

QIZHENG YIN

**TAUTOLOGICAL CYCLES
ON
CURVES AND JACOBIANS**

thesis

2013

NIJMEGEN - PARIS

To my parents

Acknowledgements

First and foremost, I would like to express my deepest gratitude to my thesis advisors: to Ben Moonen, for his style of mathematics that I admire, for his encouragement and persistence, and for everything he taught me, from the broadest thoughts to the tiniest details; and to Claire Voisin, for bringing me to this wonderful world of algebraic geometry, for her vision and guidance, and for many enlightening discussions. It has been my greatest honor and privilege to work with both of them.

I wish to thank Arnaud Beauville, Bas Edixhoven, Gerard van der Geer, Ieke Moerdijk, and Rahul Pandharipande for agreeing to serve on my thesis committee. Their insightful comments and suggestions are deeply appreciated. Special thanks to Rahul for offering me the opportunity to continue in Zürich.

Two people have made enormous contribution to this thesis: Robin de Jong, whose beautiful talk at the PCMI in the summer of 2011 inspired the whole project; and my friend and colleague Ma Li, whose brilliant computer program made this work so much more meaningful. I cannot thank them enough for what they have done.

I am also grateful to those who showed interest in this work: Pedro Luis del Ángel, Olof Bergvall, Alessandro Chiodo, Carel Faber, Gavril Farkas, Samuel Grushevsky, Daniel Huybrechts, Felix Janda, Liu Kefeng, Nicola Pagani, Dan Petersen, Aaron Pixton, Ruan Yongbin, Sergey Shadrin, Nicola Tarasca, Mehdi Tavakol, Orsola Tommasi, Dmitry Zakharov, and Zhu Shengmao. I benefited greatly from discussions with them.

Further, a large part of my mathematical debt goes to Najmuddin Fakhruddin and Alexander Polishchuk. I thank them and look forward to meeting them in the future.

I have spent nine amazing years of my university studies in five cities: Beijing, Hong Kong, Paris, Amsterdam, and Nijmegen. I would like to acknowledge the support of all the institutions involved: Peking University, Hong Kong University of Science and Technology, École normale supérieure, Université Pierre et Marie Curie (Paris VI), Universiteit van Amsterdam, and Radboud Universiteit Nijmegen. I also thank their staff members for taking pains to deal with my documents.

ACKNOWLEDGEMENTS

The past six years in Europe have been a real adventure. I met some great people, who helped me both with my studies and in daily life, and without whom I could not have survived. To name a few: Giuseppe Ancona, Yohan Brunebarbe, Alexandr Buryak, François Charles, Charles de Clercq, Fabien Cléry, Johan Commelin, Benjamin Dechant, Marcelo Gonçalves de Martino, Jérémy Guéré, Ariyan Javanpeykar, Nicolas Mascot, Romain Nguyen van yen, Arie Peterson, Loek Spitz, and Davide Cesare Veniani.

And of course there are my Chinese friends in the academic world: Chen Shuang, Chen Yuanmi, Fan Huan, Fang Xin, Feng Yongqi, Fu Lie, Gao Kun, Gong Xun, Huang Gang, Jiang Zhi, Jin Fangzhou, Jin Xin, Li Chen, Li Xinyi, Liang Yongqi, Liu Shoumin, Lu Zhenkai, Ma Yue, Qin Botao, Qiu Yanqi, Shan Peng, Shen Dali, Shen Junliang, Shen Mingmin, Sun Fei, Tan Zhihong, Wang Yu, Xie Junyi, Yang Xiye, Zhang Chao, Zhang Letao, Zhang Xiaoxing, Zhao Zhichen, Zhao Xiaolei, Zhou Fan, Zhu Bo, and many more. They have been my teachers most of the time, and they make me feel at home wherever I go.

Finally, a secret thank you to my childhood friends in Beijing and around the world. They might have no idea what I am doing, but back then it was them who kept my childhood dream alive.

I dedicate this thesis to my parents, who give me the chance and the reason to live in this world. Also to my wife Zhiyue, who brings me love, joy and the beautiful picture on the back cover of the booklet. I could take thousands of photos with streets and houses, but none would be as vivid as her drawing.

Bedankt allemaal. Merci à tous. 谢谢大家。

Qizheng



Contents

Acknowledgements	iii
1. Overview of the thesis	1
<i>Here we discuss the main themes of the thesis, explain key concepts and ideas, summarize results, and set up notation and conventions.</i>	
2. Preliminaries	7
<i>This chapter provides the basics for the rest of the thesis. We briefly review the classical theory of algebraic cycles on an abelian scheme, combining three aspects: Chow theory, motives and cohomology. Then we specialize to the case of a relative Jacobian, where certain structures can be reconstructed geometrically.</i>	
2.1. The Chow ring of an abelian scheme	7
2.2. Motivic interpretation and cohomological realization	13
2.3. An illustration: the Dutch house	18
2.4. Case of a relative Jacobian	20
3. A tale of two tautological rings (I)	25
<i>For a relative pointed curve, we prove that the tautological ring of the base variety (in the sense of Mumford, Faber, etc.) is a subring of the tautological ring of the relative Jacobian (in the sense of Beauville, Polishchuk, etc.). The \mathfrak{sl}_2-action on the Jacobian side produces relations between tautological classes, leading to various theoretical results and numerical evidence towards the Faber conjectures (on the tautological ring of the moduli space of smooth pointed curves).</i>	
3.1. Moduli side: tautological ring and the Faber conjectures	25
3.2. Jacobian side: tautological ring and its generators	27
3.3. Application: an identity of Morita	32
3.4. Relations in the tautological rings	35

4. A tale of two tautological rings (II)	43
<i>We study symmetric powers of a relative pointed curve, and we prove that the tautological ring of the infinite symmetric power is a polynomial ring over the tautological ring of the relative Jacobian. As a result, the Gorenstein property for the universal Jacobian implies the same property for symmetric powers of the universal curve. We give positive results in low genus cases.</i>	
4.1. Symmetric powers of a relative curve	43
4.2. Tautological rings of the infinite symmetric power	49
4.3. Gorenstein properties	57
5. Tautological classes on a Jacobian variety	63
<i>This chapter specializes to the case of a Jacobian variety. We determine the generic behavior of the tautological ring in codimension 2, and give examples in higher codimensions. As applications, we obtain simple proofs of (i) Green and Griffiths' result on the generic non-vanishing of the Faber-Pandharipande cycle; (ii) Sebastian's result on Voevodsky's smash-nilpotence conjecture for 1-cycles on abelian varieties.</i>	
5.1. The tautological ring over a field	63
5.2. Codimension 2 and a result of Green and Griffiths	65
5.3. Examples in higher codimensions	74
5.4. Application: smash-nilpotent 1-cycles on an abelian variety	77
6. Tautological classes in higher Griffiths groups	83
<i>We consider tautological classes modulo equivalences induced by S. Saito's filtration on Chow groups. For the Jacobian of a very general complex pointed curve, we detect non-trivial classes in Saito's higher Griffiths groups. This improves a result of Ikeda.</i>	
6.1. Motivation	83
6.2. Saito's filtration and higher Griffiths groups	85
6.3. Detecting non-trivial classes in higher Griffiths groups	89
Notes	99
Bibliography	105
Index	113
Samenvatting - Résumé	115
Curriculum vitae	117

Overview of the thesis

Here we discuss the main themes of the thesis, explain key concepts and ideas, summarize results, and set up notation and conventions.

Algebraic cycles

The study of algebraic cycles is one of the most fascinating subjects in algebraic geometry. It illustrates a nice mixture of topology, geometry and algebra, with many deep questions and conjectures.

The construction goes roughly as follows: consider an algebraic variety X , which is defined by some polynomial equations. Then a (closed) subvariety of X is defined by some more polynomial equations. The group of algebraic cycles on X is nothing but the formal abelian group generated by all subvarieties of X . With this definition, however, one does not have enough operations to study those cycles. The solution is to introduce some kind of equivalence between cycles: to allow them to move a bit, so that every pair of cycles can intersect. The least one can do is to work modulo rational equivalence, *i.e.* to allow cycles to move along a projective line. The resulting group is called the Chow group, and it carries a ring structure given by the intersection, hence the Chow ring. One may also work modulo other coarser equivalences (*e.g.* algebraic, homological and numerical equivalences) to allow more flexibility.

The Chow ring can sometimes be viewed as the algebraic counterpart of the cohomology ring, the latter being a more geometric notion. Their interactions are particularly interesting: on one hand, the Hodge conjecture predicts that cohomology should in some sense be controlled by Chow. On the other hand, the Bloch-Beilinson-Murre conjecture predicts the converse. The formal aspect of these conjectures can be explained via the language of (Chow) motives. Roughly speaking, the category of motives captures all information about Chow, and has a functor towards cohomology, called realization.

I. OVERVIEW OF THE THESIS

Tautological (cycle) classes

It is a well-known phenomenon that the Chow ring of a variety can be enormous as soon as its cohomology becomes complicated. So in that case, it seems hopeless to study the Chow ring in integrity. Nevertheless, for particular types of varieties there exist many interesting, geometrically constructed cycle classes. People often call them tautological classes — as if one gets them for free. With a bit of luck, one can find a certain degree of finiteness in those classes, as well as other nice enumerative properties. Moreover, if the definition of tautological classes is good, one also expects most classes found in nature to be tautological.

In this thesis we shall encounter two such notions of tautological classes. The first is for the moduli space of curves and dates back to Mumford in 1983. The second concerns the Jacobian of a curve and was first introduced by Beauville in 2004.

Tautological ring: curve side

A smooth curve C carries a canonical divisor class K , which is the first Chern class of its cotangent bundle. Consider a family of such curves $p: C \rightarrow S$, and glue the K 's together. Then by self intersecting K and pushing forward to S , we obtain the so-called kappa classes $\kappa_i = p_*(K^{i+1})$ on S . The tautological ring $\mathcal{R}(S)$ is just the ring generated by these kappa classes, and all classes in $\mathcal{R}(S)$ are called tautological.

The universal model of all such S 's is the moduli space \mathcal{M}_g , which is some kind of a parameter space for smooth curves of genus g . In this case, the study of the tautological ring $\mathcal{R}(\mathcal{M}_g)$ has two major motivations. The first is the Mumford conjecture, now proven by Madsen and Weiss. It roughly says that when the genus g goes to infinity, the limit of the cohomology ring $H(\mathcal{M}_g)$ coincides with $\mathcal{R}(\mathcal{M}_g)$, *i.e.* it consists only of tautological classes.

The second motivation comes from the Faber conjectures, which predict that $\mathcal{R}(\mathcal{M}_g)$ behaves like the (algebraic) cohomology of a smooth complete variety. In other words, it should be a Gorenstein ring and have Poincaré duality. The difficulty is to construct sufficiently many relations between tautological classes. Due to many failed attempts for $g \geq 24$, as well as counterexamples in other contexts, nowadays people tend not to believe the conjectures. But the evidence remains unconvincing.

One may consider similar Gorenstein properties for the tautological rings of the universal curve over \mathcal{M}_g and its powers. There is also a pointed version, *i.e.* over the moduli of pointed curves $\mathcal{M}_{g,1}$.

Tautological ring: Jacobian side

The Jacobian J of a smooth curve C is the group of isomorphism classes of line bundles on C with trivial first Chern class. With J being an abelian variety, its Chow ring carries a second ring structure

brought by the addition map, and also the action of the multiplication by $N \in \mathbb{Z}$.

By choosing a point $x_0 \in C$, one can embed the curve C into J and the former becomes a 1-dimensional cycle on J . We first declare the class of this cycle to be tautological, and then apply both ring structures as well as the multiplication by N . The eventual output is called the tautological ring $\mathcal{T}(J)$, and its elements tautological classes. One can prove that $\mathcal{T}(J)$ is in fact finitely generated (with respect to only the intersection product) by writing down an explicit set of generators. Also as is shown in this thesis, the story generalizes to a family of pointed curves, or equivalently, to over the moduli space $\mathcal{M}_{g,1}$.

Polishchuk made a great contribution to the study of this notion. He applied powerful tools that are developed for the Chow ring of abelian varieties, such as the Beauville decomposition, the Fourier transform, and the Lefschetz decomposition. He then used the last tool to construct relations between tautological classes. For the generic curve (and modulo algebraic equivalence), he conjectures that they provide all the relations.

Polishchuk's approach also brings a motivic touch to the subject. The hidden background is the so-called motivic Lefschetz decomposition, and the relations he obtained are simply determined by the motive of the Jacobian J .

Connecting the two sides

The first half of the thesis (Chapters 3 and 4) builds connections between the two tautological rings, showing that not only do they share the same name, but they are essentially the same thing. More precisely, consider a family of pointed curves $C \rightarrow S$ and the corresponding family of Jacobians $J \rightarrow S$. First by applying Polishchuk's machinery, we prove that the restriction of the tautological ring $\mathcal{T}(J)$ to S is exactly the (pointed) tautological ring $\mathcal{R}(S)$. The picture becomes even clearer when we consider the symmetric powers $C^{[n]} \rightarrow S$ for all $n \geq 0$, and their tautological rings $\mathcal{R}(C^{[n]})$. Using techniques developed by Moonen and Polishchuk, we show that when n goes to infinity, the limit of $\mathcal{R}(C^{[n]})$ is actually a polynomial ring over $\mathcal{T}(J)$.

This revelation has some significant consequences. As is mentioned above, the motivic Lefschetz decomposition produces many relations between tautological classes (in $\mathcal{T}(J)$, $\mathcal{R}(S)$, $\mathcal{R}(C^{[n]})$, etc.). Using these relations, we can confirm the conjectural Gorenstein property for various tautological rings over $S = \mathcal{M}_{g,1}$ with some low values of g . But more importantly, we may ask the following: are all relations of motivic nature? This question has a fairly strong flavor of geometry, and a positive answer to it would in general conflict with the Gorenstein property (for $\mathcal{R}(\mathcal{M}_{24})$, $\mathcal{R}(\mathcal{M}_{20,1})$, etc.). It is probably the first time one can think of a geometric reason (rather than just numerical evidence) why the Faber conjectures may not hold.

I. OVERVIEW OF THE THESIS

Detecting non-trivial tautological classes

The second half of the thesis (Chapters 5 and 6) addresses the following question: given a tautological class, can one determine if it is zero or not in the Chow groups? Conventional methods of detecting non-trivial cycles (in Chow) consist of two steps. First one applies the cycle class map to see if the cohomology class of the cycle is non-zero. If it is zero, then a second chance is to apply the Abel-Jacobi map, which goes from homologically trivial cycles to the intermediate Jacobian. But if this still returns zero, then the problem becomes very difficult. Modern Hodge theory has developed infinitesimal invariants to detect those Abel-Jacobi trivial cycles on a very general fiber of a family of varieties. However, most of the invariants are extremely difficult to compute.

In our specific case, we found a very simple way of detecting non-trivial tautological classes on the generic (or a very general) Jacobian. The idea is to exploit the fact that the Jacobians of certain singular curves (called of compact type) are still abelian varieties. Using a degeneration argument due to Fakhruddin, we are reduced to compute a certain invariant for those singular curves, and the complexity of the computation is almost nothing.

Further, to illustrate the simplicity and effectiveness of our method, we also carry out detections in the context of S. Saito's higher Griffiths groups.

Main results

- (i) Let $C \rightarrow S$ be a relative pointed curve and $J \rightarrow S$ be the relative Jacobian. We define the tautological ring $\mathcal{T}(J)$ (Definition 3.4), describe its generators (Theorem 3.6), and show that its restriction to S is the tautological ring $\mathcal{R}(S)$ (Corollary 3.8). We produce relations in $\mathcal{T}(J)$ (Construction 3.12), which lead to a new proof of the generation statement in the Faber conjectures (Theorem 3.15), as well as confirm the conjectures for $\mathcal{M}_{g,1}$ with $g \leq 19$ (Theorem 3.17). Further we raise the question whether all relations should be motivic (Conjecture 3.19).
- (ii) We define the tautological ring(s) $\mathcal{R}(C^{[\infty]})$ of the infinite symmetric power of C (Definition 4.12), and show that $\mathcal{R}(C^{[\infty]})$ is a polynomial ring over $\mathcal{T}(J)$ (Theorem 4.14). For the universal curve (resp. Jacobian) $\mathcal{C}_{g,1}$ (resp. $\mathcal{J}_{g,1}$) over $\mathcal{M}_{g,1}$, an analogue of the Gorenstein property for $\mathcal{T}(\mathcal{J}_{g,1})$ is stated (Speculation 4.24), and its connections with the Gorenstein property for $\mathcal{R}(\mathcal{C}_{g,1}^{[n]})$ established (Theorem 4.27). We confirm these properties for $g \leq 7$ (Theorem 4.26), leaving $g = 8$ as the critical case.
- (iii) Using Fakhruddin's degeneration argument (Lemma 5.7), we detect non-trivial codimension 2 tautological classes on the generic Jacobian (Theorem 5.4). As a consequence we obtain a simple and characteristic free proof of the generic non-vanishing of the Faber-Pandharipande cycle

(Corollary 5.11; due to Green and Griffiths over \mathbb{C}). We also explore some higher codimensional cases (Proposition 5.12). Further, as an independent application of the tautological ring, we document a simple proof of Sebastian's result on the Voevodsky conjecture for 1-cycles on abelian varieties (Theorem 5.21).

- (iv) On a very general Jacobian, we develop an invariant for detecting non-trivial classes in Saito's higher Griffiths groups (Proposition 6.10). We compute the invariant for the Beauville components of the curve class, proving the generic non-vanishing of those components with a sharp bound on the genus (Theorem 6.13). This improves a previous result of Ikeda.

Notation and conventions

- (i) Let k be a field. When there is no ambiguity, we simply write k for $\text{Spec}(k)$. By a variety we mean a separated, reduced scheme of finite type over k . If S is a smooth connected variety over k , we denote by \mathcal{V}_S the category of smooth projective schemes over S . If an object X/S in \mathcal{V}_S has connected fibers, we write $\dim(X/S)$ for the relative dimension of X over S .
- (ii) For a smooth variety X over k , we denote by $\text{CH}^i(X)$ the Chow group of codimension i cycles on X with \mathbb{Q} -coefficients, and by $\text{CH}(X) := \bigoplus_i \text{CH}^i(X)$ the Chow ring of X equipped with the intersection product (\cdot) . We also write $\text{CH}_{\text{alg}}(X)$ (resp. $\text{CH}_{\text{hom}}(X)$ and $\text{CH}_{\text{num}}(X)$) for the ideal of cycles that are algebraically (resp. homologically and numerically) equivalent to zero. Throughout, all cycles groups are with \mathbb{Q} -coefficients.
- (iii) The word *generic* is taken in the schematic sense. Over an uncountable field k (e.g. $k = \mathbb{C}$), the term *very general* is often used, which means outside a countable union of Zariski-closed proper subsets of the base variety.
- (iv) We write $\mathfrak{sl}_2 := \mathbb{Q} \cdot e + \mathbb{Q} \cdot f + \mathbb{Q} \cdot h$, with $[e, f] = h, [h, e] = 2e$ and $[h, f] = -2f$.

Preliminaries

This chapter provides the basics for the rest of the thesis. We briefly review the classical theory of algebraic cycles on an abelian scheme, combining three aspects: Chow theory, motives and cohomology. Then we specialize to the case of a relative Jacobian, where certain structures can be reconstructed geometrically.

2.1. The Chow ring of an abelian scheme

In this section we focus on the Chow theory side of the story, while the motivic and cohomological aspects will be discussed in the next section. References will be given and proofs will be omitted.

We work over a base variety S that is a smooth and connected over a field k . Let A/S be an abelian scheme, *i.e.* a smooth proper group scheme over S with (geometrically) connected fibers. Write $d := \dim(S/k)$ and $g := \dim(A/S)$, and we assume $g > 0$. The (abelian) group structure on A gives the following maps

$$\begin{aligned} \mu: A \times_S A &\rightarrow A, \\ [N]: A &\rightarrow A \text{ for } N \in \mathbb{Z}, \end{aligned}$$

called the *addition* and the *multiplication by N* respectively. We write $\mathrm{pr}_1, \mathrm{pr}_2: A \times_S A \rightarrow A$ for the two projections.

We also recall the notion of a polarization. To every abelian scheme A/S we may associate a dual abelian scheme $A^t := \mathrm{Pic}_{A/S}^0$, such that $(A^t)^t \simeq A$. Then a homomorphism $f: A \rightarrow A'$ induces a dual homomorphism $f^t: (A')^t \rightarrow A^t$. A homomorphism $\lambda: A \rightarrow A^t$ is called symmetric if $\lambda^t = \lambda$. On $A \times_S A^t$ we have the Poincaré line bundle trivialized along the zero sections, and we denote it by \mathcal{P} .

2. PRELIMINARIES

Definition 2.1. A *polarization*^{note 1} of A/S is a symmetric isogeny $\lambda: A \rightarrow A^t$ such that the dual of the pull-back of \mathcal{P} via $(\text{id}_A, \lambda): A \rightarrow A \times_S A^t$ is relatively ample over S . If λ is an isomorphism, then it is called a principal polarization.

The polarization λ induces an element

$$(2.1) \quad L_\lambda := ((\text{id}_A, \lambda)^*(\mathcal{P}))^{-1/2} \in \text{Pic}(A) \otimes_{\mathbb{Z}} \mathbb{Q},$$

which is relatively ample over S , symmetric (*i.e.* $[-1]^*(L_\lambda) = L_\lambda$) and trivialized along the zero section (*i.e.* $[0]^*(L_\lambda) = \mathcal{O}_A$). Moreover, the map $\text{id}_A \times_S \lambda: A \times_S A \rightarrow A \times_S A^t$ gives the identity

$$(2.2) \quad (\text{id}_A \times_S \lambda)^*(\mathcal{P}) = \text{pr}_1^*(L_\lambda) \otimes \text{pr}_2^*(L_\lambda) \otimes \mu^*(L_\lambda)^{-1} \text{ in } \text{Pic}(A \times_S A) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

The central object of our interest is the Chow ring $\text{CH}(A) = \bigoplus_{i=0}^{g+d} \text{CH}^i(A)$. It possesses two different ring structures: one given by the usual intersection product (\cdot) , the other given by the *Pontryagin product* $(*)$. The latter is defined by

$$\begin{aligned} \text{CH}^i(A) \times \text{CH}^{i'}(A) &\xrightarrow{*} \text{CH}^{i+i'-g}(A) \\ (\alpha, \beta) &\mapsto \mu_*(\text{pr}_1^*(\alpha) \cdot \text{pr}_2^*(\beta)). \end{aligned}$$

The unit of $(\text{CH}(A), *)$ is the class of the zero section $o: S \rightarrow A$, denoted by $[o] := [o(S)] \in \text{CH}^g(A)$. In general the two products do not commute with each other, *i.e.* we have $(\alpha \cdot \beta) * \gamma \neq \alpha \cdot (\beta * \gamma)$.

