

# ALGEBRAIC CYCLES AND HITCHIN SYSTEMS

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ABSTRACT. The purpose of this paper is to study motivic aspects of the Hitchin system for  $GL_n$ . Our results include the following. (a) We prove the motivic decomposition conjecture of Corti–Hanamura for the Hitchin system; in particular, the decomposition theorem associated with the Hitchin system is induced by algebraic cycles. This yields an unconditional construction of the motivic perverse filtration for the Hitchin system, which lifts the cohomological/sheaf-theoretic perverse filtration. (b) We prove that the inverse of the relative Hard Lefschetz symmetry is induced by a relative algebraic correspondence, confirming the relative Lefschetz standard conjecture for the Hitchin system. (c) We show a strong perversity bound for the normalized Chern classes of a universal bundle with respect to the motivic perverse filtration; this specializes to the sheaf-theoretic result obtained earlier by Maulik–Shen. (d) We prove a  $\chi$ -independence result for the relative Chow motives associated with Hitchin systems.

Our methods combine Fourier transforms for compactified Jacobian fibrations associated with integral locally planar curves, nearby and vanishing cycle techniques, and a Springer-theoretic interpretation of parabolic Hitchin moduli spaces.

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## 0. INTRODUCTION

The topology of the Hitchin system has been intensively studied over decades. In [45], Ngô proved the fundamental lemma of the Langlands program by reducing it to the study of the decomposition theorem [7] for the Hitchin system; the topological mirror symmetry conjecture of Hausel–Thaddeus [26] was proven by comparing the decomposition theorem for Hitchin

systems associated with the Langlands dual groups  $SL_n$  and  $PGL_n$  [23, 35]; the decomposition theorem for the Hitchin system further yields a perverse filtration on the singular cohomology of the Hitchin moduli space, whose interaction with the non-abelian Hodge theory is the main theme of the P=W conjecture [9], recently proven [37, 24, 38]; the decomposition theorem for the Hitchin system also appears naturally in enumerative geometry, particularly in the study of BPS invariants [15, 36, 30, 17].

Most of these developments relied heavily on topological and sheaf-theoretic approaches. On the other hand, hints from several recent results [22, 33, 29, 38] suggest that there should be a motivic theory hidden behind the cohomological structures mentioned above. The purpose of this work is to study motivic aspects of the decomposition theorem and the perverse filtration for the Hitchin system. The upshot is that the geometry of the Hitchin system provides us powerful tools to construct algebraic cycles on the moduli of Higgs bundles, lifting many cohomological statements to the motivic level.

**0.1. Hitchin systems.** We work over the complex numbers  $\mathbb{C}$ . Let  $\Sigma$  be a nonsingular projective curve of genus  $g \geq 2$ . Let  $M_{n,d}$  be the moduli space of stable Higgs bundles with coprime rank  $n$  and degree  $d$  on  $\Sigma$ ; it carries the structure of an integrable system, known as the *Hitchin system*:

$$(1) \quad f : M_{n,d} \rightarrow B,$$

which is a proper and surjective morphism to an affine space  $B$ . We use  $R_f$  to denote the relative dimension of (1). By the decomposition theorem of Beilinson, Bernstein, Deligne, and Gabber [7], the derived pushforward  $Rf_*\mathbb{Q}_{M_{n,d}} \in D_c^b(B)$  admits a decomposition into (shifted) semisimple perverse sheaves:

$$(2) \quad Rf_*\mathbb{Q}_{M_{n,d}}[\dim M_{n,d} - R_f] \simeq \bigoplus_{i=0}^{2R_f} {}^p\mathcal{H}^i \left( Rf_*\mathbb{Q}_{M_{n,d}}[\dim M_{n,d} - R_f] \right) [-i] \in D_c^b(B)$$

where the semisimple summands on the right-hand side satisfy the Lefschetz symmetry

$$(3) \quad \sigma^i : {}^p\mathcal{H}^{R_f-i} \left( Rf_*\mathbb{Q}_{M_{n,d}}[\dim M_{n,d} - R_f] \right) \xrightarrow{\simeq} {}^p\mathcal{H}^{R_f+i} \left( Rf_*\mathbb{Q}_{M_{n,d}}[\dim M_{n,d} - R_f] \right)$$

induced by cup-product with  $i$ -th power of a relative ample class  $\sigma$ . The decomposition (2) is an important invariant for the Hitchin system (1) and has played a crucial role in all the stories mentioned above. For example, recent resolutions of the P=W conjecture show that the (cohomological) perverse filtration

$$(4) \quad P_0H^*(M_{n,d}, \mathbb{Q}) \subset P_1H^*(M_{n,d}, \mathbb{Q}) \subset \cdots \subset H^*(M_{n,d}, \mathbb{Q})$$

induced by (2) matches the double-indexed weight filtration associated with the corresponding character variety via non-abelian Hodge theory:

$$P_kH^*(M_{n,d}, \mathbb{Q}) = W_{2k}H^*(M_{\text{char}}, \mathbb{Q}).$$

Furthermore, the Lefschetz symmetry (3) matches the *curious Hard Lefschetz* for the character variety [25, 40].

Our first result is that both the decomposition (2) and the filtration (4) are motivic in a strong sense. In particular, this confirms the *motivic decomposition conjecture* of Corti–Hanamura [16] for the Hitchin system.

**Theorem 0.1** (Motivic decomposition). *There exists a decomposition of the relative diagonal into orthogonal projectors:*

$$[\Delta_{M_{n,d}/B}] = \sum_{i=0}^{2R_f} \mathfrak{r}_i, \quad \mathfrak{r}_i \circ \mathfrak{r}_i = \mathfrak{r}_i, \quad \mathfrak{r}_i \circ \mathfrak{r}_j = 0, \quad i \neq j,$$

which induces a sheaf-theoretic decomposition (2). Here  $[\Delta_{M_{n,d}/B}]$  and  $\mathfrak{r}_i$  lie in the Chow group of the relative product  $M_{n,d} \times_B M_{n,d}$ , and the composition is taken as relative correspondences.

Our method also yields immediately the relative Lefschetz standard conjecture for the Hitchin system, which shows the algebraicity of the inverse of (3).

**Theorem 0.2** (Relative Lefschetz). *For any relative ample class  $\sigma$  and  $i > 0$ , the inverse of the Lefschetz symmetry (3) is induced by an algebraic cycle  $\mathfrak{Z}_{\sigma,i}$  on  $M_{n,d} \times_B M_{n,d}$ :*

$$\mathfrak{Z}_{\sigma,i} : {}^p\mathcal{H}^{R_f+i} \left( Rf_* \mathbb{Q}_{M_{n,d}}[\dim M_{n,d} - R_f] \right) \xrightarrow{\simeq} {}^p\mathcal{H}^{R_f-i} \left( Rf_* \mathbb{Q}_{M_{n,d}}[\dim M_{n,d} - R_f] \right).$$

Under the language of [16], we obtain from Theorem 0.1 a decomposition in the category of relative Chow motives over  $B$ :

$$(5) \quad h(M_{n,d}) = \bigoplus_i h_i(M_{n,d}) \in \text{CHM}(B)$$

with

$$h(M_{n,d}) = (M_{n,d}, [\Delta_{M_{n,d}/B}], 0), \quad h_i(M_{n,d}) = (M_{n,d}, \mathfrak{r}_i, 0)$$

whose homological realization under the Corti–Hanamura functor

$$\text{CHM}(B) \rightarrow D_c^b(B)$$

recovers (2). As a consequence, we obtain a motivic perverse filtration

$$(6) \quad P_\bullet h(M_{n,d}) \subset h(M_{n,d});$$

see Section 2.1. This motivic perverse filtration specializes to both the cohomological perverse filtration (4) and a perverse filtration on the Chow groups of  $M_{n,d}$ ; the latter has not yet been much studied.

**0.2. Motivic strong perversity of Chern classes.** Let  $\mathcal{U}$  be a universal Higgs bundle over  $\Sigma \times M_{n,d}$ . After normalization, it induces a canonical Chern character

$$\widetilde{\text{ch}}(\mathcal{U}) := \sum_k \widetilde{\text{ch}}_k(\mathcal{U}) \in \text{CH}^*(\Sigma \times M_{n,d}), \quad \widetilde{\text{ch}}_k(\mathcal{U}) \in \text{CH}^k(\Sigma \times M_{n,d})$$

which is independent on the choice of the universal bundle  $\mathcal{U}$ . It was proven in [37] that the cohomology class  $\widetilde{\text{ch}}_k(\mathcal{U})$  has strong perversity  $k$ , *i.e.*, its action on the direct image complex

$$\widetilde{\text{ch}}_k(\mathcal{U}) : Rf_*\mathbb{Q}_{\Sigma \times M_{n,d}} \rightarrow Rf_*\mathbb{Q}_{\Sigma \times M_{n,d}}[2k]$$

via cup-product satisfies that

$$(7) \quad \widetilde{\text{ch}}_k(\mathcal{U}) : {}^{\mathbf{P}}\tau_{\leq i} Rf_*\mathbb{Q}_{\Sigma \times M_{n,d}} \rightarrow ({}^{\mathbf{P}}\tau_{\leq i+k} Rf_*\mathbb{Q}_{\Sigma \times M_{n,d}})[2k].$$

Here, we also use  $f$  to denote the product morphism  $\Sigma \times M_{n,d} \rightarrow \Sigma \times B$  and consider the corresponding perverse filtration. Specializing to global cohomology, the equation (7) yields

$$\widetilde{\text{ch}}_k(\mathcal{U}) : P_i H^*(\Sigma \times M_{n,d}, \mathbb{Q}) \rightarrow P_{i+k} H^{*+2k}(\Sigma \times M_{n,d}, \mathbb{Q}),$$

which further implies the P=W conjecture; we refer to [37, Section 2] for details on the reduction.

In this paper, we show that the sheaf-theoretic strong perversity (7) for Chern classes can be lifted motivically using the motivic perverse filtration (6).

**Theorem 0.3** (Motivic strong perversity of Chern classes). *We have for any  $i, k \in \mathbb{Z}$ :*

$$\widetilde{\text{ch}}_k(\mathcal{U}) : P_i h(\Sigma \times M_{n,d}) \rightarrow P_{i+k} h(\Sigma \times M_{n,d})(k) \in \text{CHM}(\Sigma \times B).$$

*Here the action is given by cup-product and  $(k)$  stands for the  $k$ -th Tate twist.*

**0.3. Motivic  $\chi$ -independence.** Now we consider two degrees  $d, d'$  which are both coprime to  $n$ . The Hitchin systems associated with  $M_{n,d}, M_{n,d'}$  share the same base,

$$f_d : M_{n,d} \rightarrow B, \quad f_{d'} : M_{n,d'} \rightarrow B.$$

Here we use  $f_\bullet$  to indicate the dependence on the degree.

The following result was predicted by the Hausel–Thaddeus conjecture on the topological mirror symmetry [26], and was first proven by Groechenig–Wyss–Ziegler [23] for perverse sheaves via point counting over finite fields and Maulik–Shen [36] for Hodge modules using support theorem and the vanishing cycle functor:

$$(8) \quad Rf_{d*}\mathbb{Q}_{M_{n,d}} \simeq Rf_{d'*}\mathbb{Q}_{M_{n,d'}}.$$

This equation is an instance of the  $\chi$ -independence phenomenon: the moduli space of Higgs bundles can be viewed as the moduli space of 1-dimensional stable sheaves properly supported on the surface  $T^*\Sigma$  in the curve class  $n[\Sigma]$ , and (8) implies that the decomposition theorem is independent on the choice of the Euler characteristic  $\chi (= \deg + (1 - g)n)$  of the sheaves. This phenomenon has tight connections to enumerative geometry [46], and has been generalized to singular moduli spaces using cohomological Donaldson–Thomas theory [36, 30]. Recently,

Hoskins and Pepin Lehalleur [29] proved the  $\chi$ -independence of the Chow motives of the moduli of Higgs bundles:

$$(9) \quad h(M_{n,d}) = h(M_{n,d'}) \in \text{CHM}(\text{pt}).$$

We prove a motivic  $\chi$ -independence result for the relative Chow motives over  $B$ , which enhances both the sheaf-theoretic result (8) and the motivic result (9).

**Theorem 0.4** (Motivic  $\chi$ -independence). *Assume  $(n, d) = (n, d') = 1$ , we have an isomorphism of relative Chow motives:*

$$(10) \quad h(M_{n,d}) \simeq h(M_{n,d'}) \in \text{CHM}(B)$$

*preserving the motivic perverse filtrations*

$$P_k h(M_{n,d}) \simeq P_k h(M_{n,d'}) \in \text{CHM}(B).$$

*Remark 0.5.* (a) To the best of our knowledge, the isomorphism of the relative Chow motives (10) (even without the motivic perverse filtrations) was not known before. After pushing to a point, this recovers the isomorphism (9) of Hoskins and Pepin Lehalleur.

(b) By Proposition 2.4, we obtain that  $f_d : M_{n,d} \rightarrow B$  and  $f_{d'} : M_{n,d'} \rightarrow B$  have isomorphic motivic decompositions; *i.e.*, there exists motivic decompositions

$$h(M_{n,d}) = \bigoplus_i h_i(M_{n,d}), \quad h(M_{n,d'}) = \bigoplus_i h_i(M_{n,d'})$$

satisfying that

$$h_i(M_{n,d}) \simeq h_i(M_{n,d'}) \in \text{CHM}(B).$$

**0.4. Background and relations to other work.** In this section, we discuss briefly some background and relations to some previous work.

**0.4.1. Motivic decomposition conjecture.** Let  $f : X \rightarrow B$  be a proper morphism of nonsingular varieties. The motivic decomposition conjecture of Corti–Hanamura [16] predicts that in general the sheaf-theoretic decomposition

$$Rf_* \mathbb{Q}_X[\dim X - R_f] \simeq \bigoplus_{i=0}^{2R_f} {}^p\mathcal{H}^i(Rf_* \mathbb{Q}_X[\dim X - R_f])[-i], \quad R_f := \dim X \times_B X - \dim X$$

admits a *motivic* lifting. More precisely, as in the statement of Theorem 0.1, they conjectured that there is a decomposition of the relative diagonal cycle into orthogonal projectors

$$[\Delta_{X/B}] = \sum_{i=0}^{2R_f} \mathfrak{r}_i, \quad \mathfrak{r}_i \circ \mathfrak{r}_i = \mathfrak{r}_i, \quad \mathfrak{r}_i \circ \mathfrak{r}_j = 0, \quad i \neq j,$$

which induces a sheaf-theoretic decomposition as above. In general, the conjectural existence of the motivic decomposition is wide open. We note two major challenges in constructing such a decomposition. First, in the special case when  $X$  is a nonsingular projective variety

and  $B$  is a point, the motivic decomposition conjecture is equivalent to the existence of a Chow–Künneth decomposition for  $X$ . This was conjectured by Murre [44], and has only been verified in very limited cases. Second, even if we are able to find orthogonal projectors over a Zariski dense open subset  $U \subset B$  (e.g. the locus where  $f$  is smooth), it is unclear if these projectors over  $U$  can be extended to orthogonal projectors over  $B$ . In particular, the cycles given by the Zariski closures of the orthogonal projectors over  $U$  are not orthogonal projectors in general, and understanding the behaviour of the “boundary” is crucial.

Recently there have been some attempts to apply the theory of motivic intermediate extension of Wildeshaus to construct motivic decompositions for certain proper morphisms with *Tate fibers*; we refer to [13] and references therein for more details.

0.4.2. *Derived equivalences.* For a certain class of examples, one approach to the motivic decomposition conjecture is to use symmetries of the derived category of coherent sheaves. It was first found by Beauville [5] and Deninger–Murre [18] over 35 years ago that descending Mukai’s Fourier transform [43] from the derived category of coherent sheaves to algebraic cycles yields a motivic decomposition for an abelian scheme  $f : A \rightarrow B$ . Later, the Fourier transform was further used by Künneman [31] to construct a Lefschetz decomposition. In this case the morphism  $f$  is smooth and each summand appeared in the decomposition theorem is a (shifted) semisimple local system.

The case of abelian fibrations with singular fibers is more complicated, and the perverse  $t$ -structure comes into play. On the derived category side, Arinkin [3] extended the Fourier–Mukai transform to compactified Jacobian fibrations associated with a family of projective integral locally planar curves. Arinkin’s work was motivated by an attempt to understanding the classical limit of the geometric Langlands correspondence for Hitchin systems [3, Section 4.5]. Recently, motivated by the P=W conjecture for Hitchin moduli spaces [9, 12, 37, 24], a theory of Fourier transform for relative Chow motives was established in [38] using the Arinkin sheaf. As a consequence of the Fourier theory and Ngô’s support theorem [45], it was obtained in [38] that the motivic decomposition conjecture holds for compactified Jacobians fibrations associated with projective integral locally planar curves. We note that, for the singular fibers, this approach matches the complexity of the Arinkin sheaf with the complexity of intermediate extensions of the local systems.

0.4.3. *Hitchin systems.* For the Hitchin system (1), the techniques described above imply the motivic decomposition conjecture for the Hitchin system over a Zariski open subset  $B^{\text{ell}} \subset B$ , called *the elliptic locus*, formed by integral spectral curves. This open subset is exactly the locus over which the Hitchin fibers are integral. However, to the best of our knowledge, it is not yet known how to extend Arinkin’s derived equivalence [3] from the elliptic locus to the total Hitchin base  $B$ . Even worse, the full support theorem *fails* for (1); the results of [10] suggest that the supports of the decomposition theorem associated with (1) are very complicated and there is no complete description by far. In view of the approach of [1], the

lack of the control of the supports provides also the difficulty in proving the relative Lefschetz standard conjecture. The main point of this paper is to give a proof of these conjectures for the Hitchin system (1) in the absence of the aforementioned geometric ingredients. As a byproduct, we show that Corti–Hanamura’s motivic decomposition conjecture is preserved under specialization; see Corollary 4.2.

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## 1. MODULI OF HIGGS BUNDLES

In this section, we review the geometry of the moduli of Higgs bundles. Throughout, we fix a nonsingular projective curve  $\Sigma$  of genus  $g \geq 2$ , and fix two integers  $n, d$  with  $(n, d) = 1$ .

**1.1. Moduli spaces.** We denote by  $M_{n,d}$  the moduli space of stable Higgs bundles

$$(\mathcal{E}, \theta), \quad \theta : \mathcal{E} \rightarrow \mathcal{E} \otimes \omega_\Sigma, \quad \text{rk}(\mathcal{E}) = n, \quad \text{deg}(\mathcal{E}) = d,$$

where the stability condition is with respect to the slope  $\mu(\mathcal{E}, \theta) = \text{deg}(\mathcal{E})/\text{rk}(\mathcal{E})$ . The moduli space admits a proper morphism to an affine space

$$(11) \quad f : M_{n,d} \rightarrow B := \bigoplus_{i=1}^n H^0(\Sigma, \omega_\Sigma^{\otimes i}).$$

This map is given by calculating the characteristic polynomial of the Higgs field  $\theta$ :

$$f(\mathcal{E}, \theta) = (\text{tr}(\theta), \text{tr}(\wedge^2 \theta), \dots, \det(\theta)) \in B,$$

and is now known as the *Hitchin system*.<sup>1</sup> The moduli space  $M_{n,d}$  is a holomorphic symplectic variety and the Hitchin system  $f$  is a Lagrangian fibration [27, 28].

