THE PERIOD–INDEX PROBLEM FOR HYPER-KÄHLER VARIETIES VIA HYPERHOLOMORPHIC BUNDLES

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Abstract. We prove new bounds for the period–index problem for hyper-Kähler varieties of $K3^{[n]}$-type using projectively hyperholomorphic bundles constructed by Markman. We show that $\dim(X)$ is a bound for any X of $K3^{[n]}$-type. We also show that $\frac{1}{2}\dim(X)$ is a bound for most Brauer classes when the Picard rank of X is at least two, providing evidence for a conjecture of Huybrechts.

Contents

0. Introduction

Throughout, we work over the complex numbers $\mathbb{C}$.

0.1. **The period–index problem.** Let X be a nonsingular projective variety, and let $\alpha \in \mathrm{Br}(X)$ be a class in the Brauer group of X. Two basic invariants of α are its *period* and its *index*. The period $\mathrm{per}(\alpha)$ is nothing more than the order of α in $\mathrm{Br}(X)$, which is a torsion group, while the index is given by $\mathrm{ind}(\alpha) = \gcd\{\sqrt{\mathrm{rk}(\mathcal{A})}\}$, as $\mathcal{A}$ runs over the Azumaya algebras of class α. It is an elementary fact that $\mathrm{per}(\alpha)$ and $\mathrm{ind}(\alpha)$ share the same prime factors, and that $\mathrm{per}(\alpha)$ divides $\mathrm{ind}(\alpha)$.

In this paper, we are concerned with the *period–index problem* for the variety X, which is the problem of determining the least positive integer $e(X)$ such that the inequality

$$\mathrm{ind}(\alpha) \mid \mathrm{per}(\alpha)^{e(X)} \tag{1}$$

holds for all $\alpha \in \mathrm{Br}(X)$. The motivation for the problem comes from the theory of central simple algebras over the function field $\mathbb{C}(X)$: There is a natural inclusion $\mathrm{Br}(X) \subset \mathrm{Br}(\mathbb{C}(X))$,

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and the index of a class $\alpha \in \mathrm{Br}(X)$ coincides with $\deg(D) := \sqrt{\dim_{\mathbb{C}(X)}(D)}$, where D is the unique division algebra over $\mathbb{C}(X)$ of class α, viewed as an element of $\mathrm{Br}(\mathbb{C}(X))$. Determining the degree of a central division algebra over a field in terms of its period is a fundamental classical problem, originating in the study of Brauer groups of local and global fields from the early part of the twentieth century; for an introduction, see [1, Section 4] or [3].

If X is a surface, then $\mathrm{per}(\alpha) = \mathrm{ind}(\alpha)$ for all $\alpha \in \mathrm{Br}(X)$ by a theorem of de Jong [5]. In higher dimensions, however, little is known, beyond the fact (proved in [11]) that a bound as in (1) exists. From the longstanding period–index conjecture for the function field $\mathbb{C}(X)$ from [4], one expects

$$\mathrm{ind}(\alpha) \mid \mathrm{per}(\alpha)^{\dim(X)-1} \tag{2}$$

for Brauer classes on all nonsingular projective varieties X; moreover, there are varieties of each dimension for which the bound (2) is sharp [2]. In [8], the period–index conjecture (*i.e.*, $\mathrm{ind}(\alpha) \mid \mathrm{per}(\alpha)^2$) was proven for Brauer classes on abelian threefolds.

In [10], Huybrechts studied the period–index problem for hyper-Kähler varieties, and proved the following result concerning Lagrangian fibrations. It shows that if X admits a Lagrangian fibration, then $e(X)$ can be far smaller than the exponent $\dim(X) - 1$ occuring in the period–index conjecture, at least if $\mathrm{per}(\alpha)$ avoids finitely many primes.

Theorem 0.1 (Huybrechts [10])**.** *Let X be a hyper-Kähler variety which admits a Lagrangian fibration. Then there exists an integer N_X such that*

$$\mathrm{ind}(\alpha) \mid \mathrm{per}(\alpha)^{\frac{\dim(X)}{2}} \tag{3}$$

for all $\alpha \in \mathrm{Br}(X)$ with $\mathrm{per}(\alpha)$ coprime to N_X.

An explicit description of N_X was given in [10, (0.1)]. Huybrechts further conjectured that the equation (3) holds with $N_X = 1$ for any hyper-Kähler variety X [10, Conjecture 0.2], and verified it for the Hilbert scheme of points on a $K3$ surface [10, Theorem 0.4].

The purpose of this note is to study the period–index problem for hyper-Kähler varieties of $K3^{[n]}$-type. Using Markman's projectively hyperholomorphic bundles from [14], we are able to obtain new bounds for the period–index problem. For varieties of $K3^{[n]}$-type, our results also recover and provide a new proof of Theorem 0.1. The idea of applying projectively hyperholomorphic bundles to classical problems about Brauer groups goes back to the work of Huybrechts and Schröer [12] on Grothendieck's question for Brauer groups of analytic $K3$ surfaces.

0.2. **Hyper-Kähler varieties of $K3^{[n]}$-type.** Throughout Section 0.2, we consider X a hyper-Kähler variety of $K3^{[n]}$-type, *i.e.*, it is deformation equivalent to the Hilbert scheme of n points on a $K3$ surface. As the $K3$ surface case has been dealt with, we assume $n \geq 2$.

Our first result provides a uniform bound for any X of $K3^{[n]}$-type. For notational convenience, we set

$$I_X := \text{Index}\left(T(X) \hookrightarrow H^2(X,Z)/\text{Pic}(X)\right)$$

where $T(X) \subset H^2(X,\mathbb{Z})$ is the transcendental lattice. Note that the embedding of the lattices on the right-hand side is always of finite index.

Theorem 0.2. *Assume that X admits a primitive polarization of degree $2h$. We have*

$$\text{ind}(\alpha) \mid \text{per}(\alpha)^{\dim(X)} \tag{4}$$

for all $\alpha \in \text{Br}(X)$ with $\text{per}(\alpha)$ *coprime to $n!hI_X$.*

In Section 2.3, we discuss numerical obstructions to proving Huybrechts' conjecture (3) using our method. In particular, our approach is not expected to be sufficient to prove Huybrechts' conjecture for the Picard rank 1 case.

