote its ring of integers and residue field respectively. Assume that k has characteristic $p > 0$.

Notation 1.1.2. By a k-variety, I meant a reduced (not necessarily irreducible) separated scheme of finite type over k. (It could be shown that the theory only depends on the reduced scheme structure.) Through out the talk, X will be an open subscheme of a k-variety Y and $Z = X \backslash Y$ is the complement with the reduced scheme structure. P will always denote a topologically finite type formal scheme over $\mathcal{O}_K$ with a closed immersion $Y \hookrightarrow P_k$ into its special fiber. The generic fiber of P is the rigid analytic space P_K we are going to work on. Moreover, we require that P is **smooth** over $\mathrm{Spf}\mathcal{O}_K$ in an open neighborhood of X. To sum up, the following picture will show up very often.

$$\begin{array}{ccccccccc} X & \xrightarrow{\text{open}} & Y & \xrightarrow{\text{closed}} & P_k & \xrightarrow{\text{sp.fiber}} & P & \xleftarrow{\text{gen.fiber}} & P_K \\ & \searrow & \downarrow & \swarrow & & & \downarrow & & \downarrow \\ & & \mathrm{Spec}k & & \hookrightarrow & & \mathrm{Spf}(\mathcal{O}_K) & \longleftarrow & \mathrm{Sp}(K) \end{array} \tag{1.1.3}$$

Definition 1.1.4. A triple (X, Y, P) satisfying the conditions in Notation 1.1.2 is called a **frame**. A morphism between frames is a commutative diagram

$$\begin{array}{ccccc} X' & \xrightarrow{i'} & Y' & \xrightarrow{j'} & P' \\ \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi} \\ X & \xrightarrow{i} & Y & \xrightarrow{j} & P \end{array} \tag{1.1.5}$$

such that π is **smooth** in an open neighborhood of X'.

Given a formal model of a rigid analytic space, we can write down a specialization map $\mathrm{sp} : P_K \to P_k$ surjective onto closed points of P_k. (A deep result of Raynaud is that knowing the specialization map is equivalent to specifying a formal model.)

Definition 1.1.6. Let $S \subset P_k$ be a subscheme of P_k. The **tube** of S in P is $\mathrm{sp}^{-1}(S)$, denoted by $]S[_P$.

The following proposition gives some visualization of tubes when $P = \mathrm{Spf} A$ is affine.

Proposition 1.1.7. *If $S \subset P_k$ is a closed immersion defined by $I = (\bar{f}_1, \cdots, \bar{f}_n)$, then $]S[_P = \{x \in P_K \| |f_i(x)| < 1\}$, where f_i are liftings of $\bar{f}_i$ in A. If $S = \mathrm{Spec}(A_k)_{\bar{f}} \subset P_k$ is an open subscheme of P_k, then $]S[_P = \{x \in P_K \| |f(x)| = 1\}$, where f is a lifting of $\bar{f}$ in A. Moreover, the descriptions do not depend on choice of the liftings.*

Notation 1.1.8. Now, assume for a moment that P and hence Y is affine. Say, $P = \mathrm{Spf} A$ and Y is cut out by $\bar{f}_1, \cdots, \bar{f}_n \in A_k$ with f_i their liftings to A. Moreover, Z is a closed subscheme of Y defined by $\bar{g}_1, \cdots, \bar{g}_m \in \mathcal{O}_Y$ with g_j their liftings to A. Let $\eta, \lambda \in (0, 1)$. Denote

$$[Y]_{P,\eta} = \{x \in P_K : |f_i(x)| \le \eta\} \subset]Y[_P$$
$$U_\lambda = \{x \in]Y[_P : |g_j(x)| \ge \lambda\} \subset]Y[_P;$$
$$V_{\eta,\lambda} = [Y]_{P,\eta} \cap U_\lambda = \{x \in P_K : |f_i(x)| \le \eta, |g_j(x)| \ge \lambda\}$$

Definition 1.1.9. A neighborhood V of $]X[_P$ in $]Y[_P$ is called a **strict neighborhood** if it satisfies the following equivalent conditions:

(1) $\{V,]Z[_P\}$ is an admissible covering of $]Y[_P$.

(2) For any $\eta \in (0, 1)$, there exists $\lambda \in (0, 1)$ such that $V_{\eta,\lambda} \subseteq V \cap [Y]_{P,\eta}$.

(3) There exists sequence $\eta_n, \lambda_n \to 1^-$ as $n \to \infty$, such that V contains a standard strict neighborhood $V_{\underline{\eta},\underline{\lambda}} = \cup_n V_{\eta_n,\lambda_n}$.

The equivalence of above definitions is checked in [2, Section 1.2]. Moreover, the first definition can be generalized to non-affine case.

Exercise 1.1.10. Let $P = \mathrm{Spf}\mathcal{O}_K\langle x, y\rangle$, $Y = \mathbb{A}^1_k \hookrightarrow \mathbb{A}^2_k$ the x-axis, and $X = Y \backslash \{0\}$. Draw a picture for a standard strict neighborhood of $]X[_P$ in $]Y[_P$.

The following theorem [2, Théorème 1.3.7] is one of the most crucial techniques in the theory of rigid cohomology. (see also [5, Proposition 2.2.9])

Theorem 1.1.11 (Strong Fibration Theorem). *Let π be a morphism of frames $(X, Y', P') \to (X, Y, P)$ inducing identity on X. $\bar{X}$ is the closure of X in $P'_Y = Y \times_P P'$ and suppose that $\pi : \bar{X} \to Y$ is proper.*

$$\begin{array}{ccccc} X & \overset{i'}{\hookrightarrow} & Y' & \overset{j'}{\hookrightarrow} & P' \\ \| & & \downarrow \text{“proper”} & & \downarrow \text{“smooth”} \\ X & \overset{i}{\hookrightarrow} & Y & \overset{j}{\hookrightarrow} & P \end{array}$$

Let $\mathcal{I}' \subset \mathcal{O}_{P'}$ be the defining ideal of Y' in P', and let $\bar{\mathcal{I}}'$ be the defining ideal of Y' within P'_Y; suppose further that there exist sections $t_1, \cdots, t_d \in \Gamma(P', \mathcal{I}')$ whose reductions induce a basis of the conormal sheaf $\bar{\mathcal{I}}'/(\bar{\mathcal{I}}')^2$ on X. Put

$$P'' = P \times_{\mathcal{O}_K} \widehat{\mathbb{A}^d_{\mathcal{O}_K}} = \mathrm{Spf}\mathcal{O}_P\langle t_1, \cdots, t_d\rangle;$$

then the morphism $\phi : P' \to P''$ induces an isomorphism of some strict neighborhood of $]X[_{P'}$ within $]Y[_{P'}$ with some strict neighborhood of $]X[_{P''}$ within $]Y[_{P''}$.

Indeed, one does not have to remember the exact formulation of the theorem. Instead, one need to remember the following typical application of the theorem.