By the Beauville–Narasimhan–Ramanan correspondence [6], the Hitchin base  $B$  can be viewed as the parameter space of spectral curves; these are properly supported curves in the open surface  $\text{Tot}_\Sigma(\omega_\Sigma)$  that are degree  $n$  covers of the 0-section  $\Sigma$ . We define the elliptic locus  $B^{\text{ell}} \subset B$  to be the open subset formed by integral spectral curves. The results of [38, Theorem 0.3 and Section 4] thus establish the motivic decomposition for the restricted Hitchin system

$$f^{\text{ell}} : M_{n,d}^{\text{ell}} \rightarrow B^{\text{ell}}.$$

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<sup>1</sup>Later, when we need to consider the moduli spaces  $M_{n,d}$  with different  $d$ , we will write  $f_d$  to indicate the dependence of the Hitchin system on the degree.

As we commented in Section 0.4.2, at the level of algebraic cycles it is not easy to extend orthogonal projectors from  $B^{\text{ell}}$  to  $B$ . A plausible approach is to extend first Arinkin’s Fourier–Mukai transform from  $M_{n,d}^{\text{ell}}$  to the total space  $M_{n,d}$ , and then extend the Fourier theory of relative Chow motives of [38] as well. Even if we are able to achieve this, a potential difficulty is the complexity of the supports (see Remark 1.1 below) which makes it hard to calculate the homological realization of the projectors.

Instead, our approach in this paper is to first construct orthogonal projectors for certain parabolic Higgs moduli space which we review in the following sections; then we show that these projectors induce the desired orthogonal projectors of the original Hitchin system. This path of reduction using parabolic Higgs moduli spaces was applied in proving the P=W conjecture: it was first used in the proof of Hausel–Mellit–Minets–Schiffmann [24], and was later also used in the proof of Maulik–Shen–Yin [38, Section 5.4].

*Remark 1.1.* We can also consider the moduli space of stable  $\mathfrak{D}$ -twisted Higgs bundles

$$(\mathcal{E}, \theta), \quad \theta : \mathcal{E} \rightarrow \mathcal{E} \otimes \omega_{\Sigma}(\mathfrak{D})$$

as in [14] with  $\mathfrak{D} \geq 0$  an effective divisor. The case of  $\mathfrak{D} = 0$  recovers the Hitchin system (11) we considered above. We note that the decomposition theorem for the Hitchin system are very different when  $\mathfrak{D} = 0$  and  $\mathfrak{D} > 0$  respectively. For  $\mathfrak{D} > 0$ , Chaudouard–Laumon [14] showed that every simple summand that appeared in the decomposition theorem has full support  $B$ . This full-support property has been generalized to singular moduli spaces without coprime assumption on  $n$  and  $d$  [36] and has many applications [35, 22, 29, 37]. However, when  $D = 0$ , the behaviour of the supports for the Hitchin system is very different. In [10], the authors classified all the supports over an open subset of the Hitchin base formed by *reduced* spectral curves. However it is not well understood what are all the possible supports outside this open subset. A question raised by Mauri–Migliorini [39, Question 1.5] asked if the full support property holds for the singular moduli space of rank  $n$  and degree  $d = 0$ . In Section 4.3.5, we will discuss briefly how to extend the results of this paper to the  $\mathfrak{D}$ -twisted case.

**1.2. Parabolic Higgs bundles.** In this section, we review the parabolic Higgs moduli space (*c.f.* [24, Section 8.4]).

We fix a point  $p \in \Sigma$ . Denote by  $M^{\text{par}}$  the moduli space of stable parabolic Higgs bundles  $(\mathcal{E}, \theta, F_p)$ , where  $(\mathcal{E}, \theta)$  is a meromorphic Higgs bundle

$$(\mathcal{E}, \theta), \quad \theta : \mathcal{E} \rightarrow \mathcal{E} \otimes \omega_{\Sigma}(p), \quad \text{rk}(\mathcal{E}) = n, \quad \text{deg}(\mathcal{E}) = d,$$

$F_p$  is a complete flag of  $\mathcal{E}$  over the point  $p$  such that the residue  $\text{res}_p(\theta)$  of the Higgs field over the point  $p$  preserves the flag  $F_p$ . The stability condition here requires that any parabolic sub-Higgs bundle has a smaller slope.

We record the following lemma which is a direct consequence of the Riemann–Roch formula.

**Lemma 1.2.** *We have*

$$H^0(\Sigma, \omega_{\Sigma}) = H^0(\Sigma, \omega_{\Sigma}(p)).$$

Note that, since  $\text{res}_p(\theta)$  preserves a complete flag, its eigenvalues naturally lie in an affine space

$$\mathbb{A}^n := \{(\lambda_1, \dots, \lambda_n) \mid \lambda_i \in \mathbb{A}^1\},$$

and Lemma 1.2 ensures that the sum of the eigenvalues is always 0; in other words, the image of  $\chi_p$  lies in a codimension 1 hyperplane

$$\mathcal{A} = \left\{ (\lambda_1, \dots, \lambda_n) \mid \sum_i \lambda_i = 0 \right\} \subset \mathbb{A}^n.$$

As a result, the moduli space admits a natural morphism

$$(12) \quad \chi_p : M^{\text{par}} \rightarrow \mathcal{A} \simeq \mathbb{A}^{n-1}$$

which is in fact smooth. Each closed fiber is a moduli of stable parabolic Higgs bundles with specified eigenvalues at  $p$ ; this is a nonsingular symplectic variety.

For our purposes, we take a general line  $T (\simeq \mathbb{A}^1) \subset \mathcal{A}$  passing through the origin and restrict the family (12) over  $T$ :

$$\chi_p : M_T^{\text{par}} \rightarrow T.$$

This family connects two types of Hitchin moduli spaces which we now describe:

- (a) Let  $\eta \in T$  be a general point which represents  $n$  *distinct* eigenvalues satisfying the condition that no proper subset of the eigenvalues has sum 0. We denote by  $W_\eta$  the linear system parameterizing spectral curves with fixed eigenvalues  $\eta$  over  $p \in \Sigma$ . Let  $C \rightarrow W_\eta$  be the universal spectral curve. By the generality of  $\eta$  and Lemma 1.2, every curve in the family  $C \rightarrow W_\eta$  is integral. The fiber  $M_\eta^{\text{par}} := \chi_p^{-1}(\eta)$  fits into the following Cartesian diagram:

$$\begin{array}{ccc} M_\eta^{\text{par}} & \longrightarrow & \overline{J}_C^e \\ \downarrow & & \downarrow \\ \widetilde{W}_\eta & \longrightarrow & W_\eta. \end{array}$$

Here the right vertical arrow is the (twisted) compactified Jacobian fibration of degree  $e$ ,  $\widetilde{W}_\eta$  can be viewed as the parameter space of spectral curves lying in  $W_\eta$  with a marking over  $p \in \Sigma$ , and the bottom horizontal arrow is the projection to  $W_\eta$  which is a natural  $\mathfrak{S}_n$ -quotient.<sup>2</sup> The Hitchin fibration associated with  $M_\eta^{\text{par}}$  is the composition of a vertical and a horizontal maps in the diagram above

$$h_\eta : M_\eta^{\text{par}} \rightarrow W_\eta.$$

Let  $T^\circ \subset T$  be the Zariski open subset such that the spectral curves in  $W_\eta$  are integral for any  $\eta \in T^\circ$ . We thus obtain a morphism

$$h^\circ : M^{\text{par}, \circ} \rightarrow W^\circ$$

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<sup>2</sup>Here  $e$  is an integer determined by the degree  $d$  of the Higgs bundles and is coprime to  $n$ .

satisfying that  $M^{\text{par},\circ}$  is a compactified Jacobian fibration over a base  $\widetilde{W}^\circ$ , and  $W^\circ$  is the quotient of  $\widetilde{W}^\circ$  by the permutation group  $\mathfrak{S}_n$ .

- (b) The central fiber  $M_0^{\text{par}} := \chi_p^{-1}(0)$  together with its associated Hitchin system

$$h_0 : M_0^{\text{par}} \rightarrow W_0$$

is a classical object, known as the moduli of stable strongly parabolic Higgs bundles. The variety  $M_0^{\text{par}}$  parameterizes triples  $(\mathcal{E}, \theta, F_p)$  with similar condition as for stable parabolic Higgs bundle, but the main change is that the residue of the Higgs field preserves the flag in a stronger sense:

$$\text{res}_p(\theta) : F_{p,j} \rightarrow F_{p,j+1}.$$

In other words,  $M_0^{\text{par}}$  is the moduli of stable parabolic Higgs bundles with nilpotent residue.

In conclusion, by specializing over  $T$ , we are able to connect the geometry of a compactified Jacobian fibration to the Hitchin system associated with the moduli of strongly parabolic Higgs bundles:

$$h^\circ : M^{\text{par},\circ} \rightarrow W^\circ \rightsquigarrow h_0 : M_0^{\text{par}} \rightarrow W_0.$$

We conclude this section by further connecting the latter to the more classical Hitchin system  $f : M_{n,d} \rightarrow B$  via a correspondence following [24, Section 8.6].

**1.3. Correspondences.** We consider the closed subvariety parameterizing parabolic Higgs bundles with 0-residue at  $p$ ,

$$(13) \quad \widetilde{M}_0 = \{\text{res}_p(\theta) = 0\} \subset M_0^{\text{par}} = \{\text{res}_p(\theta) \text{ is nilpotent}\}.$$

In particular, every parabolic Higgs bundle in  $\widetilde{M}_0$  has no pole at  $p \in \Sigma$ . Hence there is a well-defined forgetful map:

$$(14) \quad \widetilde{M}_0 \rightarrow M_{n,d}, \quad (\mathcal{E}, \theta, F_p) \mapsto (\mathcal{E}, \theta)$$

where every closed fiber is isomorphic to a flag variety  $\text{Fl}_n$  parameterizing all possible choices of flags.

We summarize the discussion above in the diagram

$$(15) \quad \begin{array}{ccc} \widetilde{M}_0 & \xrightarrow{\iota} & M_0^{\text{par}} \\ \downarrow \pi & & \\ M_{n,d} & & \end{array}$$

where  $\iota$  is the closed embedding (13) and  $\pi$  is the smooth projection (14). All the three moduli spaces in (15) map to a common base  $W_0$  with all the induced natural diagrams commuting: the map from  $M_0^{\text{par}}$  is the parabolic Hitchin map  $h_0$ , the map from  $\widetilde{M}_0$  is the induced map, and the map from  $M_{n,d}$  is the Hitchin map  $f : M_{n,d} \rightarrow B$  composed with the natural inclusion of the linear subspace  $B \subset W_0$ .

Our purpose is to construct certain algebraic cycles on  $M_{n,d} \times_B M_{n,d}$  which serve as orthogonal projectors. The diagram (15) allows us to construct cycles for  $M_{n,d}$  from  $M_0^{\text{par}}$ , *i.e.*, we have

$$\Gamma : \text{CH}_*(M_0^{\text{par}} \times_{W_0} M_0^{\text{par}}) \rightarrow \text{CH}_*(M_{n,d} \times_B M_{n,d})$$

given by the formula

$$\Gamma(\alpha) := {}^t[\widetilde{M}_0] \circ \alpha \circ [\widetilde{M}_0].$$

Throughout, the Chow groups are taken with rational coefficients, and here we view  $[\widetilde{M}_0]$  to be a correspondence from  $M_{n,d}$  to  $M_0^{\text{par}}$  via (15) and  ${}^t[\widetilde{M}_0]$  to be its transpose. The change of indices of the Chow groups through  $\Gamma$  is given by the dimension formula:

$$\text{codim}(\iota) = \text{rel.dim}(\pi) = \binom{n}{2};$$

in particular,  $\Gamma$  sends  $\text{CH}_{\dim M_0^{\text{par}}}(M_0^{\text{par}} \times_{W_0} M_0^{\text{par}})$  to  $\text{CH}_{\dim M_{n,d}}(M_{n,d} \times_B M_{n,d})$ .

The following lemma gives a sufficient condition for  $\Gamma(\alpha)$  to vanish.

**Lemma 1.3.** *Suppose we are given a flat, surjective morphism  $p : Z \rightarrow W_0$  with  $Z$  irreducible and nonsingular and that there exists a factorization of  $\alpha$*

$$\alpha := \alpha_2 \circ \alpha_1 \in \text{CH}_{\dim M_0^{\text{par}}}(M_0^{\text{par}} \times_{W_0} M_0^{\text{par}})$$

such that either

$$\alpha_1 \in \text{CH}_{>\dim(M_0^{\text{par}} \times_{W_0} Z) - \binom{n}{2}}(M_0^{\text{par}} \times_{W_0} Z) \text{ or } \alpha_2 \in \text{CH}_{>\dim(M_0^{\text{par}} \times_{W_0} Z) - \binom{n}{2}}(Z \times_{W_0} M_0^{\text{par}}).$$

Then we have

$$\Gamma(\alpha) = 0 \in \text{CH}_{\dim M_{n,d}}(M_{n,d} \times_{W_0} M_{n,d}) = \text{CH}_{\dim M_{n,d}}(M_{n,d} \times_B M_{n,d}).$$

*Proof.* We prove the vanishing under the first hypothesis; the argument under the second hypothesis is identical. If we consider the composition  $\alpha_1 \circ [\widetilde{M}_0] \in \text{CH}(M_{n,d} \times_{W_0} Z)$ , the assumption on  $\alpha_1$  implies that this lives in degree strictly larger than

$$\dim(M_0^{\text{par}} \times_{W_0} Z) - 2 \binom{n}{2}.$$

The assumption on  $p$  implies that this bound equals

$$\dim M_0^{\text{par}} + \dim Z - \dim W_0 - 2 \binom{n}{2} = \dim M_{n,d} + \dim Z - \dim W_0$$

which is the dimension of  $M_{n,d} \times_{W_0} Z$ . As a result, we have the vanishing of  $\alpha_1 \circ [\widetilde{M}_0]$  so  $\Gamma(\alpha)$  vanishes as well.  $\square$

We conclude this section by giving an overview of the rest of the paper. Our strategy is to first construct projectors for compactified Jacobian fibrations and show that these algebraic cycles satisfy the desired properties as in Theorems 0.3 and 0.4 using the techniques of [38]. We will discuss this in Section 2, which solves the problems for  $h_\eta : M_\eta^{\text{par}} \rightarrow W_\eta$  of Section 1.2(a). Then applying the specialization and the correspondence  $\Gamma$ , we obtain algebraic cycles for  $M_{n,d}$ ; we further apply a Springer theory argument to show that these cycles form orthogonal projectors as desired. This is treated in Section 3. Finally, in Section 4, we show that the projectors obtained via  $\Gamma$  have the correct homological realization under the Corti–Hanamura functor.

## 2. PROJECTORS FOR COMPACTIFIED JACOBIANS

In this section, we first introduce the notion of motivic perverse filtrations. Then we treat the case of compactified Jacobians associated with locally planar curves, where our main tool is the Fourier theory developed in [38]. For our purpose, we need a slight extension of the results of [38] in Section 2.3. Then we apply these results to the parabolic Hitchin system of Section 1.2(a), which further allows us to prove analogues of Theorems 0.1, 0.3, and 0.4 for  $h_0 : M_0^{\text{par}} \rightarrow W_0$ .

**2.1. Motivic perverse filtrations.** In this section, we define a *motivic perverse filtration*, which lifts its cohomological counterpart to the theory of relative Chow motives of Corti–Hanamura.

The theory of relative Chow motives [16] is built to be compatible with the decomposition theorem, adapts well to non-proper bases, and admits natural Chow/homological realizations [21]. We refer to [38, Section 2.2] for a brief review of the theory.

We work over a nonsingular base variety  $B$ . Recall that the group of degree  $k$  relative correspondences between two nonsingular proper  $B$ -schemes  $X, Y$  is<sup>3</sup>

$$\text{Corr}_B^k(X, Y) := \text{CH}_{\dim Y - k}(X \times_B Y).$$

Compositions of relative correspondences are defined via refined intersection theory. The category of relative Chow motives  $\text{CHM}(B)$  consists of objects triples  $(X, \mathfrak{p}, m)$  where  $X$  is a nonsingular proper  $B$ -scheme,  $\mathfrak{p} \in \text{Corr}_B^0(X, X)$  is a projector, and  $m \in \mathbb{Z}$ . For example, the motive of  $X$  is

$$h(X) := (X, [\Delta_{X/B}], 0)$$

where  $\Delta_{X/B}$  is the relative diagonal. Morphisms between two motives  $M = (X, \mathfrak{p}, m)$ ,  $N = (Y, \mathfrak{q}, n)$  are

$$\text{Hom}_{\text{CHM}(B)}(M, N) := \mathfrak{q} \circ \text{Corr}_B^{n-m}(X, Y) \circ \mathfrak{p}.$$

---

<sup>3</sup>Here we assume  $Y$  irreducible. For reducible  $Y$ , we decompose  $Y = \sqcup_\alpha Y_\alpha$  and define the group of correspondences accordingly.

Note that in [38, Section 2.3.3], we also introduced relative binary correspondences

$$\mathrm{Corr}_B^k(X, Y; Z) := \mathrm{CH}_{\dim Z - k}(X \times_B Y \times_B Z)$$

where  $X, Y, Z$  are nonsingular proper  $B$ -schemes. Binary morphisms from  $M = (X, \mathfrak{p}, m)$ ,  $N = (Y, \mathfrak{q}, n)$  to  $P = (Z, \mathfrak{r}, p)$  are given by

$$\mathfrak{r} \circ \mathrm{Corr}_B^{p-m-n}(X, Y; Z) \circ (\mathfrak{p} \times \mathfrak{q}).$$

For example, the motivic cup-product

$$\cup : h(X) \times h(X) \rightarrow h(X)$$

is induced by the class of the relative small diagonal

$$[\Delta_{X/B}^{\mathrm{sm}}] \in \mathrm{Corr}_B^0(X, X; X);$$

see [38, Section 2.3.3] for more details. Finally, the degree  $k$  Chow group of  $M = (X, \mathfrak{p}, m)$  is

$$\mathrm{CH}^k(M) := \mathrm{Hom}_{\mathrm{CHM}(B)}(h(B/B), M(k))$$

where  $M(k) := (X, \mathfrak{p}, m + k)$  is the  $k$ -th Tate twist of  $M$ .