Remark 0.3. To the best of our knowledge, (4) gives the best bound currently known for $K3^{[n]}$-type varieties of Picard rank 1. In [10, Section 2.6], Huybrechts proved that the Fano variety of lines $F(Y)$ of a nonsingular cubic 4-fold Y, which is a variety of $K3^{[2]}$-type, satisfies

$$\text{ind}(\alpha) \mid \text{per}(\alpha)^5, \quad \alpha \in \text{Br}(F(Y))$$

if $\text{per}(\alpha)$ is coprime to some uniform integer depending on Y. In this case, Theorem 0.2 reads

$$\text{ind}(\alpha) \mid \text{per}(\alpha)^4$$

when $\text{per}(\alpha)$ avoids finitely many prime factors.

When X has higher Picard rank, much better bounds can be obtained. For further discussions, we need to introduce some notation. Recall the explicit description of the Brauer group

$$\text{Br}(X) = \left(\frac{H^2(X,\mathbb{Z})}{\text{Pic}(X)}\right) \otimes \mathbb{Q}/\mathbb{Z},$$

which allows us to present a Brauer class in the form of a "B-field":

$$\alpha = \left[\frac{\mathcal{B}}{\ell}\right]. \tag{5}$$

Here ℓ is a positive integer, and $\mathcal{B}$ is an element of the transcendental lattice $T(X) \subset H^2(X,\mathbb{Z})$ which is not divisible by any prime factor of ℓ. We say that the Brauer class (5) is *non-special* if

$$\gcd(q(\mathcal{B}),\ell) = 1,$$

where $q(\mathcal{B}) \in \mathbb{Z}$ is the norm with respect to the Beauville–Bogomolov–Fujiki (BBF) form $q(-,-)$ on $H^2(X,\mathbb{Z})$. If $\text{per}(\alpha)$ is coprime to I_X, we further have

$$\text{per}(\alpha) = \ell.$$

The following result is our main tool of constructing Brauer classes satisfying (3).

Theorem 0.4. *For a Brauer class* $\alpha = \left[\frac{\mathcal{B}}{\ell}\right]$ *with* $\mathrm{per}(\alpha)$ *coprime to* $n!I_X$*, if there exist classes* $\mathcal{D}, \mathcal{D}' \in \mathrm{Pic}(X)$ *and a positive integer* d *with* $\gcd(d, \ell) = 1$ *satisfying that*

(i) $q(\mathcal{D}) \equiv -q(\mathcal{B})d^2 \mod \ell$,

(ii) $\gcd\left(q(\mathcal{D}, \mathcal{D}'), q(\mathcal{B}), \ell\right) = 1$,

then (3) holds for α.

As an application, Theorem 0.4 implies immediately Theorem 0.1 for hyper-Kähler varieties of $K3^{[n]}$-type; see Section 2.5.

The next theorem is also a consequence of Theorem 0.4, which verifies (3) for most Brauer classes when the Picard rank is at least 2.

Theorem 0.5. *If* X *has Picard rank at least* 2*, then there exists an integer* N_X *such that*

$$\mathrm{ind}(\alpha) \mid \mathrm{per}(\alpha)^{\frac{\dim(X)}{2}}$$

for all non-special $\alpha \in \mathrm{Br}(X)$ *with* $\mathrm{per}(\alpha)$ *coprime to* N_X.

Here, the constant N_X is explicitly determined by I_X and the BBF form on $\mathrm{Pic}(X)$.

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1. Projectively hyperholomorphic bundles

Throughout this section, we assume that X is a hyper-Kähler variety of $K3^{[n]}$-type with $n \geq 2$, and $\alpha = \left[\frac{\mathcal{B}}{\ell}\right]$ is a Brauer class with $\mathrm{per}(\alpha)$ coprime to I_X; in particular we identify ℓ and $\mathrm{per}(\alpha)$. The main result of this section is Proposition 1.3.

1.1. **Brauer classes.** The period–index problem for X can be reduced to a geometric question concerning the existence of certain twisted vector bundles. A comparison of different notions of the index (e.g., the fact that the index of α viewed in $\mathrm{Br}(X)$ coincides with the index of α viewed in $\mathrm{Br}(\mathbb{C}(X))$) may be found in [6].

We write the Brauer class $\alpha = \left[\frac{\mathcal{B}}{\ell}\right]$ on X as in (5), and consider α-twisted vector bundles on X. The index of α is bounded above by the rank of any α-twisted vector bundle $\mathcal{F}$ on X:

$$\mathrm{ind}(\alpha) \mid \mathrm{rk}(\mathcal{F}). \tag{6}$$

Indeed, $\mathscr{H}om(\mathcal{F}, \mathcal{F})$ is an Azumaya algebra on X of class α. The following lemma is an immediate consequence.

Lemma 1.1. *Assume* $\alpha = \left[\frac{\mathcal{B}}{\ell}\right]$ *with* $\mathrm{per}(\alpha) = \ell$. *If there exists an* α*-twisted vector bundle whose rank divides* $c\ell^e$ *for positive integers* c, e *with* $\gcd(c, \ell) = 1$, *then we have*

$$\mathrm{ind}(\alpha) \mid \mathrm{per}(\alpha)^e.$$

Proof. By (6) we have

$$\mathrm{ind}(\alpha) \mid c\ell^e.$$

Since $\mathrm{ind}(\alpha)$ and $\ell = \mathrm{per}(\alpha)$ share the same prime factors, the assumption that c, ℓ are coprime implies

$$\gcd(c, \mathrm{ind}(\alpha)) = 1.$$

Hence we must have

$$\mathrm{ind}(\alpha) \mid \ell^e = \mathrm{per}(\alpha)^e.$$

□

1.2. **Projectively hyperholomorphic bundles.** Markman constructed in [14] a class of projectively hyperholomorphic bundles. They are our main sources of twisted vector bundles in view of Lemma 1.1. We briefly review the construction as follows, and refer to [14] and [15] for details.

