THE D-EQUIVALENCE CONJECTURE FOR HYPER-KÄHLER VARIETIES VIA HYPERHOLOMORPHIC BUNDLES

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Abstract. We show that birational hyper-Kähler varieties of $K3^{[n]}$-type are derived equivalent, establishing the D-equivalence conjecture in these cases. The Fourier–Mukai kernels of our derived equivalences are constructed from projectively hyperholomorphic bundles, following ideas of Markman. Our method also proves a stronger version of the D-equivalence conjecture for hyper-Kähler varieties of $K3^{[n]}$-type with Brauer classes.

Contents

0. Introduction

Throughout, we work over the complex numbers $\mathbb{C}$. We recall that the D-equivalence conjecture [5, 20] predicts that birational Calabi–Yau varieties have equivalent bounded derived categories of coherent sheaves.

Conjecture 0.1 (D-equivalence conjecture)**.** *If X, X' are nonsingular projective birational Calabi–Yau varieties, then there is an equivalence of bounded derived categories*

$$D^b(X) \simeq D^b(X').$$

The purpose of this paper is to prove Conjecture 0.1 for hyper-Kähler varieties of $K3^{[n]}$-type; these are nonsingular projective varieties deformation equivalent to the Hilbert scheme of n points on a $K3$ surface. More generally, our method reduces the D-equivalence conjecture for hyper-Kähler varieties to the construction of certain projectively hyperholomorphic bundles.

Theorem 0.2. *Conjecture 0.1 holds for any hyper-Kähler varieties of $K3^{[n]}$-type.*

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The D-equivalence conjecture has been proven by Bridgeland [6] for Calabi–Yau threefolds. For projective hyper-Kähler fourfolds, the D-equivalence conjecture holds by combining the classification results [8, 29] and the case of Mukai flops by Kawamata [20] and Namikawa [25]. However, very few cases of this conjecture are known in dimension > 4; see [26, 1] for some partial results. Using equivalences obtained from window conditions, Halpern-Leistner [13] proved the D-equivalence conjecture for any hyper-Kähler variety which can be realized as a Bridgeland moduli space of stable objects on a (possibly twisted) $K3$ surface. Theorem 0.2 generalizes Halpern-Leistner's result, but our construction of the derived equivalences is very different. We obtain explicit Fourier–Mukai kernels which rely on the theory of moduli spaces of hyper-Kähler manifolds and hyperholomorphic bundles [28, 23]; this is closer in spirit to the proposal of Huybrechts [17, Section 5.1]. It would be interesting to find connections between the two approaches.

Our method in fact proves the following stronger, twisted version of the D-equivalence conjecture involving arbitrary Brauer classes. Let $X \dashrightarrow X'$ be a birational transform between hyper-Kähler varieties of $K3^{[n]}$-type. It naturally identifies the Brauer groups of X, X': any Brauer class $\alpha \in \mathrm{Br}(X)$ induces a Brauer class $\alpha' \in \mathrm{Br}(X')$.

Theorem 0.3. *Let $X \dashrightarrow X'$ be as above, and let α be any Brauer class on X. Then there is an equivalence of bounded derived categories of twisted sheaves*

$$D^b(X, \alpha) \simeq D^b(X', \alpha').$$

Theorem 0.3 specializes to Theorem 0.2 by taking $\alpha = 0$.

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1. Moduli of Hodge isometries

Assume $n \geq 2$. We denote by Λ the $K3^{[n]}$-lattice, which is isometric to $H^2(X, \mathbb{Z})$, equipped with the Beauville–Bogomolov–Fujiki (BBF) form, for any hyper-Kähler manifold X of $K3^{[n]}$-type.[1] In particular, we have a decomposition

$$\Lambda = \Lambda_{\mathrm{K3}} \oplus \mathbb{Z}\delta, \quad \delta^2 = 2 - 2n$$

with Λ_{K3} the unimodular $K3$ lattice, so that any vector $\omega \in \Lambda$ can be expressed uniquely as

$$\omega = \omega_{\mathrm{K3}} + \lambda\delta, \quad \omega_{\mathrm{K3}} \in \Lambda_{\mathrm{K3}}, \quad \lambda \in \mathbb{Z}.$$

A marking (X, η_X) for a manifold X of $K3^{[n]}$-type is an isometry $\eta_X : H^2(X, \mathbb{Z}) \xrightarrow{\simeq} \Lambda$.

[1] When we say that X is a hyper-Kähler manifold or a manifold of $K3^{[n]}$-type, it means that X is not necessarily projective.

1.1. **Inseparable pairs.** We denote by $\mathfrak{M}_\Lambda$ the moduli space of marked manifolds (X, η_X) of $K3^{[n]}$-type; it is naturally a non-Hausdorff complex manifold whose non-separation illustrates the complexity of the birational/bimeromorphic geometry of hyper-Kähler varieties/manifolds [14].

We say that a pair $(X, \eta_X), (X', \eta_{X'})$ is *inseparable* if they represent inseparable points on the moduli space $\mathfrak{M}_\Lambda$; as a consequence of the global Torelli theorem, this is equivalent to the condition that $(X, \eta_X), (X', \eta_{X'})$ share the same period and lie in the same connected component of $\mathfrak{M}_\Lambda$.

Typical examples of inseparable pairs are given by bimeromorphic transforms. More precisely, a bimeromorphic map $X \dashrightarrow X'$ induces a natural identification $H^2(X, \mathbb{Z}) = H^2(X', \mathbb{Z})$ respecting the Hodge structures. A marking η_X for X then induces a marking $\eta_{X'}$ for X', and the pair $(X, \eta_X), (X', \eta_{X'})$ is therefore inseparable. Note that inseparable points are not necessarily induced by bimeromorphic transforms *directly.* As an example, we consider bimeromorphic X, X' as above and assume that

$$\rho : H^2(X', \mathbb{Z}) \to H^2(X, \mathbb{Z})$$

is a parallel transport respecting the Hodge structures. Then the pair

$$(X, \eta_X), \quad (X', \eta_{X'}), \quad \eta_{X'} := \eta_X \circ \rho$$

is inseparable. By [14] (see also [21, Section 3.1]), every inseparable pair arises this way.