Recall the Corti–Hanamura realization functor

$$(16) \quad \mathrm{CHM}(B) \rightarrow D_c^b(B)$$

which sends the motive  $h(X)$  for  $f : X \rightarrow B$  to the direct image complex  $Rf_*\mathbb{Q}_X$ . The latter admits a natural (sheaf-theoretic) perverse filtration induced by the perverse truncation functor.

Let  $f : X \rightarrow B$  be a proper morphism with  $X$  irreducible and nonsingular, and let  $R_f$  be the defect of semismallness  $\dim X \times_B X - \dim X$ .

**Definition 2.1.** We say that a sequence of motives

$$P_k h(X) = (X, \mathfrak{p}_k, 0), \quad 0 \leq k \leq 2R_f,$$

which are all summands of  $h(X)$ , form a *motivic perverse filtration* for  $f$ , if

- (a) (Termination)  $P_{2R_f} h(X) = h(X)$ ;
- (b) (Realization) under the Corti–Hanamura functor (16) the natural inclusion

$$P_\bullet h(X) \hookrightarrow h(X)$$

specializes to the natural inclusion induced by the perverse truncation functor

$${}^p\tau_{\leq \bullet + (\dim X - R_f)} Rf_*\mathbb{Q}_X \hookrightarrow Rf_*\mathbb{Q}_X;$$

- (c) (Semi-orthogonality) for any  $k$  we have the relation

$$\mathfrak{p}_{k+1} \circ \mathfrak{p}_k = \mathfrak{p}_k.$$

We note that (c) together with an easy induction implies

$$(17) \quad \mathfrak{p}_l \circ \mathfrak{p}_k = \mathfrak{p}_k, \quad k < l.$$

Indeed, we have

$$\mathfrak{p}_l \circ \mathfrak{p}_k = \mathfrak{p}_l \circ \mathfrak{p}_{k+1} \circ \mathfrak{p}_k = \mathfrak{p}_l \circ \mathfrak{p}_{l-1} \circ \cdots \circ \mathfrak{p}_{k+1} \circ \mathfrak{p}_k = \mathfrak{p}_{l-1} \circ \cdots \circ \mathfrak{p}_{k+1} \circ \mathfrak{p}_k = \mathfrak{p}_k.$$

On the other hand, this does not mean that  $P_k h(X)$  is a summand of  $P_l h(X)$  if  $k < l$ . So if we define directly

$$\mathfrak{r}_k := \mathfrak{p}_k - \mathfrak{p}_{k-1},$$

they are only semi-orthogonal:

$$\mathfrak{r}_l \circ \mathfrak{r}_k = 0, \quad k < l,$$

and we lose control of the other half of the orthogonality

$$\mathfrak{r}_l \circ \mathfrak{r}_k \stackrel{?}{=} 0, \quad k \geq l.$$

In particular, the  $\mathfrak{r}_k$  are not necessarily projectors. However, the following proposition in [38] shows that the existence of a motivic perverse filtration is equivalent to the existence of a motivic decomposition.

**Proposition 2.2.** *[Motivic perverse filtration = motivic decomposition] A motivic perverse filtration induces a motivic decomposition*

$$h(X) = \bigoplus_{i=0}^{2R_f} h_i(X) \in \text{CHM}(B).$$

*Proof.* The proof was given in the [38, Section 2.5.5]; we include it here for completeness. Assume that the motives  $P_k h(X) = (X, \mathfrak{p}_k, 0)$  induce a motivic perverse filtration. We consider the modification of projectors:

$$\tilde{\mathfrak{p}}_k := \mathfrak{p}_k \circ \cdots \circ \mathfrak{p}_{2R_f}, \quad \bar{\mathfrak{p}}_k := \tilde{\mathfrak{p}}_k - \tilde{\mathfrak{p}}_{k-1}.$$

Using (17), it is straightforward to verify that both  $\tilde{\mathfrak{p}}_k$  and  $\bar{\mathfrak{p}}_k$  are projectors satisfying the relations

$$\tilde{\mathfrak{p}}_l \circ \tilde{\mathfrak{p}}_k = \tilde{\mathfrak{p}}_k \circ \tilde{\mathfrak{p}}_l = \tilde{\mathfrak{p}}_k, \quad k < l.$$

In particular, we have

$$\tilde{P}_k h(X) = \tilde{P}_{k-1} h(X) \oplus h_k(X) \in \text{CHM}(B)$$

with

$$\tilde{P}_k h(X) := (X, \tilde{\mathfrak{p}}_k, 0), \quad h_k(X) := (X, \bar{\mathfrak{p}}_k, 0).$$

We see from (b) that both motives  $\tilde{P}_k h(X)$  and  $P_k h(X)$  have the same realization under (16), which is the one given by the perverse truncation functor. In particular, the projector  $\bar{\mathfrak{p}}_k$  realizes taking the corresponding perverse cohomology of the complex  $Rf_* \mathbb{Q}_X$ . This finishes the proof of the proposition.  $\square$

In the following, we further extend the equivalence between motivic perverse filtrations and motivic decompositions of Proposition 2.2 to isomorphism classes.

Assume that

$$f : X \rightarrow B, \quad f' : X' \rightarrow B$$

admit motivic perverse filtrations

$$(18) \quad P_k h(X) = (X, \mathfrak{p}_k, 0), \quad P_k h(X') = (X', \mathfrak{p}'_k, 0), \quad 0 \leq k \leq 2R_f = 2R_{f'}.$$

**Definition 2.3.** We say that an isomorphism of relative Chow motives

$$\mathfrak{C} : h(X) \xrightarrow{\cong} h(X'), \quad \mathfrak{C}^{-1} : h(X') \xrightarrow{\cong} h(X)$$

with

$$\mathfrak{C} \in \text{Corr}_B^0(X, X'), \quad \mathfrak{C}^{-1} \in \text{Corr}_B^0(X', X)$$

preserves the motivic perverse filtrations (18), if for any  $k$  we have

$$(19) \quad \mathfrak{q}'_{k+1} \circ \mathfrak{C} \circ \mathfrak{p}_k = 0, \quad \mathfrak{q}_{k+1} \circ \mathfrak{C}^{-1} \circ \mathfrak{p}'_k = 0.$$

Here

$$\mathfrak{q}_{k+1} := [\Delta_{X/B}] - \mathfrak{p}_k, \quad \mathfrak{q}'_{k+1} := [\Delta_{X'/B}] - \mathfrak{p}'_k.$$

The following proposition shows that, for two relative Chow motives, an isomorphism preserving motivic filtrations automatically yields isomorphic motivic decompositions as well.

**Proposition 2.4.** *With the notation as above, assume that there is an isomorphism between the relative Chow motives  $h(X)$  and  $h(X')$  preserving the motivic perverse filtrations (18). Then the following holds:*

(a) *For any  $k$  we have an isomorphism*

$$P_k h(X) \simeq P_k h(X') \in \text{CHM}(B).$$

*compatible with the inclusions  $P_k h(X) \hookrightarrow h(X)$  and  $P_k h(X') \hookrightarrow h(X')$ .*

(b) *The motivic decompositions of  $h(X), h(X')$  given by Proposition 2.2 are isomorphic, i.e., for any  $k$  we have an isomorphism*

$$h_k(X) \simeq h_k(X') \in \text{CHM}(B).$$

*Proof.* We first prove (a). Assume that the correspondences  $\mathfrak{C}, \mathfrak{C}^{-1}$  induce an isomorphism between  $h(X), h(X')$  preserving the perverse filtrations. We claim that the morphisms between the relative Chow motives

$$\mathfrak{p}'_k \circ \mathfrak{C} \circ \mathfrak{p}_k : P_k h(X) \rightarrow P_k h(X'), \quad \mathfrak{p}_k \circ \mathfrak{C}^{-1} \circ \mathfrak{p}'_k : P_k h(X') \rightarrow P_k h(X)$$

are inverse to each other. In fact, we have

$$\begin{aligned} \mathfrak{p}_k &= \mathfrak{p}_k \circ \mathfrak{C}^{-1} \circ \mathfrak{C} \circ \mathfrak{p}_k = \mathfrak{p}_k \circ \mathfrak{C}^{-1} \circ (\mathfrak{p}'_k + \mathfrak{q}'_{k+1}) \circ \mathfrak{C} \circ \mathfrak{p}_k \\ &= \mathfrak{p}_k \circ \mathfrak{C}^{-1} \circ \mathfrak{p}'_k \circ \mathfrak{C} \circ \mathfrak{p}_k \end{aligned}$$

where we used  $\mathfrak{q}'_{k+1} \circ \mathfrak{C} \circ \mathfrak{p}_k = 0$  in the last equation. This proves

$$\mathfrak{p}_k \circ \mathfrak{C}^{-1} \circ \mathfrak{p}'_k \circ \mathfrak{C} \circ \mathfrak{p}_k = \mathfrak{p}_k;$$

the other identity is parallel. To show that this isomorphism is compatible with  $\mathfrak{C}$  and the inclusions into  $h(X)$  and  $h(X')$ , it suffices to show that

$$\mathfrak{C} \circ \mathfrak{p}_k = \mathfrak{p}'_k \circ (\mathfrak{p}'_k \circ \mathfrak{C} \circ \mathfrak{p}_k)$$

which again follows via the condition that  $\mathfrak{C}$  preserves the motivic filtration. We have completed the proof of (a).

Now we prove (b); we follow the notation in the proof of Proposition 2.2. As in (a), it suffices to show that

$$(20) \quad (\tilde{\mathfrak{p}}_k - \tilde{\mathfrak{p}}_{k-1}) \circ \mathfrak{C}^{-1} \circ (\tilde{\mathfrak{p}}'_k - \tilde{\mathfrak{p}}'_{k-1}) \circ \mathfrak{C} \circ (\tilde{\mathfrak{p}}_k - \tilde{\mathfrak{p}}_{k-1}) = \tilde{\mathfrak{p}}_k - \tilde{\mathfrak{p}}_{k-1};$$

the other identity follows from a parallel argument. Expanding the left-hand side of (20), we obtain 8 terms, each of which is of the form

$$\pm \tilde{\mathfrak{p}}_{i_1} \circ \mathfrak{C}^{-1} \circ \tilde{\mathfrak{p}}'_{i_2} \circ \mathfrak{C} \circ \tilde{\mathfrak{p}}_{i_3}, \quad i_1, i_2, i_3 \in \{k-1, k\}.$$

We claim that

$$(21) \quad \tilde{\mathfrak{p}}_{i_1} \circ \mathfrak{C}^{-1} \circ \tilde{\mathfrak{p}}'_{i_2} \circ \mathfrak{C} \circ \tilde{\mathfrak{p}}_{i_3} = \tilde{\mathfrak{p}}_{\min\{i_1, i_2, i_3\}}.$$

Since at least two of the three indices  $i_1, i_2, i_3$  are equal, without loss of generality we treat here the case  $i_2 = i_3$ . By (17) and the first equation of (19), we have

$$\mathfrak{C} \circ \mathfrak{p}_k = \mathfrak{p}'_l \circ \mathfrak{C} \circ \mathfrak{p}_k, \quad k \leq l,$$

which further yields

$$\begin{aligned} \tilde{\mathfrak{p}}'_{i_2} \circ \mathfrak{C} \circ \tilde{\mathfrak{p}}_{i_2} &= \mathfrak{p}'_{i_2} \circ \cdots \circ \mathfrak{p}'_{2R_f} \circ \mathfrak{C} \circ \mathfrak{p}_{i_2} \circ \cdots \circ \mathfrak{p}_{2R_f} \\ &= \mathfrak{C} \circ \mathfrak{p}_{i_2} \circ \cdots \circ \mathfrak{p}_{2R_f} \\ &= \mathfrak{C} \circ \tilde{\mathfrak{p}}_{i_2}. \end{aligned}$$

Consequently, (21) follows:

$$\begin{aligned} \tilde{\mathfrak{p}}_{i_1} \circ \mathfrak{C}^{-1} \circ \tilde{\mathfrak{p}}'_{i_2} \circ \mathfrak{C} \circ \tilde{\mathfrak{p}}_{i_2} &= \tilde{\mathfrak{p}}_{i_1} \circ \mathfrak{C}^{-1} \circ \mathfrak{C} \circ \tilde{\mathfrak{p}}_{i_2} \\ &= \tilde{\mathfrak{p}}_{i_1} \circ \tilde{\mathfrak{p}}_{i_2} \\ &= \tilde{\mathfrak{p}}_{\min\{i_1, i_2\}}. \end{aligned}$$

Finally, (20) is deduced immediately from (21), which completes the proof of (b).  $\square$

In many cases, it is more natural to work with a motivic perverse filtration, rather than a motivic decomposition. This is because they carry the same information, but the extra complexity of obtaining the orthogonal projectors from the semi-orthogonal projectors, as in the proof of Proposition 2.2, is formal and not essential. This in particular is the case for

compactified Jacobian fibrations [38, Section 2.5.5] as we will also review below, where the semi-orthogonal projectors are of better form; see Theorem 2.5.

**2.2. Compactified Jacobians.** Throughout this section, we work with the general setting that  $C \rightarrow B$  is a flat family of integral projective curves of arithmetic genus  $g$  with planar singularities over an irreducible base  $B$ ; we will specialize to the case of spectral curves in Section 2.4. We assume that the total space  $C$  is nonsingular with a multi-section

$$D \subset C \rightarrow B$$

of degree  $r$  which is finite and flat over  $B$ . For any integer  $d$ , the degree  $d$  compactified Jacobian exists as a scheme étale locally over the base  $B$  since there is always a section of  $C \rightarrow B$  étale locally, while globally the degree  $d$  compactified Jacobian  $\overline{\mathcal{J}}_C^d$  exists only as an algebraic space. For convenience, we assume that for any degree  $d$ , the compactified Jacobian  $\overline{\mathcal{J}}_C^d$  is a quasi-projective variety. This is, for example, satisfied when  $C \rightarrow B$  arises as a linear system of a nonsingular surface, where the quasi-projectivity follows from the fact that the compactified Jacobian arises as the moduli space of certain semistable torsion sheaves on the surface. We further assume that  $\overline{\mathcal{J}}_C^d$  is nonsingular. In order to work with the universal family, we also need to consider the moduli stack of the degree  $d$  compactified Jacobian, which is a  $\mathbb{G}_m$ -gerbe over  $\overline{\mathcal{J}}_C^d$ . Following [38, Section 4.2], we rigidify this stack by imposing the condition that the universal family is *trivialized* along the multi-section; this gives a Deligne–Mumford stack which is a  $\mu_r$ -gerbe over the scheme:

$$\overline{\mathcal{J}}_C^d \rightarrow \overline{\mathcal{J}}_C^d;$$

see [38, Proposition 4.1]. The rigidification condition we imposed implies that there is a universal sheaf  $\mathcal{F}^d$  over  $C \times_B \overline{\mathcal{J}}_C^d$  satisfying

$$\det \left( p_{J*} \left( \mathcal{F}^d|_{\overline{\mathcal{J}}_C^d \times_C D} \right) \right) \simeq \mathcal{O}_{\overline{\mathcal{J}}_C^d} \in \text{Pic} \left( \overline{\mathcal{J}}_C^d \right).$$

Here  $p_J$  is the projection to compactified Jacobian factor.

Now we consider two integers  $d, e$ . Arinkin’s work [3] shows that the universal sheaves  $\mathcal{F}^d, \mathcal{F}^e$  induce a pair of Fourier–Mukai kernels

$$(22) \quad \mathcal{P}_{e,d} \in D^b \text{Coh} \left( \overline{\mathcal{J}}_C^e \times \overline{\mathcal{J}}_C^d \right), \quad \mathcal{P}_{e,d}^{-1} \in D^b \text{Coh} \left( \overline{\mathcal{J}}_C^d \times \overline{\mathcal{J}}_C^e \right).$$

The associated Fourier–Mukai transforms are equivalences of categories

$$\text{FM}_{\mathcal{P}_{e,d}} : D^b \text{Coh}(\overline{\mathcal{J}}_C^e)_{(-d)} \xrightarrow{\simeq} D^b \text{Coh}(\overline{\mathcal{J}}_C^d)_{(e)}, \quad \text{FM}_{\mathcal{P}_{e,d}^{-1}} = \text{FM}_{\mathcal{P}_{e,d}}^{-1},$$

where  $D^b \text{Coh}(\overline{\mathcal{J}}_C^m)_{(k)}$  is the isotypic category consisting of objects for which the action of  $\mu_r$  on the fibers is given by the character  $\lambda \mapsto \lambda^k$  of  $\mu_r$ . The objects (22) induce (Chow-theoretic)

Fourier transforms

$$\begin{aligned}\mathfrak{F}_{e,d} &:= \mathrm{td} \left( -T_{\overline{\mathcal{J}}_C^e \times_B \overline{\mathcal{J}}_C^d} \right) \cap \tau(\mathcal{P}_{e,d}) \in \mathrm{CH}_*(\overline{\mathcal{J}}_C^e \times_B \overline{\mathcal{J}}_C^d) = \mathrm{CH}_*(\overline{\mathcal{J}}_C^e \times_B \overline{\mathcal{J}}_C^d), \\ \mathfrak{F}_{e,d}^{-1} &:= \mathrm{td}(-(\pi_d \times_B \pi_e)^* T_B) \cap \tau(\mathcal{P}_{e,d}^{-1}) \in \mathrm{CH}_*(\overline{\mathcal{J}}_C^d \times_B \overline{\mathcal{J}}_C^e) = \mathrm{CH}_*(\overline{\mathcal{J}}_C^d \times_B \overline{\mathcal{J}}_C^e).\end{aligned}$$

Here  $\pi_\bullet$  is the natural projection to the base  $B$ ,  $T_{\overline{\mathcal{J}}_C^e \times_B \overline{\mathcal{J}}_C^d}$  is the virtual tangent bundle,  $\tau(-)$  is the tau-class used in the Riemann–Roch theorem of Baum–Fulton–MacPherson [4], and we identify the (rational) Chow groups of the coarse moduli spaces and the gerbes; we refer to [38, Section 4] for more details.