Consider a projective $K3$ surface S with $\mathrm{Pic}(S) = \mathbb{Z}H$, and a primitive and isotropic Mukai vector

$$v_0 := (r, mH, s) \in H^*(S, \mathbb{Z}), \quad v_0^2 = 0$$

with

$$m \in \mathbb{Z}, \quad \gcd(r, s) = 1,$$

satisfying

$$\frac{r}{\rho} \nmid \frac{1}{2}\left(\frac{mH}{\rho}\right)^2 + 1, \quad \rho := \gcd(r, m). \tag{7}$$

Clearly (7) guarantees that $r \geq 2$. By [16, Lemma 1.2], the numerical condition (7) implies that the moduli space M of stable vector bundles on S with Mukai vector v_0 is again a $K3$ surface. Furthermore, the coprime condition on r, s ensures the existence of a universal rank r bundle $\mathcal{U}$ on $M \times S$. Conjugating the Bridgeland–King–Reid (BKR) correspondence yields a vector bundle $\mathcal{U}^{[n]}$ on $M^{[n]} \times S^{[n]}$ of rank

$$\mathrm{rk}(\mathcal{U}^{[n]}) = n!r^n;$$

see [14, Lemma 7.1]. Markman further showed in [14, Section 5.6] that the characteristic class of $\mathcal{U}^{[n]}$ induces a Hodge isometry

$$\phi_{\mathcal{U}^{[n]}} : H^2(M^{[n]}, \mathbb{Q}) \to H^2(S^{[n]}, \mathbb{Q}). \tag{8}$$

Let Λ be the $K3^{[n]}$-lattice; the Hodge isometry above induces an abstract isometry $\phi \in O(\Lambda_{\mathbb{Q}})$ on the lattice $\Lambda_{\mathbb{Q}}$ up to equivalences.

We now consider the moduli space $\mathfrak{M}_\phi$ of isomorphism classes of quadruples (Y, η_Y, X, η_X) where $(Y, \eta_Y), (X, \eta_X)$ are marked hyper-Kähler manifolds of $K3^{[n]}$-type and

$$\eta_X^{-1} \circ \phi \circ \eta_Y : H^2(Y, \mathbb{Q}) \to H^2(X, \mathbb{Q})$$

is a Hodge isometry sending some Kähler class of Y to a Kähler class of X. Let $\mathfrak{M}_\phi^\circ \subset \mathfrak{M}_\phi$ be the connected component which contains the quadruple

$$(M^{[n]}, \eta_{M^{[n]}}, S^{[n]}, \eta_{S^{[n]}})$$

canonically induced by (8). A key result proved in [14], which is crucial for our purpose, is that the bundle $\mathcal{U}^{[n]}$ is a projectively hyperholomorphic bundle which deforms along diagonal twistor paths in $\mathfrak{M}_\phi^0$. In particular, for every point

$$(Y, \eta_Y, X, \eta_X) \in \mathfrak{M}_\phi^0,$$

there exists a twisted vector bundle

$$\mathcal{E} \rightsquigarrow Y \times X$$

with respect to the Brauer class

$$\alpha_{\mathcal{E}} := \left[-\frac{c_1(\mathcal{U}^{[n]})}{n!r^n} \right].$$

Here we view $H^2(Y \times X, \mathbb{Z}) = H^2(Y, \mathbb{Z}) \oplus H^2(X, \mathbb{Z})$ as a trivial local system over the moduli space $\mathfrak{M}_\phi^0$ via the markings, and therefore $c_1(\mathcal{U}^{[n]})$ is a well-defined class over $Y \times X$:

$$c_1(\mathcal{U}^{[n]}) = \alpha_Y + \alpha_X \in H^2(Y, \mathbb{Z}) \oplus H^2(X, \mathbb{Z}), \quad \alpha_Y \in H^2(Y, \mathbb{Z}), \quad \alpha_X \in H^2(X, \mathbb{Z}).$$

Further restricting $\mathcal{E}$ over a point $y \in Y$, we obtain a $[-\frac{\alpha_X}{n!r^n}]$-twisted vector bundle of rank

$$\mathrm{rk}\,(\mathcal{E}_y) = n!r^n.$$

Note that under the canonical identification

$$H^2(S^{[n]}, \mathbb{Z}) = H^2(S, \mathbb{Z}) \oplus \mathbb{Z}\delta, \quad \delta^2 = 2 - 2n,$$

the explicit formula of $\alpha_{S^{[n]}}$ over the Hilbert scheme $S^{[n]}$ is given by

$$\alpha_{S^{[n]}} = n!r^n \left(\frac{mH}{r} - \frac{\delta}{2} \right);$$

see [14, Equation (7.11)].

We now summarize the discussion above in the following proposition.

Proposition 1.2. *Let X be a variety of $K3^{[n]}$-type with a Brauer class $\alpha = \left[\frac{\mathcal{B}}{\ell}\right]$. Assume that there exist*

(i) *a projective $K3$ surface with* $\mathrm{Pic}(S) = \mathbb{Z}H$,

(ii) *two integers r, m with*

$$2r \mid m^2H^2, \quad \gcd\left(r, \frac{m^2H^2}{2r}\right) = 1,$$

satisfying (7).

(iii) *a parallel transport $\rho : H^2(S^{[n]}, \mathbb{Q}) \to H^2(X, \mathbb{Q})$ satisfying*

$$\rho\left(-\frac{mH}{r} + \frac{\delta}{2}\right) = \frac{\mathcal{B}}{\ell} + \mathcal{A} + \mathcal{D},$$

where $\mathcal{A} \in H^2(X, \mathbb{Z})$ and $\mathcal{D} \in \mathrm{Pic}(X)_{\mathbb{Q}}$,

Then there is an α-twisted vector bundle on X of rank $n!r^n$.

Proof. By (ii) we may set

$$s := \frac{m^2H^2}{2r} \in \mathbb{Z}_{>0},$$

which satisfies $\gcd(r, s) = 1$. The Mukai vector $v_0 = (r, mH, s)$ is primitive and isotropic satisfying (7), from which we can construct the $K3$ surface M as a fine moduli space, the vector bundle $\mathcal{U}$ on $M \times S$, and ultimately the vector bundle $\mathcal{U}^{[n]}$ on $M^{[n]} \times S^{[n]}$. Then the discussion before Proposition 1.2 yields a twisted vector bundle $\mathcal{E}_y$ of the desired rank with respect to the Brauer class

$$\left[-\frac{\alpha_X}{n!r^n}\right] = \left[-\frac{\rho(\alpha_{S^{[n]}})}{n!r^n}\right] = \left[\frac{\mathcal{B}}{\ell} + \mathcal{A} + D\right] = \left[\frac{\mathcal{B}}{\ell}\right] \in \left(\frac{H^2(X, \mathbb{Z})}{\mathrm{Pic}(X)}\right) \otimes \mathbb{Q}/\mathbb{Z}. \qquad \square$$

1.3. **Removing technical assumptions.** The coprime condition in Proposition 1.2(ii) is to guarantee that M is a fine moduli space over S. The following result is a strengthened version of Proposition 1.2, which proves that this coprime condition, as well as the primitivity assumption of v_0, can be completely removed. This is more convenient in practice.