Example 1.1.12. We consider for the diagonal embedding with same Y. We require P to be smooth of dimension d and $\Gamma(P, \Omega^1_P)$ has a basis $dt_1, \cdots, dt_d$ for $t_i \in \mathcal{O}_P$:

$$\begin{array}{ccccc} X & \xhookrightarrow{i} & Y & \xhookrightarrow{\Delta_j} & P \times P \\ \| & & \| & & \downarrow{\scriptstyle \mathrm{pr}_1} \\ X & \xhookrightarrow{i} & Y & \xhookrightarrow{j} & P \end{array}$$

The strict neighborhood of $]X[_{P\times P}$ in $]Y[_{P\times P}$ is isomorphic to a strict neighborhood of $]X[_{P\times\mathbb{A}^d}=]X[_P\times A^d_K[0,1)$ in $]Y[_{P\times\mathbb{A}^d}=]Y[_P\times A^d_K[0,1)$ considered as embedding $P \hookrightarrow P \times \mathbb{A}^d$ using the zero section, where $A^d_K[0,1)$ is the standard notion of d-dimensional open unit polydisc with coordinates $t_1 - t'_1, \cdots, t_d - t'_d$.

1.2 Isocrystals

I will not talk about the definition of Monsky-Washnitzer cohomology (see for example) and actually the theory of rigid cohomology has incorporate the theory of Monsky-Washnitzer cohomology as a special case.

Rigid cohomology is a hybrid of Monsky-Washnitzer cohomology and the crystalline cohomology. The Monsky-Washnitzer cohomology has Lefschetz trace formula and is intimately related to algebraic de Rham theory of characteristic p. But it has a big restriction because of its poor functoriality. It is very inconvenient to "globalize" and glue affine pieces. Moreover, another crucial obstruction in the theory of Monsky-Washnitzer cohomology is the smoothness assumption.

In contrast to the Monsky-Washnitzer cohomology, crystalline cohomology is a relatively well understood theory. It has meaningful p-torsion, good sheaf theory and finite dimensionality for proper k-varieties. But crystalline cohomology uses Grothendieck topology which makes it hard to operate and to do explicit computation. Moreover, it seems to me that one does not know any finite dimensional statement except for proper k-schemes.

The rigid cohomology was firstly introduced in [1]. It combined the idea of overconvergence from Monsky-Washnitzer cohomology and the lifting technique from crystalline cohomology. At the expense of inverting p, it built up a bridge linking these two cohomology theories and then extracted good properties from them. One of the biggest achievement of theory of rigid cohomology is to give a purely p-adic proof of Weil Conjecture ([14]).

In this subsection, we will summarize the definition and basic properties of isocrystals following the approach of [2, Chapter 2].

Notation 1.2.1. In order to simplify the notation, sheaves are always sheaves of $\mathcal{O}$-modules. The general abelian sheaves behave exactly the same except the notational complication.

We start with a strict neighborhood V of $]X[_P$ in $]Y[_P$ and a sheaf $\mathcal{F}$ over V. Then, for any strict neighborhood V' of $]X[_P$ in $]Y[_P$ contained in V, we have a canonical morphism $\mathcal{F} \to \alpha_{VV'*}\alpha^*_{VV'}\mathcal{F}$, where $\alpha_{VV'} : V' \to V$ is the natural inclusion. Define

$$j^\dagger_V \mathcal{F} \stackrel{\text{def}}{=} \varinjlim_{V' \subset V} \alpha_{VV'*}\alpha^*_{VV'}\mathcal{F} = \text{“}\mathcal{F} \otimes_{\mathcal{O}_V} (\cup_{V' \subset V} \mathcal{O}_{V'})\text{”} \tag{1.2.2}$$

Moreover, we denote $j^\dagger \mathcal{F} = \alpha_{V*} j^\dagger_V \mathcal{F}$, where $\alpha_V : V \to]Y[_P$ is the natural inclusion.

Proposition 1.2.3. *$j^\dagger_V \mathcal{F}$ satisfies the following properties:*

(1) *When taking the limit in 1.2.2, we need only to take the limit over all the standard strict neighborhood.*

(2) *Let V' be as above, then $\alpha_{VV'*} j^\dagger_{V'} \alpha^*_{VV'} \mathcal{F} \xrightarrow{\sim} \mathcal{F}$. In particular, the definition of $j^\dagger$ does not depend on the choice of V.*

(3) *The canonical map $\mathcal{F} \otimes j^\dagger_V \mathcal{O}_V \to j^\dagger_V \mathcal{F}$ is an isomorphism.*

(4) *The map $\mathcal{F} \to j^\dagger_V \mathcal{F}$ is an epimorphism. Moreover, if $\mathcal{F}$ is already a $j^\dagger_V \mathcal{O}_V$-module, the map is an isomorphism.*

(5) *Let $\pi : (X', Y', P') \to (X, Y, P)$ be a morphism of frames together with strict neighborhood V (resp. V') of $]X[_P$ (resp. $]X'[_{P'}$). Assume that $\pi_K(V') \subset V$. Let $\mathcal{F}$ be a sheaf on V. Then we have a functorial morphism $\pi^* j^\dagger_V \mathcal{F} \to j'^\dagger_{V'} \pi^* \mathcal{F}$. It is an isomorphism if $\pi^{-1}(X) = X'$. In particular, the same is true for $\pi^* j^\dagger \mathcal{F} \to j'^\dagger \pi^* \mathcal{F}$.*

(6) *If $Y = P_k$ is smooth irreducible projective k-variety and X is an open affine subset of Y, then $\Gamma(P_K, j^\dagger \mathcal{O}_{P_K})$ is a Monsky-Washnitzer overconvergent algebra associated to X.*

Now, let us define isocrystals on X overconvergent along Z. First, assume that we have a model 1.1.3. Let $\mathcal{I}$ be the ideal of the diagonal embedding $\delta : P_K \hookrightarrow P_K \times P_K$. Denote $\mathcal{P}^n = \mathcal{O}_{P_K \times P_K}/\mathcal{I}^{n+1}$. Then we can view $j'^\dagger \mathcal{P}^n$ as a polynomial algebra over $j^\dagger \mathcal{O}_{]Y[_P}$ truncated at degree n, where j is the injection of $Y \hookrightarrow P_k$ and $j' = \delta \circ j : Y \hookrightarrow P_k \times P_k$. The isomorphisms are different if we use different projection π_i.

Proposition 1.2.4. *Let $\mathcal{E}$ be a coherent $\mathcal{O}_{]X[_P}$-module. The following data are equivalent:*

(1) *A connection on $\nabla : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{O}_{]X[_P}} \Omega^1_{]X[_P}$, such that $\nabla \circ \nabla = 0$, (i.e., ∇ is integrable).*

(2) *A compatible system of $\mathcal{P}^n$ isomorphisms $\epsilon_n : \mathcal{P}^n \otimes_{\mathcal{O}_{]X[}} \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{O}_{]X[}} \mathcal{P}^n$ with $\epsilon_0 = \mathrm{id}$, whose pull-back to $P_K \times P_K \times P_K$ satisfies a cocycle condition. Here we use the convention that tensoring $\mathcal{P}^n$ on the right means using the left projection of $\mathcal{O}_{]X[_P} \to \mathcal{P}^n$ and similar for tensoring on the left.*

Definition 1.2.5. An **overconvergent (integrable) ∇-module** on $]X[_P$ overconvergent along $]Z[_P$ is a coherent (and hence locally free) $j^\dagger \mathcal{O}_{]Y[_P}$-module $\mathcal{E}$ such that there exists an isomorphism $\epsilon : \mathrm{pr}_1^* \mathcal{E} \xrightarrow{\sim} \mathrm{pr}_2^* \mathcal{E}$ on a strict neighborhood of X in $P \times P$ inducing the same isomorphism as ϵ_n modulo $j'^\dagger \mathcal{I}^n$ and identity if we pull it back using δ, i.e., $\delta^* j'^\dagger = j^\dagger \delta^*$.