1.2. **Hodge isometries.** We recall the moduli space of Hodge isometries; this was used by Buskin [9] and Markman [23] to construct algebraic cycles realizing rational Hodge isometries.

For $\phi \in O(\Lambda_{\mathbb{Q}})$, we define $\mathfrak{M}_\phi$ to be the moduli space of isomorphism classes of quadruples (X, η_X, Y, η_Y) where $(X, \eta_X), (Y, \eta_Y) \in \mathfrak{M}_\Lambda$ are the corresponding markings and

$$\eta_Y^{-1} \circ \phi \circ \eta_X : H^2(X, \mathbb{Q}) \to H^2(Y, \mathbb{Q})$$

is a Hodge isometry sending some Kähler class of X to a Kähler class of Y. We have the natural forgetful maps

$$\Pi_1 : \mathfrak{M}_\phi \to \mathfrak{M}_\Lambda, \quad (X, \eta_X, Y, \eta_Y) \mapsto (X, \eta_X),$$
$$\Pi_2 : \mathfrak{M}_\phi \to \mathfrak{M}_\Lambda, \quad (X, \eta_X, Y, \eta_Y) \mapsto (Y, \eta_Y).$$

Any connected component $\mathfrak{M}_\phi^0$ of $\mathfrak{M}_\phi$ maps to a connected component of $\mathfrak{M}_\Lambda$ via Π_i which we denote by $\mathfrak{M}_\Lambda^0$.

Lemma 1.1 ([23, Lemma 5.7])**.** *The maps $\Pi_i : \mathfrak{M}_\phi^0 \to \mathfrak{M}_\Lambda^0$ $(i = 1, 2)$ between connected components are surjective.*

Lemma 1.2. *Assume that the point (X, η_X, Y, η_Y) lies in a connected component $\mathfrak{M}_\phi^0$. Assume further that $(X, \eta_X), (X', \eta_{X'})$ form an inseparable pair such that*

$$(X', \eta_{X'}, Y, \eta_Y) \in \mathfrak{M}_\phi. \tag{1}$$

Then $(X', \eta_{X'}, Y, \eta_Y)$ *lies in the same component* $\mathfrak{M}^0_\phi$.

Note that (1) is equivalent to the condition that $\eta_Y^{-1} \circ \phi \circ \eta_{X'}$ sends some Kähler class of X' to a Kähler class of Y.

Proof. Both $(X, \eta_X), (X', \eta_{X'})$ lie in the same connected component of $\mathfrak{M}_\Lambda$ which we call $\mathfrak{M}^0_\Lambda$. We first find paths in $\mathfrak{M}^0_\Lambda$ connecting both points to $(X_0, \eta_{X_0}) \in \mathfrak{M}^0_\Lambda$ with $\mathrm{Pic}(X_0) = 0$. By Lemma 1.1, we can lift these paths to $\mathfrak{M}_\phi$, which connect (X, η_X, Y, η_Y) to $(X_0, \eta_{X_0}, Y_0, \eta_{Y_0})$, and $(X', \eta_{X'}, Y, \eta_Y)$ to $(X_0, \eta_{X_0}, Y'_0, \eta_{Y'_0})$ respectively. On one hand, by considering the projection Π_2, we know that the two points $(Y_0, \eta_{Y_0}), (Y'_0, \eta_{Y'_0})$ lie in the same connected component of $\mathfrak{M}_\Lambda$; on the other hand, the Hodge isometry condition ensures that both of them share the same period [23, Lemma 5.4] and they have trivial Picard group. By the global Torelli theorem, we must have $(Y_0, \eta_{Y_0}) = (Y'_0, \eta_{Y'_0})$. This completes the proof. □

Suppose we are given a point (X, η_X, Y, η_Y) in $\mathfrak{M}_\phi$, and Kähler classes ω_X, ω_Y on X, Y which are identified via $\eta_Y^{-1} \circ \phi \circ \eta_X$. Using this data, one can define a *diagonal twistor line* $\ell \subset \mathfrak{M}_\phi$ which lifts the twistor lines associated to (X, ω_X) and (Y, ω_Y) on $\mathfrak{M}_\Lambda$. A *generic diagonal twistor path* on $\mathfrak{M}_\phi$ is given by a chain of diagonal twistor lines such that, at each node of the chain, the associated hyper-Kähler manifolds have trivial Picard group. Generic diagonal twistor paths are used in Theorem 1.3 below to deform certain Fourier–Mukai kernels.

1.3. **Brauer groups.** Assume that X is a manifold of $K3^{[n]}$-type. Since X has no odd cohomology, the discussion in [11, Section 4.1] yields the following explicit description of the (cohomological) Brauer group:

$$\mathrm{Br}(X) = \Big(H^2(X, \mathbb{Z})/\mathrm{Pic}(X)\Big) \otimes \mathbb{Q}/\mathbb{Z}. \tag{2}$$

In particular, given a bimeromorphic map $X \dashrightarrow X'$ between manifolds of $K3^{[n]}$-type, there is a natural identification

$$\mathrm{Br}(X) = \mathrm{Br}(X')$$

since both $H^2(-, \mathbb{Z})$ and $\mathrm{Pic}(-)$ are identified for X and X'. The description (2) also allows us to present a Brauer class in the form

$$\left[\frac{\beta}{d}\right] \in \mathrm{Br}(X), \quad \beta \in H^2(X, \mathbb{Z}), \quad d \in \mathbb{Z}_{>0}; \tag{3}$$

this is referred to as the "B-field".

We note that the cohomology $H^2(X, \mathbb{Z})$ forms a trivial local system over any connected component of the moduli space $\mathfrak{M}^0_\Lambda$; therefore (3) for a single X presents a Brauer class for any point in the component $\mathfrak{M}^0_\Lambda$ containing (X, η_X).