Proposition 1.3. *Assume all the assumptions of Proposition 1.2 are satisfied, except the coprime assumption*

$$\gcd\left(r, \frac{m^2H^2}{2r}\right) = 1. \tag{9}$$

There is an α-twisted vector bundle on X whose rank divides $n!r^n$.

Proof. We proceed with the following two steps.

Step 1. We first show that, if we only remove the coprime assumption (9) from Proposition 1.2(ii) while keeping the Mukai vector $v_0 = \left(r, mH, \frac{m^2H^2}{2r}\right)$ primitive, then there exists an α-twisted vector bundle of rank $n!r^n$.

More precisely, assume that we have a $K3$ surface S and two integers r, m without (9). The strategy is that we can always achieve our goal by moving via a parallel transport to a different $K3$ surface S' which satisfies the coprime condition. The construction is as follows.

Set $s := \frac{m^2H^2}{2r}$. We first claim that we can pick $k \in \mathbb{Z}_{>0}$ such that

$$\gcd(r, s + km) = 1. \tag{10}$$

Indeed, since $v_0 = (r, mH, s)$ is primitive, we must have

$$\gcd\left(r, \gcd(s, m)\right) = 1.$$

Therefore

$$\gcd(r, s + km) = \gcd\left(r, \frac{s}{\gcd(s, m)} + k\frac{m}{\gcd(s, m)}\right).$$

Since $\frac{s}{\gcd(s,m)}, \frac{m}{\gcd(s,m)}$ are coprime, a desired k clearly exists.

Now, we pick $L \in H^2(S, \mathbb{Z})$ with

$$L \cdot H = 1, \quad L^2 > 0.$$

We set

$$s' := \frac{(mH + rkL)^2}{2r}.$$

Since $m^2H^2 = 2rs$, a direct calculation yields

$$s' = s + km + rk^2 \cdot \frac{L^2}{2} \in \mathbb{Z}_{>0},$$

and consequently

$$\gcd(r, s') = \gcd(r, s + km) = 1$$

where we used (10) in the last equation.

Finally, we find a projective $K3$ surface S' of Picard rank 1 and a parallel transport

$$\varphi : H^2(S, \mathbb{Z}) \to H^2(S', \mathbb{Z})$$

such that the class $\varphi(mH + rkL) \in H^2(S', \mathbb{Z})$ is algebraic; the existence of S' is guaranteed by the global Torelli theorem. The new $K3$ surface S' and the Mukai vector

$$v_0' := (r, \varphi(mH + rkL), s') \in H^*(S', \mathbb{Z})$$

satisfies

$$v_0'^2 = 0, \quad \gcd(r, s') = 1,$$

and (7), to which we can apply Proposition 1.2. This completes Step 1.

Step 2. In general, for a non-necessarily primitive Mukai vector

$$v_0 = \left(r, mH, \frac{m^2H^2}{2r}\right),$$

we can write

$$v_0 = av_0'$$

with $a \in \mathbb{Z}_{>0}$ and v_0' primitive. Then applying Step 1 to v_0' shows that there exists an α-twisted vector bundle with rank $n!\left(\frac{r}{a}\right)^n$, which clearly divides $n!r^n$ as desired.

We have completed the proof of the proposition. □

In the next two sections, we explore how to use Markman's projectively hyperholomorphic bundle, via Proposition 1.3, to give new effective bounds for the period–index problem. This question is closely related to the arithmetic of the BBF form on $\mathrm{Pic}(X)$.

2. Proofs of Theorems 0.2 and 0.4

In this section, we still assume that X is a hyper-Kähler variety of $K3^{[n]}$-type with $n \geq 2$. Our purpose is to prove Theorems 0.2 and 0.4. Since for both theorems we assume that $\mathrm{per}(\alpha)$ is coprime to I_X, the period $\mathrm{per}(\alpha)$ is always ℓ.

2.1. **Lattice theory.** The second cohomology of X, endowed with the BBF form $q(-,-)$, is given by

$$H^2(X,\mathbb{Z}) \simeq U^{\oplus 3} \oplus E_8(-1)^{\oplus 2} \oplus \langle 2-2n\rangle, \tag{11}$$

where U stands for the rank 2 hyperbolic lattice. The *divisibility* of a class $L \in H^2(X,\mathbb{Z})$ is the positive generator $\mathrm{div}(L)$ of the subgroup

$$\left\{q(L,L') \mid L' \in H^2(X,\mathbb{Z})\right\} \subset \mathbb{Z}.$$

We recall the following obvious fact.

Lemma 2.1. *The divisibility of any primitive class $L \in H^2(X,\mathbb{Z})$ satisfies*

$$\mathrm{div}(L) \mid 2n-2.$$

Proof. This follows directly from the definition, combined with the fact that the first two summands of the right-hand side of (11) form a unimodular lattice. □

For a given class $L \in H^2(X,\mathbb{Z})$, we use $L^\perp$ to denote the sub-lattice of $H^2(X,\mathbb{Z})$ given by the orthogonal complement of L with respect to the BBF form $q(-,-)$.