We introduce three important tools for studying the structure of $\text{CH}(A)$, namely the *Beauville decomposition*, the *Fourier transform*, and the *Lefschetz decomposition*. They play a central role in the entire thesis. For simplicity's sake, we shall from now on restrict to the case of principally polarized abelian schemes $(A/S, \lambda)$, where we identify A with A^t using the principal polarization $\lambda: A \xrightarrow{\sim} A^t$. We write

$$(2.3) \quad \begin{aligned} \ell &:= c_1(\mathcal{P}) \in \text{CH}^1(A \times_S A), \\ \theta &:= c_1(L_\lambda) \in \text{CH}^1(A) \end{aligned}$$

for the first Chern classes of \mathcal{P} (viewed as a line bundle on $A \times_S A$) and L_λ . The identity (2.2) then becomes

$$(2.4) \quad \ell = \text{pr}_1^*(\theta) + \text{pr}_2^*(\theta) - \mu^*(\theta) \text{ in } \text{CH}^1(A \times_S A).$$

Sometimes we even use the notation $(A/S, \theta)$ for the principally polarized abelian scheme, but one should always keep in mind what a polarization means.

2.1. The Chow ring of an abelian scheme

Beauville decomposition

The Chow ring $\mathrm{CH}(A)$ can be decomposed into eigenspaces according to the action of $[N]: A \rightarrow A$, for all $N \in \mathbb{Z}$. In other words, the operators $[N]^*$ (or $[N]_*$) for all integer N are simultaneously diagonalizable on $\mathrm{CH}(A)$. The precise statement is the following.

Theorem 2.2. *For $j \in \mathbb{Z}$, denote*

$$\begin{aligned} \mathrm{CH}_{(j)}^i(A) &:= \{ \alpha \in \mathrm{CH}^i(A) : [N]^*(\alpha) = N^{2i-j} \alpha \text{ for all } N \in \mathbb{Z} \} \\ &= \{ \alpha \in \mathrm{CH}^i(A) : [N]_*(\alpha) = N^{2g-2i+j} \alpha \text{ for all } N \in \mathbb{Z} \}. \end{aligned}$$

Then we have a decomposition

$$(2.5) \quad \mathrm{CH}^i(A) = \bigoplus_{j=\max\{i-g, 2i-2g\}}^{\min\{i+g, 2i\}} \mathrm{CH}_{(j)}^i(A). \quad \square$$

This result was first proven by Beauville in the case $S = k$ ([Bea86], Théorème), and later generalized by Deninger and Murre to the relative setting ([DM91], Theorem 2.19). By comparing the actions of $[N]$, we have the compatibility with the products (\cdot) and $(*)$:

$$\begin{aligned} \mathrm{CH}_{(j)}^i(A) \times \mathrm{CH}_{(l)}^k(A) &\xrightarrow{\cdot} \mathrm{CH}_{(j+l)}^{i+k}(A), \\ \mathrm{CH}_{(j)}^i(A) \times \mathrm{CH}_{(l)}^k(A) &\xrightarrow{*} \mathrm{CH}_{(j+l)}^{i+k-g}(A). \end{aligned}$$

Example 2.3. In (2.1), the fact that $[-1]^*(L_\lambda) = L_\lambda$ and $[0]^*(L_\lambda) = \mathcal{O}_A$ is equivalent to saying that $\theta = c_1(L_\lambda) \in \mathrm{CH}_{(0)}^1(A)$. To see this, write $\theta = \theta_{(0)} + \theta_{(1)} + \theta_{(2)}$ with $\theta_{(j)} \in \mathrm{CH}_{(j)}^1(A)$. Then we have $[0]^*(\theta) = \theta_{(2)} = 0$ and $[-1]^*(\theta) = \theta_{(0)} - \theta_{(1)} = \theta = \theta_{(0)} + \theta_{(1)}$, so that $\theta_{(1)} = \theta_{(2)} = 0$.

Fourier transform

The Beauville decomposition is obtained via a technique called the Fourier (or Fourier-Mukai) transform. It was first introduced by Mukai in the context of derived categories, and later adapted to Chow theory by Beauville ([Bea83] and [Bea86]). Again, the form used here is due to Deninger and Murre [DM91].

Definition 2.4. The *Fourier transform* is the \mathbb{Q} -linear endomorphism (in fact automorphism) of $\mathrm{CH}(A)$ defined by

$$\begin{aligned} \mathcal{F} : \mathrm{CH}(A) &\rightarrow \mathrm{CH}(A) \\ \alpha &\mapsto \mathrm{pr}_{2,*}(\mathrm{pr}_1^*(\alpha) \cdot \exp(\ell)). \end{aligned}$$

2. PRELIMINARIES

Proposition 2.5. *We have the following properties for \mathcal{F} :*

- (i) $\mathcal{F} : \mathrm{CH}_{(j)}^i(A) \xrightarrow{\sim} \mathrm{CH}_{(j)}^{g-i+j}(A)$;
- (ii) $\mathcal{F} \circ [N]_* = [N]^* \circ \mathcal{F}$ and $\mathcal{F} \circ [N]^* = [N]_* \circ \mathcal{F}$, for all $N \in \mathbb{Z}$;
- (iii) $\mathcal{F}(\alpha * \beta) = \mathcal{F}(\alpha) \cdot \mathcal{F}(\beta)$;
- (iv) $\mathcal{F}(\alpha \cdot \beta) = (-1)^g \mathcal{F}(\alpha) * \mathcal{F}(\beta)$;
- (v) $\mathcal{F} \circ \mathcal{F} = (-1)^g [-1]^*$, so that $\mathcal{F}^{-1} = (-1)^g [-1]^* \circ \mathcal{F}$. □

We refer to [DM91], Section 2 for a proof of the statements. Property (i) shows how one can obtain the Beauville decomposition using \mathcal{F} : just take elements in $\mathrm{CH}^i(A)$, apply \mathcal{F} , collect components of different codimensions, and apply \mathcal{F}^{-1} back. Property (ii) gives the compatibility between \mathcal{F} and $[N]$ that leads to the proof of (i), and Properties (iii) - (v) justify the name Fourier transform. One may regard $(*)$ as a convolution product, and \mathcal{F} maps the convolution to the usual inner product.

Example 2.6. Since $[o]$ is the unit of $(\mathrm{CH}(A), *)$ and since \mathcal{F} interchanges the two ring structures, we have $\mathcal{F}([o]) = [A]$. It follows that $\mathcal{F}([A]) = (-1)^g [o]$.

Lefschetz decomposition

The classical Lefschetz decomposition for a smooth projective variety X/k says that given an ample line bundle, there exist two operators L and Λ that generate an \mathfrak{sl}_2 -action on the cohomology of X . In the case of an abelian variety, Künnemann [Kün93] showed that one can obtain this \mathfrak{sl}_2 -action at the Chow theory level. Also it can easily be generalized to the relative setting.

On $\mathrm{CH}(A)$, define operators

$$\begin{aligned}
 (2.6) \quad e : \mathrm{CH}_{(j)}^i(A) &\rightarrow \mathrm{CH}_{(j)}^{i+1}(A) & \alpha &\mapsto -\theta \cdot \alpha, \\
 f : \mathrm{CH}_{(j)}^i(A) &\rightarrow \mathrm{CH}_{(j)}^{i-1}(A) & \alpha &\mapsto -\frac{\theta^{g-1}}{(g-1)!} * \alpha, \\
 h : \mathrm{CH}_{(j)}^i(A) &\rightarrow \mathrm{CH}_{(j)}^i(A) & \alpha &\mapsto (2i - j - g)\alpha.
 \end{aligned}$$

Note that we followed the convention of [Pol07], Section 1, where the operators e and f differ by a sign from the ones in [Kün93]. This avoids sign complications in the identity (2.7) below, and is more suitable for studying the case of Jacobians (see Section 2.4). We have the following Lefschetz decomposition theorem ([Kün93], Theorems 5.1 and 5.2; see also [Bea10], Theorem 4.2).

Theorem 2.7.

- (i) The operators e, f and h generate a \mathbb{Q} -linear representation of the Lie algebra \mathfrak{sl}_2 on $\mathrm{CH}(A)$.
- (ii) For i and j in the range of (2.5), we have a decomposition

$$\mathrm{CH}_{(j)}^i(A) = \bigoplus_{k=\max\{0, 2i-j-g\}}^{\min\{i, \lfloor (2i-j)/2 \rfloor, i-j+d\}} e^k \left(\mathrm{CH}_{(j), \mathrm{prim}}^{i-k}(A) \right),$$

where $\mathrm{CH}_{(j), \mathrm{prim}}^i(A)$ is the kernel of $e^{g-2i+j+1} : \mathrm{CH}_{(j)}^i(A) \rightarrow \mathrm{CH}_{(j)}^{g-i+j+1}(A)$.

- (iii) (hard Lefschetz) For $0 \leq 2i - j \leq g$, there are isomorphisms

$$e^{g-2i+j} : \mathrm{CH}_{(j)}^i(A) \xrightarrow{\sim} \mathrm{CH}_{(j)}^{g-i+j}(A).$$

□

Remark 2.8. The messy indices in (ii) and (iii) are largely due to the fact that the i for codimension is a bad choice of grading. Things will clear up after a suitable change of gradings in Section 2.2, and become even more transparent using the picture in Section 2.3.

From now on, we shall refer to the \mathfrak{sl}_2 -action on $\mathrm{CH}(A)$ defined above as *the \mathfrak{sl}_2 -action*. The Fourier transform \mathcal{F} intertwines the \mathfrak{sl}_2 -action by

$$\mathcal{F}^{-1} \circ e \circ \mathcal{F} = -f, \quad \mathcal{F}^{-1} \circ f \circ \mathcal{F} = -e, \quad \text{and} \quad \mathcal{F}^{-1} \circ h \circ \mathcal{F} = -h.$$

In fact, one can even reconstruct \mathcal{F} from \mathfrak{sl}_2 . It means if we understand the \mathfrak{sl}_2 -action on $\mathrm{CH}(A)$, then we know both the Fourier transform and the Beauville decomposition.

Proposition 2.9. *We have*

$$(2.7) \quad \mathcal{F} = \exp(e) \circ \exp(-f) \circ \exp(e) \quad \text{on} \quad \mathrm{CH}(A).$$

□

So if we represent the operators e, f by the matrices $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \in \mathfrak{sl}_2(\mathbb{Q})$, then \mathcal{F} corresponds to the matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Q})$. The proof of Proposition 2.9 is essentially the same as in [Bea04], Section 2.3 (iv) (see also [Pol08], Lemma 1.4). Further, Beauville showed that indeed the \mathfrak{sl}_2 -action lifts to a Lie group SL_2 -action on $\mathrm{CH}(A)$ ([Bea10], Theorem 4.2).

We finish this section by looking at the case of an abelian variety (*i.e.* when $S = k$). Since $d = \dim(S/k) = 0$, the Beauville decomposition becomes

$$(2.8) \quad \mathrm{CH}^i(A) = \bigoplus_{j=i-g}^i \mathrm{CH}_{(j)}^i(A).$$

2. PRELIMINARIES

Now assume a Weil cohomology theory H^\bullet for \mathcal{V}_k (see [Kle94], Section 3), a cycle class map $\text{cl}: \text{CH}^i \rightarrow H^{2i}$, and a well-defined Abel-Jacobi map $\text{aj}: \text{CH}_{\text{hom}}^i \rightarrow (\text{certain object related to } H^{2i-1})$ (e.g. singular cohomology with the classical cycle class and Abel-Jacobi maps when $k = \mathbb{C}$, or the ℓ -adic version in general; see [Cha10], Section 2). There are the following predictions^{note 2} for $\text{CH}(A)$.

Conjecture 2.10.

- (i) We have $\text{CH}_{(j)}^i(A) = 0$ for all $j < 0$.
- (ii) Let $\alpha \in \text{CH}_{(0)}^i(A)$. If $\alpha \in \text{CH}_{\text{num}}^i(A)$, i.e. α is numerically equivalent to zero (or weaker: if $\alpha \in \text{CH}_{\text{hom}}^i(A)$, i.e. α is homologically equivalent to zero), then $\alpha = 0$.
- (iii) Let $\alpha \in \text{CH}_{(1)}^i(A)$ (in particular $\alpha \in \text{CH}_{\text{hom}}^i(A)$). If $\text{aj}(\alpha) = 0$, then $\alpha = 0$.

Part (i) is known as the Beauville conjecture (see [Bea83], Section 5 and [Bea86], Section 2). We refer to [Kün93], Conjecture 8.1, [KV96], Conjecture 2.13 and [Fu10], Conjecture 1 for some equivalent forms of this conjecture. All three conjectures are only known in very special cases. They are trivially true for $\text{CH}^0(A)$, and are also true for $\text{CH}^1(A)$, the latter being a consequence of the theorem of the square. In fact, the theorem of the square says $\text{CH}_{(0)}^1(A) \simeq \text{NS}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$ and $\text{CH}_{(1)}^1(A) \simeq \text{Pic}^0(A) \otimes_{\mathbb{Z}} \mathbb{Q}$. By the Fourier transform \mathcal{F} , we also obtain Part (i) for $i = g, g-1$ and $g-2$, Part (ii) for $i = g$ and $g-1$, and Part (iii) for $i = g$. So the first remaining cases are

- (i) $\text{CH}_{(-1)}^2(A)$ for $g = 5$;
- (ii) $\text{CH}_{(0)}^2(A)$ for $g = 4$;
- (iii) $\text{CH}_{(1)}^2(A)$ for $g = 3$.

The Beauville conjecture is also known for abelian varieties defined over $\overline{\mathbb{F}}_p$. This was proven by Soulé using the action of Frobenius and consequences of the Weil conjectures ([Sou84], Corollaire 2). In fact, he proved that for $A/\overline{\mathbb{F}}_p$, we have $\text{CH}^i(A) = \text{CH}_{(0)}^i(A)$.

Recently in [Moo11], Moonen studied the Chow ring of abelian schemes with non-trivial endomorphisms. In that situation he generalized the \mathfrak{sl}_2 -action to the action of a much larger Lie algebra, including all \mathfrak{sl}_2 -tuples coming from relatively ample line bundles. In particular, he proved the following special case of Conjecture 2.10 (*loc. cit.*, Corollary 8.4 and Theorem 8.6).

Theorem 2.11. *Conjecture 2.10 is true for the \mathbb{Q} -subalgebra $\mathcal{D}(A) \subset \text{CH}(A)$ generated by divisor classes, i.e. classes in $\text{CH}^1(A)$. (Here Part (iii) is obtained for the classical and ℓ -adic Abel-Jacobi maps, under the assumption that $\text{End}^0(A)$ is simple.)* \square

2.2. Motivic interpretation and cohomological realization

The spirit of this section is to show that everything in Section 2.1 is of motivic nature. We shall rewrite all previous results in a motivic language. Since later we would like to apply cohomological methods in Chapters 5 and 6, we also list the results in cohomology in parallel.

Let $(A/S, \theta)$ be a principally polarized abelian scheme. We denote by $\pi: A \rightarrow S$ the structural map, and by $o: S \rightarrow A$ the zero section of π . Again we write $d = \dim(S/k)$ and $g = \dim(A/S)$.

Relative Chow motives

We briefly recall the definition of Chow motives, and we follow [DM91], Section 1 in the relative setting. The rough idea of a motive is to enlarge the set of maps between varieties by including all correspondences.

More precisely, given objects X/S and Y/S in \mathcal{V}_S , define the group of relative *correspondences* between X and Y over S as $\text{Corr}_S(X, Y) := \text{CH}(X \times_S Y)$. An element $\gamma \in \text{Corr}_S(X, Y)$ induces

$$\begin{aligned} \gamma_*: \text{CH}(X) &\rightarrow \text{CH}(Y) \\ \alpha &\mapsto \text{pr}_{2,*}(\text{pr}_1^*(\alpha) \cdot \gamma), \end{aligned}$$

where $\text{pr}_1: X \times_S Y \rightarrow X$ and $\text{pr}_2: X \times_S Y \rightarrow Y$ are the two projections. If $Z/S \in \mathcal{V}_S$, $\gamma \in \text{Corr}_S(X, Y)$ and $\gamma' \in \text{Corr}_S(Y, Z)$, we also define the composition of correspondences

$$\gamma' \circ \gamma := \text{pr}_{13,*}(\text{pr}_{12}^*(\gamma) \cdot \text{pr}_{23}^*(\gamma')) \in \text{Corr}_S(X, Z),$$

where $\text{pr}_{13}, \text{pr}_{12}$ and pr_{23} are the projections of $X \times_S Y \times_S Z$ to $X \times_S Z$, $X \times_S Y$ and $Y \times_S Z$ respectively. We have $(\gamma' \circ \gamma)_* = \gamma'_* \circ \gamma_*$.

The group $\text{Corr}_S(X, Y)$ is graded: if $X = \sqcup_i X_i$ such that each X_i has connected fibers, then we write $\text{Corr}_S^r(X, Y) := \oplus_i \text{CH}^{\dim(X_i/S)+r}(X_i \times_S Y)$. It follows that

$$\text{Corr}_S^r(X, Y) \times \text{Corr}_S^s(Y, Z) \xrightarrow{\circ} \text{Corr}_S^{r+s}(X, Z).$$

Finally, a projector of X/S is a correspondence $p \in \text{Corr}_S^0(X, X)$ such that $p \circ p = p$.

Definition 2.12. We define the category of relative *Chow motives* over S , denoted by \mathcal{M}_S .

- (i) The objects of \mathcal{M}_S are triples (X, p, m) with $X/S \in \mathcal{V}_S$, p a projector of X/S and $m \in \mathbb{Z}$.
- (ii) The morphisms between $M = (X, p, m)$ and $N = (Y, q, n)$ are defined by

$$\begin{aligned} \text{Hom}_{\mathcal{M}_S}(M, N) &:= q \circ \text{Corr}_S^{n-m}(X, Y) \circ p \\ &= \{\gamma \in \text{Corr}_S^{n-m}(X, Y) : q \circ \gamma \circ p = \gamma\}. \end{aligned}$$

2. PRELIMINARIES

The identity morphism of $M = (X, p, m)$ is $\text{id}_M := p \circ [\Delta_{X/S}] \circ p = p$, where $\Delta_{X/S}$ is the diagonal $X \subset X \times_S X$.

The abstract theory says that \mathcal{M}_S is a rigid, pseudo-abelian tensor category. We refer to [And04], Chapitre 2 for the terminology. Moreover, there is a functor

$$R: \mathcal{V}_S^{\text{op}} \rightarrow \mathcal{M}_S$$

that maps objects X/S to $(X, [\Delta_{X/S}], 0)$, and morphisms $f: X \rightarrow Y$ over S to $[\Gamma_f^t] \in \text{Corr}_S^0(Y, X)$, where $\Gamma_f^t \subset Y \times_S X$ is the transpose of the graph of f . Further, we define the Chow groups of $M = (X, p, m)$ to be

$$\text{CH}^i(M) := p_* (\text{CH}^{i+m}(X)).$$

Examples 2.13.

- (i) The motive $\mathbb{1}_S := R(S/S) = (S, [\Delta_{S/S}], 0)$ is the unit of \mathcal{M}_S with respect to the tensor product. For $i \in \mathbb{Z}$, define $\mathbb{1}_S(i) := (S, [\Delta_{S/S}], i)$. The motive $\mathbb{1}_S(1)$ (resp. $\mathbb{1}_S(-1)$) is called the *relative Tate motive* (resp. *Lefschetz motive*) over S . For $M \in \mathcal{M}_S$, write $M(i) := M \otimes \mathbb{1}_S(i)$. Then we have

$$\text{CH}^i(M) = \text{Hom}_{\mathcal{M}_S}(\mathbb{1}_S(-i), M) = \text{Hom}_{\mathcal{M}_S}(\mathbb{1}_S, M(i)).$$

- (ii) If $X/S \in \mathcal{V}_S$ admits a section $\sigma: S \rightarrow X$, then σ defines a projector $\pi_0 := [\sigma(S) \times_S X]$ and a motive $R^0(X/S) := (X, \pi_0, 0)$. Further assume that X/S has connected fibers with $\dim(X/S) = g$. Then we have a projector $\pi_{2g} := [X \times_S \sigma(S)]$ and a motive $R^{2g}(X/S) := (X, \pi_{2g}, 0)$. The correspondence $[S \times_S X]$ (resp. $[X \times_S S]$) induces an isomorphism $\mathbb{1}_S \simeq R^0(X/S)$ (resp. $R^{2g}(X/S) \simeq \mathbb{1}_S(-g)$).
- (iii) For $n \geq 1$, there is an \mathfrak{S}_n -action on the projector of the motive $M^{\otimes n}$. One can symmetrize (resp. alternate) the projector to obtain the n -th *symmetric* (resp. *wedge*) *product* of M , denoted by $S^n(M)$ (resp. $\wedge^n(M)$). Both can be viewed as direct summands of $M^{\otimes n}$ (see [Kim05], Section 3).

Back to the abelian scheme A/S . We now state the motivic version of the results in Section 2.1. Proofs can be found in [DM91], Theorem 3.1 and Corollary 3.2, and [Kün93], Theorems 5.1 and 5.2.

2.2. Motivic interpretation and cohomological realization

Theorem 2.14.

(i) *There is a unique decomposition $[\Delta_{A/S}] = \sum_{i=0}^{2g} \pi_i$ in $\text{Corr}_S^0(A, A)$ such that*

$$(\text{id}_A \times_S [N])^*(\pi_i) = N^i \pi_i \text{ for all } N \in \mathbb{Z}.$$

We have $\pi_i \circ \pi_j = 0$ for $i \neq j$ and $\pi_i \circ \pi_i = \pi_i$. Write $R^i(A/S) := (A, \pi_i, 0)$ and we obtain a motivic decomposition

$$(2.9) \quad R(A/S) = \bigoplus_{i=0}^{2g} R^i(A/S).$$

Moreover, there are isomorphisms $R^i(A/S) \simeq S^i(R^1(A/S))$ for $0 \leq i \leq 2g$.