1.4. **Projectively hyperholomorphic bundles.** Using the Bridgeland–King–Reid (BKR) correspondence [7], Markman constructed in [23] a class of projectively hyperholomorphic bundles which we recall here. We consider a projective $K3$ surface S with $\mathrm{Pic}(S) = \mathbb{Z}H$. Assume that r, s are two coprime integers with $r \geq 2$. Assume further that the Mukai vector

$$v_0 := (r, mH, s) \in H^*(S, \mathbb{Z})$$

is isotropic, *i.e.* $v_0^2 = 0$.[2] Let M be the moduli of stable vector bundles on S with Mukai vector v_0. Then M is again a $K3$ surface, and the coprime condition of r, s ensures the existence of a universal rank r bundle $\mathcal{U}$ on $M \times S$. Conjugating the BKR correspondence, we obtain a vector bundle $\mathcal{U}^{[n]}$ on $M^{[n]} \times S^{[n]}$ of rank

$$\mathrm{rk}(\mathcal{U}^{[n]}) = n! r^n;$$

see [23, Lemma 7.1]. This vector bundle induces a derived equivalence

$$\Phi_{\mathcal{U}^{[n]}} : D^b(M^{[n]}) \xrightarrow{\simeq} D^b(S^{[n]}). \tag{4}$$

Markman further showed in [23, Section 5.6] that the characteristic class of $\mathcal{U}^{[n]}$ induces a Hodge isometry

$$\phi_{\mathcal{U}^{[n]}} : H^2(M^{[n]}, \mathbb{Q}) \to H^2(S^{[n]}, \mathbb{Q}).$$

Under the natural identification

$$H^2(M^{[n]}, \mathbb{Q}) = H^2(M, \mathbb{Q}) \oplus \mathbb{Q}\delta, \quad H^2(S^{[n]}, \mathbb{Q}) = H^2(S, \mathbb{Q}) \oplus \mathbb{Q}\delta, \tag{5}$$

this Hodge isometry is of the form

$$\phi_{\mathcal{U}^{[n]}} = (\phi_{\mathcal{U}}, \mathrm{id}), \quad \phi_{\mathcal{U}} : H^2(M, \mathbb{Q}) \to H^2(S, \mathbb{Q}),$$

where $\phi_{\mathcal{U}}$ is the Hodge isometry of $K3$ surfaces induced by $\mathcal{U}$; see [23, Corollary 7.3].

The key results, which are summarized in the following theorem, show that the Fourier–Mukai kernel $\mathcal{U}^{[n]}$, as a projectively hyperholomorphic bundle, deforms along generic diagonal twistor paths. Moreover, at each point of the path, it induces a (twisted) derived equivalence:

Theorem 1.3 ([23, 19])**.** *There exist markings $\eta_{M^{[n]}}, \eta_{S^{[n]}}$ for the Hilbert schemes $M^{[n]}, S^{[n]}$ respectively, which induce $\phi \in O(\Lambda_{\mathbb{Q}})$ via $\phi_{\mathcal{U}^{[n]}}$, such that the connected component containing the quadruple*

$$(M^{[n]}, \eta_{M^{[n]}}, S^{[n]}, \eta_{S^{[n]}}) \in \mathfrak{M}_\phi^0$$

satisfies the following:

(a) *For every point (X, η_X, Y, η_Y) lying in the component $\mathfrak{M}_\phi^0$, there exists a twisted vector bundle $(\mathcal{E}, \alpha_{\mathcal{E}})$ on $X \times Y$, which is deformed from $\mathcal{U}^{[n]}$ along a generic diagonal twistor path.*

[2] In [23], Markman only considered the case $m = 1$; here considering large $\pm m$ is crucial for our purpose. Using [16, Proposition 2.2] (see also [30, Theorem 2.2]), Markman's argument works identically in this generality.

(b) *Using the form (3), the Brauer class in (a) is presented by*

$$\alpha_{\mathcal{E}} = \left[-\frac{c_1(\mathcal{U}^{[n]})}{\mathrm{rk}(\mathcal{U}^{[n]})}\right].$$

Here we view $H^2(X \times Y, \mathbb{Z}) = H^2(X, \mathbb{Z}) \oplus H^2(Y, \mathbb{Z})$ as a trivial local system over the moduli space $\mathfrak{M}^0_\phi$ via the markings.

(c) *Further assume that X, Y are varieties. Then the twisted bundle $(\mathcal{E}, \alpha_{\mathcal{E}})$ induces an equivalence of twisted derived categories*

$$\Phi_{(\mathcal{E},\alpha_{\mathcal{E}})} : D^b(X, \alpha_X) \xrightarrow{\simeq} D^b(Y, \alpha_Y), \quad \alpha_X = \left[\frac{a_X}{\mathrm{rk}(\mathcal{E})}\right], \quad \alpha_Y = \left[-\frac{a_Y}{\mathrm{rk}(\mathcal{E})}\right],$$

where $a_X \in H^2(X, \mathbb{Z}), a_Y \in H^2(Y, \mathbb{Z})$ are given by

$$c_1(\mathcal{U}^{[n]}) = a_X + a_Y \in H^2(X, \mathbb{Z}) \oplus H^2(Y, \mathbb{Z}).$$

Proof. (a) was proven in [23, Theorem 8.4]; Markman showed that $\mathcal{U}^{[n]}$ on $M^{[n]} \times S^{[n]}$ is projectively slope-stable hyperholomorphic in the sense of [28, 22] which allows him to deform it along diagonal twistor paths to all points in the component $\mathfrak{M}^0_\phi$.

(b) can be obtained by applying Căldăraru's result [11, Theorem 4.1] along the diagonal twistor paths; see the discussion in [19, Section 2.3].

(c) was proven in [19, Theorem 2.3]. More precisely, the condition that a twisted bundle induces a twisted derived equivalence can be characterized by cohomological properties [10, Theorem 3.2.1]. These properties are preserved along a twistor path due to the fact that the cohomology of slope-polystable hyperholomorphic bundles is invariant under hyper-Kähler rotations [27, Corollary 8.1]. Therefore we ultimately reduce the desired cohomological properties to those for $M^{[n]} \times S^{[n]}$ which are given by the original equivalence (4). □

1.5. **Birational geometry and MBM classes.** The birational geometry of hyper-Kähler varieties is governed by certain integral *primitive* cohomology classes, called the monodromy birationally minimal (MBM) classes. We refer to [3] for an introduction to these classes. In the following, we summarize some results which are needed in our proof.