Lemma 2.2. *Assume that $\mathcal{B},\mathcal{D}$ are two primitive classes in $H^2(X,\mathbb{Z})$ with $\mathcal{B}$ lying in the transcendental part $T(X) \subset H^2(X,\mathbb{Z})$ and $\mathcal{D} \in \mathrm{Pic}(X)$. There exists a class $\mathcal{A}$ lying in the sub-lattice $\mathcal{B}^\perp \cap \mathcal{D}^\perp \subset H^2(X,\mathbb{Z})$, which is of divisibility 1 in the lattice $\mathcal{B}^\perp \cap \mathcal{D}^\perp$.*

Proof. We fix an isomorphism of lattices

$$H^2(X,\mathbb{Z}) \simeq U^{\oplus 3} \oplus E_8(-1)^{\oplus 2} \oplus \mathbb{Z}\delta, \quad \delta^2 = 2-2n,$$

where the right-hand side is naturally embedded into the unimodular lattice

$$\Lambda := U^{\oplus 4} \oplus E_8(-1)^{\oplus 2},$$

known as the Markman–Mukai lattice. We consider the saturation of the sub-lattice generated by $\mathcal{B},\mathcal{D},\delta$ with the canonical primitive embedding:

$$\iota : \Lambda' := \langle \mathcal{B},\mathcal{D},\delta\rangle_{\mathrm{sat}} \hookrightarrow \Lambda.$$

On the other hand, we know from [9, Proposition 1.8] that there is a primitive embedding of abstract lattices:

$$\Lambda' \hookrightarrow U^{\oplus 3}.$$

Combined with the natural embedding

$$U^{\oplus 3} \oplus U' \hookrightarrow \Lambda$$

given by the definition of Λ, we obtain

$$\iota' : \Lambda' \hookrightarrow U^{\oplus 3} \xrightarrow{(\mathrm{id},0)} U^{\oplus 3} \oplus U' \hookrightarrow \Lambda.$$

Then the uniqueness part of [9, Corollary 1.9] implies that ι, ι' differ by an automorphism of Λ. In other words, there exists an isomorphism

$$\Lambda = U^{\oplus 3} \oplus U' \oplus E_8(-1)^{\oplus 2}$$

such that the image of $\iota : \Lambda' \hookrightarrow \Lambda$ lies completely in the first factor $U^{\oplus 3}$. A primitive vector

$$\mathcal{A} \in U' \cap H^2(X, \mathbb{Z}) \subset \Lambda$$

satisfies the desired properties. □

2.2. **Proof of Theorem 0.2.** Let $\mathcal{D} \in \mathrm{Pic}(X)$ be a primitive polarization of degree

$$q(\mathcal{D}) = 2h > 0.$$

We fix a Brauer class

$$\alpha = \left[\frac{\mathcal{B}}{\ell}\right] \in \mathrm{Br}(X), \quad \mathcal{B} \in T(X), \quad \mathrm{per}(\alpha) = \ell;$$

we further assume that

$$\gcd(\ell, n!h) = 1. \tag{12}$$

Our goal is to construct an α-twisted vector bundle on X whose rank divides $\left(2^{2n} n!\right) \cdot \ell^{2n}$. Then by Lemma 1.1 we have

$$\mathrm{ind}(\alpha) \mid \mathrm{per}(\alpha)^{2n}$$

which implies Theorem 0.2.

We choose $\mathcal{A} \in H^2(X, \mathbb{Z})$ as in Lemma 2.2 which we fix from now on. Define

$$L_u := 2\ell\mathcal{A} + 2\mathcal{B} + u\mathcal{D} \in H^2(X, \mathbb{Z}), \quad u \in \mathbb{Z}_{>0}.$$

Proposition 2.3. *There exist odd integers u_1, u_2 with*

$$\ell \mid u_1, \quad \gcd(u_2, \ell) = 1$$

satisfying that for both $i = 1, 2$:

(i) L_{u_i} *is primitive,*
(ii) $q(L_{u_i}) > 0$,
(iii) $\mathrm{div}(L_{u_i}) = 1$ *or* 2.

Proof. We complete the proof via two steps.

Step 1: finding u_1. Let u_1 be a sufficiently large odd multiple of ℓ, so that (ii) holds. We check in the following that the conditions (i, iii) are satisfied for L_{u_1}.

Note that $\frac{L_{u_1}}{2}$ is not integral since u_1 is odd. For (i) it suffices to show that $\frac{L_{u_1}}{p}$ cannot be integral for any odd prime number p.

Assume that $\frac{L_{u_1}}{p} \in H^2(X,\mathbb{Z})$ with p an odd prime number. Then by the definition of $\mathcal{A}$, there exists $L \in H^2(X,\mathbb{Z})$ such that

$$q(L,\mathcal{A}) = 1, \quad q(L,\mathcal{B}) = q(L,\mathcal{D}) = 0. \tag{13}$$

Therefore we have

$$q\left(L, \frac{L_{u_1}}{p}\right) = \frac{2\ell}{p} \in \mathbb{Z},$$

which implies

$$p \mid \ell. \tag{14}$$

Now, if we have $\ell \mid u_1$, then it cannot happen that

$$\frac{L_{u_1}}{p} \in H^2(X,\mathbb{Z}).$$

Otherwise, since p is a prime factor of ℓ by (14), this implies

$$\frac{2\mathcal{B}}{p} \in H^2(X,\mathbb{Z}),$$

contradicting the assumption that $\mathrm{per}(\alpha) = \ell$.

We then observe that (iii) holds automatically for all the L_{u_1} satisfying (i, ii).

Let L be a class satisfying (13). We have

$$q(L, L_{u_1}) = 2\ell.$$

Consequently

$$\mathrm{div}(L_{u_1}) \mid 2\ell.$$

On the other hand, by (i) and Lemma 2.1, we obtain

$$\mathrm{div}(L_{u_1}) \mid 2n-2.$$

Then the proposition follows from

$$\gcd(2\ell, 2n-2) = 2$$

which is given by the assumption (12).

Step 2: finding u_2. Let u_2 be a sufficiently large odd positive integer coprime to ℓ. By the observation above, it suffices to check that L_{u_2} is primitive.

If L_{u_2} is divisible by an odd prime number p, we deduce from (14) that p is a factor of ℓ. On the other hand, we know that

$$p \mid q(L_{u_2}, \mathcal{D}) = u_2 q(\mathcal{D}).$$

By our assumptions on ℓ, u_2, we have

$$\gcd(\ell, u_2 q(\mathcal{D})) = \gcd(\ell, 2u_2 h) = 1.$$

This is a contradiction; hence L_{u_2} has to be primitive. □

We are now ready to prove Theorem 0.2; we separate the two cases:

(A) $\ell \nmid 2q(\mathcal{B}) + 1$;
(B) $\ell \mid 2q(\mathcal{B}) + 1$.