(ii) *For $0 \leq i \leq 2g$, the Fourier transform \mathcal{F} (viewed as the correspondence $\exp(\ell) \in \text{Corr}_S(A, A)$) induces isomorphisms*

$$\mathcal{F} : R^i(A/S) \xrightarrow{\sim} R^{2g-i}(A/S)(g-i). \quad \square$$

Remark 2.15. Some people, thinking on the cohomological side, write $R^i(A/S) \simeq \wedge^i(R^1(A/S))$. However, motivically one should take the symmetric product, since there is a sign change when applying cohomological realizations (see [And05], Section 2.2).

Then we describe the motivic Lefschetz decomposition. Define correspondences

$$\begin{aligned} L &:= \Delta_*(-\theta) \in \text{Corr}_S^1(A, A), \\ \Lambda &:= -\mathcal{F}^{-1} \circ L \circ \mathcal{F} \in \text{Corr}_S^{-1}(A, A), \end{aligned}$$

where $\Delta : A \rightarrow A \times_S A$ is the diagonal map, and \mathcal{F} is viewed as a correspondence. One shows that

$$L : R^i(A/S) \rightarrow R^{i+2}(A/S)(1), \text{ and } \Lambda : R^i(A/S) \rightarrow R^{i-2}(A/S)(-1).$$

Theorem 2.16.

(i) *We have $[L, \Lambda] = \sum_{i=0}^{2g} (i-g)\pi_i$ in $\text{Corr}_S^0(A, A)$.*

(ii) *For $0 \leq i \leq 2g$, the motive $R^i(A/S)$ has a decomposition*

$$R^i(A/S) = \bigoplus_{k=\max\{0, i-g\}}^{\lfloor i/2 \rfloor} L^k(R_{\text{prim}}^{i-2k}(A/S)(-k)),$$

where $R_{\text{prim}}^i(A/S)$ is a direct summand of $R^i(A/S)$ on which L^{g-i+1} induces zero morphism.

(iii) *(hard Lefschetz) For $0 \leq i \leq g$, there are isomorphisms*

$$L^{g-i} : R^i(A/S) \xrightarrow{\sim} R^{2g-i}(A/S)(g-i). \quad \square$$

2. PRELIMINARIES

ℓ -adic realization

We shall use ℓ -adic cohomology with \mathbb{Q}_ℓ -coefficients in the general setting (here ℓ is a prime number different from the characteristic of k , not to be confused with $\ell = c_1(\mathcal{P})$). If the base field $k = \mathbb{C}$, one may instead use singular cohomology with \mathbb{Q} -coefficients. For simplicity, we assume k to be algebraically closed (or at least separably closed) when working with cohomology.

Denote by $D^b(S, \mathbb{Q}_\ell)$ the bounded derived category of \mathbb{Q}_ℓ -sheaves on S . The functor $\mathcal{V}_S \rightarrow D^b(S, \mathbb{Q}_\ell)$ sending $\phi: X \rightarrow S$ to $R\phi_* \mathbb{Q}_\ell$ extends to a \mathbb{Q} -linear tensor functor

$$\mathcal{M}_S \rightarrow D^b(S, \mathbb{Q}_\ell),$$

called the ℓ -adic realization (see [DM91], Section 1.8).

Deligne's E_2 -degeneration of the Leray spectral sequence ([Del68], Théorème 1.5) says that for $\phi: X \rightarrow S$ in \mathcal{V}_S , there is a (in general non-canonical) decomposition

$$(2.10) \quad R\phi_* \mathbb{Q}_\ell = \sum_i R^i \phi_* \mathbb{Q}_\ell[-i] \text{ in } D^b(S, \mathbb{Q}_\ell).$$

What is special in the case of an abelian scheme $\pi: A \rightarrow S$, is that one can make this decomposition canonical using the multiplication by N . More precisely, we have the following analogue of Theorems 2.14 and 2.16 (see [Del68], Remarque 2.19 and [Voi12], Corollary 2.2).

Corollary 2.17.

(i) In $D^b(S, \mathbb{Q}_\ell)$, there is a canonical multiplicative decomposition

$$R\pi_* \mathbb{Q}_\ell = \sum_{i=0}^{2g} R^i \pi_* \mathbb{Q}_\ell[-i],$$

such that $[N]^*$ acts on $R^i \pi_* \mathbb{Q}_\ell$ by multiplication by N^i , for all $N \in \mathbb{Z}$. Here by multiplicative we mean that the decomposition is compatible with the cup products on both sides. Moreover, there are isomorphisms $R^i \pi_* \mathbb{Q}_\ell \simeq \wedge^i(R^1 \pi_* \mathbb{Q}_\ell)$ for $0 \leq i \leq 2g$.

(ii) For $0 \leq i \leq 2g$, the Fourier transform \mathcal{F} gives isomorphisms

$$\mathcal{F}: R^i \pi_* \mathbb{Q}_\ell \xrightarrow{\sim} R^{2g-i} \pi_* \mathbb{Q}_\ell(g-i),$$

where $(-)$ stands for Tate twists.

(iii) The ℓ -adic cycle class $u := \text{cl}(-\theta) \in H^2(A, \mathbb{Q}_\ell(1))$ induces a map

$$R\pi_*(u): R^i \pi_* \mathbb{Q}_\ell \rightarrow R^{i+2} \pi_* \mathbb{Q}_\ell(1).$$

2.2. Motivic interpretation and cohomological realization

For $0 \leq i \leq 2g$, we have a sheaf decomposition

$$R^i \pi_* \mathbb{Q}_\ell = \bigoplus_{k=\max\{0, i-g\}}^{\lfloor i/2 \rfloor} (R\pi_*(u))^k (R_{\text{prim}}^{i-2k} \pi_* \mathbb{Q}_\ell(-k)),$$

where $R_{\text{prim}}^i \pi_* \mathbb{Q}_\ell$ is the kernel of $(R\pi_*(u))^{g-i+1} : R^i \pi_* \mathbb{Q}_\ell \rightarrow R^{2g-i+2} \pi_* \mathbb{Q}_\ell(g-i+1)$.

(hard Lefschetz) For $0 \leq i \leq g$, there are isomorphisms

$$(R\pi_*(u))^{g-i} : R^i \pi_* \mathbb{Q}_\ell \xrightarrow{\sim} R^{2g-i} \pi_* \mathbb{Q}_\ell(g-i). \quad \square$$

Remark 2.18. Unlike the case of abelian schemes, in general one does not have a multiplicative decomposition (2.10) (not even after shrinking the base variety S). See [Voi12], Section 1.2 for a counterexample.

Compatibility with the Chow ring

Finally we build connections between the Chow theory side and the motivic and cohomological sides. This is done simply by comparing the actions of the multiplication by N , for all $N \in \mathbb{Z}$.

By definition $[N]^*$ acts on $\text{CH}_{(j)}^i(A)$ by multiplication by N^{2i-j} . On the other hand, Theorem 2.14 (i) and Corollary 2.17 (i) tell that $[N]^*$ acts on $\text{CH}(R^i(A/S))$ and $R^i \pi_* \mathbb{Q}_\ell$ by multiplication by N^i . It follows immediately that

$$\text{CH}_{(j)}^i(A) \subset \text{CH}(R^{2i-j}(A/S)).$$

Similarly, we see that the ℓ -adic cycle class map $\text{cl} : \text{CH}^i(A) \rightarrow H^{2i}(A, \mathbb{Q}_\ell(i))$ factors through

$$(2.11) \quad \text{cl} : \text{CH}_{(j)}^i(A) \rightarrow H^j(S, R^{2i-j} \pi_* \mathbb{Q}_\ell(i)),$$

which is compatible with the multiplicative structures on both sides.

Very often we find it convenient to replace the grading i (for codimension) in $\text{CH}_{(j)}^i(A)$ by a new, motivic grading. We write

$$(2.12) \quad \text{CH}_{(i,j)}(A) := \text{CH}_{(j)}^{(i+j)/2}(A), \text{ or equivalently } \text{CH}_{(2i-j,j)}(A) := \text{CH}_{(j)}^i(A),$$

so that $[N]^*$ acts on $\text{CH}_{(i,j)}(A)$ also by multiplication by N^i . In other words, we set

$$\text{CH}_{(i,j)}(A) = \text{CH}(R^i(A/S)) \cap \text{CH}_{(j)}(A).$$

The Beauville decomposition (2.5) then takes the form

$$(2.13) \quad \text{CH}(A) = \bigoplus_{i,j} \text{CH}_{(i,j)}(A),$$

2. PRELIMINARIES

with $0 \leq i \leq 2g$ and $\max\{-i, i-2g\} \leq j \leq \min\{i, 2g-i\} + 2d$. Expressions of algebraic operations are simple (if not simpler):

$$\begin{aligned} \mathrm{CH}_{(i,j)}(A) \times \mathrm{CH}_{(k,l)}(A) &\xrightarrow{\cdot} \mathrm{CH}_{(i+k,j+l)}(A), \\ \mathrm{CH}_{(i,j)}(A) \times \mathrm{CH}_{(k,l)}(A) &\xrightarrow{*} \mathrm{CH}_{(i+k-2g,j+l)}(A), \\ \mathcal{F}: \mathrm{CH}_{(i,j)}(A) &\xrightarrow{\sim} \mathrm{CH}_{(2g-i,j)}(A). \end{aligned}$$

We have $\theta \in \mathrm{CH}_{(2,0)}(A)$, so that the \mathfrak{sl}_2 -action in (2.6) becomes $e: \mathrm{CH}_{(i,j)}(A) \rightarrow \mathrm{CH}_{(i+2,j)}(A)$, $f: \mathrm{CH}_{(i,j)}(A) \rightarrow \mathrm{CH}_{(i-2,j)}(A)$, and $h = (i-g)\mathrm{id}$ on $\mathrm{CH}_{(i,j)}(A)$.

To conclude, the actions of $[N]$ group together $\oplus_j \mathrm{CH}_{(i,j)}(A)$, $R^i(A/S)$ and $R^i\pi_*\mathbb{Q}_\ell$. The same goes for the primitive parts $\oplus_j \mathrm{CH}_{(i,j),\mathrm{prim}}(A)$, $R_{\mathrm{prim}}^i(A/S)$ and $R_{\mathrm{prim}}^i\pi_*\mathbb{Q}_\ell$.

2.3. An illustration: the Dutch house

Here we present a useful picture that describes the Chow ring $\mathrm{CH}(A)$ (see [Moo09], Figure 1). The picture illustrates all structures discussed in Section 2.1 while combining the motivic aspect of Section 2.2. Further, it enables us to make clear statements without complicated indices. We decide to call it the *Dutch house*, due to its resemblance to a traditional Dutch *trapgevel*.

In figure 1, the (i, j) -th block represents the component $\mathrm{CH}_{(i,j)}(A)$ in the Beauville decomposition. Then the columns read the motivic decomposition $R(A/S) = \oplus_i R^i(A/S)$, and the rows read Beauville's grading j . As a result, components with the same codimension lie on a dashed line from upper left to lower right.

It is not difficult to verify that the house shape results from the precise index range of (2.13). The width of the house depends on $g = \dim(A/S)$, while the height (without roof) depends on $d = \dim(S/k)$. In particular when $S = k$ (i.e. $d = 0$), the house reduces to the roof only. Here Figure 1 is drawn based on the universal Jacobian over $S = \mathcal{M}_{4,1}$, i.e. the moduli space of pointed curves of genus 4 (see Sections 2.4 and 3.1). In this case we have $g = 4$ and $d = \dim(\mathcal{M}_{4,1}/k) = 3g - 3 + 1 = 10$.

Remark 2.19. Note that we have not drawn the components $\mathrm{CH}_{(i,j)}(A)$ with negative j . On one hand, when $S = k$ the Beauville conjecture (Conjecture 2.10 (i)) predicts the vanishing of those components. On the other hand, the classes we shall study are all in $\mathrm{CH}_{(i,j)}(A)$ with $j \geq 0$, i.e. inside the house.

As is shown in the picture, the Fourier transform \mathcal{F} acts as the reflection over the middle vertical line. Regarding the \mathfrak{sl}_2 -action, we find that e shifts classes to the right by 2 blocks, while f shifts classes to the left by 2 blocks. Finally the middle column of the house has weight 0 with respect to h .

2.3. An illustration: the Dutch house

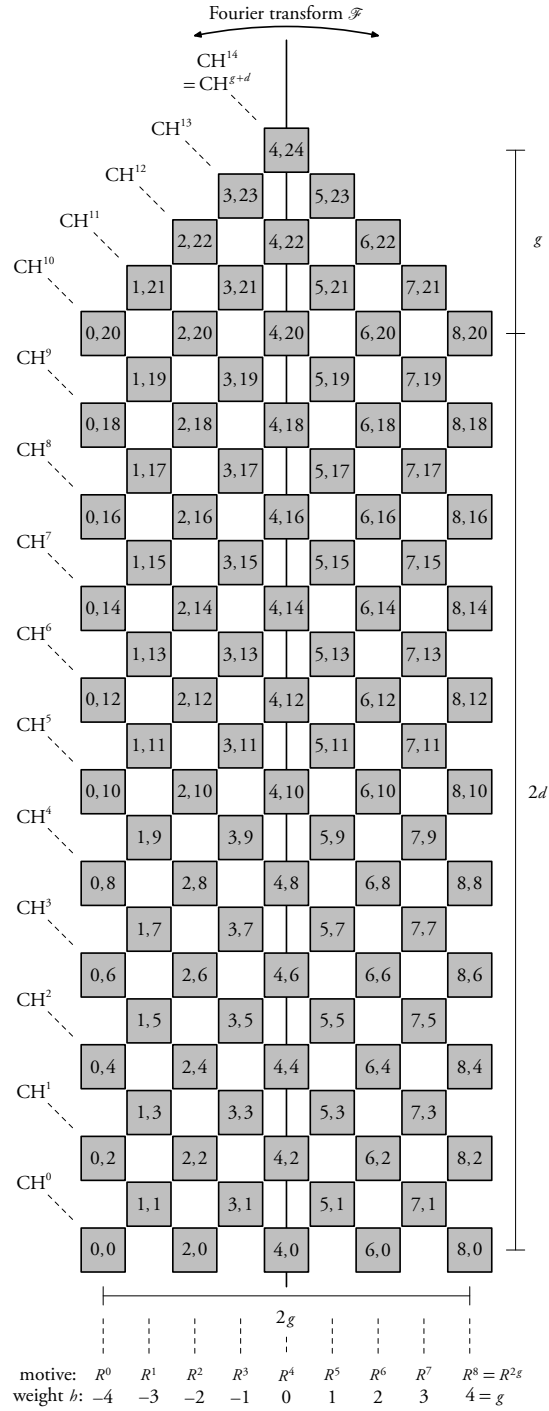


Figure 1. Mon dessin n° 1: the outside of the Dutch house ($g = 4$ and $d = 10$).

2. PRELIMINARIES

Example 2.20. The isomorphisms of motives (see Examples 2.13 (ii) and Theorem 2.14 (ii))

$$\begin{array}{ccc} R^0(A/S) & \xleftarrow[\sim]{\mathcal{F}} & R^{2g}(A/S)(g) \\ & \swarrow \sim \pi^* & \searrow \sim \pi_* \\ & R(S/S) & \end{array}$$

induce isomorphisms of \mathbb{Q} -algebras

$$(2.14) \quad \begin{array}{ccc} \left(\bigoplus_{i=0}^d \mathrm{CH}_{(0,2i)}(A), \cdot \right) & \xleftarrow[\sim]{\mathcal{F}} & \left(\bigoplus_{i=0}^d \mathrm{CH}_{(2g,2i)}(A), * \right) \\ & \swarrow \sim \pi^* & \searrow \sim \pi_* \\ & (\mathrm{CH}(S), \cdot) & \end{array}$$

The gradings are preserved as $\pi^*: \mathrm{CH}^i(S) \xrightarrow{\sim} \mathrm{CH}_{(0,2i)}(A)$ and $\pi_*: \mathrm{CH}_{(2g,2i)}(A) \xrightarrow{\sim} \mathrm{CH}^i(S)$.

In particular, the Chow ring $\mathrm{CH}(S)$ may be regarded as a \mathbb{Q} -subalgebra of $(\mathrm{CH}(A), \cdot)$ via π^* , or as a \mathbb{Q} -subalgebra of $(\mathrm{CH}(A), *)$ via π_* . In terms of the Dutch house, we may identify $\mathrm{CH}(S)$ with the 0-th column or with the $2g$ -th column of the house.

2.4. Case of a relative Jacobian

We consider the special case where A/S is the Jacobian of a relative curve. The main result is that the \mathfrak{sl}_2 -action on $\mathrm{CH}(A)$ can be reconstructed by the geometry of the curve.

As before S is a smooth connected variety of dimension d over k . Let $p: C \rightarrow S$ be a relative curve of genus g , *i.e.* a smooth projective scheme over S with geometrically connected fibers of relative dimension 1 and of genus g . We assume $g > 0$. Denote by $\pi: J \rightarrow S$ the associated relative Jacobian, where $J := \mathrm{Pic}^0(C/S)$ is an abelian scheme with $\dim(J/S) = g$. It has a canonical principal polarization, and we write $\theta \in \mathrm{CH}_{(0)}^1(J) = \mathrm{CH}_{(2,0)}(J)$ for the corresponding divisor class as defined in (2.3).

Now further assume that C/S admits a section $x_0: S \rightarrow C$. It induces a closed embedding $\iota: C \hookrightarrow J$, which sends locally a section x of C/S to the class $\iota(x) := \mathcal{O}_C(x - x_0)$. The composition $\iota \circ x_0$ is then the zero section $o: S \rightarrow J$. To summarize, we have the following diagram.

$$(2.15) \quad \begin{array}{ccc} C & \xhookrightarrow{\iota} & J \\ \downarrow p & & \downarrow \pi \\ S & \xrightarrow{o} & J \end{array} \quad \begin{array}{c} \nearrow x_0 \\ \searrow \end{array}$$

2.4. Case of a relative Jacobian

We denote $[x_0] := [x_0(S)] \in \text{CH}^1(C)$ and $[o] := [o(S)] \in \text{CH}^g(J)$, and also $[C] := [\iota(C)] \in \text{CH}^{g-1}(J)$ when there is no ambiguity. For $j \in \mathbb{Z}$, write

$$(2.16) \quad [C]_{(j)} \in \text{CH}_{(j)}^{g-1}(J) = \text{CH}_{(2g-2-j, j)}(J)$$

for the components of $[C]$ in the Beauville decomposition. We have $[C]_{(j)} = 0$ for $j < 0$ or $j > \min\{2g-2, g-1+d\}$.

We would like to reconstruct the \mathfrak{sl}_2 -action using the curve class $[C]$. The idea goes back to Riemann, who proved that when $S = k$, the class $[C]^{*(g-1)}/(g-1)! \in \text{CH}^1(J)$ is a translate of θ (see [Mum75], Lecture III). More generally, there is the following result due to Polishchuk ([Pol07b], Theorem 2.6).

Theorem 2.21. *We have $\theta = -\mathcal{F}([C]_{(0)})$ in $\text{CH}_{(2,0)}(J)$, and the \mathfrak{sl}_2 -action in (2.6) takes the form*

$$(2.17) \quad \begin{aligned} e: \text{CH}_{(i,j)}(J) &\rightarrow \text{CH}_{(i+2,j)}(J) & \alpha &\mapsto \mathcal{F}([C]_{(0)}) \cdot \alpha, \\ f: \text{CH}_{(i,j)}(J) &\rightarrow \text{CH}_{(i-2,j)}(J) & \alpha &\mapsto -[C]_{(0)} * \alpha, \\ h: \text{CH}_{(i,j)}(J) &\rightarrow \text{CH}_{(i,j)}(J) & \alpha &\mapsto (i-g)\alpha. \end{aligned} \quad \square$$

Let $\Omega_{C/S}^1$ be the relative cotangent bundle of C/S . Define classes

$$(2.18) \quad K := c_1(\Omega_{C/S}^1) \in \text{CH}^1(C), \text{ and } \psi := x_0^*(K) \in \text{CH}^1(S).$$

Note that we have $x_0^*([x_0]) = -\psi$ by adjunction. For simplicity, we keep the same notation ψ for the pull-back of the class ψ to C . The following lemma is probably known to experts, and is shown implicitly in [Pol07b], Theorem 2.6. As it will be applied many times in this thesis, we present the statement and the proof here.