Notation 1.2.6. For simplicity, we will omit saying "integrable" as all the ∇-modules we are interested in are integrable. Moreover, when X and Y are clear, we just simply say overconvergent ∇-modules.

Proposition 1.2.7. *We list several basic properties of overconvergent ∇-modules.*

(1) Being an overconvergent ∇-module can be checked both locally on X and locally on P.

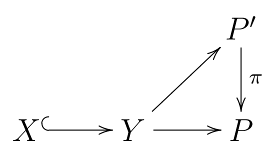
(2) Overconvergent ∇-modules have tensor products and inner $\mathcal{H}om$'s, i.e., if $\mathcal{E}$ and $\mathcal{F}$ are overconvergent ∇-modules, then the same is $\mathcal{H}om(\mathcal{E}, \mathcal{F})$, where the connection is defined to be $\nabla(\phi) = \nabla_{\mathcal{F}} \circ \phi - \phi \circ \nabla_{\mathcal{E}}$.

The following theorem is crucial in the construction of overconvergent isocrystals.

Theorem 1.2.8. *The category of overconvergent ∇-modules depends only on X and partly on Y. Precisely,*

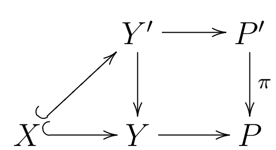
(1) [2, Proposition 2.2.17] *Given two morphisms of frames $\pi, \pi' : (X', Y', P') \to (X, Y, P)$ as in 1.1.5, if they agree on Y', then there exists a canonical isomorphism of functors $\epsilon_{\pi,\pi'} : \pi_K^* \to \pi_K'^*$ on overconvergent ∇-modules.*

(2) [2, Théorèm 2.3.1] *Given the following commutative diagram with π smooth in a neighborhood of X*



then π_K^* induces an equivalence of categories of overconvergent ∇-modules on $]X[_P$ overconvergent alone $]Y \backslash X[_P$ and overconvergent ∇-modules on $]X[_{P'}$ overconvergent alone $]Y \backslash X[_{P'}$.

(3) [2, Théorèm 2.3.5] *If we have a commutative diagram*



such that $\pi|_{Y'}$ is ***proper*** *and π is smooth on a neighborhood of X, then π_K^* induces an equivalence of categories of overconvergent ∇-modules on $]X[_P$ overconvergent alone $]Y \backslash X[_P$ and overconvergent ∇-modules on $]X[_{P'}$ overconvergent alone $]Y' \backslash X[_{P'}$.*

Construction 1.2.9. By previous theorem, we know that the category of overconvergent ∇-modules is independent of the choice of P.

Now, let $X \subset Y$ be as before. First assume that there is a smooth lifting P of Y. We define **the category of isocrystals on X overconvergent along $Y \backslash X$**, denoted by $\mathrm{Isoc}^{\dagger}(X, Y/K)$, to be a rule to associate each lifting P an overconvergent ∇-module overconvergent along $]Y\backslash X[_P$ compatible with the pull back map π_K^* described in Theorem 1.2.8. Each of the overconvergent ∇-module on P_K is called a **realization** of the isocrystal.

For general $X \subset Y$, we can work locally on Y. By gluing local open subsets, we can get the category of overconvergent isocrystals.

Moreover, for a variety X, assume that X has a compactification $\bar{X}$. we define $\mathrm{Isoc}^{\dagger}(X/K) = \mathrm{Isoc}^{\dagger}(X, \bar{X}/K)$. By Theorem 1.2.8(3), this definition does not depend on the choice of the compactification.

Remark 1.2.10. The difference between isocrystals and their realizations (or, ∇-modules) is a just psychological problem. When people say isocrystals, they tend to mean the whole family of ∇-modules although the realization is an equivalence of categories and there is no substantial difference between the two.

Definition 1.2.11. Suppose k has characteristic p. Let F be the absolute Frobenius on X and Y. An overconvergent F-isocrystal on X overconvergent along $Y\backslash X$ is an overconvergent isocrystal $\mathcal{F}$ equipped with an isomorphism $\phi : F^*\mathcal{F} \to \mathcal{F}$.

Remark 1.2.12. The definition seems to have a small ambiguity on which Frobenius lifting you use on P, the lifting of Y where you realize the isocrystal. Nevertheless, Theorem 1.2.8(1) tells us that for every lifting of Frobenius, ϕ gives a specific morphism of overconvergent ∇-modules $\phi : F^*\mathcal{F} \to \mathcal{F}$.

Notation 1.2.13. We will use $\mathcal{F}$ to denote overconvergent isocrystals with a Frobenius in order to distinguish it from $\mathcal{E}$.

1.3 Rigid Cohomology

Definition 1.3.1. Given an isocrystal on X overconvergent along $Z = Y\backslash X$. First, assume that we can find a realization $\mathcal{E}$ on P_K. Then, we define the cohomology $H^*_{\mathrm{rig}}(X/K, \mathcal{E})$ of the overconvergent isocrystal to be the hypercohomology of the de Rham complex $\mathbb{H}^*(]Y[_P, \mathcal{E} \otimes_{\mathcal{O}_{]Y[_P}} \Omega^{\bullet}_{]Y[_P})$. If Y does not have a realization, we have to use a Čech cohomology argument.

Remark 1.3.2. It can be shown that this definition does not depend on the choice of P. Indeed, for different choices P and P', by Strong Fibration Theorem 1.1.11, if we pull back along $P \times P'$ via diagonal embedding, then we ended up with comparing cohomology of $\mathcal{E}$ on some strict neighborhood V and $\mathrm{pr}^*\mathcal{E}$ on $V \times A^n_K[0, 1)$. An explicit calculation in this case showed that the two complexes are homotopic equivalence.

Definition 1.3.3. Let V be a strict neighborhood of $]X[_P$ in $]Y[_P$ and $\mathcal{E}$ an $\mathcal{O}_V$-sheaf on V. Define the subsheaf of **sections of $\mathcal{E}$ supported on** $]Z[_P$: $\Gamma^{\dagger}_{]Z[}\mathcal{E} = \mathrm{Ker}(\mathcal{E} \to j^{\dagger}_V\mathcal{E})$. By Proposition 1.2.3(3), we have an exact sequence

$$0 \to \Gamma^{\dagger}_{]Z[}\mathcal{E} \to \mathcal{E} \to j^{\dagger}_V\mathcal{E} \to 0.$$

The following lemma may give you some idea about how the $\Gamma^\dagger_{]Z[}$ functors work.