Let X be a variety of $K3^{[n]}$-type. We consider its birational Kähler cone $\mathcal{BK}_X$ and the positive cone $\mathcal{C}_X$:

$$\mathcal{BK}_X \subset \mathcal{C}_X \subset H^{1,1}(X, \mathbb{R}).$$

The positive cone is convex and admits a wall-and-chamber structure. The closure of the birational Kähler cone within the positive cone is a convex sub-cone [15], which inherits a wall-and-chamber structure. Furthermore, by a result of Amerik–Verbitsky [2], and independently Mongardi [24], all the walls are governed by the MBM classes.

Theorem 1.4 ([24, 2])**.** *Any wall of $\mathcal{C}_X$ is described by a hyperplane of the form*

$$\mathcal{W}^\perp := \{\omega \in H^{1,1}(X, \mathbb{R}),\ (\omega, \mathcal{W}) = 0\} \subset H^{1,1}(X, \mathbb{R})$$

with $\mathcal{W}$ an algebraic MBM class in $\mathrm{Pic}(X)$*. Here the pairing is with respect to the BBF form. Moreover, any chamber in $\mathcal{C}_X$ can be realized as the Kähler cone of a birational hyper-Kähler X' through a parallel transport $\rho : H^2(X', \mathbb{Z}) \to H^2(X, \mathbb{Z})$ respecting the Hodge structures.*

Note that any chamber in $\mathcal{BK}_X \subset \mathcal{C}_X$ is given by the pullback of the Kähler cone via a birational transform $X \dashrightarrow X'$ of hyper-Kähler varieties. By the discussions of Section 1.1, any chamber of $\mathcal{C}_X$ corresponds to a marked variety $(X', \eta_{X'})$ of $K3^{[n]}$-type such that the pair $(X, \eta_X), (X', \eta_{X'})$ is inseparable.

We also need the following boundedness result, which notably implies that wall-and-chamber structure of $\mathcal{C}_X$ is locally polyhedral; see [18, Remark 8.2.3] for a proof of the implication. The boundedness result was essentially obtained by [4], as explained in [2, Section 6.2].

Theorem 1.5 ([4, 2])**.** *There is a constant $C_0 > 0$, such that for any variety X of $K3^{[n]}$-type and any MBM class $\mathcal{W} \in H^2(X, \mathbb{Z})$ we have*

$$0 < -\mathcal{W}^2 < C_0.$$

Here the norm is with respect to the BBF form.

For any rational Hodge isometry $\phi : H^2(X, \mathbb{Q}) \to H^2(Y, \mathbb{Q})$ between varieties of $K3^{[n]}$-type, which sends an MBM class $\mathcal{W}_X$ on X to a class proportional to an MBM class $\mathcal{W}_Y$ on Y, there exist coprime integers a, b such that

$$\phi(\mathcal{W}_X) = \frac{a}{b}\mathcal{W}_Y.$$

The following is an immediate consequence of Theorem 1.5.

Corollary 1.6. *For any $X, Y, \phi, \mathcal{W}_X, \mathcal{W}_Y$ as above, we have*

$$a^2 < C_0, \quad b^2 < C_0.$$

Proof. Since ϕ is an isometry, we have

$$\frac{a^2}{b^2} = \frac{\mathcal{W}_X^2}{\mathcal{W}_Y^2}$$

By Theorem 1.5, both $-\mathcal{W}_X^2$ and $-\mathcal{W}_Y^2$ are positive integers $< C_0$. The corollary follows from the assumption that a, b are coprime. □

1.6. **Proof strategy.** We discuss the strategy of the proof of Theorem 0.3; Theorem 0.2 is then deduced as a special case.

Let X be a variety of $K3^{[n]}$-type. It suffices to prove Theorem 0.3 for a hyper-Kähler birational model X' with a birational map $X \dashrightarrow X'$ which corresponds to a chamber in $\mathcal{BK}_X$ adjacent to the Kähler cone of X. By Theorem 1.5, the wall between these two chambers is given by an algebraic MBM class $\mathcal{W} \in \mathrm{Pic}(X)$.

Now we choose a $K3$ surface S and a Mukai vector $v_0 = (r, mH, s)$ as in the beginning of Section 1.4, which yields the Hodge isometry $\phi_{\mathcal{U}^{[n]}}$. Associated to these, we have the moduli space of Hodge isometries $\mathfrak{M}_\phi$, and the component $\mathfrak{M}_\phi^0$ that contains the quadruple $(M^{[n]}, \eta_{M^{[n]}}, S^{[n]}, \eta_{S^{[n]}})$.

For the given birational X, X', by Lemma 1.1, we can complete them to a pair of quadruples

$$(Y, \eta_Y, X, \eta_X), \quad (Y', \eta_{Y'}, X', \eta_{X'}) \in \mathfrak{M}_\phi^0$$

such that the marking $\eta_{X'}$ is induced by η_X via the birational map $X \dashrightarrow X'$.[3] In particular, the pair $(X, \eta_X), (X', \eta_{X'})$ is inseparable. We note that the pair $(Y, \eta_Y), (Y', \eta_{Y'})$ is also inseparable. This is because they share the same period and lie in the same connected component of $\mathfrak{M}_\Lambda$. Moreover, by definition, ϕ^{-1} sends a Kähler class of X (resp. X') to a Kähler class of Y (resp. Y').[4] Therefore, if