Lemma 2.22. *We have the identity*

$$(2.19) \quad \iota^*(\theta) = \frac{1}{2}K + [x_0] + \frac{1}{2}\psi \text{ in } \text{CH}^1(C).$$

Proof. The goal is to calculate $\iota^*(\theta) = -\iota^*(\mathcal{F}([C]_{(0)}))$ and we start from $\iota^*(\mathcal{F}([C]))$. Consider the following three Cartesian squares

$$\begin{array}{ccccc} C \times_S C & \xrightarrow{\iota \times_S \text{id}_C} & J \times_S C & \xrightarrow{\text{pr}_2} & C \\ \text{id}_C \times_S \iota \downarrow & & \downarrow \text{id}_J \times_S \iota & & \downarrow \iota \\ C \times_S J & \xrightarrow{\iota \times_S \text{id}_J} & J \times_S J & \xrightarrow{\text{pr}_2} & J \\ \text{pr}_1 \downarrow & & \downarrow \text{pr}_1 & & \\ C & \xrightarrow{\iota} & J & & \end{array}$$

2. PRELIMINARIES

where pr_1 and pr_2 stand for the two projections in all cases. Then we have

$$\begin{aligned}
\iota^*(\mathcal{F}([C])) &= \iota^* \text{pr}_{2,*} \left(\text{pr}_1^* \iota_*([C]) \cdot \exp(\ell) \right) \\
&= \text{pr}_{2,*}(\text{id}_J \times_S \iota)^* \left((\iota \times_S \text{id}_J)_* \text{pr}_1^*([C]) \cdot \exp(\ell) \right) \\
&= \text{pr}_{2,*}(\text{id}_J \times_S \iota)^* \left((\iota \times_S \text{id}_J)_*([C \times_S J]) \cdot \exp(\ell) \right) \\
&= \text{pr}_{2,*}(\text{id}_J \times_S \iota)^* (\iota \times_S \text{id}_J)_* (\iota \times_S \text{id}_J)^* (\exp(\ell)) \\
&= \text{pr}_{2,*}(\iota \times_S \text{id}_C)_* (\text{id}_C \times_S \iota)^* (\iota \times_S \text{id}_J)^* (\exp(\ell)) \\
&= \text{pr}_{2,*}(\iota \times_S \iota)^* (\exp(\ell)) \\
&= \text{pr}_{2,*} \left(\exp((\iota \times_S \iota)^*(\ell)) \right).
\end{aligned}$$

The identity (2.4) and the theorem of the square imply (see [Pol07b], Formula (2.1))

$$(2.20) \quad (\iota \times_S \iota)^*(\ell) = [\Delta] - \text{pr}_1^*([x_0]) - \text{pr}_2^*([x_0]) - \psi,$$

where $\Delta := \Delta_{C/S} \subset C \times_S C$. It follows that

$$\begin{aligned}
(2.21) \quad \iota^*(\mathcal{F}([C])) &= \text{pr}_{2,*} \left(\exp([\Delta] - \text{pr}_1^*([x_0]) - \text{pr}_2^*([x_0]) - \psi) \right) \\
&= \text{pr}_{2,*} \left(\exp([\Delta] - \text{pr}_1^*([x_0])) \right) \cdot \exp(-[x_0] - \psi).
\end{aligned}$$

Observe that on the left-hand side of (2.21), we have

$$\iota^*(\mathcal{F}([C])) = \sum_{j=0}^{2g-2} \iota^*(\mathcal{F}([C]_{(j)})),$$

with $\iota^*(\mathcal{F}([C]_{(j)})) \in \text{CH}^{j+1}(C)$. Hence $\iota^*(\mathcal{F}([C]_{(0)}))$ is just the codimension 1 component of $\iota^*(\mathcal{F}([C]))$. Expanding the exponentials in (2.21) while keeping track of the codimension, we get

$$\begin{aligned}
\iota^*(\mathcal{F}([C]_{(0)})) &= \text{pr}_{2,*} \left(\frac{1}{2}([\Delta] - \text{pr}_1^*([x_0]))^2 \right) - \text{pr}_{2,*}([\Delta] - \text{pr}_1^*([x_0])) \cdot ([x_0] + \psi) \\
&= \text{pr}_{2,*} \left(\frac{1}{2}([\Delta] - \text{pr}_1^*([x_0]))^2 \right) \\
&= \frac{1}{2} \text{pr}_{2,*}([\Delta] \cdot [\Delta]) - \text{pr}_{2,*}([\Delta] \cdot \text{pr}_1^*([x_0])) + \frac{1}{2} \text{pr}_{2,*} \text{pr}_1^*([x_0] \cdot [x_0]).
\end{aligned}$$

The first two terms in the previous expression are easily calculated:

$$\text{pr}_{2,*}([\Delta] \cdot [\Delta]) = -K, \quad \text{and} \quad \text{pr}_{2,*}([\Delta] \cdot \text{pr}_1^*([x_0])) = [x_0].$$

2.4. Case of a relative Jacobian

For the term $\mathrm{pr}_{2,*} \mathrm{pr}_1^*([x_0] \cdot [x_0])$, consider the following Cartesian square.

$$\begin{array}{ccc} C \times_S C & \xrightarrow{\mathrm{pr}_2} & C \\ \mathrm{pr}_1 \downarrow & & \downarrow p \\ C & \xrightarrow{p} & S \end{array}$$

We have

$$\mathrm{pr}_{2,*} \mathrm{pr}_1^*([x_0] \cdot [x_0]) = p^* p_*([x_0] \cdot [x_0]) = p^* p_* x_{0,*} x_0^*([x_0]) = p^* x_0^*([x_0]) = -\psi.$$

In total we find $\iota^*(\mathcal{F}([C]_{(0)})) = -K/2 - [x_0] - \psi/2$. □

Remark 2.23. Lemma 2.22 shows that θ is the class of a relative *theta divisor* in the classical sense: that it comes from a family of *theta characteristics* (which in general only exists with \mathbb{Q} -coefficients).

A tale of two tautological rings (I)

For a relative pointed curve, we prove that the tautological ring of the base variety (in the sense of Mumford, Faber, *etc.*) is a subring of the tautological ring of the relative Jacobian (in the sense of Beauville, Polishchuk, *etc.*). The \mathfrak{sl}_2 -action on the Jacobian side produces relations between tautological classes, leading to various theoretical results and numerical evidence towards the Faber conjectures (on the tautological ring of the moduli space of smooth pointed curves).

3.1. Moduli side: tautological ring and the Faber conjectures

Let S be a smooth connected variety of dimension d over k . Consider a relative curve $p: C \rightarrow S$ of genus $g > 0$, together with a section (marked point) $x_0: S \rightarrow C$. In this section we recall the notion of the tautological ring of S . The prototype of all such varieties S is the moduli space of smooth pointed curves of genus g , for which there is a version of the Faber conjectures.

Recall the class $K = c_1(\Omega_{C/S}^1) \in \mathrm{CH}^1(C)$ defined in (2.18). For $i \geq 0$, define

$$(3.1) \quad \kappa_i := p_*(K^{i+1}) \in \mathrm{CH}^i(S).$$

We have $\kappa_0 = (2g - 2)[S]$, and it is often convenient to write $\kappa_{-1} = 0$. Also recall the class $\psi = x_0^*(K) \in \mathrm{CH}^1(S)$. Intuitively, the classes $\{\kappa_i\}$ reflect the geometry of the fibers of C/S , while ψ reflects the variation of the marked point.

Definition 3.1. The *tautological ring* of S , denoted by $\mathcal{R}(S)$, is the (graded) \mathbb{Q} -subalgebra of $\mathrm{CH}(S)$ generated by the geometrically constructed classes $\{\kappa_i\}$ and ψ . Elements in $\mathcal{R}(S)$ are called *tautological classes*.

3. A TALE OF TWO TAUTOLOGICAL RINGS (I)

The study of tautological classes was initiated by Mumford [Mum83], and later carried on extensively by Faber, Pandharipande, *etc.*, in the context of various moduli spaces. See [Fab99], [FP00], [Pan02], [Fab13] and [FP13] for an overview of the major questions.

In our situation, denote by $\mathcal{M}_{g,1}$ the moduli stack of smooth pointed curves of genus g over k ($g > 0$ as before). It is isomorphic to the universal curve \mathcal{C}_g over the moduli stack of smooth genus g curves \mathcal{M}_g . We have $\dim(\mathcal{M}_{g,1}/k) = \dim(\mathcal{M}_g/k) + 1 = 3g - 3 + 1$. We also write $p: \mathcal{C}_{g,1} \rightarrow \mathcal{M}_{g,1}$ for the universal curve over $\mathcal{M}_{g,1}$.

The stack $\mathcal{M}_{g,1}$ admits a finite cover by a smooth connected variety over k (see [Mum83], Part I, Section 2). Since we work with \mathbb{Q} -coefficients, the Chow ring $\mathrm{CH}(\mathcal{M}_{g,1})$ can be easily defined via the cover. Similarly, one can define the classes $\{\kappa_i\}$ and ψ in $\mathrm{CH}(\mathcal{M}_{g,1})$, as well as the tautological ring $\mathcal{R}(\mathcal{M}_{g,1})$. In principle, one may regard $\mathcal{M}_{g,1}$ as a smooth connected variety when talking about Chow groups with \mathbb{Q} -coefficients.

We now state the Faber conjectures^{note 3} in the context of $\mathcal{M}_{g,1}$. Roughly speaking, they predict that $\mathcal{R}(\mathcal{M}_{g,1})$ behaves like the algebraic cohomology ring of a smooth projective variety of dimension $g - 1$ over k .

Conjecture 3.2.

- (i) *The tautological ring $\mathcal{R}(\mathcal{M}_{g,1})$ is Gorenstein with socle in codimension $g - 1$. In other words, we have $\mathcal{R}^i(\mathcal{M}_{g,1}) = 0$ for $i > g - 1$ and $\mathcal{R}^{g-1}(\mathcal{M}_{g,1}) \simeq \mathbb{Q}$, and the pairing*

$$(3.2) \quad \mathcal{R}^i(\mathcal{M}_{g,1}) \times \mathcal{R}^{g-1-i}(\mathcal{M}_{g,1}) \xrightarrow{\cdot} \mathcal{R}^{g-1}(\mathcal{M}_{g,1}) \simeq \mathbb{Q}$$

is perfect for all $0 \leq i \leq g - 1$.

- (ii) *The classes $\kappa_1, \dots, \kappa_{\lfloor g/3 \rfloor}$ and ψ generate $\mathcal{R}(\mathcal{M}_{g,1})$, with no relations in codimension $i \leq \lfloor g/3 \rfloor$.*

We refer to [Fab99], Conjecture 1 for the original Faber conjectures on the tautological ring $\mathcal{R}(\mathcal{M}_g)$. The ring $\mathcal{R}(\mathcal{M}_g)$ is defined to be generated by $\{\kappa_i\}$ (without ψ), and the socle lies in codimension $g - 2$ instead of $g - 1$.

Remarks 3.3.

- (i) Looijenga proved that $\mathcal{R}^i(\mathcal{M}_{g,1})$ vanishes for $i > g - 1$, and that $\mathcal{R}^{g-1}(\mathcal{M}_{g,1})$ is at most 1-dimensional ([Loo95], Theorem). Later Faber [Fab97] showed that $\mathcal{R}^{g-1}(\mathcal{M}_{g,1})$ is indeed 1-dimensional. So regarding Part (i) of Conjecture 3.2, the difficult question is whether the pairing is perfect.
- (ii) The generation statement in Part (ii) of Conjecture 3.2 was first proven by Ionel ([Ion05], Theorem 1.5), and we shall give another proof of this fact (see Theorem 3.15). Moreover,

3.2. Jacobian side: tautological ring and its generators

the results of Boldsen ([Bol12], Theorem 1) and Looijenga ([Loo96], Theorem 2.3) imply that there are no relations in codimension $i < \lfloor g/3 \rfloor$. So we are one degree off when $g \equiv 0 \pmod{3}$.

- (iii) There should be a third conjecture that predicts the intersection numbers in (3.2), similar to the one for \mathcal{M}_g (see [Fab99], Conjecture 1 (c); several proofs known). However, the numbers could be obtained in an *ad hoc* manner by pushing forward to \mathcal{M}_g .

3.2. Jacobian side: tautological ring and its generators

Throughout this section, we work in the setting of (2.15). We define the tautological ring $\mathcal{T}(J)$ of the relative Jacobian J/S and describe its generators. The main result is that by pulling back via $\pi^*: \mathrm{CH}(S) \rightarrow \mathrm{CH}(J)$, one can identify $\mathcal{R}(S)$ with the \mathbb{Q} -subalgebra of $\mathcal{T}(J)$ located on the 0-th column of the Dutch house.

Definition 3.4. The *tautological ring* of J , denoted by $\mathcal{T}(J)$, is the smallest (graded) \mathbb{Q} -subalgebra of $(\mathrm{CH}(J), \cdot)$ such that

- (i) we have $[C] \in \mathcal{T}(J)$;
- (ii) the ring $\mathcal{T}(J)$ is stable under $[N]^*$ (or $[N]_*$), for all $N \in \mathbb{Z}$;
- (iii) the ring $\mathcal{T}(J)$ is stable under the Fourier transform \mathcal{F} .

Again, elements in $\mathcal{T}(J)$ are called *tautological classes*.

The notion of a tautological ring on the Jacobian side was introduced by Beauville [Bea04], in the context of a Jacobian variety (*i.e.* $S = k$) and modulo algebraic equivalence. Since then there have been various versions of the tautological ring. We refer to [Pol05], [Pol07], [Her07], [GK07], [FH07] and [Moo09] for the study of these rings. In the relative setting, Polishchuk considered a much bigger tautological ring, including all classes in $\pi^*(\mathrm{CH}(S))$ (see [Pol07b], Section 4). Here our minimalist definition is more suitable for studying the tautological ring of S .

Remarks 3.5.

- (i) Condition (ii) in Definition 3.4 is equivalent to saying that $\mathcal{T}(J)$ is stable under the Beauville decomposition (2.13). In particular, the ring is bigraded: write $\mathcal{T}_{(i,j)}(J) := \mathcal{T}(J) \cap \mathrm{CH}_{(i,j)}(J)$, and we have $\mathcal{T}(J) = \bigoplus_{i,j} \mathcal{T}_{(i,j)}(J)$.
- (ii) Since $\mathcal{T}(J)$ is stable under \mathcal{F} , it is also stable under the Pontryagin product $(*)$. Our choice of working primarily with the intersection product is due to historical reasons.

3. A TALE OF TWO TAUTOLOGICAL RINGS (I)

- (iii) It is immediate that $[C]_{(0)} \in \mathcal{T}_{(2g-2,0)}(J)$, and by Theorem 2.21 we have $\theta = -\mathcal{F}([C]_{(0)}) \in \mathcal{T}_{(2,0)}(J)$. This shows that $\mathcal{T}(J)$ is also stable under the \mathfrak{sl}_2 -action (2.17). So $\mathcal{T}(J)$ is stable under all structures described in Section 2.1. Alternatively, one can define $\mathcal{T}(J)$ to be the smallest (graded) \mathbb{Q} -subalgebra of $(\mathrm{CH}(J), \cdot)$ that contains $[C]$, and that is stable under the \mathfrak{sl}_2 -action. The equivalence of the definitions is implied by (2.7).

Since the two products (\cdot) and $(*)$ do not commute with each other, it is *a priori* not clear whether $\mathcal{T}(J)$ is finitely generated. Now we give an affirmative answer to this question by writing down an explicit set of generators.

Recall from (2.16) that $[C]_{(j)} \in \mathcal{T}_{(2g-2-j,j)}(J)$ for $j \in \mathbb{Z}$. Then for $i \leq j+2$ and $i+j$ even, consider the class

$$\theta^{(j-i+2)/2} \cdot [C]_{(j)} \in \mathcal{T}_{(2g-i,j)}(J).$$

Denote its Fourier dual by

$$p_{i,j} := \mathcal{F}(\theta^{(j-i+2)/2} \cdot [C]_{(j)}) \in \mathcal{T}_{(i,j)}(J).$$

As examples we have $p_{2,0} = \mathcal{F}([C]_{(0)}) = -\theta$ and $p_{0,0} = \mathcal{F}(\theta \cdot [C]_{(0)}) = g[J]$. Since $[C]_{(j)} = 0$ for $j < 0$ or $j > 2g-2$, we also know that $p_{i,j} = 0$ for $i < 0$ or $j < 0$ or $j > 2g-2$.

Figure 2 depicts the classes $\{p_{i,j}\}$ inside the Dutch house with $g = 8$. Also shown in the picture is the pull-back of the class ψ via π^* , again denoted by ψ , which lies in $\mathrm{CH}_{(0,2)}(J)$. Note that when $d = \dim(S/k)$ is small, classes that are above the roof also vanish.

By (2.17), the action of $e \in \mathfrak{sl}_2$ is the intersection with $p_{2,0}$. Also it is not difficult to see that

$$f(p_{i,j}) = p_{i-2,j}.$$

Then one of the questions is to calculate the class $f(p_{i,j} p_{k,l})$. This turns out to be the key to the following theorem.

Theorem 3.6.

- (i) *The tautological ring $\mathcal{T}(J)$ coincides with the \mathbb{Q} -subalgebra of $(\mathrm{CH}(J), \cdot)$ generated by the classes $\{p_{i,j}\}$ and ψ . In particular, the ring is finitely generated.*
- (ii) *The operator $f \in \mathfrak{sl}_2$ acts on polynomials in $\{p_{i,j}\}$ and ψ via the following differential operator of degree 2:*

$$(3.3) \quad \mathcal{D} := \frac{1}{2} \sum_{i,j,k,l} \left(\psi p_{i-1,j-1} p_{k-1,l-1} - \binom{i+k-2}{i-1} p_{i+k-2,j+l} \right) \partial p_{i,j} \partial p_{k,l} + \sum_{i,j} p_{i-2,j} \partial p_{i,j}.$$

3.2. Jacobian side: tautological ring and its generators

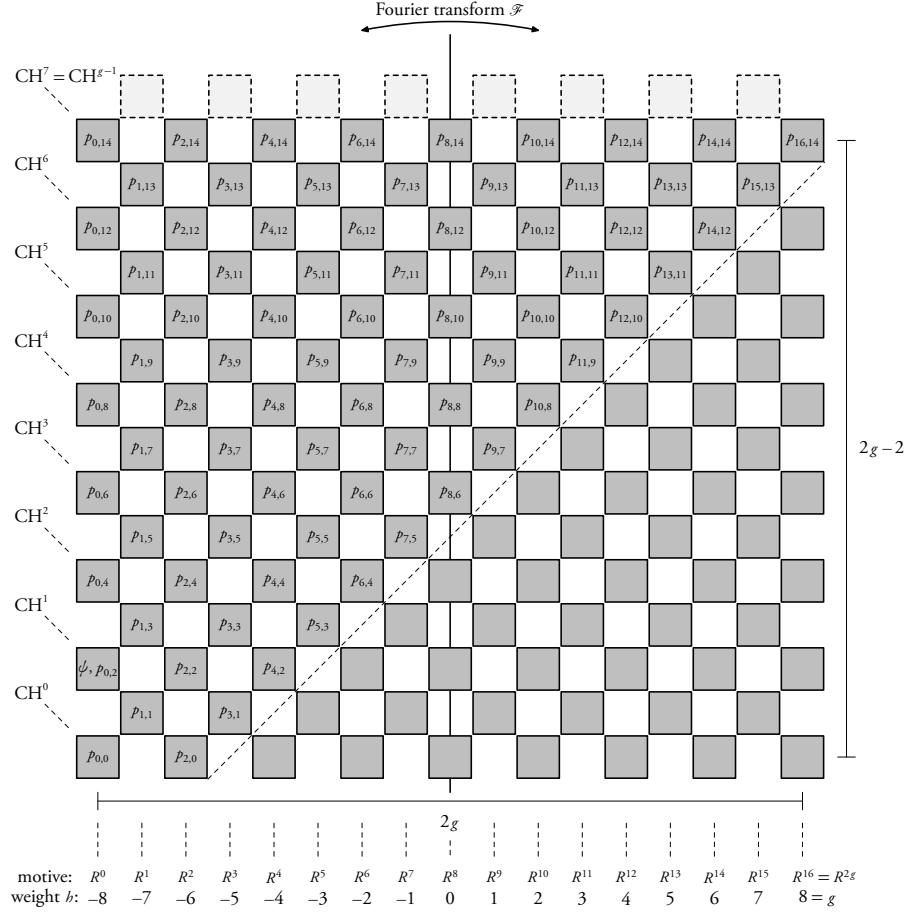


Figure 2. Mon dessin n° 2: the inside of the Dutch house ($g = 8$).

Remark 3.7. We may track the differential operator \mathcal{D} in the Dutch house. For the degree 2 part of \mathcal{D} , whenever there is a product of two classes $p_{i,j}p_{k,l}$, first find the generators to the lower-left of $p_{i,j}$ and $p_{k,l}$ by 1 block, and multiply by ψ , which yields $\psi p_{i-1,j-1}p_{k-1,l-1}$. Then look for the generator to the left of $p_{i,j}p_{k,l}$ by 2 blocks, which is $p_{i+k-2,j+l}$. For the linear part of \mathcal{D} , the generator $p_{i,j}$ is simply replaced by the one to the left of it by 2 blocks, i.e. $p_{i-2,j}$. All these operations shift classes to the left by 2 blocks.

Proof of Theorem 3.6. Suppose we have proven (ii) and that $\psi \in \mathcal{T}(J)$. Consider the \mathbb{Q} -subalgebra of $(CH(J), \cdot)$ generated by the classes $\{p_{i,j}\}$ and ψ . We denote it by $\mathcal{T}'(J)$ and we have $\mathcal{T}'(J) \subset \mathcal{T}(J)$. By definition $\mathcal{T}'(J)$ is stable under the action of $[N]^*$, for all $N \in \mathbb{Z}$. It is stable under the action of $e \in \mathfrak{sl}_2$, which is the intersection with $p_{2,0}$. Moreover, it follows from (ii) that $\mathcal{T}'(J)$ is also stable under the action of $f \in \mathfrak{sl}_2$. The identity (2.7) then shows that $\mathcal{T}'(J)$ is stable under the Fourier transform \mathcal{F} . In particular, the classes $\{[C]_{(j)}\}$ are contained in $\mathcal{T}'(J)$. Since $\mathcal{T}(J)$ is defined as

3. A TALE OF TWO TAUTOLOGICAL RINGS (I)

the smallest \mathbb{Q} -algebra that satisfies these properties, there is necessarily an equality $\mathcal{T}'(J) = \mathcal{T}(J)$, which proves (i).