Proof in the case (A). We pick u_1 given by Proposition 2.3 and consider the class

$$L_{u_1} \in H^2(X, \mathbb{Z}).$$

Since for any u we have

$$\frac{1}{2}q(L_u) \equiv 2q(\mathcal{B}) + \frac{1}{2}u^2 q(\mathcal{D}) \mod \ell,$$

our choice of u_1 and the assumption (A) ensure that

$$\ell \nmid \frac{1}{2}q(L_{u_1}) + 1. \tag{15}$$

In the following, this equation will be used to verify the condition (7) for the Mukai vectors we construct.

Case (A.1): $\mathrm{div}(L_{u_1}) = 1$. By the Eichler criterion (see [7, Lemma 3.5 and Example 3.8]) and [13, Theorem 9.8], there exist a polarized $K3$ surface (S, H) with

$$\mathrm{Pic}(S) = \mathbb{Z}H, \quad H^2 = (2n-2)\ell^2 + q(L_{u_1}),$$

and a parallel transport $\rho : H^2(S^{[n]}, \mathbb{Z}) \to H^2(X, \mathbb{Z})$ satisfying

$$\rho(H - \ell\delta) = \epsilon L_{u_1}, \quad \epsilon = 1 \text{ or } -1.$$

By setting

$$r := 4\ell^2, \quad m := -2\ell\epsilon,$$

we can check directly that (S, H, r, m) satisfies the assumptions of Proposition 1.3; here (7) is given by (15):

$$\frac{1}{2}H^2 + 1 \equiv \frac{1}{2}q(L_{u_1}) + 1 \mod \ell \quad \Longrightarrow \quad \ell \nmid \frac{1}{2}H^2 + 1.$$

Hence there exists an α-twisted vector bundle of rank $n!r^n = (2^{2n}n!) \cdot \ell^{2n}$ as desired.

Case (A.2): $\mathrm{div}(L_{u_1}) = 2$. Note that a class of divisibility 2 automatically satisfies

$$q(L_{u_1}) \equiv 2 - 2n \mod 8.$$

As in the first case, there exist a polarized $K3$ surface (S,H) with

$$\mathrm{Pic}(S) = \mathbb{Z}H, \quad H^2 = \frac{1}{4}\left((2n-2)\ell^2 + q(L_{u_1})\right),$$

and a parallel transport $\rho : H^2(S^{[n]},\mathbb{Z}) \to H^2(X,\mathbb{Z})$ satisfying

$$\rho(2H - \ell\delta) = \epsilon L_{u_1}, \quad \epsilon = 1 \text{ or } -1.$$

Setting

$$r := 4\ell^2, \quad m := -4\ell\epsilon,$$

the quadruple (S,H,r,m) satisfies the assumptions of Proposition 1.3; similarly we need (15) to guarantee (7). This constructs an α-twisted vector bundle of rank $n!r^n = (2^{2n}n!)\cdot\ell^{2n}$ as desired. □

Proof in the case (B). The proof in the case (B) is parallel; the invariants r,m are constructed by the same formulas as in the case (A). The only difference is that we replace u_1 by u_2 of Proposition 2.3. More precisely, we pick u_2 given by Proposition 2.3 and consider the class

$$L_{u_2} \in H^2(X,\mathbb{Z}).$$

Note that

$$\frac{1}{2}q(L_{u_2}) + 1 \equiv (2q(\mathcal{B})+1) + \frac{1}{2}u_2^2 q(\mathcal{D}) \equiv \frac{1}{2}u_2^2 q(\mathcal{D}) \mod \ell$$

where the second equation is given by the assumption (B). By the assumptions of u_2, ℓ, we have

$$\ell \nmid \frac{1}{2}q(L_{u_2}) + 1. \tag{16}$$

The rest of the proof is identical to that for the case (A), where (16) plays the role of (15). □

2.3. **Numerical obstructions.** In the proof above, we applied Proposition 1.3 with $r = 4\ell^2$ which gives the bound (4). A natural question is that, can we use this method to achieve the stronger bound (3)? In the following, we discuss numerical obstructions that explain why we do not expect Huybrechts' conjecture to be derived from Proposition 1.3.

If we want to get (3) for a Brauer class $\alpha = \left[\frac{\mathcal{B}}{\ell}\right]$ with ℓ a prime number, we need to set

$$r = c\ell, \quad \gcd(c,\ell) = 1.$$

The condition (iii) of Proposition 1.2 (or Proposition 1.3) implies that there exists a parallel transport

$$\rho : H^2(S^{[n]},\mathbb{Z}) \to H^2(X,\mathbb{Z})$$

satisfying

$$-2m\rho(H) - c\ell\rho(\delta) = 2c\mathcal{B} + 2c\ell\mathcal{A} + \mathcal{D}$$

with $\mathcal{A} \in H^2(X,\mathbb{Z})$ and $\mathcal{D} \in \mathrm{Pic}(X)$. The condition

$$\ell \mid m^2H^2$$

(required by Proposition 1.2 (ii)) combined with the fact that ρ is an isometry further implies

$$\ell \mid q(\mathcal{D}) + (2c)^2 q(\mathcal{B}). \tag{17}$$

If X has Picard rank 1, it is impossible for (17) to hold for all but finitely many prime numbers ℓ. To see this, we assume $\mathrm{Pic}(X) = \mathbb{Z}\mathcal{R}$, and then (17) is equivalent to finding an integer k such that

$$k^2 q(\mathcal{R}) + (2c)^2 q(\mathcal{B}) \equiv 0 \mod \ell.$$

Consequently, this implies

$$(kq(\mathcal{R}))^2 \equiv -(2c)^2 q(\mathcal{B}) q(\mathcal{R}) \mod \ell,$$

forcing the integer $-q(\mathcal{R})q(\mathcal{B})$ to be a quadratic residue modulo ℓ. This cannot always hold, even allowing ℓ to avoid finitely many primes.

We now prove Theorem 0.4 which shows that (17) is the *only* obstruction.