Lemma 1.3.4. *Let X_1 and X_2 be two open subschemes of Y with complement Z_1 and Z_2 respectively. Denote $X = X_1 \cup X_2$, $Z = Z_1 \cap Z_2$, $X' = X_1 \cap X_2$, $Z' = Z_1 \cup Z_2$ and j_1, j_2, j' the natural immersions of X_1, X_2, X' into Y. Let V be a strict neighborhood of $]X[$ in $]Y[$ and $\mathcal{E}$ be a sheaf on V. Then we have natural isomorphism of functors:*

(1) $j_1^\dagger \circ j_2^\dagger \simeq j_2^\dagger \circ j_1^\dagger \simeq j'^\dagger$.

(2) $\Gamma^\dagger_{]Z_1[} \circ \Gamma^\dagger_{]Z_2[} \simeq \Gamma^\dagger_{]Z_2[} \circ \Gamma^\dagger_{]Z_1[} \simeq \Gamma^\dagger_{]Z[}$.

Definition 1.3.5. Let $T \subset X$ be a closed subscheme, $X' = X \backslash T$ and $\mathcal{E}$ a realization of an isocrystal on $]Y[_P$. One can define $\Gamma^\dagger_{]T[}$ by transforming $\mathcal{E}$ into an isocrystal on X', i.e., taking $j^\dagger$ with respect to X'. Define the **rigid cohomology of $\mathcal{E}$ with support in T** to be $H^*_{T,\mathrm{rig}}(X, \mathcal{E}) \stackrel{\mathrm{def}}{=} \mathbb{H}^*(]Y[_P, \Gamma^\dagger_{]T[}\mathcal{E} \otimes \Omega^\bullet_{]Y[})$.

Compact supported cohomology can also be defined for overconvergent isocrystals.

Definition 1.3.6. Let V be a strict neighborhood of $]X[_P$ in $]Y[_P$. We have $\iota :]Z[_P \cap V \hookrightarrow V$. Denote $\underline{\Gamma}(\mathcal{E}) \stackrel{\mathrm{def}}{=} \mathrm{Ker}(\mathcal{E} \to \iota_*\iota^*\mathcal{E})$. Then, by Theorem B, $\mathbb{R}\underline{\Gamma}(\mathcal{E})$ is isomorphic to the two term complex: $\mathcal{E} \to \iota^*\iota^*\mathcal{E}$. The **rigid cohomology with compact supports** is defined to be $H^i_{c,\mathrm{rig}}(X/K, \mathcal{E}) = \mathbb{H}^i(V, \mathbb{R}\underline{\Gamma}(\mathcal{E} \otimes \Omega^\bullet_V))$. (Similar to rigid cohomology, this definition does not depend on the choice of lifting P.)

Now, I will list a number of theorems regarding rigid cohomology. For more detailed, one can consult [13, Section 1].

Theorem 1.3.7. *Let $\mathcal{F}$ be an overconvergent F-isocrystal on a variety X. Then the rigid cohomology $H^i_{\mathrm{rig}}(X, \mathcal{F})$ and $H^i_{c,\mathrm{rig}}(X, \mathcal{F})$ are finite dimensional K-vector spaces for all i.*

Theorem 1.3.8 (Poincaré Duality). *Let $\mathcal{F}$ be an overconvergent F-isocrystal on a smooth variety X of pure dimension d. Then for any close subscheme $T \subset X$, there are natural perfect pairings*

$$H^i_{T,\mathrm{rig}}(X/K, \mathcal{F}) \times H^{2d-i}_{c,\mathrm{rig}}(T/K, \mathcal{F}^\vee) \to K$$

Theorem 1.3.9 (Künneth Formula). *Let $\mathcal{F}_1/K$ (resp. $\mathcal{E}_F/K$) be overconvergent F-isocrystals on a k-variety X_1 (resp. X_2). Put $X = X_1 \times_k X_2$, and $\mathcal{F} = \mathcal{F}_1 \boxtimes \mathcal{F}_2 \stackrel{\mathrm{def}}{=} \mathrm{pr}_1^*\mathcal{F}_1 \otimes \mathrm{pr}_2^*\mathcal{F}_2$ where $\mathrm{pr}_i : X \to X_i$ is the natural projection. Then there is a natural isomorphism*

$$\bigoplus_{j+l=i} H^j_{c,\mathrm{rig}}(X_1/K, \mathcal{F}_1) \otimes H^l_{c,\mathrm{rig}}(X_2/K, \mathcal{F}_2) \simeq H^i_{c,\mathrm{rig}}(X/K, \mathcal{F}).$$

Using the bridge of rigid cohomology, one can carry the properties from crystalline cohomology to Monsky-Washnitzer cohomology and hence prove the finite dimensionality of Monsky-Washnitzer cohomology [3, Corollaire 3.2].

As Kedlaya has mentioned in Arizona Winter School, the rigid cohomology has a good trace formula coming from Washnitzer-Monsky cohomology. Moreover, Kedlaya used rigid cohomology to prove Weil Conjecture purely p-adically [14]. As we are short of space, we will not discuss this in details.

Remark 1.3.10. This finite dimensionality of rigid cohomology of k-varieties is firstly proved by Berthelot [3, Théorèm 3.1] for X smooth and $\mathcal{E} = \mathcal{O}_X$. Berthelot's proof consists of four important ingredients.

(1) A long exact sequence for devisage ([3, Proposition 2.5]):

$$\cdots \to H^i_T(X/K) \to H^i_S(X/K) \to H^i_{S'}(X'/K) \to \cdots$$

where $T \subset S \subset X$ are closed subschemes and $S' = S \backslash T$, $X' = X \backslash T$.

(2) Let $f : X' \to X$ be a (surjective) finite flat morphism. To check the finite dimensionality of $\mathcal{E}$ over X, it is suffice to check it for $f^*\mathcal{E}$ over X'.

(3) Gysin's isomorphism ([3, Corollaire 5.6], more generally, see [17, Theorem 4.1.1]):

$$H^i_{T,\mathrm{rig}}(X/K) \simeq H^{i-2c}_{\mathrm{rig}}(T/K)$$

for smooth pair (T, X).

(4) Comparison theorems to crystalline cohomology in smooth proper case:

$$H^i_{\mathrm{rig}}(X/K) = H^i_{\mathrm{cris}}(X, W(k)) \otimes K$$

(Of course, in terms of finite dimensionality, this could be replaced by Kiehl's finiteness theorem.)

The same strategy can not be used to prove the general finiteness theorem 1.3.7 because an isocrystal usually does not extend to a proper X. In [13], Kedlaya adopted an indirect way to approach the problem using his p-adic local monodromy theorem [11] and fibration by curves.

But one should expect a more direct approach to the problem using Berthelot's strategy. To this end, one need the following semistable reduction conjecture. The terminology in the theorem will be explained in later talks.