The correspondences  $\mathfrak{F}_{e,d}, \mathfrak{F}_{e,d}^{-1}$  are of mixed degrees, and we have decompositions

$$\begin{aligned}\mathfrak{F}_{e,d} &= \sum_{i \geq 0} (\mathfrak{F}_{e,d})_i, \quad (\mathfrak{F}_{e,d})_i \in \mathrm{Corr}_B^{i-g}(\overline{\mathcal{J}}_C^e, \overline{\mathcal{J}}_C^d) \\ \mathfrak{F}_{e,d}^{-1} &= \sum_{i \geq 0} (\mathfrak{F}_{e,d}^{-1})_i, \quad (\mathfrak{F}_{e,d}^{-1})_i \in \mathrm{Corr}_B^{i-g}(\overline{\mathcal{J}}_C^d, \overline{\mathcal{J}}_C^e).\end{aligned}$$

For a fixed  $d$ , in order to define a motivic perverse filtration associated with  $\pi_d : \overline{\mathcal{J}}_C^d \rightarrow B$  using the Fourier transform, we need to choose a degree  $e$ , whose associated compactified Jacobian fibration will serve as the “dual” side; see Step 1 of [38, Section 4.4]. In this paper, we choose this degree uniformly to be 1; this choice will be important for applying the Abel–Jacobi map  $C \rightarrow \overline{\mathcal{J}}_C^1$  to build a connection between the Fourier transform and the normalized Chern classes of a universal bundle for a Hitchin-type moduli space. We define

$$(23) \quad \mathfrak{p}_k := \sum_{i \leq k} (\mathfrak{F}_{1,d})_i \circ (\mathfrak{F}_{1,d}^{-1})_{2g-i} \in \mathrm{Corr}_B^0(\overline{\mathcal{J}}_C^d, \overline{\mathcal{J}}_C^d).$$

The following theorem proven in [38] confirmed the motivic decomposition conjecture for the compactified Jacobian fibration associated with  $C \rightarrow B$ , with the projectors constructed above.

**Theorem 2.5** ([38, Corollary 4.5(i)]). *For every  $0 \leq k \leq 2g$ , the self-correspondence  $\mathfrak{p}_k$  is a projector for  $\pi_d : \overline{\mathcal{J}}_C^d \rightarrow B$ , and the motives*

$$P_k h(\overline{\mathcal{J}}_C^d) := (\overline{\mathcal{J}}_C^d, \mathfrak{p}_k, 0) \in \mathrm{CHM}(B), \quad 0 \leq k \leq 2g$$

*form a motivic perverse filtration. Moreover, every submotive  $P_k h(\overline{\mathcal{J}}_C^d)$  admits a canonical orthogonal complement  $Q_{k+1} h(\overline{\mathcal{J}}_C^d)$ :*

$$h(\overline{\mathcal{J}}_C^d) = P_k h(\overline{\mathcal{J}}_C^d) \oplus Q_{k+1} h(\overline{\mathcal{J}}_C^d), \quad Q_{k+1} h(\overline{\mathcal{J}}_C^d) = (\overline{\mathcal{J}}_C^d, \mathfrak{q}_{k+1}, 0)$$

*with*

$$\mathfrak{q}_{k+1} := [\Delta_{\overline{\mathcal{J}}_C^d/B}] - \mathfrak{p}_k = \sum_{i \geq k+1} (\mathfrak{F}_{1,d})_i \circ (\mathfrak{F}_{1,d}^{-1})_{2g-i}.$$

The orthogonal complements  $Q_\bullet h(\overline{\mathcal{J}}_C^d)$  allow us to directly formulate the condition that a correspondence to  $h(\overline{\mathcal{J}}_C^d)$  has image lying in a certain piece of the motivic perverse filtration; we refer to the statements in Theorem 2.6 below as examples.

Now we consider two natural operations on the relative Chow motive  $h(\overline{\mathcal{J}}_C^d)$ . The first is induced by the correspondence  $(\mathfrak{F}_{1,d})_k$ :

$$(\mathfrak{F}_{1,d})_k : h(\overline{\mathcal{J}}_C^1)(g-k) \rightarrow h(\overline{\mathcal{J}}_C^d).$$

The second is the multiplicative structure given by the cup-product:

$$\cup : h(\overline{\mathcal{J}}_C^d) \times h(\overline{\mathcal{J}}_C^d) \rightarrow h(\overline{\mathcal{J}}_C^d).$$

The following theorem shows the compatibility between the motivic perverse filtration of Theorem 2.5 and the two operations above.

**Theorem 2.6.** *The following hold for  $\pi_d : \overline{\mathcal{J}}_C^d \rightarrow B$ .*

(a) *We have a morphism*

$$(\mathfrak{F}_{1,d})_k : h(\overline{\mathcal{J}}_C^1)(g-k) \rightarrow P_k h(\overline{\mathcal{J}}_C^d).$$

*More precisely, the following composition is zero:*

$$h(\overline{\mathcal{J}}_C^1)(g-k) \xrightarrow{(\mathfrak{F}_{1,d})_k} h(\overline{\mathcal{J}}_C^d) \rightarrow Q_{k+1} h(\overline{\mathcal{J}}_C^d).$$

(b) *The motivic perverse filtration is multiplicative:*

$$\cup : P_k h(\overline{\mathcal{J}}_C^d) \times P_l h(\overline{\mathcal{J}}_C^d) \rightarrow P_{k+l} h(\overline{\mathcal{J}}_C^d).$$

*More precisely, the following composition is zero:*

$$P_k h(\overline{\mathcal{J}}_C^d) \times P_l h(\overline{\mathcal{J}}_C^d) \xrightarrow{\cup} h(\overline{\mathcal{J}}_C^d) \rightarrow Q_{k+l+1} h(\overline{\mathcal{J}}_C^d).$$

Statement (a) was proven in [38, Corollary 4.5(iv)] by setting  $e = 1$ . However, (b) does not follow directly from the multiplicativity result [38, Corollary 4.5(iii)] (since we cannot make  $e_1 = e_2 = e_1 + e_2 = 1$ ); we postpone the proof to Section 2.3. The following result is an immediate consequence of the multiplicativity.

**Corollary 2.7.** *Assume that  $\alpha \in \mathrm{CH}^l(\overline{\mathcal{J}}_C^d)$  lies in the  $k$ -th piece of the perverse filtration, i.e.,*

$$\alpha \in \mathrm{CH}^l(P_k h(\overline{\mathcal{J}}_C^d)).$$

*Then cupping with  $\alpha$  interacts with the motivic perverse filtration as follows:*

$$\alpha : P_i h(\overline{\mathcal{J}}_C^d) \rightarrow P_{i+k} h(\overline{\mathcal{J}}_C^d)(l).$$

*Proof.* It suffices to prove the relation

$$\mathfrak{q}_{i+k+1} \circ \Delta_* \alpha \circ \mathfrak{p}_i = 0$$

on the relative product  $\overline{J}_C^d \times_B \overline{J}_C^d$ , where  $\Delta : \overline{J}_C^d \hookrightarrow \overline{J}_C^d \times_B \overline{J}_C^d$  is the diagonal embedding. Theorem 2.6(b) reads

$$\mathfrak{q}_{i+k+1} \circ [\Delta_{\overline{J}_C^d/B}^{\text{sm}}] \circ (\mathfrak{p}_k \times \mathfrak{p}_i) = 0$$

which is a relation on the relative triple product. Therefore, we have

$$\begin{aligned} 0 &= p_{23*} \left( p_1^* \alpha \cap \left( \mathfrak{q}_{i+k+1} \circ [\Delta_{\overline{J}_C^d/B}^{\text{sm}}] \circ (\mathfrak{p}_k \times \mathfrak{p}_i) \right) \right) \\ &= \mathfrak{q}_{i+k+1} \circ p_{23*} \left( p_1^* \alpha \cap \left( [\Delta_{\overline{J}_C^d/B}^{\text{sm}}] \circ (\mathfrak{p}_k \times \mathfrak{p}_i) \right) \right) \\ &= \mathfrak{q}_{i+k+1} \circ \Delta_* \mathfrak{p}_k(\alpha) \circ \mathfrak{p}_i \\ &= \mathfrak{q}_{i+k+1} \circ \Delta_* \alpha \circ \mathfrak{p}_i. \end{aligned}$$

Here  $p_{ij}, p_i$  are the natural projections from the relative triple product  $\overline{J}_C^d \times_B \overline{J}_C^d \times_B \overline{J}_C^d$  to the corresponding factors. This completes the proof.  $\square$

Next, we discuss the motivic  $\chi$ -independence phenomenon for compactified Jacobian fibrations. For two integers  $d, d'$ , we consider the correspondence

$$\mathfrak{C}_{d,d'} := \sum_{i=0}^{2g} (\mathfrak{F}_{1,d'}^{-1})_i \circ (\mathfrak{F}_{1,d}^{-1})_{2g-i} \in \text{Corr}_B^0(\overline{J}_C^d, \overline{J}_C^{d'}).$$

Recall that  $r$  is the degree of the multisection  $D \subset C$  we fixed at the beginning.

**Theorem 2.8.** *Assume  $d, d'$  are two integers coprime to  $r$ . Then  $\mathfrak{C}_{d,d'}$  induces an isomorphism*

$$\mathfrak{C}_{d,d'} : h(\overline{J}_C^d) \xrightarrow{\cong} h(\overline{J}_C^{d'})$$

*preserving the motivic perverse filtrations*

$$\mathfrak{C}_{d,d'} : P_k h(\overline{J}_C^d) \xrightarrow{\cong} P_k h(\overline{J}_C^{d'}).$$

*Moreover, the inverse of  $\mathfrak{C}_{d,d'}$  is given by  $\mathfrak{C}_{d',d}$ ,*

$$(24) \quad \mathfrak{C}_{d',d} \circ \mathfrak{C}_{d,d'} = [\Delta_{\overline{J}_C^d/B}], \quad \mathfrak{C}_{d,d'} \circ \mathfrak{C}_{d',d} = [\Delta_{\overline{J}_C^{d'}/B}].$$

Both Theorem 2.6(b) and Theorem 2.8 are proven in the next section, which rely on a generalized version of the Fourier vanishing established in [38].

**2.3. Proofs of Theorem 2.6(b) and Theorem 2.8.** In [38, Section 4.4], we establish a series of relations between the Fourier components, known as Fourier vanishing (FV). For two integers  $d, e$ , this vanishing states that

$$(FV) \quad (\mathfrak{F}_{e,d}^{-1})_i \circ (\mathfrak{F}_{e,d})_j = 0, \quad i + j < 2g.$$

In this section, we prove a generalized Fourier vanishing given as follows:

**Proposition 2.9.** *Let  $d, d', e, e'$  be integers. Then if  $e$  is coprime to  $r$ , we have*

$$(FV1) \quad (\mathfrak{F}_{e',d}^{-1})_i \circ (\mathfrak{F}_{e,d})_j = 0, \quad i + j < 2g.$$

*If  $d$  is coprime to  $r$ , we have*

$$(FV2) \quad (\mathfrak{F}_{e,d})_i \circ (\mathfrak{F}_{e,d'}^{-1})_j = 0, \quad i + j < 2g.$$

*Proof.* The proof of (FV1) follows closely that of (FV) in [38, Proof of Corollary 4.5(i)], where we used the Adams operations  $\psi^N(-)$  of [2, Section 4] to scale the Poincaré sheaf  $\mathcal{P}_{e,d}$  in  $K$ -theory. By applying the tau-class  $\tau(-)$  to the  $K$ -theory classes supported in codimension  $g$ :

$$\mathcal{P}_{e,d}^{-1} \circ \psi^N(\mathcal{P}_{e,d}) \in K_*(\overline{\mathcal{J}}_C^e \times_B \overline{\mathcal{J}}_C^e)$$

for infinitely many  $N$ , we were able to obtain (FV) for each pair of  $i, j$  satisfying  $i + j < 2g$ . The crucial point here is that the functor  $\tau(-)$  of [20] is only covariant under proper *representable* morphisms of quotient stacks. By choosing

$$N \equiv 1 \pmod{r}$$

we made sure that the class

$$p_{12}^* \psi^N(\mathcal{P}_{e,d}) \otimes p_{23}^* \mathcal{P}_{e,d}^{-1} \in K_*(\overline{\mathcal{J}}_C^e \times_B \overline{\mathcal{J}}_C^d \times_B \overline{\mathcal{J}}_C^e)_{(d,0,-d)}$$

descends to a class in  $K_*(\overline{\mathcal{J}}_C^e \times_B \overline{\mathcal{J}}_C^d \times_B \overline{\mathcal{J}}_C^e)$  in order to apply  $\tau(-)$  to the proper representable morphism

$$p_{13} : \overline{\mathcal{J}}_C^e \times_B \overline{\mathcal{J}}_C^d \times_B \overline{\mathcal{J}}_C^e \rightarrow \overline{\mathcal{J}}_C^e \times_B \overline{\mathcal{J}}_C^e.$$

Here  $p_{ij}$  are the natural projections to the corresponding factors.

Now, under the assumptions of (FV1), we consider the  $K$ -theory classes supported in codimension  $g$ :

$$\mathcal{P}_{e',d}^{-1} \circ \psi^N(\mathcal{P}_{e,d}) \in K_*(\overline{\mathcal{J}}_C^e \times_B \overline{\mathcal{J}}_C^{e'}).$$

Since  $(e, r) = 1$ , we have  $ae \equiv 1 \pmod{r}$  for some integer  $a$ . Then by taking

$$N = aN', \quad N' \equiv e' \pmod{r},$$

we get an infinite sequence of classes

$$p_{12}^* \psi^N(\mathcal{P}_{e,d}) \otimes p_{23}^* \mathcal{P}_{e',d}^{-1} \in K_*(\overline{\mathcal{J}}_C^e \times_B \overline{\mathcal{J}}_C^d \times_B \overline{\mathcal{J}}_C^{e'})_{(ae'd,0,-d)}$$

which descend to  $K_*(\overline{\mathcal{J}}_C^e \times_B \overline{\mathcal{J}}_C^d \times_B \overline{\mathcal{J}}_C^{e'})$ . The argument of [38, Proof of Corollary 4.5(i)] then applies *verbatim* to yield (FV1).

For the proof of (FV2), we can either repeat the whole argument by switching the composition order of the Fourier transform and its inverse, or simply observe that the transpose of

$$(\mathfrak{F}_{e,d})_i \circ (\mathfrak{F}_{e,d'}^{-1})_j$$

is precisely

$$(25) \quad (\mathfrak{F}_{d',e}^{-1})_j \circ (\mathfrak{F}_{d,e})_i.$$

This is an immediate consequence of the symmetry in Arinkin's formula [3, (1.1)] for the Poincaré sheaf; see also [38, Footnote 12]. Therefore, (FV2) follows from (FV1) and (25) by replacing  $e$  with  $d$ ,  $e'$  with  $d'$ ,  $d$  with  $e$ , and by switching  $i$  and  $j$ .  $\square$

We are ready to prove Theorems 2.6(b) and 2.8. We begin with the latter.

*Proof of Theorem 2.8.* We first prove the identities (24). We expand

$$(26) \quad \begin{aligned} [\Delta_{\overline{J}_C/B}^d] &= \mathfrak{F}_{1,d} \circ \mathfrak{F}_{1,d'}^{-1} \circ \mathfrak{F}_{1,d'} \circ \mathfrak{F}_{1,d}^{-1} \\ &= \sum_{i,j,l,m} (\mathfrak{F}_{1,d})_i \circ (\mathfrak{F}_{1,d'}^{-1})_j \circ (\mathfrak{F}_{1,d'})_l \circ (\mathfrak{F}_{1,d}^{-1})_m. \end{aligned}$$

For degree reasons the terms in (26) vanish unless  $i + j + l + m = 4g$ . Applying the Fourier vanishing (FV2) with  $d, d'$  coprime to  $r$ , we find that the nonvanishing terms in (26) must also satisfy  $i + j \geq 2g$ ,  $l + m \geq 2g$ . Hence both inequalities are actual equalities and we have

$$[\Delta_{\overline{J}_C/B}^d] = \left( \sum_{i=0}^{2g} (\mathfrak{F}_{1,d})_i \circ (\mathfrak{F}_{1,d'}^{-1})_{2g-i} \right) \circ \left( \sum_{l=0}^{2g} (\mathfrak{F}_{1,d'})_l \circ (\mathfrak{F}_{1,d}^{-1})_{2g-l} \right) = \mathfrak{C}_{d',d} \circ \mathfrak{C}_{d,d'},$$

which proves the first identity in (24). The second identity follows by symmetry.

It remains to verify that  $\mathfrak{C}_{d,d'}, \mathfrak{C}_{d',d}$  preserve the motivic perverse filtrations. This amounts to showing the relations (19) where  $\mathfrak{p}_k, \mathfrak{q}_{k+1}$  (resp.  $\mathfrak{p}'_k, \mathfrak{q}'_{k+1}$ ) denote the projectors for  $\overline{J}_C^d$  (resp.  $\overline{J}_C^{d'}$ ). By symmetry it suffices to prove the first identity:

$$\mathfrak{q}'_{k+1} \circ \mathfrak{C}_{d,d'} \circ \mathfrak{p}_k = 0.$$

The left-hand side can be expressed as

$$(27) \quad \left( \sum_{i \geq k+1} (\mathfrak{F}_{1,d'})_i \circ (\mathfrak{F}_{1,d'}^{-1})_{2g-i} \right) \circ \left( \sum_{j=0}^{2g} (\mathfrak{F}_{1,d'})_j \circ (\mathfrak{F}_{1,d}^{-1})_{2g-j} \right) \circ \left( \sum_{m \leq k} (\mathfrak{F}_{1,d})_m \circ (\mathfrak{F}_{1,d}^{-1})_{2g-m} \right).$$

This time applying (FV1) with  $e = 1$  to (27), we see that the nonvanishing terms must satisfy

$$2g - i + j \geq 2g, \quad 2g - j + m \geq 2g, \quad i \geq k + 1, \quad m \leq k,$$

or, equivalently,

$$k \geq m \geq j \geq i \geq k + 1.$$

This is absurd, and hence (27) vanishes.  $\square$

The proof of Theorem 2.6(b) is slightly more complicated but follows a similar pattern. For two integers  $e_1, e_2$ , recall from [38, Proof of Corollary 4.5(iii)] the Chow-theoretic convolution kernel<sup>4</sup>

$$\mathfrak{K}_{e_1, e_2, d} \in \text{Corr}_B^{\geq -g}(\overline{J}_C^{e_1}, \overline{J}_C^{e_2}; \overline{J}_C^{e_1+e_2}),$$

<sup>4</sup>The convolution kernel was denoted  $\mathfrak{C}_{e_1, e_2, d}$  in [38].

which satisfies

$$(28) \quad \mathfrak{F}_{e_1+e_2,d} \circ \mathfrak{K}_{e_1,e_2,d} \circ \left( \mathfrak{F}_{e_1,d}^{-1} \times \mathfrak{F}_{e_2,d}^{-1} \right) = [\Delta_{\mathcal{J}_C/B}^{\text{sm}}].$$

*Proof of Theorem 2.6(b).* We need to show the vanishing

$$(29) \quad \mathfrak{q}_{k+l+1} \circ [\Delta_{\mathcal{J}_C/B}^{\text{sm}}] \circ (\mathfrak{p}_k \times \mathfrak{p}_l) = 0 \in \text{Corr}_B^0(\overline{\mathcal{J}}_C^d, \overline{\mathcal{J}}_C^d; \overline{\mathcal{J}}_C^d).$$