Statement (ii) follows essentially from [Pol07b], Formula (2.9). We only need to translate the notation carefully. Following Polishchuk, we write $\eta := K/2 + [x_0] + \psi/2$, which by (2.19) is equal to $\iota^*(\theta)$. We also have $f = -\tilde{X}_{2,0}(C)/2$ in his notation. Define operators $\tilde{p}_{i,j}$ on $\text{CH}(J)$ by $\tilde{p}_{i,j}(\alpha) := p_{i,j} \cdot \alpha$. Then the fact that

$$p_{i,j} = \mathcal{F}(\theta^{(j-i+2)/2} \cdot [C]_{(j)}) = \mathcal{F}(\iota_*(\eta^{(j-i+2)/2})_{(j)})$$

is translated into

$$\tilde{p}_{i,j} = \frac{1}{i!} \tilde{X}_{0,i}(\eta^{(j-i+2)/2}).$$

We apply Formula (2.9) in *loc. cit.* and find

$$\begin{aligned} [f, \tilde{p}_{i,j}] &= -\frac{1}{2 \cdot i!} [\tilde{X}_{2,0}(C), \tilde{X}_{0,i}(\eta^{(j-i+2)/2})] \\ (3.4) \quad &= \frac{1}{(i-1)!} \tilde{X}_{1,i-1}(\eta^{(j-i+2)/2}) - \frac{1}{(i-2)!} \tilde{X}_{0,i-2}(\eta^{(j-i+4)/2}). \end{aligned}$$

Note that the second equality of (3.4) also involves the fact that $\tilde{X}_{i,0}(C) = 0$ for $i \leq 1$ (see *loc. cit.*, Lemma 2.8), and that $x_0^*(\eta) = x_0^*\iota^*(\theta) = o^*(\theta) = 0$. We continue to calculate

$$\begin{aligned} [[f, \tilde{p}_{i,j}], \tilde{p}_{k,l}] &= \frac{1}{(i-1)!k!} [\tilde{X}_{1,i-1}(\eta^{(j-i+2)/2}), \tilde{X}_{0,k}(\eta^{(l-k+2)/2})] \\ &\quad - \frac{1}{(i-2)!k!} [\tilde{X}_{0,i-2}(\eta^{(j-i+4)/2}), \tilde{X}_{0,k}(\eta^{(l-k+2)/2})]. \end{aligned}$$

By applying the same formula, we have $[\tilde{X}_{0,i-2}(\eta^{(j-i+4)/2}), \tilde{X}_{0,k}(\eta^{(l-k+2)/2})] = 0$, and

$$\begin{aligned} [\tilde{X}_{1,i-1}(\eta^{(j-i+2)/2}), \tilde{X}_{0,k}(\eta^{(l-k+2)/2})] &= k\psi \tilde{X}_{0,k-1}(\eta^{(l-k+2)/2}) \tilde{X}_{0,i-1}(\eta^{(j-i+2)/2}) \\ &\quad - k\tilde{X}_{0,i+k-2}(\eta^{(j-i+l-k+4)/2}). \end{aligned}$$

In total, we obtain

$$\begin{aligned} [f, \tilde{p}_{i,j}, \tilde{p}_{k,l}] &= \frac{1}{(i-1)!(k-1)!} (\psi \tilde{X}_{0,k-1}(\eta^{(l-k+2)/2}) \tilde{X}_{0,i-1}(\eta^{(j-i+2)/2}) \\ &\quad - \tilde{X}_{0,i+k-2}(\eta^{(j-i+l-k+4)/2})) \\ (3.5) \quad &= \psi \tilde{p}_{k-1,l-1} \tilde{p}_{i-1,j-1} - \binom{i+k-2}{i-1} \tilde{p}_{i+k-2,j+l} \\ &= \psi \tilde{p}_{i-1,j-1} \tilde{p}_{k-1,l-1} - \binom{i+k-2}{i-1} \tilde{p}_{i+k-2,j+l}. \end{aligned}$$

3.2. Jacobian side: tautological ring and its generators

On the other hand, since $f([J]) = 0$, we have

$$(3.6) \quad [f, \tilde{p}_{i,j}][J] = f(p_{i,j}) = p_{i-2,j}.$$

The relations (3.5) and (3.6) imply that for any polynomial P in $\{p_{i,j}\}$ and ψ , we have

$$f(P(\{p_{i,j}\}, \psi)) = \mathcal{D}(P(\{p_{i,j}\}, \psi)),$$

where \mathcal{D} is the differential operator defined in (3.3) (see [Pol07], Section 3).

It remains to prove that $\psi \in \mathcal{T}(J)$. To see this, we apply \mathcal{D} to the class $p_{1,1}^2 \in \mathcal{T}(J)$, which gives

$$\mathcal{D}(p_{1,1}^2) = \psi p_{0,0}^2 - \begin{pmatrix} 0 \\ 0 \end{pmatrix} p_{0,2} = g^2 \psi - p_{0,2}.$$

Hence $\psi = (\mathcal{D}(p_{1,1}^2) + p_{0,2})/g^2 \in \mathcal{T}(J)$. □

Corollary 3.8. *For $i \geq 0$, there is the identity*

$$(3.7) \quad p_{0,2i} = \pi^* \left(\frac{1}{2^{i+1}} \sum_{0 \leq j \leq i} \binom{i+1}{j+1} \psi^{i-j} \kappa_j + \psi^i \right).$$

Moreover, we have the following isomorphisms of \mathbb{Q} -algebras (similar to those of (2.14)).

$$(3.8) \quad \begin{array}{ccc} (\bigoplus_{i=0}^d \mathcal{T}_{(0,2i)}(J), \cdot) & \xleftarrow[\sim]{\mathcal{F}} & (\bigoplus_{i=0}^d \mathcal{T}_{(2g,2i)}(J), *) \\ & \swarrow \sim \quad \nwarrow \sim & \\ & (\mathcal{R}(S), \cdot) & \end{array}$$

$\pi^* \quad \pi_*$

In particular, the tautological ring $\mathcal{R}(S)$ may be regarded as a \mathbb{Q} -subalgebra of $(\mathcal{T}(J), \cdot)$ via π^* .

Proof. By (2.14) we have $p_{0,2i} = \mathcal{F}(\theta^{i+1} \cdot [C]_{(2i)}) = \pi^* \pi_*(\theta^{i+1} \cdot [C]_{(2i)}) = \pi^* \pi_*(\theta^{i+1} \cdot [C])$, hence it suffices to calculate $\pi_*(\theta^{i+1} \cdot [C])$. Then (2.19) and the projection formula imply that

$$\begin{aligned} \pi_*(\theta^{i+1} \cdot [C]) &= p_* \left(\left(\frac{1}{2} K + [x_0] + \frac{1}{2} \psi \right)^{i+1} \right) \\ &= \sum_{\substack{j+k+l=i+1 \\ j,k,l \geq 0}} \frac{(i+1)!}{j!k!l!} \frac{1}{2^{j+l}} p_*(K^j \cdot [x_0]^k \cdot \psi^l). \end{aligned}$$

Again by applying the projection formula to $p: C \rightarrow S$ and $x_0: S \rightarrow C$, we find

$$p_*(K^j \cdot [x_0]^k \cdot \psi^l) = \psi^l \cdot p_*(K^j \cdot [x_0]^k) = \begin{cases} \psi^l \cdot \kappa_{j-1} & \text{if } k = 0, \\ \psi^l \cdot x_0^*(K^j \cdot [x_0]^{k-1}) = (-1)^{k-1} \psi^i & \text{if } k \geq 1, \end{cases}$$

3. A TALE OF TWO TAUTOLOGICAL RINGS (I)

with the convention $\kappa_{-1} = 0$. It follows that

$$\begin{aligned}
\pi_*(\theta^{i+1} \cdot [C]) &= \sum_{\substack{j+l=i+1 \\ j, l \geq 0}} \frac{(i+1)!}{j!l!} \frac{1}{2^{i+1}} \psi^l \kappa_{j-1} + \sum_{\substack{j+k+l=i+1 \\ j, k, l \geq 0}} \frac{(i+1)!}{j!k!l!} \frac{1}{2^{j+l}} (-1)^{k-1} \psi^i \\
&\quad - \sum_{\substack{j+l=i+1 \\ j, l \geq 0}} \frac{(i+1)!}{j!l!} \frac{1}{2^{i+1}} (-1) \psi^i \\
&= \frac{1}{2^{i+1}} \sum_{0 \leq j \leq i} \binom{i+1}{j+1} \psi^{i-j} \kappa_j + \left(\frac{1}{2} - 1 + \frac{1}{2}\right)^{i+1} \psi^i + \left(\frac{1}{2} + \frac{1}{2}\right)^{i+1} \psi^i \\
&= \frac{1}{2^{i+1}} \sum_{0 \leq j \leq i} \binom{i+1}{j+1} \psi^{i-j} \kappa_j + \psi^i,
\end{aligned}$$

which proves the identity (3.7).

Now since $\oplus_{i=0}^d \mathcal{T}_{(0,2i)}(J)$ is generated by $\{p_{0,2i}\}$ and ψ , we have one inclusion $\oplus_{i=0}^d \mathcal{T}_{(0,2i)}(J) \subset \pi^*(\mathcal{R}(S))$. For the other inclusion, it follows from (3.7) and induction on j that one can also express $\pi^*(\kappa_j)$ as linear combinations of $\{p_{0,2i}\}$ and ψ . So we have $\oplus_{i=0}^d \mathcal{T}_{(0,2i)}(J) = \pi^*(\mathcal{R}(S))$, and the rest follows from (2.14). \square

3.3. Application: an identity of Morita

As an application we prove an identity of Morita. The proof has the advantage of being purely algebraic, which holds over fields of arbitrary characteristic. But as everything else in this thesis, it only works with \mathbb{Q} -coefficients.

The identity reveals some connection between $\mathcal{T}^{g+1}(J)$ and $\mathcal{R}^1(S)$. Consider the class $[C]_{(1)} \in \mathcal{T}_{(2g-3,1)}(J)$ and its Fourier dual $\mathcal{F}([C]_{(1)}) \in \mathcal{T}_{(3,1)}(J)$. Then we have $[C]_{(1)} \cdot \mathcal{F}([C]_{(1)}) \in \mathcal{T}_{(2g,2)}(J)$, and Morita answered what the image under π_* of this class is. We refer to [HR01], Theorem 1 for the original statement, which (with \mathbb{Q} -coefficients) is equivalent to the following.

Theorem 3.9. *We have the identity*

$$\pi_*([C]_{(1)} \cdot \mathcal{F}([C]_{(1)})) = \frac{1}{6} \kappa_1 + g \psi \text{ in } \mathcal{R}^1(S).$$

Remark 3.10. The class $\kappa_1/12$ is equal to λ_1 , which stands for the first Chern class of the Hodge bundle $p_*(\Omega_{C/S}^1)$. Hence the right-hand side is also equal to $2\lambda_1 + g\psi$.

Proof of Theorem 3.9. The identity is trivial for $g = 1$ since both sides are then zero. So we assume $g \geq 2$. Recall that $\mathcal{F}([C]_{(1)}) = p_{3,1}$. By (3.8) we have

$$(3.9) \quad \pi^* \pi_*([C]_{(1)} \cdot p_{3,1}) = \mathcal{F}([C]_{(1)} \cdot p_{3,1}) = -p_{3,1} * [C]_{(1)}.$$

3.3. Application: an identity of Morita

Therefore it suffices to express $-p_{3,1} * [C]_{(1)}$ in terms of $p_{0,2}$ and ψ , and then apply (3.7).

The first step is to express $[C]_{(1)}$ in terms of the classes $\{p_{i,j}\}$. By definition we have

$$f(p_{3,1}) = p_{1,1}, \quad \text{and} \quad f e(p_{1,1}) = e f(p_{1,1}) - h(p_{1,1}) = (g-1)p_{1,1}.$$

So $f(p_{3,1} - e(p_{1,1})/(g-1)) = 0$, which implies

$$e^{g-2} \left(p_{3,1} - \frac{1}{g-1} e(p_{1,1}) \right) = e^{g-2}(p_{3,1}) - \frac{1}{g-1} e^{g-1}(p_{1,1}) = 0.$$

Apply \mathcal{F}^{-1} to the previous equation, and we find

$$(-1)^{g-2} f^{g-2}([C]_{(1)}) - \frac{(-1)^{g-1}}{g-1} f^{g-1}(-e)([C]_{(1)}) = 0,$$

so that

$$f^{g-2}([C]_{(1)}) - \frac{1}{g-1} f^{g-1} e([C]_{(1)}) = 0.$$

On the other hand, by (2.7) we have

$$\begin{aligned} p_{3,1} &= \mathcal{F}([C]_{(1)}) = \exp(e) \exp(-f) \exp(e)([C]_{(1)}) \\ &= \exp(e) \exp(-f)([C]_{(1)} + e([C]_{(1)})) \\ &= \exp(e) \left(\frac{(-1)^{g-3}}{(g-3)!} f^{g-3}([C]_{(1)}) + \frac{(-1)^{g-2}}{(g-2)!} f^{g-2}([C]_{(1)}) \right. \\ &\quad \left. + \frac{(-1)^{g-2}}{(g-2)!} f^{g-2} e([C]_{(1)}) + \frac{(-1)^{g-1}}{(g-1)!} f^{g-1} e([C]_{(1)}) \right) \\ &= \exp(e) \left(\frac{(-1)^{g-3}}{(g-3)!} f^{g-3}([C]_{(1)}) + \frac{(-1)^{g-2}}{(g-2)!} f^{g-2} e([C]_{(1)}) \right) \\ &= \frac{(-1)^{g-3}}{(g-3)!} f^{g-3}([C]_{(1)}) + \frac{(-1)^{g-2}}{(g-2)!} f^{g-2} e([C]_{(1)}). \end{aligned}$$

Note that for $g = 2$, we ignore the f^{g-3} term and the rest of the argument still works. Then apply \mathcal{F} to both sides, which gives

$$\begin{aligned} (-1)^{g+1} [C]_{(1)} &= \frac{1}{(g-3)!} e^{g-3}(p_{3,1}) - \frac{1}{(g-2)!} e^{g-2}(p_{1,1}) \\ &= \frac{1}{(g-3)!} p_{2,0}^{g-3} p_{3,1} - \frac{1}{(g-2)!} p_{2,0}^{g-2} p_{1,1}. \end{aligned}$$

Now that $[C]_{(1)}$ is expressed in terms of $\{p_{i,j}\}$, we have

$$[C]_{(1)} \cdot p_{3,1} = \frac{(-1)^{g+1}}{(g-3)!} p_{2,0}^{g-3} p_{3,1}^2 - \frac{(-1)^{g+1}}{(g-2)!} p_{2,0}^{g-2} p_{1,1} p_{3,1}.$$

3. A TALE OF TWO TAUTOLOGICAL RINGS (I)

Apply \mathcal{F} one more time to get the class we want:

$$\begin{aligned}
-p_{3,1} * [C]_{(1)} &= \mathcal{F} \left(\frac{(-1)^{g+1}}{(g-3)!} p_{2,0}^{g-3} p_{3,1}^2 - \frac{(-1)^{g+1}}{(g-2)!} p_{2,0}^{g-2} p_{1,1} p_{3,1} \right) \\
&= \exp(e) \exp(-f) \exp(e) \left(\frac{(-1)^{g+1}}{(g-3)!} e^{g-3}(p_{3,1}^2) - \frac{(-1)^{g+1}}{(g-2)!} e^{g-2}(p_{1,1} p_{3,1}) \right) \\
&= \frac{(-1)^g}{g!} f^g \left(\frac{(-1)^{g+1}}{(g-3)!} e^{g-3}(p_{3,1}^2) - \frac{(-1)^{g+1}}{(g-2)!} e^{g-2}(p_{1,1} p_{3,1}) \right) \\
&= -\frac{1}{g!(g-3)!} f^g e^{g-3}(p_{3,1}^2) + \frac{1}{g!(g-2)!} f^g e^{g-2}(p_{1,1} p_{3,1}).
\end{aligned}$$

Expressions such as $f^g e^{g-3}(p_{3,1}^2)$ and $f^g e^{g-2}(p_{1,1} p_{3,1})$ can be computed via a combinatorial formula for \mathfrak{sl}_2 -representations. Here we state this formula as a lemma, since we shall use it again later.

Lemma 3.11. *Consider a \mathbb{Q} -linear representation $\mathfrak{sl}_2 \rightarrow \text{End}_{\mathbb{Q}}(V)$. Let $\alpha \in V$ such that $h(\alpha) = \lambda \cdot \alpha$. Then for all $r, s \geq 0$ we have*

$$(3.10) \quad f^s e^r(\alpha) = \sum_{t=0}^{\min(r,s)} (-1)^t \frac{s!}{(s-t)!} \frac{r!}{(r-t)!} \binom{\lambda + r - s + t - 1}{t} e^{r-t} f^{s-t}(\alpha). \quad \square$$

Note that the binomial coefficients should be taken in the generalized sense. We refer to [Moo09], Lemma 2.4 for the proof (where the operator \mathcal{D} in *loc. cit.* is $-f$).

Proof of Theorem 3.9 (continued). It follows from (3.10) that

$$f^g e^{g-3}(p_{3,1}^2) = \frac{g!(g-3)!}{3!} f^3(p_{3,1}^2), \text{ and } f^g e^{g-2}(p_{1,1} p_{3,1}) = \frac{g!(g-2)!}{2!} f^2(p_{1,1} p_{3,1}),$$

so that

$$-p_{3,1} * [C]_{(1)} = -\frac{1}{6} f^3(p_{3,1}^2) + \frac{1}{2} f^2(p_{1,1} p_{3,1}).$$

By Theorem 3.6 we know that $f = \mathcal{D}$ on polynomials in $\{p_{i,j}\}$ and ψ , so it remains to calculate

$$\begin{aligned}
\mathcal{D}^3(p_{3,1}^2) &= \mathcal{D}^2(\psi p_{0,2}^2 - 6p_{4,2} + 2p_{1,1} p_{3,1}) \\
&= \mathcal{D}((2g-2)\psi p_{0,2} - 6p_{2,2} + 2(g\psi p_{0,2} - p_{2,2} + p_{1,1}^2)) \\
&= \mathcal{D}(2(2g-1)\psi p_{0,2} - 8p_{2,2} + 2p_{1,1}^2) \\
&= 2g(2g-1)\psi - 8p_{0,2} + 2(g^2\psi - p_{0,2}) \\
&= 2g(3g-1)\psi - 10p_{0,2},
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{D}^2(p_{1,1} p_{3,1}) &= \mathcal{D}(g\psi p_{2,0} - p_{2,2} + p_{1,1}^2) \\
&= g^2\psi - p_{0,2} + (g^2\psi - p_{0,2}) \\
&= 2g^2\psi - 2p_{0,2}.
\end{aligned}$$

Altogether we have

$$-p_{3,1} * [C]_{(1)} = \frac{g}{3}\psi + \frac{2}{3}p_{0,2} = \frac{g}{3}\psi + \frac{2}{3}\pi^*\left(\frac{1}{4}\kappa_1 + g\psi\right) = \pi^*\left(\frac{1}{6}\kappa_1 + g\psi\right),$$

where the middle equality follows from (3.7). Combining with the starting point (3.9), we find

$$\pi^*\pi_*([C]_{(1)} \cdot p_{3,1}) = \pi^*\left(\frac{1}{6}\kappa_1 + g\psi\right).$$

Therefore $\pi_*([C]_{(1)} \cdot p_{3,1}) = \kappa_1/6 + g\psi$ by the isomorphisms (3.8). \square

3.4. Relations in the tautological rings

After describing the generators of $\mathcal{T}(J)$, we construct relations between them. The crucial tool is the \mathfrak{sl}_2 -action on $\text{CH}(J)$, together with its explicit form on $\mathcal{T}(J)$ (Theorem 3.6). The idea goes back to Polishchuk (see [Pol05], Theorem 0.1).

Thanks to the isomorphisms (3.8), we identify $\mathcal{R}(S)$ with $\oplus_{i=0}^d \mathcal{T}_{(0,2i)}(J)$ via the map π^* . Then by restricting everything to $\mathcal{R}(S)$, we obtain relations between the generators of $\mathcal{R}(S)$. This suggest a new approach to studying the structure of $\mathcal{R}(\mathcal{M}_{g,1})$ and the Faber conjectures (Conjecture 3.2). Various theoretical aspects and numerical evidence will be discussed.

Further, by pushing relations forward to \mathcal{M}_g , we obtain some new evidence regarding the original Faber conjectures on the tautological ring of \mathcal{M}_g .

Producing relations

We explain how the \mathfrak{sl}_2 -action produces relations between classes in $\mathcal{T}(J)$ (resp. $\mathcal{R}(S)$). As a consequence, we give a new proof of the generation statement in Conjecture 3.2 (ii), *i.e.* that $\mathcal{R}(\mathcal{M}_{g,1})$ is generated by the classes $\kappa_1, \dots, \kappa_{\lfloor g/3 \rfloor}$ and ψ .

Construction 3.12. Theorem 3.6 shows that the space of polynomial relations between the classes $\{p_{i,j}\}$ and ψ is stable under the action of \mathcal{D} . In other words, if P is a polynomial in $\{p_{i,j}\}$ and ψ , then $P(\{p_{i,j}\}, \psi) = 0$ implies $\mathcal{D}(P(\{p_{i,j}\}, \psi)) = 0$. Now consider monomials

$$\alpha = \psi^s p_{i_1, j_1}^{r_1} p_{i_2, j_2}^{r_2} \cdots p_{i_m, j_m}^{r_m} \text{ with } I := r_1 i_1 + r_2 i_2 + \cdots + r_m i_m > 2g.$$

By definition $\alpha \in \text{CH}(R^I(J/S))$. But since $I > 2g$, we know from the motivic decomposition (2.9) that $R^I(J/S) = 0$. In terms of the Dutch house, the class α is simply outside the house. It follows that we have relations

$$\alpha = 0, \quad \mathcal{D}(\alpha) = 0, \quad \mathcal{D}^2(\alpha) = 0, \quad \dots$$

3. A TALE OF TWO TAUTOLOGICAL RINGS (I)

This argument leads to the following formal definition. Let i, j run through all integers such that $i \leq j + 2$ and that $i + j$ is even. Consider the ring

$$\mathcal{A} := \mathbb{Q}[\{x_{i,j}\}, y] / \langle x_{0,0} - g, \{x_{i,j}\}_{i < 0}, \{x_{i,j}\}_{j < 0}, \{x_{i,j}\}_{j > 2g-2} \rangle.$$

In other words, the ring \mathcal{A} is a polynomial ring in variables $\{x_{i,j}\}$ and y , with the convention that $x_{0,0} = g$ and $x_{i,j} = 0$ for $i < 0$ or $j < 0$ or $j > 2g - 2$ (same as the classes $\{p_{i,j}\}$). We introduce a bigrading $\mathcal{A} = \bigoplus_{i,j} \mathcal{A}_{(i,j)}$ by the requirements that $x_{i,j} \in \mathcal{A}_{(i,j)}$ and $y \in \mathcal{A}_{(0,2)}$. Define operators E, F and H on \mathcal{A} by

$$\begin{aligned} E: \mathcal{A}_{(i,j)} &\rightarrow \mathcal{A}_{(i+2,j)} & \alpha &\mapsto x_{2,0} \cdot \alpha, \\ F: \mathcal{A}_{(i,j)} &\rightarrow \mathcal{A}_{(i-2,j)} & \alpha &\mapsto F(\alpha), \\ H: \mathcal{A}_{(i,j)} &\rightarrow \mathcal{A}_{(i,j)} & \alpha &\mapsto (i - g)\alpha, \end{aligned}$$

where

$$(3.11) \quad F := \frac{1}{2} \sum_{i,j,k,l} \left(y x_{i-1,j-1} x_{k-1,l-1} - \binom{i+k-2}{i-1} x_{i+k-2,j+l} \right) \partial x_{i,j} \partial x_{k,l} + \sum_{i,j} x_{i-2,j} \partial x_{i,j}.$$

It is not difficult to verify that the operators above generate a \mathbb{Q} -linear representation $\mathfrak{sl}_2 \rightarrow \text{End}_{\mathbb{Q}}(\mathcal{A})$. Theorem 3.6 can then be reformulated as the existence of a surjective morphism of \mathfrak{sl}_2 -representations $\mathcal{A} \rightarrow \mathcal{T}(J)$, which maps $x_{i,j}$ to $p_{i,j}$ and y to ψ .