Theorem 1.3.11. *Let $\mathcal{F}$ be an overconvergent F-isocrystal on a k-variety X. Then after an alteration $\pi : X' \to X$, with $\bar{X}' \backslash X'$ a simple normal crossing divisor, $\pi^*\mathcal{F}$ can be extended to a log-isocrystal on X' with logarithmic poles along $\bar{X}' \backslash X'$.*

Semistable Reduction of overconvergent F-isocrystals

II: Semistable reduction Problem

We will keep the notation from the previous talk. In this talk we will discuss Kedlaya's approach to the semistable reduction conjecture of F-isocrystals ([5, 6, 7, 8]).

2.1 Statement of the Semistable Reduction Conjecture

Notation 2.1.1. Let k be a **perfect** field of characteristic p. K is the fraction field of the Witt vectors $\mathcal{O}_K = W(k)$ of k. Let $\mathcal{R}_K$ be the Robba ring over K.

Let us first recall the standard p-adic local monodromy conjecture. (see [11] or [12, Theorem 7.2.5])

Theorem 2.1.2. *Let $\mathcal{F}$ be a (σ, ∇)-module on over $\mathcal{R}_K$. Then $\mathcal{F}$ is quasi-unipotent, i.e. after pulling back along a finite étale map, $\mathcal{F}$ becomes unipotent.*

The aim of semistable reduction conjecture is to generalize this theorem to higher dimensional case, where one expect to replace finite étale map by an alteration. However, the quasi-unipotence is too strong a condition to be proved. Indeed, Kedlaya, in [5], interpreted the local unipotence in terms of the logarithmic extension. We will explain that in more detail in next subsection.

Log-isocrystal

We first gave the setup for the F-log-isocrystals. This construction is due to the work of Shiho [15, 16]. However, rather than going into the definition of log-scheme, we gave an intuitive explanation of this concept.

Definition 2.1.3. Let V be a (quasi-affinoid) rigid space over another rigid space W and $x_1, \cdots, x_n \in \Gamma(V, \mathcal{O})$ whose zero loci are smooth and meet transversely. Then $\Omega^{1,\log}_{V/W} \stackrel{\text{def}}{=} \Omega^1_{V/W} + \sum \mathcal{O}_V \frac{dx_i}{x_i}$. A log-$\nabla$-module is a locally free coherent module $\mathcal{F}$ over V together with a connection $\nabla : \mathcal{F} \to \mathcal{F} \otimes \Omega^{1,\log}_{V/W}$. The residue of $\mathcal{F}$ along the zero locus $V(x_i)$ of x_i is defined to be the endomorphism of $\mathcal{F}|_{V(x_i)}$ given by

$$\begin{array}{ccccc}
\mathcal{F} & \xrightarrow{\nabla} & \mathcal{F} \otimes \Omega^{1,\log}_{V/W} & \longrightarrow & \mathcal{F} \otimes \frac{dx_i}{x_i}\mathcal{O}_V \\
\downarrow & & & & \downarrow \\
\mathcal{F}|_{V(x_i)} & & \xrightarrow{\text{res.}} & & \mathcal{F}|_{V(x_i)}
\end{array}$$

Hypothesis 2.1.4. Recall the notations from 1.1.2, we further assume that $Y = P_k$ and $D = Y\backslash X$ is a simple normal crossing divisor. Assume further that there exists $x_1, \cdots, x_n \in \Gamma(P, \mathcal{O})$ whose reduction on Y defines the divisor D, i.e., $D = \cup V(\overline{x_i})$.

Remark 2.1.5. It is shown in [15, 16] that one can also talk about overconvergent log-isocrystals. The whole theory works well as in Berthelot's theory of overconvergent isocrystal. The definition of overconvergence of log-isocrystal is to similarly pull back to the diagonal embedding of X in $P \times P$, but one should be more careful about the log structure. I will not spend time on that. For details, one can consult Shiho's paper [15, 16] or [5, Section 6].

Definition 2.1.6. We say that an overconvergent isocrystal $\mathcal{F}$ on X extends to a log-isocrystal on Y, if $\mathcal{F}$ admits an extension to Y together with an integrable connection defined as in 2.1.3, such that the residue map along each of the divisor $V(x_i)$ is **nilpotent**.

Alteration

Now, we define the alteration. For details, one can consult [5, Section 3.1] or [4, Theorem 4.1].

Definition 2.1.7. For an irreducible k-variety X, an **alteration** is a proper dominant map $f : X' \to X$ with X' irreducible and f generically finite étale.

Theorem 2.1.8 (alteration)**.** *Let X be an irreducible k-variety and Z a proper closed subset of X. Then there exists an alteration $X' \to X$ such that X' admits a projective smooth compactification $\overline{X'}$ and $\overline{X'}\backslash X'$ is a simple normal crossing divisor.*

Conjecture 2.1.9 (Semistable Reduction Conjecture)**.** *Let k be a perfect field and $\mathcal{F}$ an overconvergent F-isocrystal on a k-variety X. Then after an alteration $\pi : X' \to X$, with $\bar{X}'\backslash X'$ a simple normal crossing divisor, $\pi^*\mathcal{F}$ can be extended to a log-isocrystal on X' with logarithmic poles along $\bar{X}'\backslash X'$.*

2.2 Unipotence Versus Logarithmic Extension

Before going further, let us first clarify the relationship between unipotence and logarithmic extension. This indicates the generalization from Theorem 2.1.2 to Conjecture 2.1.9.

Definition 2.2.1. Recall the setup from 2.1.3. Let I be an interval in $[0, +\infty)$. Consider $V \times A_K^n(I)$ over W, where the second factor has coordinates $t_1, \cdots, t_n$. Let $\mathrm{LNM}_{V\times A_K^n(I)/W}$ denote the category of log-∇-modules on $V \times A_K^n(I)$ relative to W with respect to $t_1, \cdots, t_n$, such that the residue along each $V(t_i)$ is **nilpotent** for $i = 1, \cdots, n$.

We say that a ∇-module $\mathcal{F} \in \mathrm{LNM}_{V\times A_K^n(I)/W}$ has **constant monodromy** if it is the pull back of a ∇-module on V. We say a that ∇-module has **unipotent monodromy** if it admits a filtration whose subquotients have constant monodromy. Let $\mathrm{ULNM}_{V\times A_K^n(I)/W}$ denote all the log-∇-modules on $V \times A_K^n(I)$ relative to W with respect to $t_1, \cdots, t_n$.

Theorem 2.2.2. *Let* $\mathcal{F} \in \mathrm{LNM}_{V\times A^n_K(a,1)/K}$ *be a* convergent ∇*-module. Then it extends to a log-*∇*-module on* $V \times A^n_K[0,1)$ *if and only if* $\mathcal{F}$ *has unipotent monodromy. Moreover, this extension is unique if it exists and* $\mathcal{F}$ *is constant if and only if all the residues are zero.*

Proof. This is [5, Proposition 3.6.9]. The proof consists of the following ingredients.

(1) The extension implies the unipotence because of the convergence condition on $\mathcal{F}$. This is always satisfied in the case of overconvergent isocrystals where the norm of ∇ is bounded by the overconvergence as it should give a Taylor isomorphism $p_1^*\mathcal{F} \simeq p_2^*\mathcal{F}$. Then, roughly speaking, start from a section $s \in \Gamma(V \times A^n_K[0,1), \mathcal{F})$, we know that $\sum \frac{1}{I!}\partial^I s \cdot t^I$ is a horizontal section converging on the whole polydisc. We then quotient out this section and proceed the same work and finally prove the unipotence.