(6) ϕ^{-1} does not send $\mathcal{W}$ to a class on Y that is proportional to an MBM class,

there must be a point on the wall separating the Kähler cones of X, X' which is sent to the interior of a chamber of the positive cone $\mathcal{C}_Y$. In particular, there exists a hyper-Kähler birational model Y'' of Y with a marking $(Y'', \eta_{Y''})$ such that the pair $(Y, \eta_Y), (Y'', \eta_{Y''})$ is inseparable and

$$(Y'', \eta_{Y''}, X, \eta_X), \quad (Y'', \eta_{Y''}, X', \eta_{X'}) \in \mathfrak{M}_\phi.$$

Furthermore, by Lemma 1.2, both points lie in the connected component we started with:

$$(Y'', \eta_{Y''}, X, \eta_X), \quad (Y'', \eta_{Y''}, X', \eta_{X'}) \in \mathfrak{M}_\phi^0.$$

By Theorem 1.3(b, c), we obtain Brauer classes $\alpha_X, \alpha_{Y''}$ on X, Y'' respectively, such that

$$D^b(Y'', \alpha_{Y''}) \simeq D^b(X, \alpha_X), \quad D^b(Y'', \alpha_{Y''}) \simeq D^b(X', \alpha_{X'}).$$

Here the Brauer classes $\alpha_X, \alpha_{Y''}$ only depend on the markings $(X, \eta_X), (Y'', \eta_{Y''})$ respectively, and the Brauer class $\alpha_{X'}$ is induced by α_X. Combining both equivalences yields

$$D^b(X, \alpha_X) \simeq D^b(X', \alpha_{X'})$$

whose Fourier–Mukai kernel is the composition of two (twisted) hyperholomorphic bundles.

In the next section, we show that for any pair X, X' as above with a Brauer class $\alpha \in \mathrm{Br}(X)$ and an algebraic MBM class $\mathcal{W} \in \mathrm{Pic}(X)$, a careful choice of the $K3$ surface S and the Mukai vector $v_0 = (r, mH, s)$ as in Section 1.4 can simultaneously ensure that the condition (6) holds and the induced Brauer class is as desired:

$$\alpha_X = \alpha. \tag{7}$$

This completes the proof of Theorem 0.3.

[3]Here we would like X, X' to be deformed from $S^{[n]}$ later in Section 2.

[4]Here we suppress the markings and still use ϕ to denote the Hodge isometry $H^2(Y, \mathbb{Q}) \to H^2(X, \mathbb{Q})$ for notational convenience.

Remark 1.7. For a general birational transform $X \dashrightarrow X'$ of varieties of $K3^{[n]}$-type, which do not correspond to adjacent chambers in the birational Kähler cone $\mathcal{BK}_X$, our proof realizes the derived equivalence

$$D^b(X, \alpha) \simeq D^b(X', \alpha')$$

via two sequences of varieties $X_1, \ldots, X_{t-1}$ and $Y_1, \ldots, Y_t$, with each X_i birational to X, X', such that

$$(8) \quad D^b(X, \alpha) \simeq D^b(Y_1, \alpha_{Y_1}) \simeq D^b(X_1, \alpha_{X_1}) \simeq D^b(Y_2, \alpha_{Y_2}) \simeq \cdots \\ \simeq D^b(Y_{t-1}, \alpha_{Y_{t-1}}) \simeq D^b(X_{t-1}, \alpha_{X_{t-1}}) \simeq D^b(Y_t, \alpha_{Y_t}) \simeq D^b(X', \alpha').$$

Each of the derived equivalences in (8) is induced by a (twisted) hyperholomorphic bundle.

2. Proof of Theorem 0.3

From now on, we fix a variety X of $K3^{[n]}$-type, a Brauer class $\alpha \in \mathrm{Br}(X)$, and an algebraic MBM class $\mathcal{W} \in \mathrm{Pic}(X)$ as in Section 1.6. In particular, the variety X has Picard rank ≥ 2.[5] Using (2) and (3), we present the Brauer class α by a class in the rational transcendental lattice $T(X)_{\mathbb{Q}} \subset H^2(X, \mathbb{Q})$:

$$\alpha = \left[-\frac{\mathcal{B}}{d}\right], \quad \mathcal{B} \in T(X), \quad d \in \mathbb{Z}_{>0}.$$

Up to adjusting $-\frac{\mathcal{B}}{d}$ by an integral class in $T(X) \subset H^2(X, \mathbb{Z})$, we may further assume that the class $\mathcal{B}$ satisfies

$$\mathcal{B}^2 = 2e > 0.$$

2.1. **Divisor classes.** Recall that the divisibility $\mathrm{div}(\omega)$ of a class $\omega \in H^2(X, \mathbb{Z})$ is the positive generator of the subgroup

$$\{(\omega, \mu),\ \mu \in H^2(X, \mathbb{Z})\} \subset \mathbb{Z}$$

Lemma 2.1. *There exists a class $\mathcal{A} \in \mathrm{Pic}(X)$ such that*

$$(\mathcal{A}, \mathcal{W}) \neq 0, \quad \mathrm{div}(\mathcal{A}) = 1.$$

Proof. We pick a marking identifying $H^2(X, \mathbb{Z})$ with a $K3^{[n]}$-lattice $\Lambda = \Lambda_{\mathrm{K3}} \oplus \mathbb{Z}\delta$. For any $g \in O(\Lambda)$, since $g(\delta)^{\perp}$ is a unimodular $K3$-lattice, any primitive vector $\omega \in g(\delta)^{\perp} \subset \Lambda$ satisfies $\mathrm{div}(\omega) = 1$. We would like to choose g so that there exists $\mathcal{A} \in g(\delta)^{\perp} \cap \mathrm{Pic}(X)$ satisfying $(\mathcal{A}, \mathcal{W}) \neq 0$. In other words, we want

$$g(\delta)^{\perp} \cap \mathrm{Pic}(X) \neq \mathcal{W}^{\perp} \cap \mathrm{Pic}(X).$$

If we base change to $\mathbb{C}$, the set of $g \in O(\Lambda)_{\mathbb{C}}$ such that

$$g(\delta)^{\perp} \cap \mathrm{Pic}(X)_{\mathbb{C}} \neq \mathcal{W}^{\perp} \cap \mathrm{Pic}(X)_{\mathbb{C}}$$

[5]Theorem 0.3 is automatically true if X has Picard rank 1, since any birational transform $X \dashrightarrow X'$ is necessarily an isomorphism.

is open in the Zariski topology. Furthermore, it is nonempty since X has Picard rank ≥ 2. Since $O(\Lambda)$ is Zariski-dense in $O(\Lambda)_{\mathbb{C}}$, we can find $g \in O(\Lambda)$ satisfying this condition as well. ☐

Up to replacing $\mathcal{A}$ by $-\mathcal{A}$, we may assume

$$C_1 := (\mathcal{A}, \mathcal{W}) > 0$$

which we fix from now on.