2.4. **Proof of Theorem 0.4.** As before, we write the Brauer class α as

$$\alpha = \left[\frac{\mathcal{B}}{\ell}\right] \in \mathrm{Br}(X), \quad \mathcal{B} \in T(X), \quad \mathrm{per}(\alpha) = \ell.$$

Let $\mathcal{D} \in \mathrm{Pic}(X)$ be a class satisfying

$$q(\mathcal{D}) \equiv -q(\mathcal{B})d^2 \mod \ell, \quad \gcd(d, \ell) = 1. \tag{18}$$

Recall the class

$$L_u = 2\ell\mathcal{A} + 2\mathcal{B} + u\mathcal{D} \in H^2(X, \mathbb{Z}).$$

We may assume that $q(\mathcal{D}) > 0$ and $\frac{\mathcal{D}}{2}$ is not integral; otherwise we modify it by a multiple of ℓ times an ample class. We pick an odd $u \in \mathbb{Z}_{>0}$ such that

$$du \equiv 2 \mod \ell, \qquad q(L_u) > 0, \tag{19}$$

and fix it from now on.

We note the following lemma; it is the *only* place where (18) is essentially used.

Lemma 2.4. *With u as chosen above, we have*

$$\ell \mid q(L_u).$$

Proof. A direct calculation yields

$$q(L_u) \equiv 4q(\mathcal{B}) + u^2 q(\mathcal{D}) \mod \ell.$$

Combining with (18) and the assumption (19) above, the right-hand side is

$$q(\mathcal{B})\left(4 - (du)^2\right) \equiv 0 \mod \ell.$$ □

Next, we show that the assumption (ii) ensures that the class L_u is primitive.

Lemma 2.5. *The class L_u is primitive, and the divisibility of L_u is either* 1 *or* 2.

Proof. We write

$$L_u = aL'_u$$

with $a \in \mathbb{Z}_{>0}$ an odd integer and L'_u primitive. The same argument as in Proposition 2.3 shows

$$a \cdot \operatorname{div}(L'_u) = \operatorname{div}(L_u) \mid 2\ell, \quad \operatorname{div}(L'_u) \mid 2n-2,$$

which further implies

$$\operatorname{div}(L'_u) = 1 \text{ or } 2, \quad a \mid \ell.$$

Recall the class $\mathcal{D}'$ which satisfies the assumptiom (ii) in Theorem 0.4. We must have

$$a \mid q(L_u, \mathcal{D}') = uq(\mathcal{D}, \mathcal{D}');$$

combined with

$$a \mid q(L_u, \mathcal{B}) = 4q(\mathcal{B}),$$

we conclude that

$$a \mid \gcd\left(q(\mathcal{D}, \mathcal{D}'), q(\mathcal{B}), \ell\right) = 1 \implies a = 1.$$

□

The remainder of the proof is parallel to that of Theorem 0.2; we note that in both cases 1 and 2 below, the condition (7) is satisfied automatically due to Lemma 2.4.

Case 1: $\operatorname{div}(L_u) = 1$**.** There exist a polarized $K3$ surface (S, H) with

$$\operatorname{Pic}(S) = \mathbb{Z}H, \quad H^2 = (2n-2)\ell^2 + q(L_u),$$

and a parallel transport $\rho : H^2(S^{[n]}, \mathbb{Z}) \to H^2(X, \mathbb{Z})$ satisfying

$$\rho\left(H - \ell\delta\right) = \epsilon L_u, \quad \epsilon = 1 \text{ or } -1.$$

By setting

$$r := 4\ell, \quad m := -2\epsilon,$$

we can check directly that (S, H, r, m) satisfies the assumptions of Proposition 1.3. Hence there exists an α-twisted vector bundle whose rank divides $n!r^n = \left(2^{2n}n!\right) \cdot \ell^n$, and the theorem in this case follows from Lemma 1.1.

Case 2: $\operatorname{div}(L_u) = 2$**.** There exist a polarized $K3$ surface (S, H) with

$$\operatorname{Pic}(S) = \mathbb{Z}H, \quad 4H^2 = (2n-2)\ell^2 + q(L_u),$$

and a parallel transport $\rho : H^2(S^{[n]}, \mathbb{Z}) \to H^2(X, \mathbb{Z})$ satisfying

$$\rho\left(2H - \ell\delta\right) = \epsilon L_u, \quad \epsilon = 1 \text{ or } -1.$$

By setting

$$r := 4\ell, \quad m := -4\epsilon,$$

we can check directly that (S, H, r, m) satisfies the assumptions of Proposition 1.3. Hence there exists an α-twisted vector bundle whose rank divides $n!r^n = \left(2^{2n}n!\right) \cdot \ell^n$ as desired.

The proof of Theorem 0.4 is completed. □

2.5. **Theorem 0.1 for $K3^{[n]}$-type revisited.** We explain in this section that Theorem 0.1 for $K3^{[n]}$-type varieties is an immediate consequence of Theorem 0.4.

Assume that X is of $K3^{[n]}$-type admitting a Lagrangian fibration. It has Picard rank ≥ 2, and the BBF form of the rank 2 lattice spanned by a polarization and the pullback of the hyperplane class of the base is

$$Q(x,y) = C_X \cdot xy,$$

where $C_X \in \mathbb{Z}$ is an intrinsic invariant of X. If ℓ is coprime to C_X, it is clear that

$$Q(x,y) \equiv -q(\mathcal{B})d^2 \mod \ell, \quad \gcd(d,\ell) = 1$$

has an integral solution. Moreover, there exist integral x', y' such that

$$\gcd(Q(x',y'),\ell) = 1.$$

Therefore Theorem 0.1 for X follows from Theorem 0.4 with

$$N_X := C_X \cdot n! I_X.$$

This proves Theorem 0.1 for $K3^{[n]}$-type varieties. □

We have explained in (2.3) that (18) may not be solvable if X has Picard rank 1. In fact, except the case of Lagrangian fibrations as we discussed above, the equation (18) is not solvable for most rank 2 Picard lattices.

To see a typical example, we consider the quadratic form

$$Q(x,y) = 2x^2 - 10y^2.$$

If $\ell = 9$ and $q(\mathcal{B}) \equiv 3 \mod 9$, then (18) is not solvable. Indeed, a solution to

$$Q(x,y) \equiv 0 \mod 3$$

must satisfy that

$$x \equiv y \equiv 0 \mod 3;$$

in this case $Q(x,y)$ has to be a multiple of 9. We discuss this phenomenon geometrically in Section 3.2; in particular, for Picard rank 2, this is essentially the *only* case where (18) is not solvable, and the *non-special* condition in Theorem 0.5 rules it out.