(2) Conversely, we use the following key lemma coming from cohomology computation:

Lemma 2.2.3. *If* I *is an open interval of* $(0,1)$ *or interval of the form* $[0,a)$ *with* $a < 1$*, there exists an equivalence of categories* $\mathrm{ULNM}_{V\times A^n_K[0,0]/W} \to \mathrm{ULNM}_{V\times A^n_K(I)/W}$*, where* $\mathrm{ULNM}_{V\times A^n_K[0,0]/W}$ *means overconvergence modules on* V *together with* n *commutative nilpotent operators.*

□

Using this theorem, one can freely translate between unipotence and logarithmic extension. In particular, the unipotence is easy to work with as it is just extension of constant modules. In contrast, the logarithmic extension is a global concept that behave well functorially.

Generalization Versus Unipotence

The following theorem [5, Theorem 3.4.3] is an interesting phenomena that the unipotence is determine only by the generic fiber.

Theorem 2.2.4. *Let* I *is an open interval of* $(0,1)$ *or interval of the form* $[0,a)$ *with* $a < 1$*. Let* A *be an integral affinoid* K*-algebra and* $V = \mathrm{Max}(A)$*. Let* L *be a complete archimedean field containing* A *(typically* $(\mathrm{Frac}A)^\wedge$*). Let* $\mathcal{E}$ *be a* ∇*-module over* $V \times A^n_K(I)$*, and* $\mathcal{F} = \mathcal{E}\widehat{\otimes}_A L$*. Then,* $\mathcal{F}$ *is constant (unipotent)* ***if and only if*** $\mathcal{E}$ *is constant (unipotent).*

Corollary 2.2.5. *The semistable reduction on smooth variety is insensitive to codimension 2 locus.*

Proof. The theorem is true essentially because one can use Taylor series to find horizontal sections and the convergence of Taylor series depends on the norm. The corollary follows by passing to the generic point of the divisor. □

Local to Global

The first step of going from local to global is to observe that the tube of an irreducible smooth divisor $D = V(\bar{f})$ in X looks like $Q \times A^1_K[0,1)$, where Q is $V(f)$ on P. Thus, an isocrystal on X gave a ∇-module $\mathcal{F}$ on $Q \times A^1_K(\epsilon, 1]$. We use the thin piece $Q \times A^1_K(\epsilon, 1)$ to talk about monodromy along Z.

Thus, according to Theorem 2.2.2, if $\mathcal{F}$ has unipotent monodromy along Z then we can extend $\mathcal{F}$ to a log-∇-module over X.

There is a subtlety here: say we are in the situation of $X = \mathbb{A}^2$ and $D = (\mathbb{A}^1_x \cup \mathbb{A}^1_y)$. We begin with an overconvergent ∇-module over $X \backslash D$. We know that **on $X \backslash \mathbb{A}^1_x$, $\mathcal{F}$ has unipotent monodromy along** $\mathbb{A}^1_y$, so we can extent $\mathcal{F}$ to a log-∇-module over $]X \backslash \mathbb{A}^1_x[$. However, apriori, we do not know that the extended thing is still overconvergent to the area of $](0,0)[$. One has to do some work to solve this problem and the answer is of course affirmative.

Remark 2.2.6. Up to now, we have not use Frobenius yet. The equivalence between unipotence and logarithmic extension works without Frobenius.

2.3 Valuational Approach

One of the reason that Kedlaya abandoned the approach by discussing every divisor separately is because if, along one divisor, it requires to do some finite étale extension, then he had no control on the ramification along other divisors.

Riemann-Zariski Space

Definition 2.3.1. Let F be a field finitely generated over k, then any k-valuation ($v(k) = 0$) is of the form $v : F^\times \to \mathbb{R}^n$ where $\mathbb{R}^n$ is endowed with the lexicographic order.

A valuation coming from an irreducible smooth divisor is called **divisorial** valuation.

The minimal n is called the **rank** (or **height**) of v.

If v is of height 1 and the residue field of v is algebraic over k, then v is called **minimal**.

Two valuations v_1, v_2 are considered equivalent if $v_1(x) > 0 \Leftrightarrow v_2(x)$, $\forall x \in F^\times$.

Definition 2.3.2. The Riemann-Zariski space T_F is the space of all equivalent classes of valuations on F. It has two topology generated by the following subsets as base:

(1) Zariski: $\{v \in T_F | v(x_1) \geq 0, \cdots, v(x_n) \geq 0\}$, for all $x_1, \cdots, x_n \in F^\times$.

(2) Patch: $\{v \in T_F | v(x_1) \geq 0, \cdots, v(x_n) \geq 0; v(y_1) > 0, \cdots v(y_m) > 0\}$, for all $x_1, \cdots, x_n, y_1, \cdots, y_m \in F^\times$.

We will use the Patch topology later on.

Theorem 2.3.3. *T_F is Hausdorff and compact with respect to the Patch topology and hence quasi-compact with respect to the Zariski topology.*

Definition 2.3.4. If $F = k(X)$, we say that v is **centered** on X if there exists a point $x \in X$, such that $v(\mathcal{O}_{X,x}) \geq 0$, or equivalently, $v(\mathcal{O}_{X,x}) \subseteq \mathfrak{o}_v$.

Semistable Reduction at a Valuation

Definition 2.3.5. Let v be a valuation on $k(X)$ and $\mathcal{F}$ an overconvergent isocrystal on X, we say that $\mathcal{F}$ has a **semistable reduction at** v if there exists an alteration $f : X' \to X$ and a compactification $X' \hookrightarrow \overline{X'}$ such that $f^*\mathcal{F}$ extends to a log-∇-module on a neighborhood $X' \subset V$ in $\overline{X'}$ and any extension of v to $F' = k(X')$ is **centered on** V.

Theorem 2.3.6. *To prove the Semistable Reduction Conjecture 2.1.9, it is enough to prove the semistable reduction for all valuations* $v \in T_{k(X)}$.

Proof. The key fact is that, for any V as given in 2.3.5, the valuations that centered on V form an open subset of $T_{k(X')}$, and hence, it proves the semistable reduction at an open subset of $T_{k(X)}$ (because $k(X')/k(X)$ is a finite separable extension).

Thus, use the compactness of the Riemann-Zariski spaces, we can prove the theorem in a way similar to the proof of Chow Lemma. Indeed, we need to show that $\mathcal{F}$ extends across all the missing divisors "generically" (see Theorem 2.2.4). □

Minimal Valuation

Theorem 2.3.7. *It is enough to prove the Semistable Reduction Conjecture 2.1.9 at minimal valuation.*

Proof. The reduction to the height 1 case is dealt in [6, Section 4.2]. Here, **we first time use the Frobenius.**

The reduction to the case when the residue field is algebraic over k is dealt in [6, Section 4.3]. This is essentially using Theorem 2.2.4. □

Semistable Reduction of overconvergent F-isocrystals

III: Swan Conductor

3.1 Spectral Norms and Swan Conductor

Why Swan conductor?