By (28) we have

$$(30) \quad \begin{aligned} \mathfrak{q}_{k+l+1} \circ [\Delta_{\mathcal{J}_C/B}^{\text{sm}}] \circ (\mathfrak{p}_k \times \mathfrak{p}_l) &= \mathfrak{q}_{k+l+1} \circ \mathfrak{F}_{e_1+e_2,d} \circ \mathfrak{K}_{e_1,e_2,d} \circ \left( (\mathfrak{F}_{e_1,d}^{-1} \circ \mathfrak{p}_k) \times (\mathfrak{F}_{e_2,d}^{-1} \circ \mathfrak{p}_l) \right) \\ &= \left( \sum_{i_3 \geq k+l+1} (\mathfrak{F}_{1,d})_{i_3} \circ (\mathfrak{F}_{1,d}^{-1})_{2g-i_3} \right) \circ \left( \sum_{j_3} (\mathfrak{F}_{e_1+e_2,d})_{j_3} \right) \circ \mathfrak{K}_{e_1,e_2,d} \\ &\quad \circ \left( \left( \left( \sum_{j_1} (\mathfrak{F}_{e_1,d}^{-1})_{j_1} \right) \circ \left( \sum_{i_1 \leq k} (\mathfrak{F}_{1,d})_{i_1} \circ (\mathfrak{F}_{1,d}^{-1})_{2g-i_1} \right) \right) \right) \\ &\quad \times \left( \left( \left( \sum_{j_2} (\mathfrak{F}_{e_2,d}^{-1})_{j_2} \right) \circ \left( \sum_{i_2 \leq l} (\mathfrak{F}_{1,d})_{i_2} \circ (\mathfrak{F}_{1,d}^{-1})_{2g-i_2} \right) \right) \right). \end{aligned}$$

So far the choices of  $e_1, e_2$  have been arbitrary. But if we choose  $e_1, e_2$  such that  $e_1 + e_2$  is coprime to  $r$ , then by applying (FV1) with  $e = 1$  or  $e = e_1 + e_2$  we see that the nonvanishing terms in (30) must satisfy

$$j_1 + i_1 \geq 2g, \quad j_2 + i_2 \geq 2g, \quad 2g - i_3 + j_3 \geq 2g.$$

This implies

$$j_1 \geq 2g - k, \quad j_2 \geq 2g - l, \quad j_3 \geq k + l + 1.$$

Therefore we have

$$(31) \quad \mathfrak{q}_{k+l+1} \circ [\Delta_{\mathcal{J}_C/B}^{\text{sm}}] \circ (\mathfrak{p}_k \times \mathfrak{p}_l) = \mathfrak{q}_{k+l+1} \circ \left( \sum_{j_3 \geq k+l+1} (\mathfrak{F}_{e_1+e_2,d})_{j_3} \right) \circ \mathfrak{K}_{e_1,e_2,d} \\ \circ \left( \left( \left( \sum_{j_1 \geq 2g-k} (\mathfrak{F}_{e_1,d}^{-1})_{j_1} \right) \circ \mathfrak{p}_k \right) \times \left( \left( \sum_{j_2 \geq 2g-l} (\mathfrak{F}_{e_2,d}^{-1})_{j_2} \right) \circ \mathfrak{p}_l \right) \right)$$

Comparing degrees, we find that the right-hand side of (31) lies in

$$\text{Corr}_B^{\geq -g+(k+l+1-g)+(g-k)+(g-l)}(\overline{\mathcal{J}}_C^d, \overline{\mathcal{J}}_C^d; \overline{\mathcal{J}}_C^d) = \text{Corr}_B^{\geq 1}(\overline{\mathcal{J}}_C^d, \overline{\mathcal{J}}_C^d; \overline{\mathcal{J}}_C^d),$$

while the left-hand side lies in  $\text{Corr}_B^0(\overline{\mathcal{J}}_C^d, \overline{\mathcal{J}}_C^d; \overline{\mathcal{J}}_C^d)$ . This shows the desired vanishing (29).  $\square$

**2.4. Parabolic Higgs bundles: general residue.** Theorems 2.5, 2.6, and 2.8 in Section 2.2 are analogues of our main theorems in the general setting of compactified Jacobians. For our purpose, we specialize to the case where  $C \rightarrow B$  arises as a family of spectral curves. The main results of this section are Theorems 2.11, 2.12, and 2.13 which treat the case for the Hitchin system associated with parabolic Higgs bundles with general residue.

For the curve  $\Sigma$ , we consider nonsingular surface

$$S := \text{Tot}_\Sigma(\omega_\Sigma(p)).$$

Now we work with a family of curves  $C \rightarrow B$  as in Section 2.2, imposing the extra condition that they lie in the linear system  $|n\Sigma|$  in the surface  $S$ ; in particular, every curve  $C_b \subset S$  is a degree  $n$  finite cover over the zero-section  $\Sigma$ . We have a natural evaluation map

$$\text{ev} : C \rightarrow S \times B.$$

The composition of  $\text{ev}$  with the projection  $S \times B \rightarrow \Sigma \times B$  gives the spectral cover together with a (degree  $n$ ) multisection:

$$\pi : C \rightarrow \Sigma \times B, \quad D := \pi^{-1}(p \times B).$$

For each curve  $C_b$  with a degree  $n$  spectral cover  $\pi_b : C_b \rightarrow \Sigma$ , the multisection over this fiber is  $D_b = \pi_b^{-1}(p)$ .

Next, we introduce the normalized Chern character of Section 0.2. Let  $M^{\text{mero}}$  be the moduli space of stable meromorphic Higgs bundles on the curve  $\Sigma$ , which parameterizes slope stable (meromorphic) Higgs bundles:

$$(\mathcal{E}, \theta), \quad \theta : \mathcal{E} \rightarrow \mathcal{E} \otimes \omega_\Sigma(p), \quad \text{rk}(\mathcal{E}) = n, \quad \text{deg}(\mathcal{E}) = d.$$

Here we call them meromorphic Higgs bundles since they can be viewed as Higgs bundles which are allowed to have a simple pole at  $p$ . The variety  $M^{\text{mero}}$  is nonsingular, and the coprime condition  $(n, d) = 1$  ensures the existence of a universal bundle  $\mathcal{U}$  on  $\Sigma \times M^{\text{mero}}$ . The reason we work first with this moduli space is that all the moduli spaces in Section 1.2, including the original moduli space  $M_{n,d}$ , admit natural morphisms to  $M^{\text{mero}}$ , and the pullback of the universal bundle gives a universal bundle on each moduli space. So the normalized Chern character

$$\widetilde{\text{ch}}(\mathcal{U}) \in \text{CH}^*(\Sigma \times M^{\text{mero}}),$$

yields the normalized Chern character for each moduli space of Section 1.2. This strategy was taken in [38, Section 5.4].

In cohomology, the normalization is given by a class

$$\delta := \pi_\Sigma^* \delta_\Sigma + \pi_M^* \delta_M \in H^2(\Sigma \times M^{\text{mero}}, \mathbb{Q}),$$

with  $\pi_\bullet$  the natural projections, satisfying that

$$c_1(\mathcal{U}) + \delta \in H^1(\Sigma, \mathbb{Q}) \otimes H^1(M^{\text{mero}}, \mathbb{Q}) \subset H^2(\Sigma \times M^{\text{mero}}, \mathbb{Q});$$

this determines  $\delta$  completely. Then the normalized Chern character is defined to be

$$\widetilde{\text{ch}}(\mathcal{U}) := \text{ch}(\mathcal{U}) \cup \exp(\delta) \in H^*(\Sigma \times M^{\text{mero}}, \mathbb{Q}),$$

which is canonical and independent of the choice of  $\mathcal{U}$ . We refer to [12, Section 0.3] for more details.

For our purpose, we need to lift  $\delta$  (associated with a universal bundle  $\mathcal{U}$ ) canonically to a divisor

$$(32) \quad \delta := \pi_\Sigma^* D_\Sigma + \pi_M^* D_M \in \text{CH}^1(\Sigma \times M^{\text{mero}}).$$

We first determine  $D_M$  using the marked point  $p \in \Sigma$ :

$$D_M := c_1(\mathcal{U})|_{p \times M^{\text{mero}}} \in \text{CH}^1(M^{\text{mero}}).$$

Now we determine  $D_\Sigma$ , which is eventually written as a linear combination of the point class of  $p$  and the canonical divisor  $K_\Sigma$  on the curve  $\Sigma$ . Pick a nonsingular spectral curve  $C_b \subset \text{Tot}_\Sigma(\omega_\Sigma(p))$ ; a Higgs bundle in  $M^{\text{mero}}$  associated with the spectral curve  $C_b$  can be represented by a line bundle  $\mathcal{L}_{C_b}$  on  $C_b$  of degree  $l(n, d) \in \mathbb{Z}$ , where  $l(n, d)$  is a number only depending on  $n, d$  and is coprime to  $n$ . In particular, we have

$$c_1(\mathcal{L}_{C_b}) - \frac{l(n, d)}{n} \cdot D_b = c_1(\mathcal{L}_{C_b}) - \frac{l(n, d)}{n} \cdot \pi_b^*[p] = 0 \in H^2(C_b, \mathbb{Q}).$$

Finally, there is a unique constant  $\lambda_0 \in \frac{1}{n}\mathbb{Z}$  such that

$$c_1(\mathcal{U}) - \frac{l(n, d)}{n} \cdot \pi_b^*[p] + \lambda_0 \cdot K_\Sigma = 0 \in H^2(\Sigma, 0).$$

We set

$$D_\Sigma := -\frac{l(n, d)}{n} \cdot [p] + \lambda_0 \cdot K_\Sigma,$$

in (32), which defines the normalized Chern character

$$(33) \quad \widetilde{\text{ch}}(\mathcal{U}) := \text{ch}(\mathcal{U}) \cup \exp(\delta) \in \text{CH}^*(\Sigma \times M^{\text{mero}}).$$

We denote by  $\widetilde{\text{ch}}_k(\mathcal{U})$  its degree  $k$  part:

$$\widetilde{\text{ch}}(\mathcal{U}) = \sum_k \widetilde{\text{ch}}_k(\mathcal{U}), \quad \widetilde{\text{ch}}_k(\mathcal{U}) \in \text{CH}^k(\Sigma \times M^{\text{mero}}).$$

By definition, the normalized Chern character (33) does not depend on the choice of the universal bundle  $\mathcal{U}$ , and its image under the cycle class map recovers the normalized Chern character in cohomology.

Next, we give an explicit description of the normalized Chern character for the moduli space  $M_\eta^{\text{par}}$  for a general  $\eta \in T$  as in Section 1.2(a); it admits a compactified Jacobian fibration over  $\widetilde{W}_\eta$ . For notational convenience, we denote by  $C \rightarrow \widetilde{W}_\eta$  the family of spectral curves. Therefore we have

$$M_\eta^{\text{par}} = \overline{\mathcal{J}}_C^e, \quad e = l(n, d).$$

We consider the diagram

$$(34) \quad \begin{array}{ccc} C & \xrightarrow{\text{AJ}} & \overline{J}_C^1 \\ \downarrow \pi & & \\ \Sigma \times \widetilde{W}_\eta & & \end{array}$$

where the horizontal map is the Abel–Jacobi map

$$\text{AJ} : C \rightarrow \overline{J}_C^1, \quad (x \in C_b) \mapsto \mathbf{m}_{x/C_b}^\vee.$$

The diagram (34) yields a relative correspondence over  $\widetilde{W}_\eta$ :

$$[C] \in \text{CH}_{\dim C}(\Sigma \times \overline{J}_C^1).$$

**Proposition 2.10** ([38]). *Let  $\mathcal{U}$  be a universal bundle over  $\Sigma \times \overline{J}_C^e = \Sigma \times M_\eta^{\text{par}}$ . Then we have*

$$\widetilde{\text{ch}}(\mathcal{U}) = \mathfrak{F}_{1,d} \circ [C] \in \text{CH}^*(\Sigma \times \overline{J}_C^d).$$

*Proof.* Applying the Grothendieck–Riemann–Roch (GRR) formula to [38, Proposition 4.7] as in the proof of [38, Proposition 5.1(a)], we express the Chern character of the universal 1-dimensional torsion free sheaf over the universal spectral curve  $C$  in terms of  $\mathfrak{F}_{1,d}$ . Then the result follows from applying further the GRR formula along  $\pi : C \rightarrow \Sigma \times \widetilde{W}_\eta$ , expressing the Chern character of a universal bundle over  $\Sigma$  in terms of the Chern character of a universal torsion free sheaf over  $C$  [38, Section 5].

Note that the term  $-\frac{l(n,d)}{n} \cdot [p]$  in the normalization comes from the line bundle  $\mathcal{N}$  of [38, Proposition 4.7], and the second term  $\lambda_0 \cdot K_C$  arise from the Todd contribution of the GRR calculation along  $\pi : C \rightarrow \Sigma \times \widetilde{W}_\eta$ .  $\square$

The morphism  $h_\eta : M_\eta^{\text{par}} \rightarrow W_\eta$  can be written as a composition

$$M_\eta^{\text{par}} = \overline{J}_C^e \rightarrow \widetilde{W}_\eta \rightarrow W_\eta$$

where the first map is the compactified Jacobian fibration associated with  $C \rightarrow \widetilde{W}$ , and the second is a finite map given by  $\mathfrak{S}_n$ -quotient.

**Theorem 2.11.** *The morphism  $h_\eta : M_\eta^{\text{par}} \rightarrow W_\eta$  admits a motivic perverse filtration, where the projectors are induced by those for the compactified Jacobian fibration  $\overline{J}_C^e \rightarrow \widetilde{W}_\eta$  given in Theorem 2.5.*

*Proof.* By Theorem 2.5, we have the semi-orthogonal projectors for the map  $\overline{J}_C^e \rightarrow \widetilde{W}_\eta$ ,

$$\mathfrak{p}_k \in \text{CH}_{\dim \overline{J}_C^e}(\overline{J}_C^e \times_{\widetilde{W}_\eta} \overline{J}_C^e), \quad 0 \leq k \leq 2g.$$

Then under the natural morphism

$$\text{CH}_{\dim \overline{J}_C^e}(\overline{J}_C^e \times_{\widetilde{W}_\eta} \overline{J}_C^e) \rightarrow \text{CH}_{\dim \overline{J}_C^e}(\overline{J}_C^e \times_{W_\eta} \overline{J}_C^e)$$

induced by  $\widetilde{W}_\eta \rightarrow W_\eta$ , we obtain semi-orthogonal projectors for the latter:

$$\mathfrak{p}_k \in \mathrm{CH}_{\dim \overline{\mathcal{J}}_C^e}(\overline{\mathcal{J}}_C^e \times_{W_\eta} \overline{\mathcal{J}}_C^e), \quad 0 \leq k \leq 2g.$$

Finally, we need to check that these projectors have the correct homological realization, which follows from the fact that the pushforward along the finite map

$$D_c^b(\widetilde{W}_\eta) \rightarrow D_c^b(W_\eta)$$

is  $t$ -exact with respect to the perverse  $t$ -structure.  $\square$

The following theorem determines the perversity of the normalized Chern character.

**Theorem 2.12.** *Taking cup-product with the normalized Chern character satisfies*

$$\widetilde{\mathrm{ch}}_k(\mathcal{U}) : P_i h(\Sigma \times \overline{\mathcal{J}}_C^e) \rightarrow P_{i+k} h(\Sigma \times \overline{\mathcal{J}}_C^e)(k) \in \mathrm{CHM}(\Sigma \times W_\eta).$$

Here the motivic perverse filtration for  $\Sigma \times \overline{\mathcal{J}}_C^e \rightarrow \Sigma \times W_\eta$  is given by the pullback of the motivic perverse filtration for  $h_\eta : \overline{\mathcal{J}}_C^e \rightarrow W_\eta$ .

*Proof.* As in the proof of Theorem 2.11, it suffices to prove the corresponding statement in the category  $\mathrm{CHM}(\Sigma \times \widetilde{W}_\eta)$ . Then pushing forward along  $\Sigma \times \widetilde{W}_\eta \rightarrow \Sigma \times W_\eta$  yields the desired statement over  $W_\eta$ .

Recall the perverse filtration on  $\mathrm{CH}_*(\overline{\mathcal{J}}_C^e)$  induced by the motivic perverse filtration associated with the compactified Jacobian fibration  $\overline{\mathcal{J}}_C^e \rightarrow \widetilde{W}_\eta$ . We first note that

$$\widetilde{\mathrm{ch}}_k(\mathcal{U}) \in P_k \mathrm{CH}^k(\Sigma \times \overline{\mathcal{J}}_C^e).$$

This is a direct consequence of Proposition 2.10:

$$\widetilde{\mathrm{ch}}_k(\mathcal{U}) = (\mathfrak{F}_{1,d})_k \circ [C]$$

combined with Theorem 2.6(a). Then applying Corollary 2.7 to

$$\Sigma \times \overline{\mathcal{J}}_C^e \rightarrow \Sigma \times \widetilde{W}_\eta$$

completes the proof of the theorem.  $\square$

The next theorem concerning the  $\chi$ -independence phenomenon is immediately deduced from Theorem 2.8 by pushing forward along  $\widetilde{W}_\eta \rightarrow W_\eta$ .

**Theorem 2.13.** *Assume  $e, e'$  are two integers coprime to  $n$ . Then  $\mathfrak{C}_{e,e'}$  from Theorem 2.8 induces an isomorphism*

$$\mathfrak{C}_{e,e'} : h(\overline{\mathcal{J}}_C^e) \xrightarrow{\cong} h(\overline{\mathcal{J}}_C^{e'}) \in \mathrm{CHM}(W_\eta)$$

preserving the motivic perverse filtrations

$$\mathfrak{C}_{e,e'} : P_k h(\overline{\mathcal{J}}_C^e) \xrightarrow{\cong} P_k h(\overline{\mathcal{J}}_C^{e'}).$$

Moreover, the inverse of  $\mathfrak{C}_{e,e'}$  is given by  $\mathfrak{C}_{e',e}$ ,

$$\mathfrak{C}_{e',e} \circ \mathfrak{C}_{e,e'} = [\Delta_{\overline{\mathcal{J}}_C^e/W_\eta}], \quad \mathfrak{C}_{e,e'} \circ \mathfrak{C}_{e',e} = [\Delta_{\overline{\mathcal{J}}_C^{e'}/W_\eta}].$$

**2.5. Parabolic Higgs bundles: nilpotent residue.** In the previous section, we have established the desired results for

$$h_\eta : M_\eta^{\text{par}} \rightarrow W_\eta$$

associated with parabolic Higgs bundles with general residue. The proofs in fact give the results for the family

$$h^\circ : M^{\text{par}, \circ} \rightarrow W^\circ$$

over  $T^\circ$  introduced in Section 1.2(a), which puts together the fibrations  $h_\eta$  for  $\eta \in T^\circ$ . By applying the specialization map

$$\eta \in T^\circ \rightsquigarrow 0 \in T$$

for Chow groups, all the results concerning relations of algebraic cycles hold for

$$h_0 : M_0^{\text{par}} \rightarrow W_0.$$

**Theorem 2.14.** *The morphism  $h_0 : M_0^{\text{par}} \rightarrow W_0$  admits a motivic perverse filtration, where the projectors are induced by specializing the projectors of  $h_\eta : M_\eta^{\text{par}} \rightarrow \widetilde{W}_\eta$  with  $\eta \in T^\circ$  given in Theorem 2.11.*

As we discussed above, the projectors we obtain from specialization automatically satisfy the conditions (a, c) in the definition of motivic perverse filtration. It remains to prove (b), *i.e.*, the homological realization recovers the perverse truncation functor. This will be proven in Section 4.1.