Denote by $\text{Mon}_{(i,j)}$ the set of monomials in $\{x_{i,j}\}$ (excluding $x_{0,0}$ and all $x_{i,j}$ that vanish in \mathcal{A}) and y that belong to $\mathcal{A}_{i,j}$. Note that we set $\text{Mon}_{(0,0)} = \{1\}$ as an exception. Then consider the quotient ring

$$(3.12) \quad \tilde{\mathcal{T}} := \mathcal{A} / \langle \{F^\nu(\text{Mon}_{(i,j)})\}_{i > 2g, \nu \geq 0} \rangle.$$

The ring $\tilde{\mathcal{T}}$ inherits from \mathcal{A} a bigrading $\tilde{\mathcal{T}} = \bigoplus_{i,j} \tilde{\mathcal{T}}_{(i,j)}$. The operators E, F and H induce operators on $\tilde{\mathcal{T}}$, which we denote by e, f and h . Again we obtain a representation $\mathfrak{sl}_2 \rightarrow \text{End}_{\mathbb{Q}}(\tilde{\mathcal{T}})$. Moreover, since $e^{g+1} = f^{g+1} = 0$, we can formally define the Fourier transform \mathcal{F} on $\tilde{\mathcal{T}}$ by

$$\mathcal{F} := \exp(e) \circ \exp(-f) \circ \exp(e).$$

We also define the subring $\tilde{\mathcal{R}} = \bigoplus_i \tilde{\mathcal{T}}_{(0,2i)}$, with the grading $\tilde{\mathcal{R}} = \bigoplus_i \tilde{\mathcal{R}}^i$ such that $\tilde{\mathcal{R}}^i := \tilde{\mathcal{T}}_{(0,2i)}$. Then we have

$$\begin{aligned} \tilde{\mathcal{R}}^i &= \mathcal{A}_{(0,2i)} / \langle \{F^I(\text{Mon}_{(2I,2i)})\}_{I > g} \rangle \\ &= \mathcal{A}_{(0,2i)} / \langle F^{g+1}(\text{Mon}_{(2g+2,2i)}) \rangle. \end{aligned}$$

3.4. Relations in the tautological rings

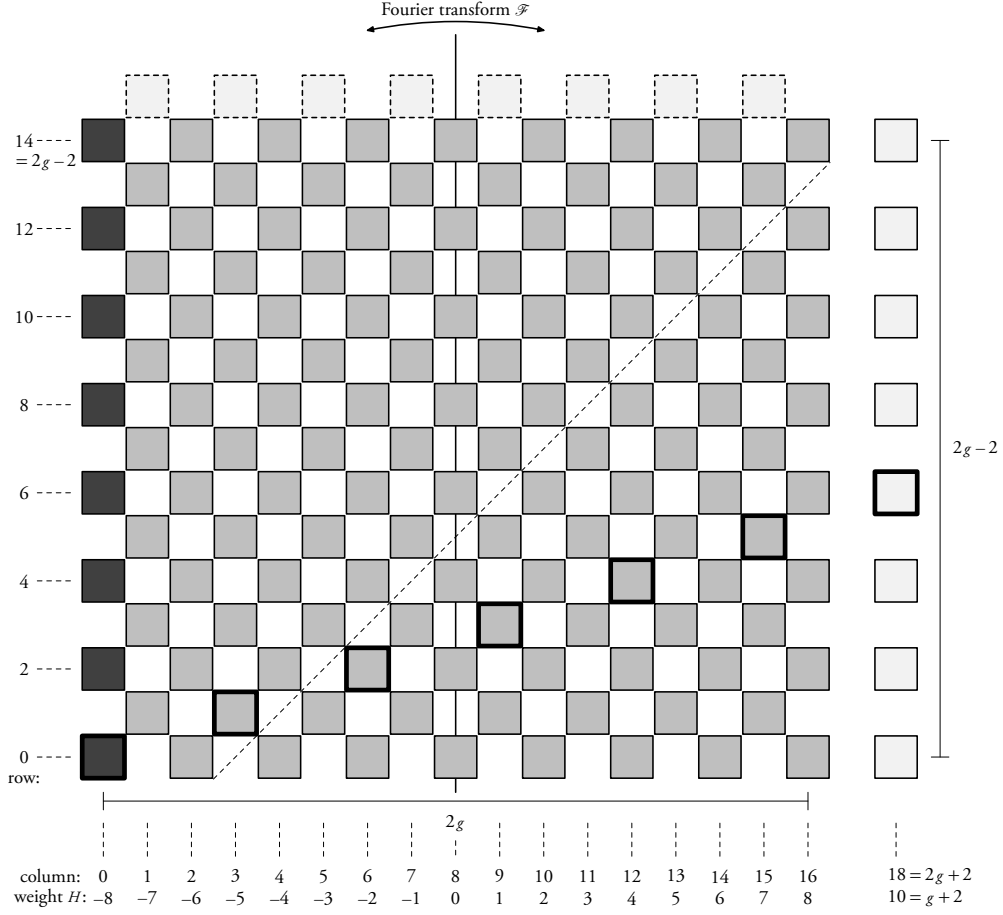


Figure 3. Producing relations ($g = 8$).

Figure 3 illustrates the construction of $\tilde{\mathcal{R}}$: take monomials on the $(2g + 2)$ -th column of the Dutch house (white blocks), and then apply $g + 1$ times the operator F to obtain relations between the generators (black blocks).

To summarize this formal approach, we have the following proposition.

Proposition 3.13. *For all S as in (2.15) (including $S = \mathcal{M}_{g,1}$), we have surjective maps*

$$(3.13) \quad \Phi: \tilde{\mathcal{T}} \twoheadrightarrow \mathcal{T}(J), \text{ and } \Phi|_{\tilde{\mathcal{R}}}: \tilde{\mathcal{R}} \twoheadrightarrow \mathcal{R}(S),$$

which map $x_{i,j}$ to $p_{i,j}$ and y to ψ . □

From now on, we will concentrate on the structure of $\tilde{\mathcal{R}}$. We start with a lemma, which shows one can already eliminate certain monomials that produce trivial relations.

Lemma 3.14. *For all $\alpha \in \text{Mon}_{(2g+2,2i)}$ of the form $\alpha = x_{2,0} \cdot \beta$, we have $F^{g+1}(\alpha) = 0$.*

3. A TALE OF TWO TAUTOLOGICAL RINGS (I)

Proof. This follows directly from (3.10). In fact, we have $F^{g+1}(\alpha) = F^{g+1}(x_{2,0} \cdot \beta) = F^{g+1}E(\beta)$. Then by applying (3.10) with $\lambda = g$, $r = 1$ and $s = g + 1$, we find

$$F^{g+1}E(\beta) = EF^{g+1}(\beta) - (g+1)(g+1-g-1+1-1)F^g(\beta) = EF^{g+1}(\beta).$$

On the other hand $F^{g+1}(\beta) \in \mathcal{A}_{(-2,2i)} = 0$, which implies $F^{g+1}(\alpha) = 0$. \square

As a result, if we write $\text{mon}_{(2g+2,2i)} \subset \text{Mon}_{(2g+2,2i)}$ for the subset of monomials without $x_{2,0}$ as a factor, then we have

$$(3.14) \quad \tilde{\mathcal{R}}^i = \mathcal{A}_{(0,2i)} \setminus \langle F^{g+1}(\text{mon}_{(2g+2,2i)}) \rangle.$$

The bold blocks in Figure 3 describe the lower bound of i such that $\text{mon}_{(2g+2,2i)}$ is non-empty. In numerical terms, we have $x_{3,1}^{2i} \in \mathcal{A}_{(6i,2i)}$ and that $6i > 2g$ implies $i > g/3$. So $\text{mon}_{(2g+2,2i)} = \emptyset$ for all $i \leq \lfloor g/3 \rfloor$.

Theorem 3.15. *The elements $x_{0,2}, \dots, x_{0,2\lfloor g/3 \rfloor}$ and y generate $\tilde{\mathcal{R}}$, with no relations in $\tilde{\mathcal{R}}^i$ for $i \leq \lfloor g/3 \rfloor$.*

Remark 3.16. Combining with (3.7), Theorem 3.15 implies the generation statement in Conjecture 3.2 (ii). Further, it would solve Conjecture 3.2 (ii) completely if the map $\Phi|_{\tilde{\mathcal{R}}}: \tilde{\mathcal{R}} \rightarrow \mathcal{R}(\mathcal{M}_{g,1})$ in (3.13) is also injective, *i.e.* an isomorphism.

Proof of Theorem 3.15. The second part is immediate after (3.14) and the fact that $\text{mon}_{(2g+2,2i)} = \emptyset$ for all $i \leq \lfloor g/3 \rfloor$. For the first part, our goal is to relate all $x_{0,2i}$ with $i > g/3$ to the elements $x_{0,2}, \dots, x_{0,2\lfloor g/3 \rfloor}$ and y , and the idea is to use specific monomials to get these relations.

We proceed by induction. Suppose all $\{x_{0,2j}\}_{g/3 < j < i}$ can be expressed in terms of the elements $x_{0,2}, \dots, x_{0,2\lfloor g/3 \rfloor}$ and y . Then consider the monomial $x_{3,1}^{2i} \in \mathcal{A}_{(6i,2i)}$. Apply $3i$ -times the operator F and we get $F^{3i}(x_{3,1}^{2i}) \in \mathcal{A}_{(0,2i)}$, which vanishes in $\tilde{\mathcal{R}}$. On the other hand, the explicit expression (3.11) implies

$$F^{3i}(x_{3,1}^{2i}) = cx_{0,2i} + \alpha,$$

where α is a polynomial in $\{x_{0,2j}\}_{j < i}$ and y . It only remains to prove that the coefficient c is non-zero.

The observation is the following: when we apply the operator F , the minus sign occurs every time two factors $(x_{i,j}, x_{k,l})$ are merged into one $(x_{i+k-j, j+l})$. Then if we start from $x_{3,1}^{2i}$ and arrive at $x_{0,2i}$, no matter how we proceed we have to do the merging $(2i-1)$ -times. This means that all non-zero summands of c are of the form $(-1)^{2g-1}$ times a positive number, hence negative. Therefore the sum c is negative as well. \square

We have also tried to see if we could recover Looijenga's result using our relations, *i.e.* to prove that $\tilde{\mathcal{R}}^i = 0$ for $i > g-1$ and $\tilde{\mathcal{R}}^{g-1} \simeq \mathbb{Q}$. However, there seems to be some combinatorial difficulty^{note 4} that we do not yet know how to resolve.

Numerical evidence

My colleague Li Ma made a C++ program that computes the ring $\tilde{\mathcal{R}}$ for a given genus g . The program calculates relations obtained in Construction 3.12 and outputs the dimension of each component $\tilde{\mathcal{R}}^i$.

Meanwhile, based on an algorithm developed by Liu and Xu (see [LX12], Section 3), Bergvall computed all intersection numbers in the pairing (3.2) for many values of g (see [Ber11], Section 4.2). As a result, it gives the dimensions of the Gorenstein quotient $\mathcal{G}(\mathcal{M}_{g,1})$ of $\mathcal{R}(\mathcal{M}_{g,1})$. Here the ring $\mathcal{G}(\mathcal{M}_{g,1})$ is the quotient of $\mathcal{R}(\mathcal{M}_{g,1})$ by the homogeneous ideal generated by all classes of pure codimension that have zero pairing with all classes of the opposite codimension. Note that this procedure of obtaining the dimensions of $\mathcal{G}(\mathcal{M}_{g,1})$ is formal and does not involve computation of actual relations in $\mathcal{R}(\mathcal{M}_{g,1})$.

There are surjective maps $\tilde{\mathcal{R}} \twoheadrightarrow \mathcal{R}(\mathcal{M}_{g,1}) \twoheadrightarrow \mathcal{G}(\mathcal{M}_{g,1})$. Our computation shows that for $g \leq 19$, the dimensions of $\tilde{\mathcal{R}}$ and $\mathcal{G}(\mathcal{M}_{g,1})$ are the same, which implies isomorphisms $\tilde{\mathcal{R}} \simeq \mathcal{R}(\mathcal{M}_{g,1}) \simeq \mathcal{G}(\mathcal{M}_{g,1})$. In particular, we can confirm the following (for $g \leq 9$ this has been obtained independently by Bergvall; see [Ber11], Section 4.4).

Theorem 3.17. *Conjecture 3.2 is true for $g \leq 19$.* □

However, the computer result is negative for $g = 20$ and some greater values of g . There the dimensions of $\tilde{\mathcal{R}}$ are simply not symmetric. Again by comparing with the dimensions of $\mathcal{G}(\mathcal{M}_{g,1})$, we know exactly how many relations are missing. The numbers are listed in Table 1 below.

The computation also involves an observation similar to Faber and Zagier's approach. Consider the dimension of $\tilde{\mathcal{R}}^i$ for $g/3 < i \leq \lfloor (g-1)/2 \rfloor$, which we denote by $d(g, i)$. Note that $i \leq \lfloor (g-1)/2 \rfloor$ corresponds to the lower half of the conjectural Gorenstein ring. We know that $\tilde{\mathcal{R}}^i$ is spanned by the image of $\text{Mon}_{(0,2i)}$, and the cardinal $\# \text{Mon}_{(0,2i)}$ is $p(0) + \dots + p(i) =: \phi(i)$, where $p(-)$ is the partition function. Then consider the difference $\phi(i) - d(g, i)$, which reflects the minimal number of relations needed to obtain $\tilde{\mathcal{R}}^i$ from $\mathcal{A}_{(0,2i)}$. We have observed an interesting phenomenon: that $\phi(i) - d(g, i)$ seems to be a function of $3i - g - 1$. More precisely, if we set

$$b(3i - g - 1) := \phi(i) - d(g, i),$$

then based on our data for $g \leq 28$, we obtain the following values of the function b .

n	0	1	2	3	4	5	6	7	8	9	10	11
$b(n)$	1	2	3	6	10	14	22	33	45	64	90	119

Bergvall did the same calculation with the dimensions of the Gorenstein quotient $\mathcal{G}(\mathcal{M}_{g,1})$ (see [Ber11], Table 4.3). There he obtained the same values as the function $b(n)$ for $n \leq 9$. For $n = 10$,

3. A TALE OF TWO TAUTOLOGICAL RINGS (I)

however, he observed some unexpected fact: the component $\mathcal{G}^{12}(\mathcal{M}_{25,1})$ yields the number 91 instead of 90, while further components $\mathcal{G}^{13}(\mathcal{M}_{28,1})$ and $\mathcal{G}^{14}(\mathcal{M}_{31,1})$ give the number 90. Similar things happen to $n = 11$, where the component $\mathcal{G}^{13}(\mathcal{M}_{27,1})$ yields the number 120 instead of 119, while further components seem to give the number 119. Note that the defect occurs only in the middle codimension, *i.e.* $(g-1)/2$ for g odd.

Another aspect is to guess what this function $b(n)$ could be. Recall Faber and Zagier's function $a(n)$ for \mathcal{M}_g (see [Fab99], Section 4 and [LX12], Section 2).

n	0	1	2	3	4	5	6	7	8	9	10	11
$a(n)$	1	1	2	3	5	6	10	13	18	24	33	41

It has been suggested by Bergvall and Faber that the first values of $b(n)$ satisfy

$$(3.15) \quad b(n) = \sum_{\substack{i=0 \\ i \not\equiv 2 \pmod{3}}}^n a(n-i) = a(n) + a(n-1) + a(n-3) + a(n-4) + \dots.$$

Here we observe that (3.15) is compatible with the dimensions of $\tilde{\mathcal{R}}$, but conflicts with the Gorenstein property.

Pushing forward to \mathcal{M}_g

Recall that the tautological ring $\mathcal{R}(\mathcal{M}_g)$ is defined to be the \mathbb{Q} -subalgebra of $\text{CH}(\mathcal{M}_g)$ generated by the classes $\{\kappa_i\}$. The original Faber conjectures predict that $\mathcal{R}(\mathcal{M}_g)$ is Gorenstein with socle in codimension $g-2$, and that it is generated by $\kappa_1, \dots, \kappa_{\lfloor g/3 \rfloor}$ with no relations in codimension $i \leq \lfloor g/3 \rfloor$ (see [Fab99], Conjecture 1).

Let $q: \mathcal{M}_{g,1} \rightarrow \mathcal{M}_g$ be the map that forgets the marked point. Then we have $q_*(\mathcal{R}(\mathcal{M}_{g,1})) = \mathcal{R}(\mathcal{M}_g)$. In fact, consider the diagram

$$\begin{array}{ccc} \mathcal{C}_{g,1} & \xrightarrow{\tilde{q}} & \mathcal{C}_g \\ \begin{array}{c} \nearrow x_0 \\ p \downarrow \end{array} & & \downarrow p_0 \\ \mathcal{M}_{g,1} & \xrightarrow{q} & \mathcal{M}_g \end{array}$$

where \mathcal{C}_g (resp. $\mathcal{C}_{g,1}$) is the universal curve over \mathcal{M}_g (resp. $\mathcal{M}_{g,1}$), and x_0 is the section of p that gives the marked point. The classes $\{\kappa_i\}$ (resp. K) are defined on \mathcal{M}_g (resp. \mathcal{C}_g), and we keep the

3.4. Relations in the tautological rings

g	$\mathcal{M}_{g,1}$	\mathcal{M}_g
≤ 19	OK	OK
20	codim 10: 1 missing	OK
21	codim 11: 1 missing	OK
22	codim 11: 1 missing	OK
23	codim 12: 3 missing	OK
24	codim 13: 2 missing codim 12: 4 missing	codim 12: 1 missing
25	codim 13: 5 missing codim 12: 1 missing	codim 12: 1 missing
26	codim 14: 6 missing codim 13: 6 missing	codim 13: 1 missing
27	codim 15: 3 missing codim 14: 11 missing codim 13: 1 missing	codim 14: 1 missing codim 13: 1 missing
28	codim 15: 10 missing codim 14: 10 missing	codim 14: 2 missing

Table 1. Computer output for $g \leq 28$.

same notation for their pull-back to $\mathcal{M}_{g,1}$ (resp. $\mathcal{C}_{g,1}$). Then for $\psi^s \kappa_{i_1}^{r_1} \cdots \kappa_{i_m}^{r_m} \in \mathcal{R}(\mathcal{M}_{g,1})$, we have

$$\begin{aligned}
 q_*(\psi^s \kappa_{i_1}^{r_1} \cdots \kappa_{i_m}^{r_m}) &= q_*(\psi^s \cdot q^*(\kappa_{i_1}^{r_1} \cdots \kappa_{i_m}^{r_m})) \\
 &= q_*(\psi^s) \cdot \kappa_{i_1}^{r_1} \cdots \kappa_{i_m}^{r_m} \\
 &= q_* x_0^*(K^s) \cdot \kappa_{i_1}^{r_1} \cdots \kappa_{i_m}^{r_m} \\
 &= q_* x_0^* \tilde{q}^*(K^s) \cdot \kappa_{i_1}^{r_1} \cdots \kappa_{i_m}^{r_m} \\
 &= p_{0,*}(\tilde{q} \circ x_0)_*(\tilde{q} \circ x_0)^*(K^s) \cdot \kappa_{i_1}^{r_1} \cdots \kappa_{i_m}^{r_m} \\
 &= p_{0,*}(K^s) \cdot \kappa_{i_1}^{r_1} \cdots \kappa_{i_m}^{r_m} \\
 &= \kappa_{s-1} \kappa_{i_1}^{r_1} \cdots \kappa_{i_m}^{r_m},
 \end{aligned}
 \tag{3.16}$$

with the convention that $\kappa_{-1} = 0$. Hence $q_*(\psi^s \kappa_{i_1}^{r_1} \cdots \kappa_{i_m}^{r_m}) \in \mathcal{R}(\mathcal{M}_g)$.

The identities (3.7) and (3.16) allow us to push relations in $\mathcal{R}(\mathcal{M}_{g,1})$ forward to $\mathcal{R}(\mathcal{M}_g)$. We used another computer program to do the work. Then for $g \leq 23$, we obtain a new proof of the well-known result of Faber-Zagier, Faber and Pandharipande-Pixton (see [Fab13], Lecture 1 and [PP13], Section 0.4).

3. A TALE OF TWO TAUTOLOGICAL RINGS (I)

Theorem 3.18. *For $g \leq 23$, the ring $\mathcal{R}(\mathcal{M}_g)$ is Gorenstein with socle in codimension $g - 2$. \square*

In this situation, the rest of the Faber conjectures are also true by explicit calculation. Note that when $20 \leq g \leq 23$, the missing relations in $\mathcal{R}(\mathcal{M}_{g,1})$ do not affect the Gorenstein property of $\mathcal{R}(\mathcal{M}_g)$. From $g = 24$ on, however, the computer result is again negative. Our computation for $g \leq 28$ suggests that we obtain exactly the same set of relations as the Faber-Zagier relations (see [Fab13], Lecture 1 and [PP13], Section 0.2). Notably in the crucial case of $g = 24$, we have not found the missing relation in codimension 12. It is not yet known whether in theory we obtain the same relations.

We summarize in Table 1 the computer result for both $\mathcal{M}_{g,1}$ and \mathcal{M}_g . Note that all codimensions have been calculated for $g \leq 24$. For $g \geq 25$, we only calculated a range near the middle codimension, presuming that the tautological rings behave well near the top.