Recall first what we achieved so far. Our aim is to prove the semistable reduction conjecture and we have shown that it is suffice to prove the conjecture for each single valuation, i.e., for any valuation v on $k(X)$, there exists an alteration $X' \to X$ such that the pull back of the isocrystal extends to a log-isocrystal on a subscheme $U \hookrightarrow \bar{X}$, on which the valuation v is centered.

Strictly speaking, we have not done anything crucial yet but just interpreted the problem in another fashion. It is natural to try to reduce the problem to theorems similar to the p-adic local monodromy theorem. But that kind of theorems can basically only deal with things look like annuli, and are hard to generalize to a global version. A strategy to get around this difficulty is to study the Swan conductor, a numerical criterion for logarithmic extension. When the Swan conductor is 0, one can make a tame base change and get a logarithmic extension over the corresponding divisor. We will study how it behaves when we change the divisors. This will eventually give the proof of semistable reduction conjecture in the surface case.

Spectral norms

Definition 3.1.1. If F is a differential field of order 1 equipped with a non-archimedean norm $|\cdot|$ and V a differential module with differential operator ∂. Then the *spectral norm of ∂ on V* is defined to be $|\partial|_{V,\mathrm{sp}} = \lim_{n\to\infty} |\partial^n|_V^{1/n}$. If F is a differential field equipped with $m+1$ differentials $\partial_1, \cdots, \partial_{m+1}$, then we define the *scale* of V to be

$$\max\left\{\frac{|\partial_j|_{V,\mathrm{sp}}}{|\partial_j|_{F,\mathrm{sp}}} : j = 1, \cdots, m+1\right\}.$$

If V_i, $i = 1, \cdots, n$ are the Jordan-Hölder factors of V, we define the *scale multiset* of V to be the set consists of scales of V_i with multiplicity $\dim_F V_i$ for all i.

Definition 3.1.2. Let $\eta \in (\eta_0, 1)$ and L_η the completion of the fraction field of $\Gamma(A_K^1[\eta, \eta], \mathcal{O})$ with respect to the Gauss norm. we consider $\mathcal{E}_\eta = \mathcal{E}|_{L_\eta} = \mathcal{E} \otimes L_\eta$ which inherits differential operators $\partial_j = \partial/\partial B_j$ for $j = 1, \cdots, m$ and $\partial_{m+1} = \partial_t$. There is a natural norm $|\cdot|_\eta$ on $\mathcal{E}_\eta$. Define the *radius multiset* $S(\mathcal{E}, \eta)$ to be the reciprocal of the scale multiset of $\mathcal{E}_\eta$ with respect to these differential operators and the norm on $|\cdot|_\eta$. Also, define *generic radius of convergence* $T(\mathcal{E},\eta)$ to be the smallest element in $S(\mathcal{E},\eta)$.

We only care about the behavior of $|\partial_i|_{\mathrm{sp},\rho}$ when $\rho \to 1$.

Proposition 3.1.3. *The function* $f(r) = \log T(\mathcal{E}, e^{-r})$ *on* $(0, -\log\eta_0)$ *is a piecewise linear concave function with slopes in* $(1/\mathrm{rank}\mathcal{E}!)\mathbb{Z}$*. It is linear in a neighborhood of 0. Moreover, there exists* $j \in \{1, \cdots, m+1\}$ *such that* ∂_j *is dominant for* $\mathcal{E}$*.*

Break decomposition and Swan conductor

Definition 3.1.4. As a consequence of 3.1.3, there exists a $b_{KSK} \in \mathbb{Q}_{\geq 0}$ and $\eta_0 \in (0,1)$ such that $T(\mathcal{E},\eta) = \eta^{b_{KSK}}$ for all $\eta \in (\eta_0, 1)$. This b_{KSK} is call the *(differential) highest ramification break* of $\mathcal{E}$.

Theorem 3.1.5. *For some* $\eta_0 \in (0,1)$*, there exists a unique decomposition of* (ϕ,∇)*-modules* $\mathcal{E} = \bigoplus_{b\in\mathbb{Q}_{\geq 0}} \mathcal{E}_b$ *over* $A^1_K(\eta_0, 1)$*, where each of* $\mathcal{E}_b$ *is of pure slope* b*, i.e., the radius multiset* $S(\mathcal{E},\eta)$ *consists only elements* η^b*.*

Definition 3.1.6. By previous theorem, there exists a multiset $\{b_1, \cdots, b_d\}$ such that for all η sufficiently close to 1, $S(\mathcal{E},\eta) = \{\eta^{b_1}, \cdots, \eta^{b_d}\}$. Define the *(differential) Swan conductor* of $\mathcal{E}$ (resp. ρ), denoted by $\mathrm{Swan}(\mathcal{E})$ (resp. $\mathrm{Swan}(\rho)$), as $b_1 + \cdots + b_d$.

Examples

Example 3.1.7. Let $k = \mathbb{F}_p(x)((t))$ and the standard Artin-Scheier extension l/k given by $s^p - s = t^{-n}$, where $p \nmid n > 0$. One can convert the extension l/k as an extension $\mathcal{R}_L/\mathcal{R}_K$ induced by the Cohen ring extension and taking the overconvergent part. Then, any character $\chi : \mathrm{Gal}(l/k) \to \mathbb{Q}_p$ gives rise to an overconvergent isocrystal $\mathcal{F} \subset \pi_*\mathcal{R}_L$ over $A^1_K(\eta, 1)$ for some $\eta \to 1^-$.

Then, the Swan conductor of $\mathcal{F}$ is n.

Example 3.1.8. If the extension l/k is given by $s^p - s = xt^{-mp}$, where x is a typical element in $\mathbb{F}_p(x)$. Then the Swan conductor of $\mathcal{F}$ is mp. However, this time the conduction actually comes from the other direction ∂_x.

3.2 Variation Between Divisors

What do I mean by variation between divisors

Let us take an example to illustrate this. We begin with $X = \mathbb{A}^2 \backslash (\mathbb{A}^1_x \cup \mathbb{A}^1_y)$. The two missing divisors are defined by $x = 0$ and $y = 0$ respectively.

We can blow up the original point and get a exceptional divisor D, whose definition function is $x = 0, y = 0$. On this divisor, $v_D(x) = 1$ and $v_D(y) = 1$. We note that D

also intersects the proper transforms of $\mathbb{A}^1_x$ and $\mathbb{A}^1_y$. We can further blow-up the intersection points, and hence get divisors over which $v(x) = 2, v(y) = 1$ or $v(x) = 1, v(y) = 2$.

We keep doing blowing-ups and we can get all divisors, over which $v(x) = a, v(y) = b$ for any $a, b \in \mathbb{N}$. We are interested in the behaviors of (properly normalized) Swan conductors along these divisors as a/b varies.

Convexity of Swan conductor

Example 3.2.1. Let us start with an example.

Over $X = \mathbb{A}^2 \backslash (\mathbb{A}^1_x \cup \mathbb{A}^1_y)$, we consider the Artin-Scheier extension defined by $s^p - s = \frac{1}{x^2} + \frac{1}{y}$ and we get the corresponding isocrystals.