Proposition 2.2. *For any $N > 0$, there exists a class $\mathcal{D} \in \mathrm{Pic}(X)$ of divisibility 1, satisfying*

$$\mathcal{D}^2 > N, \quad (\mathcal{D}, \mathcal{W}) = C_1.$$

Proof. Since X has Picard rank ≥ 2, we have $\mathcal{W}^\perp \cap \mathrm{Pic}(X) \neq 0$. Pick an element

$$\omega \in \mathcal{W}^\perp \cap \mathrm{Pic}(X), \quad \omega^2 > 0.$$

Then for large enough $t \in \mathbb{Z}_{>0}$, we have

$$(\mathcal{A} + t\omega, \mathcal{W}) = C_1, \quad (\mathcal{A} + t\omega)^2 > N.$$

It suffices to show that there exist infinitely many choices of $t \in \mathbb{Z}_{>0}$ satisfying

$$\mathrm{div}(\mathcal{A} + t\omega) = 1.$$

We pick an integral class $\mu \in H^2(X, \mathbb{Z})$ such that

$$(\mathcal{A}, \mu) = 1, \quad (\omega, \mu) \neq 0;$$

then we pick another integral class $\nu \in H^2(X, \mathbb{Z})$ such that

$$(\mathcal{A}, \nu) = 0, \quad (\omega, \nu) \neq 0.$$

We claim that for sufficiently large $t \in \mathbb{Z}_{>0}$ with $1 + t(\omega, \mu)$ a prime number, the class $\mathcal{A} + t\omega$ must have divisibility 1. This follows immediately from the observation that

$$\mathrm{div}(\mathcal{A} + t\omega) \,|\, 1 + t(\omega, \mu), \quad \mathrm{div}(\mathcal{A} + t\omega) \,|\, (\omega, \nu).$$ ☐

2.2. **Mukai vectors.** We construct the $K3$ surface S and the Mukai vector v_0 of Section 1.6.

By Proposition 2.2, we can find $\mathcal{D} \in \mathrm{Pic}(X)$ with

$$\mathrm{div}(\mathcal{D}) = 1, \quad (\mathcal{D}, \mathcal{W}) = C_1 > 0, \quad \mathcal{D}^2 = 2g > 2C_0C_1, \tag{9}$$

where C_0 is the constant in Theorem 1.5. Repeating the same argument as in Proposition 2.2, we also find $t \in \mathbb{Z}_{>0}$ such that

$$\mathrm{div}(\mathcal{D} + 4gtd\mathcal{B}) = 1.$$

Let (S, H) be a primitively polarized $K3$ surface of Picard rank 1 of degree

$$H^2 = 2g\left(1 + 4gt^2d^4(n-1) + 16gt^2d^2e\right) > 0.$$

We observe that both classes

$$H - 2gtd^2\delta \in \mathrm{H}^2(S^{[n]}, \mathbb{Z}), \quad \mathcal{D} + 4gtd\mathcal{B} \in H^2(X, \mathbb{Z})$$

are of divisibility 1 and have the same norm, where we have used that $(\mathcal{D}, \mathcal{B}) = 0$ since $\mathcal{B}$ is transcendental. Therefore, by [12, Example 3.8] and [21, Theorem 9.8], there is a parallel transport

$$\rho : H^2(S^{[n]}, \mathbb{Z}) \to H^2(X, \mathbb{Z})$$

satisfying

$$\rho(H - 2gtd^2\delta) = \epsilon(\mathcal{D} + 4gtd\mathcal{B}), \tag{10}$$

where $\epsilon = \pm 1$ is a sign determined by the orientation.

We now consider the Mukai vector

$$v_0 := \left(16gt^2d^4,\ \epsilon \cdot 4td^2H,\ 1 + 4gt^2d^4(n-1) + 16gt^2d^2e\right),$$

which clearly satisfies

$$\gcd\left(16gt^2d^4, 1 + 4gt^2d^4(n-1) + 16gt^2d^2e\right) = 1, \quad 16gt^2d^4 \geq 2, \quad v_0^2 = 0.$$

The moduli space M of stable vector bundles on S with Mukai vector v_0 is a $K3$ surface of Picard rank 1 with a universal bundle $\mathcal{U}$ on $M \times S$ which we fix from now on. Also fixed are the markings $\eta_{M^{[n]}}, \eta_{S^{[n]}}$ as in Theorem 1.3, as well as the induced marking

$$\eta_X := \eta_{S^{[n]}} \circ \rho^{-1} : H^2(X, \mathbb{Z}) \xrightarrow{\simeq} \Lambda.$$

Proposition 2.3. *Let $S, M, \mathcal{U}$ be as above.*

(a) *The primitive polarization $\widehat{H}$ of M satisfies $\widehat{H}^2 = H^2$.*
(b) *Let $s \in S$ be a point. Assume that the vector bundle $\mathcal{U}|_s$ has Mukai vector*

$$\widehat{v}_0 = (16gt^2d^4, k\widehat{H}, \widehat{s}) \in H^*(M, \mathbb{Z}).$$

Then we have

$$\gcd(16gt^2d^4, k) = 4td^2.$$

Proof. (a) follows from [30, Appendix A]. For (b), we note that [30, Theorem 2.2] implies that the Mukai vector $\widehat{v}_0$ is primitive with $\widehat{v}_0^2 = 0$. Using (a), we deduce that

$$\widehat{s} = \left(\frac{k}{4td^2}\right)^2 \left(1 + 4gt^2d^4(n-1) + 16gt^2d^2e\right) \in \mathbb{Z}.$$