We finish this chapter by some speculations. Since the ring $\tilde{\mathcal{R}}$ is in general not Gorenstein, there are two main possibilities.

- (i) *The ring $\mathcal{R}(\mathcal{M}_{g,1})$ is still Gorenstein.* In this case we need to find other ways that produce the missing relations. Earlier works (see [Moo09], Section 2.13) suggest that our approach is closely related to Brill-Noether theory, while the relation is not yet clear. However, it seems that a new kind of geometry will be needed to prove the Gorenstein property.
- (ii) *The ring $\mathcal{R}(\mathcal{M}_{g,1})$ is isomorphic to $\tilde{\mathcal{R}}$.* As seen in Theorem 2.16, the \mathfrak{sl}_2 approach of producing relations is of motivic nature. We find it somewhat reasonable to expect all relations to be motivic, rather than believing the existence of non-motivic relations. Also, it is conjectured that on the generic Jacobian variety, the \mathfrak{sl}_2 -action is the only source of relations between tautological classes (see [Pol05], Introduction). The idea goes back to Beauville (see [Bea04], Section 5.5), and there have been various results on the non-triviality of certain tautological classes. We refer to [Cer83], [Fak96], [Ike03] and [Voi13] for more details. See also Chapters 5 and 6 for some discussions.

For the moment, it might be worthwhile to re-state the second case as a conjecture, although it contradicts the Faber conjectures for both $\mathcal{M}_{g,1}$ and \mathcal{M}_g .

Conjecture 3.19. *We have an isomorphism $\Phi|_{\tilde{\mathcal{R}}}: \tilde{\mathcal{R}} \xrightarrow{\sim} \mathcal{R}(\mathcal{M}_{g,1})$.*

A tale of two tautological rings (II)

We study symmetric powers of a relative pointed curve, and we prove that the tautological ring of the infinite symmetric power is a polynomial ring over the tautological ring of the relative Jacobian. As a result, the Gorenstein property for the universal Jacobian implies the same property for symmetric powers of the universal curve. We give positive results in low genus cases.

4.1. Symmetric powers of a relative curve

We turn our attention to symmetric powers of a relative curve, which are closely related to the relative Jacobian. Again we work in the setting of (2.15). For $n \geq 1$, denote by $p^n: C^n \rightarrow S$ (resp. $p^{[n]}: C^{[n]} \rightarrow S$) the n -th power (resp. symmetric power) of C relative to S . The quotient map $\sigma_n: C^n \rightarrow C^{[n]}$ induces a canonical isomorphism of \mathbb{Q} -algebras

$$(4.1) \quad \sigma_n^*: \mathrm{CH}(C^{[n]}) \xrightarrow{\sim} \mathrm{CH}(C^n)^{\mathfrak{S}_n}.$$

Also $\sigma_{n,*} \circ \sigma_n^*: \mathrm{CH}(C^{[n]}) \rightarrow \mathrm{CH}(C^{[n]})$ is the multiplication by $n!$.

Define maps $\varphi_n: C^{[n]} \rightarrow J$ and $\phi_n := \varphi_n \circ \sigma_n: C^n \rightarrow J$, which send locally n sections x_1, \dots, x_n of C/S to the class $\mathcal{O}_C(x_1 + \dots + x_n - nx_0)$ (in particular $\varphi_1 = \phi_1 = \iota: C \hookrightarrow J$). To summarize, we have the following diagram.

$$(4.2) \quad \begin{array}{ccccc} & & \phi_n & & \\ & \nearrow & & \searrow & \\ C^n & \xrightarrow{\sigma_n} & C^{[n]} & \xrightarrow{\varphi_n} & J \\ & \searrow p^n & \downarrow p^{[n]} & & \downarrow \pi \\ & & & & S \end{array}$$

4. A TALE OF TWO TAUTOLOGICAL RINGS (II)

For convenience we set $C^0 = C^{[0]} = S$ and $\varphi_0 = \phi_0 = o: S \rightarrow J$.

It is not difficult to see that the fibers of φ_n are projective spaces (if non-empty), and it follows from Riemann-Roch that $C^{[n]}$ is a \mathbb{P}^{n-g} -bundle over J when $n \geq 2g - 1$. In this case the Chow ring $\text{CH}(C^{[n]})$ can be described in terms of $\text{CH}(J)$, following [Ful98], Section 3.3.

A nice way to interpret this fact and to treat all symmetric powers at once is to consider the *infinite symmetric power* $C^{[\infty]}$, which is defined as the ind-scheme

$$C^{[\infty]} := \varinjlim (S = C^{[0]} \hookrightarrow C \hookrightarrow C^{[2]} \hookrightarrow C^{[3]} \hookrightarrow \dots).$$

Here the transition maps $\varepsilon_n: C^{[n-1]} \hookrightarrow C^{[n]}$ are given by adding a copy of x_0 (in particular $\varepsilon_1 = x_0: S \rightarrow C$). We write $p^{[\infty]}: C^{[\infty]} \rightarrow S$ for the structural map.

The collection of $\varphi_n: C^{[n]} \rightarrow J$ induces a map $\varphi: C^{[\infty]} \rightarrow J$. Moreover, both structures μ and $[N]$ on J can be lifted to $C^{[\infty]}$: the addition maps $\mu_{m,n}: C^{[m]} \times_S C^{[n]} \rightarrow C^{[m+n]}$ give rise to

$$\mu: C^{[\infty]} \times_S C^{[\infty]} \rightarrow C^{[\infty]},$$

while the diagonal maps $\Delta_N: C^{[n]} \rightarrow C^{[Nn]}$ give rise to

$$[N]: C^{[\infty]} \rightarrow C^{[\infty]}, \text{ for } N \geq 0.$$

Again μ and $[N]$ are called the *addition* and the *multiplication by N* respectively.

In this section we review the Chow theories of $C^{[\infty]}$. They were first introduced by Kimura and Vistoli [KV96] in the case $S = k$, and then generalized to the relative setting by Moonen and Polishchuk [MP10].

Chow homology and Chow cohomology

Unlike the case of J , there are (at least) two different notions of Chow groups for the ind-scheme $C^{[\infty]}$. One is graded by relative dimension and the other by codimension.

If X/S is an object in \mathcal{V}_S , we write $\text{CH}_i(X/S)$ for the Chow group of cycles on X that are of relative dimension i over S (i.e. of dimension $i + d$ over k , where $d = \dim(S/k)$). When there is no ambiguity (since we always work relatively over S), we drop the S and abbreviate to $\text{CH}_i(X)$. We also distinguish the two gradings on $\text{CH}(J)$: we write $\text{CH}^\bullet(J) := (\oplus_i \text{CH}^i(J), \cdot)$ and $\text{CH}_\bullet(J) := (\oplus_i \text{CH}_i(J), *)$.

Definition 4.1.

- (i) The *Chow homology* of $C^{[\infty]}$ of relative dimension i over S is the direct limit

$$\text{CH}_i(C^{[\infty]}) := \varinjlim (\text{CH}_i(S) \rightarrow \text{CH}_i(C) \rightarrow \text{CH}_i(C^{[2]}) \rightarrow \text{CH}_i(C^{[3]}) \rightarrow \dots),$$

4.1. Symmetric powers of a relative curve

where the transition maps are $\varepsilon_{n,*} : \mathrm{CH}_i(C^{[n-1]}) \rightarrow \mathrm{CH}_i(C^{[n]})$. We write

$$\mathrm{CH}_\bullet(C^{[\infty]}) := \bigoplus_i \mathrm{CH}_i(C^{[\infty]}).$$

(ii) The *Chow cohomology* of $C^{[\infty]}$ of codimension i is the inverse limit

$$\mathrm{CH}^i(C^{[\infty]}) := \varprojlim (\mathrm{CH}^i(S) \leftarrow \mathrm{CH}^i(C) \leftarrow \mathrm{CH}^i(C^{[2]}) \leftarrow \mathrm{CH}^i(C^{[3]}) \leftarrow \dots),$$

where the transition maps are $\varepsilon_n^* : \mathrm{CH}^i(C^{[n]}) \rightarrow \mathrm{CH}^i(C^{[n-1]})$. We write

$$\mathrm{CH}^\bullet(C^{[\infty]}) := \bigoplus_i \mathrm{CH}^i(C^{[\infty]}).$$

(iii) The Chow homology (resp. cohomology) of $C^{[\infty]} \times_S C^{[\infty]}$ is defined similarly.

Remark 4.2. For all $n \geq 1$, the transition maps $\varepsilon_{n,*}$ (resp. ε_n^*) are injective (resp. surjective). In fact, one can construct a correspondence $\Gamma_n \in \mathrm{Corr}_S(C^{[n]}, C^{[n-1]})$ satisfying $\Gamma_{n,*} \circ \varepsilon_{n,*} = \mathrm{id}$ and $\varepsilon_n^* \circ \Gamma_n^* = \mathrm{id}$ (see [KV96], Remark 1.9).

By definition, an element in $\mathrm{CH}_i(C^{[\infty]})$ is represented by some $\alpha \in \mathrm{CH}_i(C^{[n]})$ for some n . An element in $\mathrm{CH}^i(C^{[\infty]})$ is a sequence $\alpha = (\alpha_n)_{n \geq 0}$ with $\alpha_n \in \mathrm{CH}^i(C^{[n]})$, such that $\varepsilon_n^*(\alpha_n) = \alpha_{n-1}$.

Both $\mathrm{CH}_\bullet(C^{[\infty]})$ and $\mathrm{CH}^\bullet(C^{[\infty]})$ are equipped with a ring structure: on $\mathrm{CH}^\bullet(C^{[\infty]})$ there is the intersection product (\cdot) , while on $\mathrm{CH}_\bullet(C^{[\infty]})$ there is the *Pontryagin product* $(*)$, defined by

$$\begin{aligned} \mathrm{CH}_i(C^{[\infty]}) \times \mathrm{CH}_{i'}(C^{[\infty]}) &\xrightarrow{*} \mathrm{CH}_{i+i'}(C^{[\infty]}) \\ (\alpha, \beta) &\mapsto \mu_*(\alpha \times_S \beta). \end{aligned}$$

Here if $\alpha \in \mathrm{CH}_i(C^{[m]}) \subset \mathrm{CH}_i(C^{[\infty]})$ and $\beta \in \mathrm{CH}_{i'}(C^{[n]}) \subset \mathrm{CH}_{i'}(C^{[\infty]})$, we set

$$\alpha \times_S \beta := \mathrm{pr}_1^*(\alpha) \cdot \mathrm{pr}_2^*(\beta) \in \mathrm{CH}_{i+i'}(C^{[m]} \times_S C^{[n]}) \subset \mathrm{CH}_{i+i'}(C^{[\infty]} \times_S C^{[\infty]}),$$

where $\mathrm{pr}_1 : C^{[m]} \times_S C^{[n]} \rightarrow C^{[m]}$ and $\mathrm{pr}_2 : C^{[m]} \times_S C^{[n]} \rightarrow C^{[n]}$ are the two projections. One verifies easily that $(*)$ is well-defined. The unit of $\mathrm{CH}_\bullet(C^{[\infty]})$ is $[S] \in \mathrm{CH}_0(S) \subset \mathrm{CH}_0(C^{[\infty]})$.

Note that unlike the case of $\mathrm{CH}(J)$, one cannot define both ring structures on the same object $\mathrm{CH}_\bullet(C^{[\infty]})$ or $\mathrm{CH}^\bullet(C^{[\infty]})$. We do, however, have a *cap product*

$$(4.3) \quad \mathrm{CH}^\bullet(C^{[\infty]}) \times \mathrm{CH}_\bullet(C^{[\infty]}) \xrightarrow{\cap} \mathrm{CH}_\bullet(C^{[\infty]}),$$

which sends $\alpha = (\alpha_m)_{m \geq 0} \in \mathrm{CH}^i(C^{[\infty]})$ and $\beta \in \mathrm{CH}_{i'}(C^{[n]}) \subset \mathrm{CH}_{i'}(C^{[\infty]})$ to $\alpha \cap \beta := \alpha_n \cdot \beta \in \mathrm{CH}_{i'-i}(C^{[n]}) \subset \mathrm{CH}_{i'-i}(C^{[\infty]})$. Again one verifies that (\cap) is well-defined.

For $N \geq 0$, the multiplication by N induces

$$[N]_* : \mathrm{CH}_\bullet(C^{[\infty]}) \rightarrow \mathrm{CH}_\bullet(C^{[\infty]}), \text{ and } [N]^* : \mathrm{CH}^\bullet(C^{[\infty]}) \rightarrow \mathrm{CH}^\bullet(C^{[\infty]}).$$

Further, the map $\varphi : C^{[\infty]} \rightarrow J$ induces morphisms of \mathbb{Q} -algebras

$$\varphi_* : \mathrm{CH}_\bullet(C^{[\infty]}) \rightarrow \mathrm{CH}_\bullet(J), \text{ and } \varphi^* : \mathrm{CH}^\bullet(J) \rightarrow \mathrm{CH}^\bullet(C^{[\infty]}).$$

4. A TALE OF TWO TAUTOLOGICAL RINGS (II)

Connections with the Jacobian

We build connections between the Chow theories of the infinite symmetric power and the Jacobian. The easier part is the Chow cohomology $\mathrm{CH}^\bullet(C^{[\infty]})$.

Recall the class $\psi = x_0^*(K) \in \mathrm{CH}^1(S)$ defined in (2.18). Again we simply write ψ for the pull-back of ψ to the schemes on which we work. For $n \geq 1$, define

$$(4.4) \quad \mathcal{O}_{C^{[n]}}(1) := \mathcal{O}_{C^{[n]}}(\varepsilon_n(C^{[n-1]}) + n\psi), \text{ and } \xi_n := c_1(\mathcal{O}_{C^{[n]}}(1)) \in \mathrm{CH}^1(C^{[n]}).$$

We set $\xi_0 = 0$. Then for $n \geq 1$ we have $\varepsilon_n^*(\xi_n) = \xi_{n-1}$, so we obtain a class

$$\xi := (\xi_n)_{n \geq 0} \in \mathrm{CH}^1(C^{[\infty]}).$$

There is an alternative description of $\mathcal{O}_{C^{[n]}}(1)$ (essentially due to Schwarzenberger [Schw63]; see also [MP10], Section 1). Let \mathcal{L} be the pull-back of the Poincaré line bundle \mathcal{P} via $\iota \times_S \mathrm{id}_J : C \times_S J \rightarrow J \times_S J$. For $n \geq 0$, define the sheaf

$$E_n := \mathrm{pr}_{2,*}(\mathrm{pr}_1^*(\mathcal{O}_C(nx_0)) \otimes \mathcal{L}),$$

where $\mathrm{pr}_1 : C \times_S J \rightarrow C$ and $\mathrm{pr}_2 : C \times_S J \rightarrow J$ are the two projections. There is a canonical isomorphism $C^{[n]} \simeq \mathbb{P}(E_n)$, under which $\mathcal{O}_{C^{[n]}}(1)$ corresponds to the line bundle $\mathcal{O}_{\mathbb{P}(E_n)}(1)$.

When $n \geq 2g - 1$, the sheaf E_n is locally free over J (i.e. a vector bundle). Then we have an isomorphism of \mathbb{Q} -algebras

$$(4.5) \quad \mathrm{CH}(C^{[n]}) = \mathrm{CH}(\mathbb{P}(E_n)) \simeq \mathrm{CH}(J)[\xi_n] / \langle P(\xi_n) \rangle,$$

where $P(\xi_n)$ is a polynomial in ξ_n of degree $n - g + 1$ with coefficients in $\mathrm{CH}(J)$. The following result is merely a reinterpretation of this fact ([MP10], Theorem 1.4).

Theorem 4.3.

- (i) When $n \geq \max\{2g, i + g + 1\}$, the transition map $\varepsilon_n^* : \mathrm{CH}^i(C^{[n]}) \rightarrow \mathrm{CH}^i(C^{[n-1]})$ becomes an isomorphism.
- (ii) The map $\varphi^* : \mathrm{CH}^\bullet(J) \rightarrow \mathrm{CH}^\bullet(C^{[\infty]})$ induces an isomorphism of \mathbb{Q} -algebras

$$(4.6) \quad \Phi : \mathrm{CH}^\bullet(J)[t] \xrightarrow{\sim} \mathrm{CH}^\bullet(C^{[\infty]}),$$

which sends $\alpha \in \mathrm{CH}^\bullet(J)$ to $\varphi^*(\alpha)$ and t to the class ξ . □

4.1. Symmetric powers of a relative curve

Now we switch to the Chow homology $\mathrm{CH}_\bullet(C^{[\infty]})$. We begin by introducing two classes $\Gamma \in \mathrm{CH}_g(C^{[\infty]})$ and $L \in \mathrm{CH}_1(C^{[\infty]})$.

Choose an integer $n \geq 2g + d$ (Recall that $d = \dim(S/k)$), and we identify $C^{[n]}$ with $\mathbb{P}(E_n)$. Consider the short exact sequence

$$0 \rightarrow \mathcal{O}_{C^{[n]}}(-1) \rightarrow \varphi_n^*(E_n) \rightarrow Q \rightarrow 0,$$

where Q is a vector bundle of rank $n - g$ on $C^{[n]}$, called the universal quotient bundle of $\varphi_n^*(E_n)$. Define

$$\Gamma := c_{n-g}(Q) \in \mathrm{CH}_g(C^{[n]}) \subset \mathrm{CH}_g(C^{[\infty]}).$$

One can show that Γ is independent of the choice of n .

This time take $n \geq 2g + d + 1$. Define

$$L := \varphi_n^*([o]) \cdot c_{n-g-1}(Q) \in \mathrm{CH}_1(C^{[n]}) \subset \mathrm{CH}_1(C^{[\infty]}),$$

where $[o] \in \mathrm{CH}^g(J) = \mathrm{CH}_0(J)$ is the class of the zero section. Again one can show that L is independent of n . The two classes Γ and L are crucial in relating $\mathrm{CH}_\bullet(C^{[\infty]})$ with $\mathrm{CH}_\bullet(J)$.

Theorem 4.4.

(i) *The map*

$$(4.7) \quad \begin{aligned} s: \mathrm{CH}_\bullet(J) &\rightarrow \mathrm{CH}_\bullet(C^{[\infty]}) \\ \alpha &\mapsto \varphi^*(\alpha) \cap \Gamma \end{aligned}$$

is a section of $\varphi_: \mathrm{CH}_\bullet(C^{[\infty]}) \rightarrow \mathrm{CH}_\bullet(J)$. It respects the Pontryagin products on both sides and satisfies $s \circ [N]_* = [N]_* \circ s$ for all $N \geq 0$.*

(ii) *The section s induces an isomorphism of \mathbb{Q} -algebras*

$$(4.8) \quad \Psi: \mathrm{CH}_\bullet(J)[t] \xrightarrow{\sim} \mathrm{CH}_\bullet(C^{[\infty]}),$$

which sends $\alpha \in \mathrm{CH}_\bullet(J)$ to $s(\alpha)$ and t to the class L .

(iii) *Under the isomorphism Ψ , the push-forward φ_* is the evaluation at zero, and the action of $\xi \cap$ is the derivation d/dt . \square*

The proof of the theorem is a delicate intersection theory calculation. See [MP10], Proposition 1.6 and Theorem 1.11.

4. A TALE OF TWO TAUTOLOGICAL RINGS (II)

Remark 4.5. For all $N \geq 0$, we have (see [MP10], Lemma 1.3)

$$[N]^*(\xi) = N\xi, \quad [N]_*(L) = NL \quad \text{and} \quad [N]_*(\Gamma) = N^{2g}\Gamma.$$

Moreover, there are explicit expressions for the classes L and Γ (see *loc. cit.*, Corollary 1.13):

$$(4.9) \quad L = \frac{\log(1 + \psi \cdot [C]) - \log\left(1 + \psi \cdot s\left([\iota(C)]\right)\right)}{\psi},$$

and for any $N \geq 2$

$$(4.10) \quad \Gamma = \frac{1}{g!(N-1)^g} \left(\frac{\log(1 + \psi \cdot [N]_*([C])) - N \log(1 + \psi \cdot [C])}{N\psi} \right)^g.$$

Here we distinguish $[C] \in \text{CH}_1(C) \subset \text{CH}_1(C^{[\infty]})$ from $[\iota(C)] \in \text{CH}_1(J)$.

Fourier transform

Further, the Fourier transform \mathcal{F} can be lifted to $C^{[\infty]}$. Recall that $\ell = c_1(\mathcal{P}) \in \text{CH}^1(J \times_S J)$. We refer to [KV96], Theorems 3.13 and 3.18 for the proof of the following result.

Definition-Theorem 4.6.

(i) *Define classes*

$$\begin{aligned} \ell_{\infty, \infty} &:= (\varphi \times_S \varphi)^*(\ell) \in \text{CH}^1(C^{[\infty]} \times_S C^{[\infty]}), \\ \xi \times_S \xi &:= \text{pr}_1^*(\xi) \cdot \text{pr}_2^*(\xi) \in \text{CH}^2(C^{[\infty]} \times_S C^{[\infty]}), \end{aligned}$$

where $\text{pr}_1, \text{pr}_2: C^{[\infty]} \times_S C^{[\infty]} \rightarrow C^{[\infty]}$ are the two projections. Then the expression

$$(4.11) \quad \exp(\ell_{\infty, \infty} + \xi \times_S \xi)$$

is an upper correspondence (see [KV96], Definition 3.2). It induces an isomorphism of \mathbb{Q} -algebras

$$\mathcal{F}: \text{CH}_\bullet(C^{[\infty]}) \xrightarrow{\sim} \text{CH}^\bullet(C^{[\infty]}),$$

again called the Fourier transform.

(ii) We have $\mathcal{F}(L) = \xi$, and a commutative diagram

$$\begin{array}{ccc} \text{CH}_\bullet(J) & \xrightarrow[\sim]{\mathcal{F}} & \text{CH}^\bullet(J) \\ \downarrow s & & \downarrow \varphi^* \\ \text{CH}_\bullet(C^{[\infty]}) & \xrightarrow[\sim]{\mathcal{F}} & \text{CH}^\bullet(C^{[\infty]}). \end{array}$$

Also \mathcal{F} satisfies $\mathcal{F} \circ [N]_* = [N]^* \circ \mathcal{F}$, for all $N \geq 0$.