**Theorem 2.15.** *Let  $\mathcal{U}$  be a universal bundle over  $\Sigma \times M_0^{\text{par}}$ . Then we have*

$$\widetilde{\text{ch}}_k(\mathcal{U}) : P_i h(\Sigma \times M_0^{\text{par}}) \rightarrow P_{i+k} h(\Sigma \times M_0^{\text{par}})(k) \in \text{CHM}(\Sigma \times W_0).$$

*Proof.* Since the statement can be expressed in terms of a relation of algebraic cycles:

$$\mathfrak{q}_{i+k+1} \circ \Delta_* \widetilde{\text{ch}}_k(\mathcal{U}) \circ \mathfrak{p}_i = 0$$

with  $\Delta : M_0^{\text{par}} \hookrightarrow M_0^{\text{par}} \times_{W_0} M_0^{\text{par}}$  the relative diagonal, it follows from specializing the corresponding relation for  $h_\eta : M_\eta^{\text{par}} \rightarrow W_\eta$  established in Theorem 2.12.  $\square$

Similarly, by a straightforward specialization argument, we obtain from Theorem 2.13 the following theorem concerning the  $\chi$ -independence.

**Theorem 2.16.** *Assume  $d, d'$  are two integers coprime to  $n$ . Let  $M_0^{\text{par}}, M_0^{\text{par}'}$  be the parabolic moduli spaces associated with the degrees  $d, d'$  respectively. Then there is an isomorphism*

$$h(M_0^{\text{par}}) \simeq h(M_0^{\text{par}'})$$

*preserving the motivic perverse filtrations*

$$P_k h(M_0^{\text{par}}) \simeq P_k h(M_0^{\text{par}'}).$$

*Remark 2.17.* We conclude this section by noting that the motivic perverse filtration we introduced in Theorem 2.14 is independent of the choice of the 1-dimensional family  $T$ . Indeed, we consider the Hitchin system associated with the total space of the moduli of stable parabolic Higgs bundles

$$M^{\text{par}} \rightarrow W$$

in Section 1.2. The Fourier transform can be defined over a Zariski open subset  $W^\circ$  of the base  $W$  formed by integral spectral curves, and the projectors  $\mathfrak{p}_k$  we obtained in Theorem 2.15 is the specialization of the projectors over  $W^\circ$ . Here we choose to work with a 1-dimensional family  $T$  for convenience (*e.g.* in the sheaf-theoretic argument of Section 4).

### 3. PROJECTORS AND SPRINGER THEORY

In this section, we study the correspondences given by (15) connecting  $M_0^{\text{par}}$  and  $M_{n,d}$ . Our main purpose is to prove Proposition 3.5, which yields a candidate for the motivic perverse filtration (42) for  $f : M_{n,d} \rightarrow B$ . For the approach, we reduce the Higgs case to a *local model* where the correspondence (15) is understood via classical Springer theory. This lifts several cohomological results of [24, Section 8] motivically. Springer theoretic interpretations of parabolic Hitchin moduli spaces have previously been used in [19] in the study of Kac polynomials.

**3.1. Springer theory.** For our purpose, we consider the case  $G = \text{PGL}_n$ . We denote by  $B, T$  the Borel subgroup given by upper triangular matrices and the maximal torus given by diagonal matrices respectively. We use  $\mathfrak{g}, \mathfrak{b}, \mathfrak{t}$  to denote the corresponding Lie algebra. The Weyl group is the permutation group  $\mathfrak{S}_n$  which acts on  $\mathfrak{t}$  naturally. Let

$$\mathfrak{c} := \mathfrak{g} // G \simeq \mathfrak{t} // \mathfrak{S}_n$$

be the categorical quotient, and we have a natural morphism

$$\chi : \mathfrak{g} \rightarrow \mathfrak{c}$$

which can be viewed as calculating the characteristic polynomial of a matrix. We consider the Grothendieck–Springer resolution over the base  $\mathfrak{c}$ :

$$\begin{array}{ccc} \tilde{\mathfrak{g}} & \xrightarrow{q} & \mathfrak{g} \\ & \searrow \chi & \swarrow \chi \\ & \mathfrak{c} & \end{array}$$

The fibers over  $0 \in \mathfrak{c}$  recovers the symplectic resolution of the nilpotent cone

$$q_0 : \tilde{\mathcal{N}}_0 \rightarrow \mathcal{N}_0$$

by the total cotangent bundle of the flag variety  $\tilde{\mathcal{N}} = T^*\text{Fl}$ , while the fibers over a general  $\eta \in \mathfrak{c}$  (*e.g.* the corresponding matrix has distinct eigenvalues) is a natural quotient of the Weyl group:

$$q_\eta : \tilde{\mathcal{N}}_\eta \rightarrow \mathcal{N}_\eta = \tilde{\mathcal{N}}_\eta / \mathfrak{S}_n.$$

From this description, we see that any element  $w \in \mathfrak{S}_n$  induces a self-correspondence of  $\widetilde{\mathcal{N}}_\eta$ . As we will illustrate in the next section, the local analog of the correspondence (15) is (the  $G$ -equivariant version of) the following:

$$(35) \quad \begin{array}{ccc} \mathrm{Fl} & \xrightarrow{\iota} & \widetilde{\mathcal{N}}_0 = T^*\mathrm{Fl} \\ \downarrow \pi & & \\ \mathrm{pt} & & \end{array}$$

where  $\iota$  is the closed embedding of the 0-section. The following lemma collects a few standard results in Springer theory, which concerns the self-correspondences

$$(36) \quad [\mathrm{Fl}] \circ {}^t[\mathrm{Fl}] \in \mathrm{CH}_*(\widetilde{\mathcal{N}}_0 \times \widetilde{\mathcal{N}}_0) = \mathrm{Corr}^*(\widetilde{\mathcal{N}}_0, \widetilde{\mathcal{N}}_0)$$

and

$$(37) \quad {}^t[\mathrm{Fl}] \circ [\mathrm{Fl}] \in \mathrm{CH}_*(\mathrm{pt}) = \mathrm{Corr}^*(\mathrm{pt}, \mathrm{pt}).$$

**Lemma 3.1.** (a) *The self-correspondence (36) of  $\widetilde{\mathcal{N}}_0$  is the specialization of the self-correspondence of  $\widetilde{\mathcal{N}}_\eta$  induced by the longest element  $w_0$  in the Weyl group  $\mathfrak{S}_n$ .*

(b) *The self-correspondence of (37) of a point is*

$$\epsilon n! \cdot \mathrm{id} \in \mathrm{Corr}^0(\mathrm{pt}, \mathrm{pt}).$$

Here  $\epsilon = \pm 1$  is a sign dependent on  $n$ .

(c) *There is a natural Weyl group  $\mathfrak{S}_n$  action on the cohomology  $H^*(\widetilde{\mathcal{N}}_0, \mathbb{Q})$ ; the anti-invariant part is 1-dimensional concentrated in the top degree cohomology:*

$$H^*(\widetilde{\mathcal{N}}_0, \mathbb{Q})^{\mathrm{sgn}} = \mathbb{Q} \left[ 2 \binom{n}{2} \right].$$

Here the shift is given by the dimension  $\dim \mathrm{Fl} = \binom{n}{2}$ . Furthermore, the cohomological correspondences:

$$[\mathrm{Fl}] : H^*(\widetilde{\mathcal{N}}_0, \mathbb{Q}) \rightarrow H^{*- \binom{n}{2}}(\mathrm{pt}, \mathbb{Q}) = \mathbb{Q}, \quad {}^t[\mathrm{Fl}] : H^{*- \binom{n}{2}}(\mathrm{pt}, \mathbb{Q}) = \mathbb{Q} \rightarrow H^*(\widetilde{\mathcal{N}}_0, \mathbb{Q})$$

are projecting to the anti-invariant part, and the natural inclusion from the anti-invariant part respectively.

*Remark 3.2.* The correspondences  $[\mathrm{Fl}]$ ,  ${}^t[\mathrm{Fl}]$  are naturally  $G$ -equivariant and relative over  $\mathcal{N}_0$ . All the statements above concerning the correspondences hold  $G$ -equivariantly and relatively. For example, the statement of (c) actually is obtained from a sheaf-theoretic result: the Weyl group  $\mathfrak{S}_n$  acts on the derived push-forward

$$Rq_{0*} \mathbb{Q}_{\widetilde{\mathcal{N}}_0} \in D_c^b(\mathcal{N}_0)$$

induced by the  $\mathfrak{S}_n$ -structure of  $q_\eta : \widetilde{\mathcal{N}}_\eta \rightarrow \mathcal{N}_\eta$ . The correspondences

$$[\mathrm{Fl}] : Rq_{0*} \mathbb{Q}_{\widetilde{\mathcal{N}}_0} \rightarrow \mathbb{Q}_0 \left[ -2 \binom{n}{2} \right], \quad {}^t[\mathrm{Fl}] : \mathbb{Q}_0 \left[ -2 \binom{n}{2} \right] \rightarrow Rq_{0*} \mathbb{Q}_{\widetilde{\mathcal{N}}_0} \in D_c^b(\mathcal{N}_0)$$

are projecting to the anti-invariant part and the natural inclusion from the anti-invariant part respectively. Then the cohomological statement of Lemma 3.1(c) is obtained by further pushing forward the sheaf-theoretic statements to a point.

As a consequence of Lemma 3.1(a), if we take a self-correspondence of  $\mathcal{N}_\eta$ , pull it back along

$$q_\eta : \widetilde{\mathcal{N}}_\eta \times \widetilde{\mathcal{N}}_\eta \rightarrow \mathcal{N}_\eta \times \mathcal{N}_\eta,$$

and specialize it to a self-correspondence of  $\widetilde{\mathcal{N}}_0$ , then this resulting self-correspondence commute with (36). In the next section, we will use a global version of this result to analyze the correspondence

$$(38) \quad [\widetilde{M}_0] \circ {}^t[\widetilde{M}_0] \in \text{Corr}_{W_0}^*(M_0^{\text{par}}, M_0^{\text{par}}).$$

**3.2. Hitchin moduli spaces.** Recall the moduli space  $M^{\text{mero}}$  of stable meromorphic Higgs bundles of Section 2.4. It admits a natural evaluation map at the point  $p \in \Sigma$ :

$$(39) \quad \text{ev}_p : M^{\text{mero}} \rightarrow \mathfrak{g}/G$$

as we introduce below.

We denote by  $M^{\text{ab}}$  the *abelian* Hitchin moduli space

$$M^{\text{ab}} = \text{Pic}^0(\Sigma) \times H^0(\Sigma, \omega_\Sigma(p))$$

which parameterize rank 1 Higgs bundles. There is a natural action of  $M^{\text{ab}}$  on  $M^{\text{mero}}$  by

$$(\mathcal{L}, \sigma) \cdot (\mathcal{E}, \theta) = (\mathcal{L} \otimes \mathcal{E}, \sigma + \theta), \quad (\mathcal{L}, \sigma) \in M^{\text{ab}}, \quad (\mathcal{E}, \theta) \in M^{\text{mero}}$$

where the sum  $\sigma + \theta$  is defined in the sense that we view both  $\sigma, \theta$  as morphisms

$$\mathcal{L} \otimes \mathcal{E} \rightarrow \mathcal{L} \otimes \mathcal{E} \otimes \omega_\Sigma(p).$$

The quotient space

$$\widehat{M}^{\text{mero}} := M^{\text{mero}}/M^{\text{ab}}$$

is a nonsingular Deligne–Mumford stack, known as the moduli space of stable meromorphic  $\text{PGL}_n$ -Higgs bundles, which admits a natural evaluation map at the point  $p$ . The evaluation map (39) for the original moduli space the composition of the quotient map and the  $\text{PGL}_n$ -evaluation map:

$$M^{\text{mero}} \twoheadrightarrow \widehat{M}^{\text{mero}} \xrightarrow{\text{ev}_p} \mathfrak{g}/G.$$

**Proposition 3.3.** *The morphism (39) is smooth and surjective.*

*Proof.* The quotient map  $M^{\text{mero}} \twoheadrightarrow \widehat{M}^{\text{mero}}$  is smooth. Therefore, the smoothness of the composition follows from the smoothness of the second evaluation map which has been proven in [35, Propositin 4.1] via deformation theory.  $\square$

*Remark 3.4.* We do not work directly with the evaluation map associated with  $\mathrm{GL}_n$  for the following reasons. First, Lemma 1.2 implies that the image of  $\mathrm{ev}_p$  for a  $\mathrm{GL}_n$ -Higgs bundle has trace 0; in particular the trace lies naturally in

$$\mathfrak{g} = \{\mathrm{tr} = 0\} \subset \mathfrak{gl}_n.$$

Second, when we pass from the moduli stack of  $\mathrm{GL}_n$ -stable Higgs bundle to the corresponding moduli space, we need to rigidify the automorphism group  $\mathbb{C}^* \subset \mathrm{GL}_n$ . Working with the evaluation map associated with  $G = \mathrm{PGL}_n$  is more convenient which resolves both issues.

Now we may use the evaluation map to pull back natural morphisms in Springer theory. As a first example, the pullback of (39) via the ( $G$ -equivariant) Groethendieck–Springer resolution recovers the parabolic moduli space

$$\begin{array}{ccc} M^{\mathrm{par}} & \xrightarrow{\mathrm{ev}_p} & \tilde{\mathfrak{g}}/G \\ \downarrow & & \downarrow q \\ M^{\mathrm{mero}} & \xrightarrow{\mathrm{ev}_p} & \mathfrak{g}/G. \end{array}$$

This further yields the specialization

$$M_\eta^{\mathrm{par}} \xrightarrow{\mathrm{ev}_p} \tilde{\mathcal{N}}_\eta/G \rightsquigarrow M_0^{\mathrm{par}} \xrightarrow{\mathrm{ev}_p} \tilde{\mathcal{N}}_0/G.$$

Finally, the pullback along  $\mathrm{ev}_p$  of the  $G$ -equivariant version of the diagram (35)

$$(40) \quad \begin{array}{ccc} \mathrm{Fl}/G & \xrightarrow{\iota} & \tilde{\mathcal{N}}_0/G \\ \downarrow \pi & & \\ \mathrm{pt}/G & & \end{array}$$

recovers the diagram (15). Since all the maps  $\mathrm{ev}_p$  are obtained via base change from (39), they are all smooth and surjective by Proposition 3.3.

Recall the  $\mathfrak{S}_n$ -quotient map

$$M_\eta^{\mathrm{par}} \rightarrow \bar{J}_C^e$$

of Section 1.2(a) for general  $\eta \in T$ .

**Proposition 3.5.** *Assume that the self-correspondence*

$$\mathfrak{Z}_\eta \in \mathrm{Corr}_{W_\eta}^*(M_\eta^{\mathrm{par}}, M_\eta^{\mathrm{par}})$$

*is pulled back from  $\bar{J}_C^e \times_{W_\eta} \bar{J}_C^e$ . Let*

$$\mathfrak{Z}_0 \in \mathrm{Corr}_{W_0}^*(M_0^{\mathrm{par}}, M_0^{\mathrm{par}})$$

*be the specialization of  $\mathfrak{Z}_\eta$ . Then (38) commutes with  $\mathfrak{Z}_0$ , i.e., we have*

$$\left([\tilde{M}_0] \circ {}^t[\tilde{M}_0]\right) \circ \mathfrak{Z}_0 = \mathfrak{Z}_0 \circ \left([\tilde{M}_0] \circ {}^t[\tilde{M}_0]\right) \in \mathrm{Corr}_{W_0}^*(M_0^{\mathrm{par}}, M_0^{\mathrm{par}}).$$

*Proof.* The correspondence (15) is induced by the pullback of the correspondence (40) via the evaluation maps, we obtain that

$$(41) \quad \text{ev}^* \left( [\text{Fl}/G] \circ {}^t[\text{Fl}/G] \right) = [\widetilde{M}_0] \circ {}^t[\widetilde{M}_0].$$

Here the evaluation map is given by

$$\text{ev} : M_0^{\text{par}} \times_{W_0} M_0^{\text{par}} \rightarrow \widetilde{\mathcal{N}}_0/G \times \widetilde{\mathcal{N}}_0/G.$$

The  $G$ -equivariant version of Lemma 3.1(a) then implies that the correspondence (41) is the specialization of a self-correspondence of  $M_\eta^{\text{par}}$  induced by  $w_0 \in \mathfrak{S}_n$ . The latter commutes with the pullback of any self-correspondence of  $\overline{J}_C^e = M_\eta^{\text{par}}/\mathfrak{S}_n$ , which proves the proposition.  $\square$

Recall the morphism  $\Gamma$  from Section 1.3. From now on, we use

$$\mathfrak{p}_k^{\text{par}} \in \text{Corr}_{W_0}^0(M_0^{\text{par}}, M_0^{\text{par}}), \quad k = 0, 1, \dots, 2(\dim M_0^{\text{par}} - \dim W_0)$$

to denote the projectors associated with the motivic perverse filtration of Theorem 2.14.

**Proposition 3.6.** *The following statements hold.*

(a) *We have*

$$\Gamma([\Delta_{M_0^{\text{par}}/W_0}]) = \epsilon n! \cdot [\Delta_{M_{n,d}/B}].$$

Here  $\epsilon = \pm 1$  is as in Lemma 3.1(b).