Therefore, we have that k is divisible by $4td^2$, which shows

$$4td^2 \mid \gcd(16gt^2d^4, k).$$

On the other hand, if $\frac{k}{4td^2}$ is not coprime to $16gt^2d^4$, the Mukai vector $\widehat{v}_0$ is divisible by their common factor. This contradicts the fact that $\widehat{v}_0$ is primitive. □

2.3. **End of proof.** We complete the proof using the $K3$ surface S, the Mukai vector v_0, and the universal bundle $\mathcal{U}$ constructed in the last section. This gives the vector bundle $\mathcal{U}^{[n]}$ on $M^{[n]} \times S^{[n]}$. We write

$$c_1(\mathcal{U}^{[n]}) = a_{M^{[n]}} + a_{S^{[n]}} \in H^2(M^{[n]}, \mathbb{Z}) \oplus H^2(S^{[n]}, \mathbb{Z})$$

with

$$a_{M^{[n]}} \in H^2(M^{[n]}, \mathbb{Z}), \quad a_{S^{[n]}} \in H^2(S^{[n]}, \mathbb{Z}).$$

Recall the natural identification

$$H^2(S^{[n]}, \mathbb{Z}) = H^2(S, \mathbb{Z}) \oplus \mathbb{Z}\delta. \tag{11}$$

By [23, Equation (7.11)], we can present the class $a_{S^{[n]}}$ using (11):

$$a_{S^{[n]}} = \mathrm{rk}(\mathcal{U}^{[n]}) \cdot \left(\frac{\epsilon \cdot 4td^2 H}{16gt^2d^4} - \frac{\delta}{2} \right) \in H^2(S^{[n]}, \mathbb{Z}).$$

Via the parallel transport ρ and (10), we obtain

$$\begin{aligned} \rho\left(\frac{a_{S^{[n]}}}{\mathrm{rk}(\mathcal{U}^{[n]})}\right) &= \rho\left(\frac{\epsilon \cdot H}{4gtd^2} - \frac{\delta}{2}\right) \\ &= \rho\left(\epsilon\left(\frac{H}{4gtd^2} - \frac{\delta}{2}\right) + (\epsilon - 1)\frac{\delta}{2}\right) \\ &= \frac{\epsilon \cdot \rho(H - 2gtd^2\delta)}{4gtd^2} + \frac{(\epsilon-1)}{2}\rho(\delta) \\ &= \frac{\mathcal{D} + 4gtd\mathcal{B}}{4gtd^2} + \frac{(\epsilon-1)}{2}\rho(\delta) \\ &= \frac{\mathcal{B}}{d} + [\text{class in } \mathrm{Pic}(X)_{\mathbb{Q}}] + [\text{class in } H^2(X, \mathbb{Z})] \in H^2(X, \mathbb{Q}). \end{aligned}$$

Hence, by Theorem 1.3(c), we have

$$\alpha_X = \left[-\rho\left(\frac{a_{S^{[n]}}}{\mathrm{rk}(\mathcal{U}^{[n]})}\right)\right] = \left[-\frac{\mathcal{B}}{d}\right] = \alpha.$$

To complete the proof, it remains to address (6). This is given by the following proposition.

Proposition 2.4. *Let $\mathfrak{M}^0_\phi$ be the connected component of the moduli space of Hodge isometries constructed from $S, M, \mathcal{U}$ as above. For any quadruple*

$$(Y, \eta_Y, X, \eta_X) \in \mathfrak{M}^0_\phi$$

with $X, \mathcal{W}$ fixed as above, the class $\phi^{-1}(\mathcal{W}) \in H^2(Y, \mathbb{Q})$ is not proportional to any MBM class on Y. Here we suppress the markings and still use ϕ to denote the Hodge isometry $H^2(Y, \mathbb{Q}) \to H^2(X, \mathbb{Q})$ for notational convenience.

Proof. The main idea of the argument is that, for our choice of the Mukai vector v_0, by a calculation of Buskin [9], the rational Hodge isometry ϕ^{-1} is conjugate to a reflection by a vector of large norm. By Corollary 1.6, we then show that it cannot send $\mathcal{W}$ to a class proportional to an MBM class.

The details are as follows. Since the MBM classes are deformation invariant, we only need to treat the Hodge isometry

$$\phi_{\mathcal{U}^{[n]}} : H^2(M^{[n]}, \mathbb{Q}) \to H^2(S^{[n]}, \mathbb{Q})$$

which can be further simplified under the identification (5):

$$(\phi_{\mathcal{U}}, \mathrm{id}) : H^2(M, \mathbb{Q}) \oplus \mathbb{Q}\delta \to H^2(S, \mathbb{Q}) \oplus \mathbb{Q}\delta.$$

Assume that

$$\phi^{-1}_{\mathcal{U}^{[n]}}(\rho^{-1}(\mathcal{W})) = \frac{b}{a}\mathcal{W}' \tag{12}$$

with $\mathcal{W}'$ an MBM class on $M^{[n]}$ and a, b coprime. We write

$$\rho^{-1}(\mathcal{W}) = \mathcal{W}_{\mathrm{K3}} + \lambda\delta, \quad \mathcal{W}_{\mathrm{K3}} \in H^2(S, \mathbb{Z}), \quad \lambda \in \mathbb{Z}.$$

The equation (12) implies that $\phi_{\mathcal{U}}^{-1}(a\mathcal{W}_{\mathrm{K3}})$ is an integral class. By the formula right before [9, Conclusion 3.8], the integrality forces the pairing

$$(H, a\mathcal{W}_{\mathrm{K3}}) \in \mathbb{Z}$$

to be divisible by

$$\frac{16gt^2d^4}{\gcd\left(16gt^2d^4, 4td^2k\right)} = g,$$

where we have used Proposition 2.3(b) in the last equation.