(i) The Swan conductor along $D(y = 0)$: Since $1/x^2$ can be resolved, the Swan conductor is determined by $1/y$, which is 1.

(ii) The Swan conductor along $D(x = 0)$: Since $1/y$ can be resolved, the Swan conductor is determined by $1/x^2$, which is 2.

(iii) The Swan conductor along a divisor where $v(x) = a, v(y) = b$. (a and b coprime.)

Case 1: $b > 2a$. The dominant term will be $1/y$, and the Swan conductor will be b.

Case 2: $b < 2a$. The dominant term will be $1/x^2$, and the Swan conductor will be $2a$.

To normalize it and see the pattern how the Swan conductor vary with respect to a/b, we need to divide the Swan conductor by for example $v(y) = b$. Thus, set $r = a/b$ and we get

$$\mathrm{Swan}(D_r, \mathcal{F}) = \begin{cases} 1 & r < 1/2 \\ 2r & r > 1/2 \end{cases}$$

The above phenomenon can be proved in a very general setting [9, Theorem 6.0.1].

Theorem 3.2.2. *Let $\bar{X}$ be a smooth irreducible variety over k. Let $D_1, \cdots, D_n$ be divisors on $\bar{X}$ meeting transversely at a point z. Let $t_1, \cdots, t_n$ be parameters of $D_1, \cdots, D_n$ at z. Let $\mathcal{F}$ be an overconvergent F-isocrystal over $X = \bar{X} \backslash \{D_1 \cup, \cdots, \cup D_n\}$. For $R = (r_1, \cdots, r_{n-1}) \in \mathbb{Q}^{n-1}_{\geq 0}$, let v_R be the divisorial valuation given by $t_i \sim t_n^i$ for $i = 1, \cdots, n-1$ and let $\mathrm{Swan}_{t_m}(\mathcal{F}, R)$ be the differential Swan conductor of $\mathcal{F}$ along v_R (i.e., along the divisor of the corresponding blow-ups).*

Then the function $R \mapsto \mathrm{Swan}_{t_m}(\mathcal{F}, R)$ is continuous, piecewise linear and convex. Moreover, all its "turning points" are rational numbers.

3.3 Semistable Reduction over Surfaces

The semistable reduction of overconvergent F-isocrystals over surfaces is proved in two steps: [7] for monomial valuations and [8] for infinitely singular valuations.

The two kinds of valuations

The following description is copied from [8, Section 1.2].

For surfaces over an algebraically closed field, the valuations of height 1 and residual transcendence degree 0 come in two types: the monomial (or Abhyankar) valuations and the nonmonomial (or infinitely singular) valuations. Monomial valuations have the following structure: in suitable local coordinates $x; y$, they are determined by the fact that $v(x)$ and $v(y)$ are linearly independent over the rationals. Nonmonomial valuations admit no such nice local description, and hence are more complicated to work with.

If one imagines valuations as being determined by sequences of points on successive blowups of the surface, the distinction between the two types of valuations can be made as follows. For monomial valuations, after some point the center of each blowup is always at the intersection of the exceptional divisor of the previous blowup with the proper transform of some earlier exceptional divisor. For infinitely singular valuations, the opposite is true: infinitely often, the center of a blowup occurs at a point on the previous exceptional divisor that does not meet the proper transforms of earlier exceptional divisors.

Semistable reduction at monomial valuations

This is carried out in [7]. The essential input is [10] where Kedlaya proved that at the monomial valuation, one can make a finite étale extension and get a unipotent isocrystal. In particular, it has highest ramification break 0. Then, by the piecewise linearity and rationality of the Swan conductor, the ramification break is 0 at a neighborhood around that valuation v. Hence, we can make the whole picture algebraic and prove the semistable reduction at a monomial valuation.

Semistable reduction at infinitely singular valuations

Contrary to the monomial case, infinitely singular valuations appear at the end of the valuation tree. So, there is no way to use any continuity argument. In [8], Kedlaya used an alternative approach. He invoked the convexity of the Swan conductors on valuation tree and showed that, at a neighborhood of the infinitely singular valuation, the Swan conductor is actually constant. After doing some alteration, one can get a break decomposition or make the module decomposable. This requires some trick.

References

[1] Pierre Berthelot. Géométrie rigide et cohomologie des variétés algébriques de caractéristique p. *Mém. Soc. Math. France (N.S.)*, (23):3, 7–32, 1986. Introductions aux cohomologies p-adiques (Luminy, 1984).

[2] Pierre Berthelot. *Cohomologie rigide et cohomologie rigide à support propre. Première partie.* IRMAR, 1996.

[3] Pierre Berthelot. Finitude et pureté cohomologique en cohomologie rigide. *Invent. Math.*, 128(2):329–377, 1997. With an appendix in English by Aise Johan de Jong.

[4] A. J. de Jong. Smoothness, semi-stability and alterations. *Inst. Hautes Études Sci. Publ. Math.*, (83):51–93, 1996.

[5] Kiran S. Kedlaya. Semistable reduction for overconvergent F-isocrystals, I: Unipotence and logarithmic extensions. arXiv: `math.NT/0405069`.

[6] Kiran S. Kedlaya. Semistable reduction for overconvergent F-isocrystals, II: A valuation-theoretic approach. arXiv: `math.NT/0508191`.

[7] Kiran S. Kedlaya. Semistable reduction for overconvergent F-isocrystals, III: Local semistable reduction at monomial valuations. arXiv: `math.NT/0609645`.

[8] Kiran S. Kedlaya. Semistable reduction for overconvergent F-isocrystals, IV: The case of surfaces. available online: `http://math.mit.edu/~kedlaya/papers/`.

[9] Kiran S. Kedlaya. Swan conductors for p-adic differential modules, II: Global variation. arXiv: `math.NT/0705.0031`.

[10] Kiran S. Kedlaya. The p-adic local monodromy theorem for fake annuli. arXiv: `math/0507496`.

[11] Kiran S. Kedlaya. A p-adic local monodromy theorem. *Ann. of Math. (2)*, 160(1):93–184, 2004.

[12] Kiran S. Kedlaya. Slope filtrations revisited. *Doc. Math.*, 10:447–525 (electronic), 2005.

[13] Kiran S. Kedlaya. Finiteness of rigid cohomology with coefficients. *Duke Math. J.*, 134(1):15–97, 2006.

[14] Kiran S. Kedlaya. Fourier transforms and p-adic 'Weil II'. *Compos. Math.*, 142(6):1426–1450, 2006.

[15] Atsushi Shiho. Crystalline fundamental groups. I. Isocrystals on log crystalline site and log convergent site. *J. Math. Sci. Univ. Tokyo*, 7(4):509–656, 2000.

[16] Atsushi Shiho. Crystalline fundamental groups. II. Log convergent cohomology and rigid cohomology. *J. Math. Sci. Univ. Tokyo*, 9(1):1–163, 2002.

[17] Nobuo Tsuzuki. On the Gysin isomorphism of rigid cohomology. *Hiroshima Math. J.*, 29(3):479–527, 1999.