(b) *We have the vanishing*

$$\Gamma(\mathfrak{p}_k^{\text{par}}) = 0, \quad k < \binom{n}{2}.$$

(c) *We have the stabilization*

$$\Gamma(\mathfrak{p}_k^{\text{par}}) = [\Delta_{M_{n,d}/B}], \quad k \geq 2(\dim M_0^{\text{par}} - \dim W_0) - \binom{n}{2}.$$

(d) *We have the semi-orthogonality relation*

$$\Gamma(\mathfrak{p}_l^{\text{par}}) \circ \Gamma(\mathfrak{p}_k^{\text{par}}) = \epsilon n! \cdot \Gamma(\mathfrak{p}_k^{\text{par}}), \quad k < l.$$

*Proof.* (a) follows from pulling back the local statement of Lemma 3.1(b). (b) and (c) follow from Lemma 1.3 and the construction of  $\mathfrak{p}_k^{\text{par}}$  in terms of the specialization of algebraic cycles of the form (23). (d) is a consequence of part (a) and Proposition 3.5:

$$\begin{aligned} \Gamma(\mathfrak{p}_l^{\text{par}}) \circ \Gamma(\mathfrak{p}_k^{\text{par}}) &= {}^t[\widetilde{M}_0] \circ \mathfrak{p}_l^{\text{par}} \circ \left( [\widetilde{M}_0] \circ {}^t[\widetilde{M}_0] \right) \circ \mathfrak{p}_k^{\text{par}} \circ [\widetilde{M}_0] \\ &= \left( {}^t[\widetilde{M}_0] \circ [\widetilde{M}_0] \right) \circ {}^t[\widetilde{M}_0] \circ \left( \mathfrak{p}_l^{\text{par}} \circ \mathfrak{p}_k^{\text{par}} \right) \circ [\widetilde{M}_0] \\ &= \epsilon n! \cdot {}^t[\widetilde{M}_0] \circ \mathfrak{p}_k^{\text{par}} \circ [\widetilde{M}_0] \\ &= \epsilon n! \cdot \Gamma(\mathfrak{p}_k^{\text{par}}). \end{aligned}$$

Here we have used that every  $\mathfrak{p}_k^{\text{par}}$  satisfies the assumption of  $\mathfrak{J}_0$  in Proposition 3.5, and we applied (a) to deduce the third identity.  $\square$

By a direct dimension calculation, we have

$$(\dim M_0^{\text{par}} - \dim W_0) - \binom{n}{2} = R_f.$$

Therefore, after rescaling and relabelling as

$$\mathfrak{p}_k := \frac{\epsilon}{n!} \cdot \Gamma \left( \mathfrak{p}_{k - \binom{n}{2}}^{\text{par}} \right),$$

we obtain the projectors

$$(42) \quad \mathfrak{p}_k \in \text{Corr}_B^0(M_{n,d}, M_{n,d}), \quad k = 0, 1, \dots, 2R_f.$$

In Section 4, we will prove that (42) forms a motivic perverse filtration; as for Theorem 2.14, we only need to show that the homological realization of  $\mathfrak{p}_k$  recovers the perverse truncation functor.

#### 4. SHEAF-THEORETIC OPERATIONS

The purpose of this section is to prove a general specialization result for the decomposition theorem (Proposition 4.1). Then we apply it to the Hitchin systems we studied in the previous sections and complete the proofs of all the main theorems.

**4.1. Specializations.** In this section, we finish the proof of Theorem 2.14 by addressing the compatibility of the Chow and sheaf-theoretic specialization maps, and their relationship with the perverse truncation.

Our setup is the following: let  $T$  be a nonsingular curve and let  $\pi : B \rightarrow T$  be a smooth morphism. Consider two proper morphisms  $f : X \rightarrow B$ ,  $g : Y \rightarrow B$  such that the compositions  $\pi \circ f : X \rightarrow T$ ,  $\pi \circ g : Y \rightarrow T$  are smooth:

$$\begin{array}{ccc} X & & Y \\ & \searrow f & \swarrow g \\ & B & \\ & \downarrow \pi & \\ & T & \end{array}$$

We think of  $f : X \rightarrow B$ ,  $g : Y \rightarrow B$  as two families of proper morphisms over  $T$ .

Let  $\eta \in T$  (resp.  $0 \in T$ ) be a general (resp. the special) point. Suppose we are given a correspondence

$$\mathfrak{z}_\eta \in \text{Corr}_{B_\eta}^*(X_\eta, Y_\eta) = \text{CH}_*(X_\eta \times_{B_\eta} Y_\eta).$$

Then by the specialization map for Chow groups via

$$X_\eta \rightarrow B_\eta \quad \rightsquigarrow \quad X_0 \rightarrow B_0,$$

we obtain a correspondence

$$\mathfrak{z}_0 \in \text{Corr}_{B_0}^*(X_\eta, Y_\eta) = \text{CH}_*(X_0 \times_{B_0} Y_0).$$

Also recall from [16, Lemma 2.21] that the cycle class  $\text{cl}(\mathfrak{Z}_\eta) \in H_{2*}^{\text{BM}}(X_\eta \times_{B_\eta} Y_\eta, \mathbb{Q})$  induces a morphism in  $D_c^b(B_\eta)$ :

$$(43) \quad \text{cl}(\mathfrak{Z}_\eta) : Rf_{\eta*} \mathbb{Q}_{X_\eta} \rightarrow Rg_{\eta*} \mathbb{Q}_{Y_\eta}[*],$$

where we ignore the precise shifting index. Similarly, we have a morphism in  $D_c^b(B_0)$ :

$$(44) \quad \text{cl}(\mathfrak{Z}_0) : Rf_{0*} \mathbb{Q}_{X_0} \rightarrow Rg_{0*} \mathbb{Q}_{Y_0}[*].$$

**Proposition 4.1.** *Let  $\mathfrak{Z}_\eta, \mathfrak{Z}_0$  be as above. For  $k \in \mathbb{Z}$ , if*

$$(45) \quad {}^p\mathcal{H}^k(\text{cl}(\mathfrak{Z}_\eta)) : {}^p\mathcal{H}^k(Rf_{\eta*} \mathbb{Q}_{X_\eta}) \rightarrow {}^p\mathcal{H}^k(Rg_{\eta*} \mathbb{Q}_{Y_\eta}[*])$$

*is an isomorphism (resp. zero), then the same holds for*

$$(46) \quad {}^p\mathcal{H}^k(\text{cl}(\mathfrak{Z}_0)) : {}^p\mathcal{H}^k(Rf_{0*} \mathbb{Q}_{X_0}) \rightarrow {}^p\mathcal{H}^k(Rg_{0*} \mathbb{Q}_{Y_0}[*]).$$

*Proof.* Let  $\bar{\mathfrak{Z}} \in \text{CH}_*(X \times_B Y)$  be the Zariski closure of (the spread-out of)  $\mathfrak{Z}_\eta \in \text{CH}_*(X_\eta \times_{B_\eta} Y_\eta)$ . The cycle class  $\text{cl}(\bar{\mathfrak{Z}}) \in H_{2*}^{\text{BM}}(X \times_B Y, \mathbb{Q})$  induces a morphism in  $D_c^b(B)$ :

$$(47) \quad \text{cl}(\bar{\mathfrak{Z}}) : Rf_* \mathbb{Q}_X \rightarrow Rg_* \mathbb{Q}_Y[*].$$

Then by construction and proper base change, the morphism  $\text{cl}(\mathfrak{Z}_0)$  in (44) is obtained by applying  $i^*$  to (47) for the inclusion  $i : B_0 \hookrightarrow B$ :

$$\text{cl}(\mathfrak{Z}_0) = i^* \text{cl}(\bar{\mathfrak{Z}}) : Rf_{0*} \mathbb{Q}_{X_0} \rightarrow Rg_{0*} \mathbb{Q}_{Y_0}[*].$$

We give an alternative construction of (44) via nearby and vanishing cycles following [11, 8]. Recall the distinguished triangle

$$(48) \quad i^*[-1] \rightarrow \psi_B[-1] \xrightarrow{\text{can}} \phi_B \xrightarrow{+1}$$

where  $\psi_B, \phi_B : D_c^b(B) \rightarrow D_c^b(B_0)$  are the nearby and vanishing cycle functors respectively. Since  $X$  is smooth over  $T$ , we have  $\phi_X \mathbb{Q}_X \simeq 0 \in D_c^b(X_0)$  and by proper base change for  $\phi$ ,

$$\phi_B Rf_* \mathbb{Q}_X \simeq Rf_{0*} \phi_X \mathbb{Q}_X \simeq 0 \in D_c^b(B_0).$$

Similarly, we have

$$\phi_B Rg_* \mathbb{Q}_Y \simeq Rg_{0*} \phi_Y \mathbb{Q}_Y \simeq 0 \in D_c^b(B_0).$$

We now apply the distinguished triangle (48) to the morphism  $\text{cl}(\bar{\mathfrak{Z}})$  in (47), which yields a commutative diagram

$$(49) \quad \begin{array}{ccc} Rf_{0*} \mathbb{Q}_{X_0} & \xrightarrow{\text{cl}(\mathfrak{Z}_0)} & Rg_{0*} \mathbb{Q}_{Y_0}[*] \\ \downarrow \simeq & & \downarrow \simeq \\ \psi_B Rf_* \mathbb{Q}_X & \xrightarrow{\psi_B \text{cl}(\bar{\mathfrak{Z}})} & \psi_B Rg_* \mathbb{Q}_Y[*]. \end{array}$$

Here both columns are isomorphisms since the vanishing cycles are trivial, and the crucial point is that the bottom morphism  $\psi_B \text{cl}(\bar{\mathfrak{Z}})$  is fully determined by the morphism  $\text{cl}(\mathfrak{Z}_\eta)$  in (43).

Finally, we recall the fact that  $\psi_B[-1]$  is perverse  $t$ -exact, and hence commutes with taking perverse cohomology. Therefore, if (45) is an isomorphism (resp. zero), then

$${}^p\mathcal{H}^k(\psi_{B\text{cl}}(\bar{3})) : {}^p\mathcal{H}^k(\psi_B Rf_* \mathbb{Q}_X) \rightarrow {}^p\mathcal{H}^k(\psi_B Rg_* \mathbb{Q}_Y[*])$$

is also an isomorphism (resp. zero). By the diagram (49) the same holds for (46).  $\square$

A useful consequence of Proposition 4.1 is that the validity of the Corti–Hanamura motivic decomposition conjecture is preserved under specialization. This also completes the proof of Theorem 2.14.

**Corollary 4.2.** *Let  $X \xrightarrow{f} B \xrightarrow{\pi} T$  be as above, with  $f$  proper and  $\pi, \pi \circ f$  smooth. If the motivic decomposition conjecture holds for  $X_\eta \rightarrow B_\eta$ , then the specializations of the projectors provide a motivic decomposition for  $X_0 \rightarrow B_0$ .*

*Proof.* The (semi-)orthogonal projectors of  $X_\eta \rightarrow B_\eta$  specialize immediately to (semi-)orthogonal projectors of  $X_0 \rightarrow B_0$ , while the relative diagonal class  $[\Delta_{X_\eta/B_\eta}]$  specializes to  $[\Delta_{X_0/B_0}]$ . Now Proposition 4.1 guarantees that the homological realization of the specialized projectors recovers the decomposition theorem for  $X_0 \rightarrow B_0$ .  $\square$

**4.2. Correspondences.** For our purpose, we connect the decomposition theorem associated with the Hitchin maps:

$$h_0 : M^{\text{par}} \rightarrow W_0, \quad f : M_{n,d} \rightarrow B \subset W_0.$$

The diagram (15) yields sheaf-theoretic correspondences

(50)

$$[\widetilde{M}_0] : Rh_{0*} \mathbb{Q}_{M_0^{\text{par}}} \rightarrow Rf_* \mathbb{Q}_{M_{n,d}} \left[ -2 \binom{n}{2} \right], \quad {}^t[\widetilde{M}_0] : Rf_* \mathbb{Q}_{M_{n,d}} \left[ -2 \binom{n}{2} \right] \rightarrow Rh_{0*} \mathbb{Q}_{M_0^{\text{par}}}$$

which take places in the derived category  $D_c^b(W_0)$ .

The following lemma is a global version of Lemma 3.1(c).

**Lemma 4.3.** *There is a natural Weyl group  $\mathfrak{S}_n$  action on the object*

$$Rh_{0*} \mathbb{Q}_{M_0^{\text{par}}} \in D_c^b(W_0);$$

*the anti-invariant part is*

$$\left( Rh_{0*} \mathbb{Q}_{M_0^{\text{par}}} \right)^{\text{sgn}} = Rf_* \mathbb{Q}_{M_{n,d}} \left[ -2 \binom{n}{2} \right] \in D_c^b(W_0).$$

*Furthermore, under this isomorphism the correspondences (50) are projecting to the anti-invariant part and the natural inclusion from the anti-invariant part respectively.*

*Proof.* By Proposition 3.3, the diagram (15) is the pullback of local case (35) along smooth morphisms  $\text{ev}_p$ . Therefore, Lemma 4.3 follow from pulling back the  $G$ -equivariant version of Lemma 3.1(c) (see Remark 3.2).  $\square$

**4.3. Proofs of Theorems 0.1, 0.2, 0.3, 0.4.** In this section, we complete the proofs of all the main theorems.

4.3.1. *Proof of Theorem 0.1.* We start with the proof of Theorem 0.1. By Proposition 2.2, it suffices to show that the correspondences we constructed in (42) form a motivic perverse filtration. By Proposition 3.6, it remains to check that the homological realization

$$\mathfrak{p}_k Rf_* \mathbb{Q}_{M_{n,d}} \rightarrow Rf_* \mathbb{Q}_{M_{n,d}}$$

recovers the perverse truncation

$${}^{\mathfrak{p}}\tau_{\leq k + \dim B} Rf_* \mathbb{Q}_{M_{n,d}} \hookrightarrow Rf_* \mathbb{Q}_{M_{n,d}}.$$

We know that this statement holds for  $h_0 : M_0^{\text{par}} \rightarrow W_0$  and  $\mathfrak{p}_k^{\text{par}}$  by Theorem 2.14 (proven in Section 4.1). Now, we consider the correspondence of  $\mathfrak{p}_k$  given by the composition:

$$\mathfrak{p}_k : Rf_* \mathbb{Q}_{M_{n,d}} \xrightarrow{[\tilde{M}_0]} Rh_{0*} \mathbb{Q}_{M_0^{\text{par}}} \left[ 2 \binom{n}{2} \right] \xrightarrow{\mathfrak{p}_k^{\text{par}}} Rh_{0*} \mathbb{Q}_{M_0^{\text{par}}} \left[ 2 \binom{n}{2} \right] \xrightarrow{![\tilde{M}_0]} Rf_* \mathbb{Q}_{M_{n,d}}.$$

By Lemma 4.3, we have a direct sum decomposition induced by the  $\mathfrak{S}_n$ -action:

$$(51) \quad Rh_{0*} \mathbb{Q}_{M_0^{\text{par}}} \left[ 2 \binom{n}{2} \right] = (Rh_{0*} \mathbb{Q}_{M_0^{\text{par}}})^{\text{sgn}} \left[ 2 \binom{n}{2} \right] \oplus \text{others},$$

and applying  $\mathfrak{p}_k$  to  $Rf_* \mathbb{Q}_{M_{n,d}}$  is induced by applying  $\mathfrak{p}_k^{\text{par}}$  on the first factor above. In particular, it coincides with the perverse truncation. This completes the proof.  $\square$

4.3.2. *Proof of Theorem 0.2.* We first note the following result of [1, Proposition 6.4].

**Fact 1.** The compactified Jacobian fibration

$$\bar{J}_C^e \rightarrow W_\eta$$

of Section 1.2(a) satisfies the relative Lefschetz standard conjecture.

More precisely, this is due to the resolutions of the Lefschetz standard conjecture for abelian varieties [32, 31] and the fact that the decomposition theorem associated with a compactified Jacobian fibration associated with integral locally planar curves has *full support* by Ngô support theorem [45].

Our strategy is to reduce the case of  $f : M_{n,d} \rightarrow B$  to  $\bar{J}_C^e \rightarrow W_\eta$ . Let  $\sigma \in H^2(M_{n,d}, \mathbb{Q})$  be a relative ample class. By [34], we can write  $\sigma$  in terms of tautological classes as follows. We first recall that we have a canonical isomorphism

$$(52) \quad H^2(M_{n,d}, \mathbb{Q}) = H^2(\widehat{M}_{n,d}, \mathbb{Q}) \oplus H^2(J_\Sigma, \mathbb{Q})$$

where  $\widehat{M}_{n,d}$  is the  $\text{PGL}_n$ -Higgs moduli space and  $J_\Sigma$  is the Jacobian variety of the curve  $\Sigma$ ; see [9, Section 2.4]. The first summand on the right-hand side of (52) spans a 1-dimensional subspace, generated by the relative ample tautological class given by the Künneth component

of normalized second Chern character  $\widetilde{\text{ch}}_2(\mathcal{U})$  with respect to the point class  $[\text{pt}] \in H^2(\Sigma, \mathbb{Q})$  on the curve. This class can be lifted to the  $(\mathbb{Q})$ -divisor class

$$c_2([\Sigma]) := \pi_{M*} \left( \pi_{\Sigma}^*[\Sigma] \cup \widetilde{\text{ch}}_2(\mathcal{U}) \right) \in \text{Pic}(M_{n,d})_{\mathbb{Q}}.$$

Therefore, under the decomposition (52), we may present the relative ample class  $\eta$  as

$$(53) \quad \sigma = c_2([\Sigma]) + \theta$$

where  $\theta$  is induced by an ample divisor on  $J_{\Sigma}$ . The right-hand side is the pullback of a divisor on  $M^{\text{mero}}$  with the same tautological presentation (53). As in Section 2.4, pulling back this class yields relative ample classes on each variety  $M_{\eta}^{\text{par}}$  with  $\eta \in T$ .

In conclusion, we obtain the following fact.

**Fact 2.** There exists a divisor  $\sigma_0^{\text{par}} \in \text{Pic}(M_0^{\text{par}})_{\mathbb{Q}}$  satisfying

$$(54) \quad \pi^* \sigma = \iota^* \sigma_0^{\text{par}}$$

where  $\iota, \pi$  are defined in the diagram (15). Moreover, this divisor  $\sigma_0^{\text{par}}$  is obtained as the specialization of a relative ample divisor

$$\sigma_{\eta}^{\text{par}} \in \text{Pic}(M_{\eta}^{\text{par}})_{\mathbb{Q}}$$

for general  $\eta$ .