On the other hand, we have

$$(H, a\mathcal{W}_{\mathrm{K3}}) = (H, \rho^{-1}(a\mathcal{W})) = (\rho(H), a\mathcal{W}) = \epsilon(\mathcal{D}, a\mathcal{W}) + [\text{integer divisible by } g],$$

where the last equality uses (10). In particular, we find

$$g \,|\, (\mathcal{D}, a\mathcal{W}) = aC_1.$$

Combined with Corollary 1.6, this implies

$$g \leq a^2 C_1 < C_0 C_1$$

which contradicts our choice of $\mathcal{D}$ in (9). This shows that (12) cannot hold, which proves the proposition. □

In conclusion, both (6) and (7) are settled by our choice of the $K3$ surface S and the Mukai vector v_0; the proof of Theorem 0.3 is now complete. □

References

[1] N. Addington, W. Donovan, and C. Meachan, *Moduli spaces of torsion sheaves on $K3$ surfaces and derived equivalences,* J. Lond. Math. Soc. (2) 93 (2016), no. 3, 846–865.

[2] E. Amerik and M. Verbitsky, *Rational curves on hyperkähler manifolds,* Int. Math. Res. Not. IMRN (2015), no. 23, 13009–13045.

[3] E. Amerik and M. Verbitsky, *Rational curves and MBM classes on hyperähler manifolds: a survey,* Rationality of varieties, 75–96, Progr. Math., 342, Birkhäuser/Springer, Cham, 2021.

[4] A. Bayer, B. Hassett, and Y. Tschinkel, *Mori cones of holomorphic symplectic varieties of $K3$ type,* Ann. Sci. Éc. Norm. Supér. (4) 48 (2015), no. 4, 941–950.

[5] A. Bondal and D. Orlov, *Semiorthogonal decomposition for algebraic varieties,* arXiv:alg-geom/9506012.

[6] T. Bridgeland, *Flops and derived categories,* Invent. math. 147 (2002), no. 3, 613–632.

[7] T. Bridgeland, A. King, and M. Reid, *The McKay correspondence as an equivalence of derived categories,* J. Amer. Math. Soc. 14 (2001), no. 3, 535–554.

[8] D. Burns, Y. Hu, and T. Luo, *HyperKähler manifolds and birational transformations in dimension 4,* Vector bundles and representation theory (Columbia, MO, 2002), 141–149, Contemp. Math., 322, American Mathematical Society, Providence, RI, 2003.

[9] N. Buskin, *Every rational Hodge isometry between two $K3$ surfaces is algebraic,* J. Reine Angew. Math. 755 (2019), 127–150.

[10] A. Căldăraru, *Derived categories of twisted sheaves on Calabi–Yau manifolds,* Ph.D. Thesis, Cornell University, 2000.

[11] A. Căldăraru, *Non-fine moduli spaces of sheaves on $K3$ surfaces,* Int. Math. Res. Not. IMRN (2002), no. 20, 1027–1056.

[12] V. Gritsenko, K. Hulek, and G. K. Sankaran, *Moduli spaces of irreducible symplectic manifolds,* Compos. Math. 146 (2010), no. 2, 404–434.

[13] D. Halpern-Leistner, *Derived Θ-stratifications and the D-equivalence conjecture,* arXiv:2010.01127.

[14] D. Huybrechts, *Compact hyper-Kähler manifolds: basic results,* Invent. Math. 135 (1999), no. 1, 63–113.

[15] D. Huybrechts, *The Kähler cone of a compact hyperkähler manifold,* Math. Ann. 326 (2003), no. 3, 499–513.

[16] D. Huybrechts, *Derived and abelian equivalence of $K3$ surfaces,* J. Algebraic Geom. 17 (2008), no. 2, 375–400.

[17] D. Huybrechts, *Hyperkähler manifolds and sheaves,* Proceedings of the International Congress of Mathematicians, Volume II, 450–460, Hindustan Book Agency, New Delhi, 2010.

[18] D. Huybrechts, *Lectures on K3 surfaces,* Cambridge Stud. Adv. Math., 158, Cambridge University Press, Cambridge, 2016, xi+485 pp.

[19] G. Kapustka and M. Kapustka, *Constructions of derived equivalent hyper-Kähler fourfolds,* arXiv: 2312.14543v4.

[20] Y. Kawamata, *D-equivalence and K-equivalence,* J. Differential Geom. 61 (2002), no. 1, 147–171.

[21] E. Markman, *A survey of Torelli and monodromy results for holomorphic-symplectic varieties,* Complex and differential geometry, 257–322, Springer Proc. Math., 8, Springer, Heidelberg, 2011.

[22] E. Markman, *The Beauville–Bogomolov class as a characteristic class,* J. Algebraic Geom. 29 (2020), no. 2, 199–245.

[23] E. Markman, *Rational Hodge isometries of hyper-Kähler varieties of $K3^{[n]}$-type are algebraic,* Compos. Math. 160 (2024), no. 6, 1261–1303.

[24] G. Mongardi, *A note on the Kähler and Mori cones of hyperkähler manifolds,* Asian J. Math. 19 (2015), no. 4, 583–591.

[25] Y. Namikawa, *Mukai flops and derived categories,* J. Reine Angew. Math. 560 (2003), 65–76.

[26] D. Ploog, *Equivariant equivalences for finite group actions,* Adv. Math. 216 (2007), no. 1, 62–74.

[27] M. Verbitsky, *Hyperholomorphic bundles over a hyper-Kähler manifold,* J. Algebraic Geom. 5 (1996), no. 4, 633–669.

[28] M. Verbitsky, *Hyperholomorphic sheaves and new examples of hyperkaehler manifolds,* Hyper-Kähler manifolds, by D. Kaledin, and M. Verbitsky, Mathematical Physics (Somerville), 12, International Press, Somerville, MA, 1999.

[29] J. Wierzba and J. Wiśniewski, *Small contractions of symplectic* 4*-folds,* Duke Math. J. 120 (2003), no. 1, 65–95.

[30] K. Yoshioka, *Stability and the Fourier–Mukai transform. II,* Compos. Math. 145 (2009), no. 1, 112–142.

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