3. Proof of Theorem 0.5

Let $n \geq 2$. We assume that X is a hyper-Kähler variety of $K3^{[n]}$-type of Picard rank

$$\mathrm{rk}(\mathrm{Pic}(X)) = \mu \geq 1.$$

Our purpose is to show that Theorem 0.5 is a consequence of Theorem 0.4. Since we assume that the Brauer class $\alpha = [\frac{\mathcal{B}}{\ell}]$ is non-special, the assumption (ii) of Theorem 0.4 is always satisfied.

As before, we further assume that $\mathrm{per}(\alpha)$ is coprime to I_X so that $\mathrm{per}(\alpha) = \ell$. We study in this section the congruence equation

$$q(\mathcal{D}) \equiv -q(\mathcal{B})d^2 \mod \ell, \quad \gcd(d, \ell) = 1, \tag{20}$$

which completes the proof.

3.1. **Rational points.** We write the Brauer class $\alpha = \left[\frac{\mathcal{B}}{\ell}\right]$ as before. In this section, we reduce the existence of $\mathcal{D}, d$ satisfying (20) to the existence of a nonsingular rational point on a quadric hypersurface.

We use Q to denote the BBF form on $\mathrm{Pic}(X)$ which is non-degenerate, and we set

$$b := -q(\mathcal{B}) \in \mathbb{Z}.$$

We consider

$$V := \{Q(x_1, \cdots, x_\mu) - bw^2 = 0\} \subset \mathbb{P}^\mu.$$

Note that V actually defines a scheme over $\mathbb{Z}$, which further induces a variety V_p over $\overline{\mathbb{F}}_p$ for every prime p.

Proposition 3.1. *Assume that for every prime factor p of ℓ there is an $\mathbb{F}_p$-rational point lying in the Zariski open subset*

$$V_p \cap \{w \neq 0\} \subset V_p$$

which is a nonsingular point of the $\overline{\mathbb{F}}_p$-variety V_p. Then there exists $\mathcal{D} \in \mathrm{Pic}(X)$ and d satisfying (20).

Proof. We proceed the proof by the following steps. We assume that

$$\ell = p_1^{e_1} p_2^{e_2} \cdots p_t^{e_t}$$

with p_i pairwise distinct primes.

Step 1. Our purpose is to find integers x_i and integer w with $\gcd(w, \ell) = 1$ such that

$$Q(x_1, \cdots, x_\mu) - bw^2 \equiv 0 \mod \ell. \tag{21}$$

These integers satisfy automatically

$$Q(x_1, \cdots, x_\mu) - bw^2 \equiv 0 \mod p_j^{e_j}, \quad \gcd(w, p_j) = 1 \tag{22}$$

for every $i \in \{1, 2, \cdots, t\}$. Conversely, if we can find solutions

$$x_{i,1}, x_{i,2}, \cdots, x_{i,\mu}, w_i \in \mathbb{Z}$$

to (22) for every i, then the Chinese Remainder Theorem implies that there exists a solution to the congruence equations

$$x_j \equiv x_{i,j} \mod p_j^{e_j}, \quad w \equiv w_j \mod p_j^{e_j},$$

which further gives a solution to (21).

Step 2. By Step 1, from now on we only focus on a prime $p \in \{p_1, p_2, \cdots, p_t\}$ and consider the equation

$$Q(x_1, \cdots, x_\mu) - bw^2 \equiv 0 \mod p^e, \quad \gcd(w, p) = 1. \tag{23}$$

If $V_p \cap \{w \neq 0\}$ has an $\mathbb{F}_p$-rational point, we may assume that there are integers x_i, w satisfying

$$Q(x_1, \cdots, x_\mu) - bw^2 \equiv 0 \mod p, \quad \gcd(w, p) = 1.$$

Furthermore, if this rational point is nonsingular on V_p, then by Hensel's lemma this $\mathbb{F}_p$-point can be lifted to a $\mathbb{Z}_p$-point. In particular, we can find a solution to the congruence equation (23). □

3.2. **Proof of Theorem 0.5.** We first revisit the example in Section 2.5 from the perspective of Proposition 3.1; recall that

$$Q(x, y) = 2x^2 - 10y^2, \quad p = 3, \quad \ell = p^2 = 9, \quad q(\mathcal{B}) = 12.$$

The variety V_3 is given by

$$V_3 = \{2x^2 - 10y^2 = 0\} \subset \mathbb{P}^2_{x,y,z}$$

where $[0 : 0 : 1]$ is the only rational point. Geometrically, the variety V_3 is the union of 2 lines, but both lines are not defined over $\mathbb{F}_3$. The intersection of the two lines is the unique rational point, which is singular. Note that this is essentially the only bad situation: the conic is degenerate and the only rational point is the node.

Now we prove Theorem 0.5.

To illustrate the idea, we first treat the $\mu = 2$ case. Since $Q(x, y)$ is non-degenerate, there are only finitely many primes p satisfying that the affine curve

$$\{Q(x, y) = 0\} \setminus \{(0, 0)\} \subset \mathbb{A}^2$$

is singular over $\overline{\mathbb{F}}_p$. Let C_Q be the product of all these primes. Then for any prime p coprime to C_Q and any b coprime to p, the projective curve V_p is nonsingular. So it is a nonsingular conic, and is isomorphic to $\mathbb{P}^1$. Hence its affine part $\{z \neq 0\}$ must contain a rational point. By Proposition 3.1, the theorem holds with the constant $N_X := C_Q \cdot n!$.

The general case is similar. Consider C_Q given by the product of all primes p satisfying that the affine cone

$$\{Q(x_1, x_2, \cdots, x_\mu) = 0\} \subset \mathbb{A}^\mu$$

has singular point other than $(0, 0, \cdots, 0)$ over $\overline{\mathbb{F}}_p$, and again set

$$N_X := C_Q \cdot n! I_X.$$

To complete the proof, we show that when $\gcd(p, N_X) = 1$ the equation

$$Q(x_1, x_2, \cdots, x_\mu) \equiv bw^2 \mod p^e, \quad \gcd(w, p) = 1$$

always has a solution. Since $\gcd(b,p)=1$, the variety V_p is a nonsingular quadric hypersurface in $\mathbb{P}^\mu$; therefore, the assumption of Proposition 3.1 is satisfied, which implies the existence of desired $\mathcal{D}, d$. □

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