Note that in general the cup-product with respect to an algebraic cycle  $E$  is induced by a correspondence given by the  $\Delta_* E$ , where  $\Delta$  is the (relative) diagonal. Now we consider the correspondences

$$\Delta_* \sigma^i \in \text{Corr}_B^*(M_{n,d}, M_{n,d}), \quad \Delta_*(\sigma_0^{\text{par}})^i \in \text{Corr}_{W_0}^*(M_0^{\text{par}}, M_0^{\text{par}}).$$

**Fact 3.** We have

$$\Gamma \left( \Delta_*(\sigma_0^{\text{par}})^i \right) = \epsilon n! \cdot \Delta_* \sigma^i \in \text{Corr}_B^*(M_{n,d}, M_{n,d}).$$

In particular, after the homological realization we obtain the commutative diagram

$$\begin{array}{ccc} Rf_* \mathbb{Q}_{M_{n,d}} & \xrightarrow{\epsilon n! \cdot (\cup \sigma^i)} & Rf_* \mathbb{Q}_{M_{n,d}}[2i] \\ \downarrow [\widetilde{M}_0] & & \uparrow {}^t[\widetilde{M}_0] \\ Rh_{0*} \mathbb{Q}_{M_0^{\text{par}}} [2\binom{n}{2}] & \xrightarrow{\cup (\sigma_0^{\text{par}})^i} & Rh_{0*} \mathbb{Q}_{M_0^{\text{par}}} [2\binom{n}{2} + 2i]. \end{array}$$

*Proof of Fact 3.* It suffices to show the following relation of correspondences

$$(55) \quad {}^t[\widetilde{M}_0] \circ \Delta_*(\sigma_0^{\text{par}})^i = \Delta_* \sigma^i \circ {}^t[\widetilde{M}_0] \in \text{Corr}_{W_0}^*(M_0^{\text{par}}, M_{n,d}).$$

Then by Proposition 3.6(a), we have

$$\Gamma(\Delta_*(\sigma_0^{\text{par}})^i) = {}^t[\widetilde{M}_0] \circ \Delta_*(\sigma_0^{\text{par}})^i \circ [\widetilde{M}_0] = \Delta_* \sigma^i \circ {}^t[\widetilde{M}_0] \circ [\widetilde{M}_0] = \epsilon n! \cdot \Delta_* \sigma^i,$$

which proves Fact 3. To see the relation (55), we expand the left-hand side

$$\begin{aligned}
{}^t[\widetilde{M}_0] \circ \Delta_*(\sigma_0^{\text{par}})^i &= p_{13*} \delta^! \left( \Delta_*(\sigma_0^{\text{par}})^i \times {}^t[\widetilde{M}_0] \right) \\
&= p_{13*} \delta^! \left( \left( q_2^*(\sigma_0^{\text{par}})^i \cap [\Delta_{M_0^{\text{par}}/W_0}] \right) \times {}^t[\widetilde{M}_0] \right) \\
&= p_{13*} \delta^! \left( [\Delta_{M_0^{\text{par}}/W_0}] \times \left( r_1^*(\sigma_0^{\text{par}})^i \cap {}^t[\widetilde{M}_0] \right) \right) \\
&= r_1^*(\sigma_0^{\text{par}})^i \cap {}^t[\widetilde{M}_0] = {}^t(\iota^*(\sigma_0^{\text{par}})^i \cap [\widetilde{M}_0]).
\end{aligned}$$

Here  $\delta^!$  is the refined Gysin pullback with respect to the regular (absolute diagonal) embedding  $\Delta : M_0^{\text{par}} \rightarrow M_0^{\text{par}} \times M_0^{\text{par}}$ , and  $p_{13} : M_0^{\text{par}} \times_{W_0} M_0^{\text{par}} \times_{W_0} M_{n,d} \rightarrow M_0^{\text{par}} \times_{W_0} M_{n,d}$ ,  $q_2 : M_0^{\text{par}} \times_{W_0} M_0^{\text{par}} \rightarrow M_0^{\text{par}}$ , and  $r_1 : M_0^{\text{par}} \times_{W_0} M_{n,d} \rightarrow M_0^{\text{par}}$  are the natural projections. Similarly, we compute the right-hand side of (55):

$$\begin{aligned}
\Delta_* \sigma^i \circ {}^t[\widetilde{M}_0] &= p_{13*} \delta^! \left( {}^t[\widetilde{M}_0] \times \Delta_* \sigma^i \right) \\
&= p_{13*} \delta^! \left( {}^t[\widetilde{M}_0] \times \left( r_1^* \sigma^i \cap [\Delta_{M_{n,d}/B}] \right) \right) \\
&= p_{13*} \delta^! \left( \left( q_2^* \sigma^i \cap {}^t[\widetilde{M}_0] \right) \times [\Delta_{M_{n,d}/B}] \right) \\
&= q_2^* \sigma^i \cap {}^t[\widetilde{M}_0] = {}^t(\pi^* \sigma^i \cap [\widetilde{M}_0]).
\end{aligned}$$

This time  $\delta^!$  is the refined Gysin pullback with respect to  $\Delta : M_{n,d} \rightarrow M_{n,d} \times M_{n,d}$ , and  $p_{13}, q_2, r_1$  are the corresponding natural projections. Hence (55) follows from (54).  $\square$

Now we construct the cycle  $\mathfrak{Z}_{\sigma,i}$  for the relative ample class  $\sigma$  on  $M_{n,d}$ . We start with the cycle given by Fact 1 for  $\overline{J}_C^e$ . Its pullback to  $M_\eta^{\text{par}}$  yields a correspondence

$$\mathfrak{Z}_{\sigma_\eta^{\text{par}},i} \in \text{Corr}_{W_\eta}^*(M_\eta^{\text{par}}, M_\eta^{\text{par}})$$

which gives the inverse of the Lefschetz symmetry induced by  $(\sigma_\eta^{\text{par}})^i$  for general  $\eta$ . Moreover, since the cycle is pulled back from a self-correspondence of  $\overline{J}_C^e = M_\eta^{\text{par}}/\mathfrak{S}_n$ , its induced isomorphism respects the isotypic decomposition with respect to the  $\mathfrak{S}_n$ -action.

By Proposition 4.1, its specialization

$$(56) \quad \mathfrak{Z}_{\sigma_0^{\text{par}},i} \in \text{Corr}_{W_0}^*(M_0^{\text{par}}, M_0^{\text{par}})$$

gives the inverse of the Lefschetz symmetry induced by  $(\sigma_0^{\text{par}})^i$  and respects the decomposition given by the  $\mathfrak{S}_n$ -structure. In particular, the correspondence (56) induces an isomorphism on the perverse cohomology sheaves of the first summand of (51).

Finally, we set

$$\mathfrak{Z}_{\sigma,i} := \Gamma \left( \mathfrak{Z}_{\sigma_0^{\text{par}},i} \right) \in \text{Corr}_B^*(M_{n,d}, M_{n,d}).$$

As in the proof of Theorem 0.1 in Section 4.3.1, the diagram of Fact 3 together with Lemma 4.3 implies that the sheaf-theoretic operator  $\cup \sigma^i$  coincides with the restriction of the operator  $\cup (\sigma_0^{\text{par}})^i$  to the anti-invariant component with respect to the  $\mathfrak{S}_n$ -equivariant structure.

Since the inverse operator (56) of  $\cup(\sigma_0^{\text{par}})^i$  also respects the decomposition (51), we conclude that the homological realization of  $\mathfrak{Z}_{\sigma,i}$  gives the inverse of  $\cup\sigma^i$  as desired.  $\square$

4.3.3. *Proof of Theorem 0.3.* We consider the diagram

$$(57) \quad \begin{array}{ccc} \Sigma \times \widetilde{M}_0 & \xrightarrow{\iota} & \Sigma \times M_0^{\text{par}} \\ \downarrow \pi & & \\ \Sigma \times M_{n,d} & & \end{array}$$

given by the base change of (15). For convenience, we use the same notation for this diagram as for (15); *e.g.* we still use  $[\widetilde{M}_0]$  to denote the correspondence from  $\Sigma \times M_{n,d}$  to  $\Sigma \times M_0^{\text{par}}$ .

By the construction of the normalized Chern character, we have

$$\pi^* \widetilde{\text{ch}}(\mathcal{U}) = \iota^* \widetilde{\text{ch}}(\mathcal{U}).$$

Here the normalized Chern character on the left-hand side lies over  $\Sigma \times M_{n,d}$  and the normalized Chern character on the right-hand side lies over  $\Sigma \times M_0^{\text{par}}$ ; their match after pulling back to  $\Sigma \times \widetilde{M}_0$  follows from the fact that both normalized Chern characters are pulled back from  $\Sigma \times M^{\text{mero}}$ . By the same argument as for Fact 3 in Section 4.3.2, we have

$$(58) \quad \Gamma \left( \Delta_* \widetilde{\text{ch}}_k(\mathcal{U}) \right) = \epsilon n! \cdot \Delta_* \widetilde{\text{ch}}_k(\mathcal{U}) \in \text{Corr}_{\Sigma \times B}^*(\Sigma \times M_{n,d}, \Sigma \times M_{n,d})$$

where the two sides concern the normalized Chern characters on  $\Sigma \times M_0^{\text{par}}$  and  $\Sigma \times M_{n,d}$  respectively. Recall the projectors  $\mathfrak{p}_k$  from (15) and we use the same notation to denote the pullback of these projectors to the varieties in (57). We define

$$\mathfrak{q}_{k+1} := \Delta_{\Sigma \times M_{n,d}/\Sigma \times B} - \mathfrak{p}_k = \frac{\epsilon}{n!} \cdot \Gamma \left( \Delta_{\Sigma \times M_0^{\text{par}}/\Sigma \times B} - \mathfrak{p}_{k-\binom{n}{2}}^{\text{par}} \right),$$

where the second identity is a consequence of Proposition 3.6(a).

For our purpose, we want to prove the relation:

$$\mathfrak{q}_{i+k+1} \circ \Delta_* \widetilde{\text{ch}}_k(\mathcal{U}) \circ \mathfrak{p}_i = 0;$$

using (58) this can be rewritten as

$$(59) \quad \Gamma \left( \mathfrak{q}_{i+k+1-\binom{n}{2}}^{\text{par}} \right) \circ \Gamma \left( \Delta_* \widetilde{\text{ch}}_k(\mathcal{U}) \right) \circ \Gamma \left( \mathfrak{p}_{i-\binom{n}{2}}^{\text{par}} \right) = 0.$$

Since all the correspondences

$$\mathfrak{q}_{i+k+1-\binom{n}{2}}^{\text{par}}, \quad \Delta_* \widetilde{\text{ch}}_k(\mathcal{U}), \quad \mathfrak{p}_{i-\binom{n}{2}}^{\text{par}} \in \text{Corr}_{\Sigma \times W_0}^*(\Sigma \times M_0^{\text{par}}, \Sigma \times M_0^{\text{par}})$$

are given by specializations of cycles pulled back from correspondences for  $\overline{\mathcal{J}}_C^e$ , by Proposition 3.5 the relation (59) follows from the relation

$$\mathfrak{q}_{i+k+1-\binom{n}{2}}^{\text{par}} \circ \Delta_* \widetilde{\text{ch}}_k(\mathcal{U}) \circ \mathfrak{p}_{i-\binom{n}{2}}^{\text{par}} = 0 \in \text{Corr}_{\Sigma \times W_0}^*(\Sigma \times M_0^{\text{par}}, \Sigma \times M_0^{\text{par}}).$$

This is given by Theorem 2.15. The proof is completed.  $\square$

4.3.4. *Proof of Theorem 0.4.* We denote by

$$\mathfrak{C} \in \mathrm{Corr}_{W_0}^0(M_0^{\mathrm{par}}, M_0^{\mathrm{par}'}), \quad \mathfrak{C}' \in \mathrm{Corr}_{W_0}^0(M_0^{\mathrm{par}'}, M_0^{\mathrm{par}})$$

the correspondences constructed in Theorem 2.16 inducing an isomorphism

$$h(M_0^{\mathrm{par}}) \simeq h(M_0^{\mathrm{par}'}).$$

Then an identical argument as above (using Proposition 3.5) implies that

$$\bar{\mathfrak{C}} := \frac{\epsilon}{n!} \cdot \mathfrak{C} \in \mathrm{Corr}_B^0(M_{n,d}, M_{n,d'}), \quad \bar{\mathfrak{C}}' := \frac{\epsilon}{n!} \cdot \mathfrak{C}' \in \mathrm{Corr}_B^0(M_{n,d'}, M_{n,d})$$

induces an isomorphism as desired.  $\square$

4.3.5. *Remarks on the twisted case.* As in Remark 1.1, one can also consider the  $\mathfrak{D}$ -twisted case with  $\mathfrak{D} > 0$ . Here we discuss briefly the possible strategies to prove the main theorems in the  $\mathfrak{D}$ -twisted case.

The decomposition theorem associated with  $f : M_{n,d}^{\mathfrak{D}} \rightarrow B^{\mathfrak{D}}$  has full support; see Remark 1.1 and [14]. In this case, Theorem 0.2 follows directly from [1, Proposition 6.4] as Fact 1 above, and the argument of Section 4.3.3 is not needed.

On the other hand, due to the failure of the  $\mathfrak{D}$ -twisted version of Lemma 1.2, for general  $\eta \in T$  we can only guarantee that all the spectral curves are reduced, but there may exist non-integral ones. Therefore, the extension of Arinkin's Fourier–Mukai transform over reduced locally planar curves [41, 42] may be needed to construct the projectors for the motivic decomposition.

## 5. QUESTIONS AND CONJECTURES

In this section, we discuss some conjectures and open questions concerning algebraic cycles for the Hitchin system.

5.1. **Multiplicativity.** A central question raised by de Cataldo, Hausel, and Migliorini in accompany with their discovery of the P=W conjecture (see [9, Introduction]) is the *multiplicativity* of the perverse filtration:

$$(60) \quad \cup : P_k H^*(M_{n,d}, \mathbb{Q}) \times P_l H^*(M_{n,d}, \mathbb{Q}) \rightarrow P_{k+l} H^*(M_{n,d}, \mathbb{Q}).$$

This phenomenon now is a consequence of the resolutions of the P=W conjecture, whose proofs rely heavily on the tautological generation result of Markman [34].

In general, the multiplicativity holds for Leray filtrations associated with proper morphisms, but fails for the perverse filtration. The main reason is that the (sheaf-theoretic) cup-product does not interact nicely with the perverse truncation functor in general.

For the Hitchin system, we expect that the sheaf-theoretic enhancement of (60) holds.

**Conjecture 5.1.** *The cup-product interacts with the perverse truncation functor as follows*

$$\cup : {}^p\tau_{\leq k} Rf_* \mathbb{Q}_{M_{n,d}} \times {}^p\tau_{\leq l} Rf_* \mathbb{Q}_{M_{n,d}} \rightarrow {}^p\tau_{\leq k+l-(\dim M_{n,d}-R_f)} Rf_* \mathbb{Q}_{M_{n,d}}.$$

By the proof of Theorem 0.1, the perverse filtration and its sheaf-theoretic enhancement are governed by the algebraic cycles  $\mathfrak{p}_k, \mathfrak{q}_k$ . We propose the following conjecture which further strengthens Conjecture 5.1; this also provides an explanation of the multiplicativity phenomenon in terms of *tautological relations* concerning  $\mathfrak{p}_k, \mathfrak{q}_k$ .

**Conjecture 5.2.** *The motivic perverse filtration (5) is multiplicative with respect to cup-product, i.e., the following relations hold*

$$\mathfrak{q}_{k+l+1} \circ [\Delta_{M_{n,d}/B}^{\text{sm}}] \circ (\mathfrak{p}_k \times \mathfrak{p}_l) = 0 \in \text{CH}_*(M_{n,d} \times_B M_{n,d} \times_B M_{n,d}).$$

Here  $\Delta_{M_{n,d}/B}^{\text{sm}}$  is the relative small diagonal in the relative triple product.

By Theorem 2.6(b), Conjecture 5.2 holds for the parabolic Hitchin system  $h_\eta : M_\eta^{\text{par}} \rightarrow W_\eta$  for general  $\eta$  or the elliptic part of the Hitchin system  $f^{\text{ell}} : M_{n,d}^{\text{ell}} \rightarrow B^{\text{ell}}$ . From the perspective of this paper, the main difficulty in approaching Conjecture 5.2 is to understand the small diagonal cycle for  $M_{n,d}$  using parabolic moduli spaces.

**5.2. Refined motivic decomposition.** Support theorems yield rich structures for the decomposition theorem associated with the Hitchin system. For example, by studying the decomposition theorem over the locus  $B^{\text{red}} \subset B$  of the Hitchin base formed by reduced spectral curves, de Cataldo, Heinloth, and Migliorini showed in [10] that every partition of  $n$  contributes a support of  $Rf_*\mathbb{Q}_{M_{n,d}}$ . Therefore, following the philosophy of Corti–Hanamura [16], we expect to have the motives which lift the contribution of the support indexed by each partition.

**Question 5.3.** *Can we find algebraic cycles which yield a refinement of the motivic decomposition (5) based on supports of the decomposition theorem for  $f : M_{n,d} \rightarrow B$ ?*

The conjectural refinement of Question 5.3 is mysterious for the following reasons. As we commented in Remark 1.1, if we consider the decomposition theorem for the  $\mathfrak{D}$ -twisted Hitchin system, the support theorem behaves very differently for  $\mathfrak{D} = 0$  and  $\mathfrak{D} > 0$ . In particular, the decomposition theorem has full support when  $\mathfrak{D} > 0$ . Therefore, the conjectural motives of Question 5.3, which lift the contribution of smaller supports of the decomposition theorem, should only exist for  $\mathfrak{D} = 0$ .

**5.3. Singular moduli spaces.** When the degree  $d$  is not coprime to the rank  $n$ , the Hitchin moduli space  $M_{n,d}$  is singular. Many properties concerning the decomposition theorem and the perverse filtration in the coprime setting are expected to be generalized to the non-coprime setting. We expect that there should exist algebraic cycles which induce these sheaf-theoretic structures.

A challenging question in this direction is to construct relative *BPS motives* for all  $M_{n,d}$  without the coprime assumption. More precisely, from the perspective of cohomological

Donaldson–Thomas theory the role played by the constant sheaf  $\mathbb{Q}_{n,d}$  should be replaced by a mysterious perverse sheaf, called the BPS sheaf

$$\Phi_{\text{BPS},d} \in \text{Perv}(M_{n,d}),$$

when  $M_{n,d}$  is possibly singular. This perverse sheaf recovers the (shifted) constant sheaf  $\mathbb{Q}_{M_{n,d}}[\dim M_{n,d}]$  when  $(n,d) = 1$ , but is in general more complicated than the intersection cohomology complex  $\text{IC}_{M_{n,d}}$ ; see [46] or [36, Section 0.4]. The sheaf-theoretic  $\chi$ -independence result (8) is generalized by Kinjo–Koseki [30] using the BPS perverse sheaf:

$$(61) \quad Rf_{d*}\Phi_{\text{BPS},d} \simeq Rf_{d'}*\Phi_{\text{BPS},d'}.$$

**Question 5.4.** *Can we lift the sheaf-theoretic  $\chi$ -independence identity (61) to relative Chow motives over the Hitchin base  $B$ ?*

When  $(n,d) \neq 1$ , we have

$$\Phi_{\text{BPS},d} = \text{IC}_{M_{n,d}} \oplus \varphi_d$$

where  $\varphi_d$  is a perverse sheaf supported on the singular locus of  $M_{n,d}$ . It would be interesting to see how the extra summand  $\varphi_d$  is detected by algebraic cycles. The BPS motive which lifts  $Rf_{d*}\Phi_{\text{BPS},d}$  may be viewed as the motivic Gopakumar–Vafa invariant associated with the local Calabi–Yau 3-fold  $X := \text{Tot}_\Sigma(\omega_\Sigma \oplus \mathcal{O}_\Sigma)$ .

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