4.2. Tautological rings of the infinite symmetric power

(iii) The inverse $\mathcal{F}^{-1}: \mathrm{CH}^\bullet(C^{[\infty]}) \xrightarrow{\sim} \mathrm{CH}_\bullet(C^{[\infty]})$ is given by the lower correspondence (see *loc. cit.*, Definition 3.17)

$$(4.12) \quad (-1)^g \exp(-\ell_{\infty, \infty}) \cap \left(\exp^* \left((L \times_S L) * (\Gamma \times_S \Gamma) \right) \right),$$

where \exp^* means taking the exponential power series with the product $(*)$. \square

So in terms of the isomorphisms Φ and Ψ in (4.6) and (4.8), the Fourier transform between $\mathrm{CH}_\bullet(J)$ and $\mathrm{CH}^\bullet(J)$ extends constantly to an isomorphism between the two polynomial algebras $\mathrm{CH}_\bullet(J)[t]$ and $\mathrm{CH}^\bullet(J)[t]$.

Remark 4.7. There is a (somewhat) explicit description of the class $\ell_{\infty, \infty} \in \mathrm{CH}^1(C^{[\infty]} \times_S C^{[\infty]})$. For $m, n \geq 0$, define

$$\ell_{m, n} := (\varphi_m \times_S \varphi_n)^*(\ell) \in \mathrm{CH}^1(C^{[m]} \times_S C^{[n]}).$$

We have $\ell_{m, 0} = \ell_{0, n} = 0$. Also we have seen in (2.20) that

$$\ell_{1, 1} = [\Delta] - \mathrm{pr}_1^*([x_0]) - \mathrm{pr}_2^*([x_0]) - \psi \text{ in } \mathrm{CH}^1(C \times_S C),$$

where $\Delta = \Delta_{C/S} \subset C \times_S C$, and $\mathrm{pr}_1, \mathrm{pr}_2: C \times_S C \rightarrow C$ are the two projections. Let $\mathrm{pr}_i: C^n \rightarrow C$ be the i -th projection, for $1 \leq i \leq n$. Then for all $m, n \geq 1$, we have identities (see [KV96], Proposition-Definition 3.10)

$$(4.13) \quad (\phi_m \times_S \phi_n)^*(\ell) = (\sigma_m \times_S \sigma_n)^*(\ell_{m, n}) = \sum_{i=1}^m \sum_{j=1}^n (\mathrm{pr}_i \times_S \mathrm{pr}_j)^*(\ell_{1, 1}),$$

which hold in $\mathrm{CH}^1(C^m \times_S C^n)^{\mathfrak{S}_m \times \mathfrak{S}_n}$.

4.2. Tautological rings of the infinite symmetric power

We define the tautological rings $\mathcal{R}(C^n)$ and $\mathcal{R}(C^{[n]})$ in the style of Faber and Pandharipande (see [FP05], Section 0.1). The collection $\{\mathcal{R}(C^{[n]})\}$ then leads to the definition of the tautological rings $\mathcal{R}_\bullet(C^{[\infty]})$ and $\mathcal{R}^\bullet(C^{[\infty]})$. The main result is a tautological analogue of Theorems 4.3 and 4.4, saying that both $\mathcal{R}_\bullet(C^{[\infty]})$ and $\mathcal{R}^\bullet(C^{[\infty]})$ are polynomial algebras over the tautological ring $\mathcal{T}(J)$.

Let $m, n \in \mathbb{Z}$, $m \geq 1$ and $n \geq 0$. Consider maps

$$T = (T_1, \dots, T_n): C^m \rightarrow C^n,$$

such that each $T_i: C^m \rightarrow C$ is a projection of C^m onto one of its factors. When $n = 0$, we set $T: C^m \rightarrow C^0 = S$ to be the structural map. These maps T are called *tautological maps*. There is a one-to-one correspondence between the set of tautological maps and the set of maps between $\{1, \dots, n\}$ and $\{1, \dots, m\}$ (which still makes sense when $n = 0$, where the map becomes $\emptyset \rightarrow \{1, \dots, m\}$).

4. A TALE OF TWO TAUTOLOGICAL RINGS (II)

Definition 4.8. For $n \geq 0$, the system of *tautological rings*^{note 5} of C^n is the collection of smallest (graded) \mathbb{Q} -subalgebras $\{\mathcal{R}(C^n) \subset \text{CH}(C^n)\}$, such that

- (i) we have $[x_0] \in \mathcal{R}(C)$;
- (ii) the system is stable under pull-backs and push-forwards via all tautological maps T .

Elements in $\mathcal{R}(C^n)$ are called *tautological classes*.

Example 4.9. Recall the classes $K \in \text{CH}^1(C)$ and $\psi \in \text{CH}^1(S)$ defined in (2.18), and also the classes $\kappa_i \in \text{CH}^i(S)$ in (3.1). Since the diagonal map $\Delta: C \rightarrow C^2$ is tautological, we have by adjunction

$$(4.14) \quad K = -\Delta^* (\Delta_* ([C])) \in \mathcal{R}^1(C),$$

which implies $\kappa_i = p_*(K^{i+1}) \in \mathcal{R}^i(C^0)$ for all $i \geq 0$. Moreover, we have

$$(4.15) \quad p_*([x_0]^2) = p_* x_{0,*} x_0^*([x_0]) = x_0^*([x_0]) = -\psi \in \mathcal{R}^1(C^0),$$

so that $\psi \in \mathcal{R}^1(C_0)$.

We now describe a set of generators for each of the tautological rings $\mathcal{R}(C^n)$. For $1 \leq j \leq n$, let $\text{pr}_j: C^n \rightarrow C$ be the projection to the j -th factor, and for $1 \leq k < l \leq n$, let $\text{pr}_{k,l}: C^n \rightarrow C^2$ be the projection to the k -th and l -th factors. Define classes

$$(4.16) \quad \begin{aligned} K_j &:= \text{pr}_j^*(K) \in \mathcal{R}^1(C^n), \\ [x_{0,j}] &:= [\text{pr}_j^{-1}(x_0(S))] = \text{pr}_j^*([x_0]) \in \mathcal{R}^1(C^n), \\ [\Delta_{k,l}] &:= [\text{pr}_{k,l}^{-1}(\Delta)] = \text{pr}_{k,l}^*([\Delta]) \in \mathcal{R}^1(C^n), \end{aligned}$$

where $\Delta = \Delta_{C/S} \in C^2$. Further, we keep the same notation $\{\kappa_i\}$ and ψ for the pull-backs of the classes $\{\kappa_i\}$ and ψ to C^n , for all $n \geq 1$.

Proposition 4.10. For $n \geq 0$, the ring $\mathcal{R}(C^n)$ is generated by the classes $\{\kappa_i\}$, ψ , $\{K_j\}$ and $\{[x_{0,j}]\}$ (if $n \geq 1$), and $\{[\Delta_{k,l}]\}$ (if $n \geq 2$). In particular, when $n = 0$ the ring $\mathcal{R}(C^0)$ coincides with the tautological ring $\mathcal{R}(S)$ in Definition 3.1. \square

The proof is a careful calculation using the projection formula, and can be found in [Loo95], Proposition 2.1 (see also Remark 4.30 below).

We switch to symmetric powers $C^{[n]}$. For $n \geq 1$, we identify $\text{CH}(C^{[n]})$ with the symmetric (i.e. \mathfrak{S}_n -invariant) part $\text{CH}(C^n)^{\mathfrak{S}_n}$ by (4.1). Under this identification, the push-forward map $\sigma_{n,*}: \text{CH}(C^n) \twoheadrightarrow \text{CH}(C^{[n]})$ is the symmetrization.

4.2. Tautological rings of the infinite symmetric power

Definition 4.11. For $n \geq 1$, the *tautological ring* of $C^{[n]}$ is the symmetric part

$$\mathcal{R}(C^{[n]}) := \mathcal{R}(C^n)^{\mathfrak{S}_n}.$$

We also set $\mathcal{R}(C^{[0]}) = \mathcal{R}(C^0) = \mathcal{R}(S)$. Again, elements in $\mathcal{R}(C^{[n]})$ are called *tautological classes*.

It follows from Proposition 4.10 that for $n \geq 2$, the ring $\mathcal{R}(C^{[n]})$ is generated by the classes $\{\kappa_i\}$, ψ , and the elementary symmetric polynomials in $\{K_j\}$, $\{[x_{0,j}]\}$ and $\{[\Delta_{k,l}]\}$.

Passing to the infinite symmetric power

We first observe that the rings $\mathcal{R}(C^{[n]})$ are stable under pull-backs and push-forwards via the maps $\varepsilon_n: C^{[n-1]} \hookrightarrow C^{[n]}$. In fact, the maps ε_n can be lifted to $\text{id}_{C^{n-1}} \times_S x_0: C^{n-1} \rightarrow C^n$. Then for $\alpha \in \mathcal{R}(C^{n-1})$ and $\beta \in \mathcal{R}(C^n)$, we have

$$\begin{aligned} (\text{id}_{C^{n-1}} \times_S x_0)_*(\alpha) &= \text{pr}_{1,\dots,n-1}^*(\alpha) \cdot \text{pr}_n^*([x_0]) \in \mathcal{R}(C^n), \\ (\text{id}_{C^{n-1}} \times_S x_0)^*(\beta) &= \text{pr}_{1,\dots,n-1,*}(\beta \cdot \text{pr}_n^*([x_0])) \in \mathcal{R}(C^{n-1}), \end{aligned}$$

where $\text{pr}_{1,\dots,n-1}$ (resp. pr_n) is the projection of C^n to the first $n-1$ factors (resp. n -th factor). The stability of $\mathcal{R}(C^{[n]})$ under ε_n^* and $\varepsilon_{n,*}$ allows us to pass to the infinite symmetric power $C^{[\infty]}$.

Definition 4.12.

- (i) The *tautological homology* of $C^{[\infty]}$, denote by $\mathcal{R}_\bullet(C^{[\infty]})$, is the direct limit

$$\mathcal{R}_\bullet(C^{[\infty]}) := \varinjlim (\mathcal{R}(S) \rightarrow \mathcal{R}(C) \rightarrow \mathcal{R}(C^{[2]}) \rightarrow \mathcal{R}(C^{[3]}) \rightarrow \dots),$$

where the transition maps are $\varepsilon_{n,*}: \mathcal{R}(C^{[n-1]}) \rightarrow \mathcal{R}(C^{[n]})$.

- (ii) The *tautological cohomology* of $C^{[\infty]}$, denote by $\mathcal{R}^\bullet(C^{[\infty]})$, is the inverse limit

$$\mathcal{R}^\bullet(C^{[\infty]}) := \varprojlim (\mathcal{R}(S) \leftarrow \mathcal{R}(C) \leftarrow \mathcal{R}(C^{[2]}) \leftarrow \mathcal{R}(C^{[3]}) \leftarrow \dots),$$

where the transition maps are $\varepsilon_n^*: \mathcal{R}(C^{[n]}) \rightarrow \mathcal{R}(C^{[n-1]})$.

Both $\mathcal{R}_\bullet(C^{[\infty]})$ and $\mathcal{R}^\bullet(C^{[\infty]})$ are graded, similar to $\text{CH}_\bullet(C^{[\infty]})$ and $\text{CH}^\bullet(C^{[\infty]})$. As before, elements in $\mathcal{R}_\bullet(C^{[\infty]})$ and $\mathcal{R}^\bullet(C^{[\infty]})$ are called *tautological classes*.

It is immediate that $\mathcal{R}^\bullet(C^{[\infty]})$ is stable under the intersection product (\cdot) . Since the addition map $\mu_{m,n}: C^{[m]} \times_S C^{[n]} \rightarrow C^{[m+n]}$ lifts to the identity map $C^m \times_S C^n \rightarrow C^{m+n}$, we also know that $\mathcal{R}_\bullet(C^{[\infty]})$ is stable under the Pontryagin product $(*)$. It follows that $\mathcal{R}_\bullet(C^{[\infty]})$ (resp. $\mathcal{R}^\bullet(C^{[\infty]})$) is a graded \mathbb{Q} -subalgebra of $\text{CH}_\bullet(C^{[\infty]})$ (resp. $\text{CH}^\bullet(C^{[\infty]})$).

We list several properties of $\mathcal{R}_\bullet(C^{[\infty]})$ and $\mathcal{R}^\bullet(C^{[\infty]})$, which are crucial in connecting them with the Jacobian side.

4. A TALE OF TWO TAUTOLOGICAL RINGS (II)

Proposition 4.13.

- (i) We have $\xi \in \mathcal{R}^1(C^{[\infty]})$, $\Gamma \in \mathcal{R}_g(C^{[\infty]})$ and $L \in \mathcal{R}_1(C^{[\infty]})$.
- (ii) The ring $\mathcal{R}_\bullet(C^{[\infty]})$ (resp. $\mathcal{R}^\bullet(C^{[\infty]})$) is stable under $[N]_*$ (resp. $[N]^*$), for all $N \geq 0$.
- (iii) The Fourier transform \mathcal{F} induces an isomorphism

$$\mathcal{F}: \mathcal{R}_\bullet(C^{[\infty]}) \xrightarrow{\sim} \mathcal{R}^\bullet(C^{[\infty]}).$$

- (iv) The cap product in (4.3) restricts to a map

$$\mathcal{R}^\bullet(C^{[\infty]}) \times \mathcal{R}_\bullet(C^{[\infty]}) \xrightarrow{\cap} \mathcal{R}_\bullet(C^{[\infty]}).$$

Proof. Statement (ii) follows from the fact that the diagonal map $\Delta_N: C^{[n]} \rightarrow C^{[Nn]}$ lifts to $C^n \rightarrow C^{Nn}$, which is tautological. Statement (iv) is straightforward.

For (i), by (4.4) and (4.10) we have $\xi \in \mathcal{R}^1(C^{[\infty]})$ and $\Gamma \in \mathcal{R}_g(C^{[\infty]})$. Moreover by (4.9), to show that $L \in \mathcal{R}_1(C^{[\infty]})$ it suffices to prove that $s([\iota(C)]) \in \mathcal{R}_1(C^{[\infty]})$. This is further reduced to prove that $\varphi^*([\iota(C)]) \in \mathcal{R}^{g-1}(C^{[\infty]})$ by the definition of the section s (4.7).

In fact, we can prove that for any class α in the tautological ring $\mathcal{T}(J)$ (see Definition 3.4), we have $\varphi^*(\alpha) \in \mathcal{R}^\bullet(C^{[\infty]})$. First by Theorem 3.6, we know that $(\mathcal{T}(J), \cdot)$ is generated by the classes $\{p_{i,j}\}$ and ϕ . Since $\varphi^*(\phi) \in \mathcal{R}^1(C^{[\infty]})$, it remains to prove that $\varphi^*(p_{i,j}) \in \mathcal{R}^\bullet(C^{[\infty]})$ for all possible i and j . Here we can actually calculate the pull-back of $p_{i,j}$ via $\phi_n = \varphi_n \circ \sigma_n: C^n \rightarrow J$, for all $n \geq 0$. The procedure is similar to that of Lemma 2.22: we chase through the following cartesian squares

$$\begin{array}{ccccc} C \times_S C^n & \xrightarrow{\iota \times_S \text{id}_{C^n}} & J \times_S C^n & \xrightarrow{\text{pr}_2} & C^n \\ \text{id}_C \times_S \phi_n \downarrow & & \downarrow \text{id}_J \times_S \phi_n & & \downarrow \phi_n \\ C \times_S J & \xrightarrow{\iota \times_S \text{id}_J} & J \times_S J & \xrightarrow{\text{pr}_2} & J \\ \text{pr}_1 \downarrow & & \downarrow \text{pr}_1 & & \\ C & \xrightarrow{\iota} & J & & \end{array}$$

and we find

$$(4.17) \quad \phi_n^* \left(\mathcal{F} \left(\theta^{(j-i+2)/2} \cdot [\iota(C)] \right) \right) = \text{pr}_{2,*} \left(\text{pr}_1^* \left(\iota^*(\theta)^{(j-i+2)/2} \right) \cdot \exp \left((\iota \times_S \phi_n)^*(\ell) \right) \right),$$

where $\text{pr}_1: C \times_S C^n \rightarrow C$ and $\text{pr}_2: C \times_S C^n \rightarrow C^n$ are the two projections. By definition $\phi_n^*(p_{i,j})$ is just the codimension $(i+j)/2$ component of the right-hand side of (4.17). Further by (2.19)

4.2. Tautological rings of the infinite symmetric power

and (4.13), we have explicit expressions for $\iota^*(\theta)$ and $(\iota \times_S \phi_n)^*(\ell)$ in terms of tautological classes. It follows that $\phi_n^*(p_{i,j}) \in \mathcal{R}(C^n)$, and hence $\varphi^*(p_{i,j}) \in \mathcal{R}^\bullet(C^{[\infty]})$.

Finally to prove (iii), we observe that the correspondences in (4.11) and (4.12) that define \mathcal{F} and \mathcal{F}^{-1} only involve tautological classes. \square

Connections with the Jacobian

Consider the tautological ring $\mathcal{T}(J)$ in Definition 3.4. We write $\mathcal{T}^\bullet(J) := (\mathcal{T}(J), \cdot)$ with the grading by codimension, and $\mathcal{T}_\bullet(J) := (\mathcal{T}(J), *)$ with the grading by relative dimension. Now we state and prove the main result of this section.

Theorem 4.14. *The isomorphisms Φ and Ψ in (4.6) and (4.8) restrict to isomorphisms of \mathbb{Q} -algebras*

$$(4.18) \quad \Phi|_{\mathcal{T}^\bullet(J)[t]}: \mathcal{T}^\bullet(J)[t] \xrightarrow{\sim} \mathcal{R}^\bullet(C^{[\infty]}),$$

$$(4.19) \quad \Psi|_{\mathcal{T}_\bullet(J)[t]}: \mathcal{T}_\bullet(J)[t] \xrightarrow{\sim} \mathcal{R}_\bullet(C^{[\infty]}).$$

The plan is to prove (4.19) first, and then deduce (4.18) by Fourier duality. We begin with an elementary lemma.

Lemma 4.15. *Let A be a commutative \mathbb{Q} -algebra, and B be a \mathbb{Q} -subalgebra of the polynomial algebra $A[t]$. Assume that $t \in B$, and that B is stable under derivation d/dt . Then we have*

$$B = \text{ev}(B)[t],$$

where $\text{ev}: A[t] \rightarrow A$ is the evaluation at zero.

Proof. Let $P(t) = b_0 + b_1 t + \cdots + b_n t^n$ be an element in B . Since B is stable under derivation, we have $(d/dt)^n(P(t)) = n!b_n \in B$, so that $b_n \in B$. Then since $t \in B$, we have $b_n t^n \in B$, so that $P(t) - b_n t^n \in B$. By induction, we find that all coefficients b_i are in B . It follows that $\text{ev}(B) \subset B$, and hence $\text{ev}(B)[t] \subset B$. On the other hand, we know that $b_i = \text{ev}((d/dt)^i(P(t)/i!))$, with $(d/dt)^i(P(t)/i!) \in B$. Therefore $b_i \in \text{ev}(B)$, which proves the other inclusion $B \subset \text{ev}(B)[t]$. \square

Now consider the push-forward map $\varphi_*: \text{CH}_\bullet(C^{[\infty]}) \rightarrow \text{CH}_\bullet(J)$ which, by Theorem 4.4 (iii), corresponds to the evaluation at zero.

Proposition 4.16. *We have $\varphi_*(\mathcal{R}_\bullet(C^{[\infty]})) = \mathcal{T}_\bullet(J)$.*

Proof. By Theorem 3.6, we know that $\mathcal{T}_\bullet(J)$ is generated by the classes $\{\theta^{(j-i+2)/2} \cdot [\iota(C)]_{(j)}\}$ and $o_*(\psi)$ (here $o = \varphi_0: S \rightarrow J$ is the zero section). Now consider the class $\eta = K/2 + [x_0] + \psi/2 \in \mathcal{R}(C)$, which by (2.19) is equal to $\iota^*(\theta)$. Then we have

$$\iota_*(\eta^{(j-i+2)/2}) = \theta^{(j-i+2)/2} \cdot [\iota(C)],$$

4. A TALE OF TWO TAUTOLOGICAL RINGS (II)

so that $\theta^{(j-i+2)/2} \cdot [\iota(C)]$ is in the image $\varphi_*(\mathcal{R}_\bullet(C^{[\infty]}))$ (recall that $\iota = \varphi_1$). Moreover, we have shown in Proposition 4.13 (ii) that $\mathcal{R}_\bullet(C^{[\infty]})$ is stable under $[N]_*$ for all $N \geq 0$. This implies that the components $\theta^{(j-i+2)/2} \cdot [\iota(C)]_{(j)}$ are also in the image $\varphi_*(\mathcal{R}_\bullet(C^{[\infty]}))$. Since all generators of $\mathcal{T}_\bullet(J)$ are in $\varphi_*(\mathcal{R}_\bullet(C^{[\infty]}))$, we obtain the inclusion $\mathcal{T}_\bullet(J) \subset \varphi_*(\mathcal{R}_\bullet(C^{[\infty]}))$.

To prove the reverse inclusion, we first observe that the image $\varphi_*(\mathcal{R}_\bullet(C^{[\infty]}))$, being the union of $\varphi_{n,*}(\mathcal{R}(C^{[n]}))$ for $n \geq 0$, is also equal to the union of $\phi_{n,*}(\mathcal{R}(C^n))$ for $n \geq 0$. Then it is enough to prove that for all $n \geq 0$ we have $\phi_{n,*}(\mathcal{R}(C^n)) \subset \mathcal{T}(J)$. This is done by an explicit calculation using the generators of $\mathcal{R}(C^n)$ described in Proposition 4.10.

As $\phi_{n,*}$ factors through $\phi_{n+1,*}$, we may assume $n \geq 2$. The ring $\mathcal{R}(C^n)$ is then generated by the classes $\{\kappa_i\}$, ψ , $\{K_j\}$ and $\{[x_{0,j}]\}$, and $\{[\Delta_{k,l}]\}$, as defined in (4.16). We make a change of variables

$$\eta_j := \frac{1}{2}K_j + [x_{0,j}] + \frac{1}{2}\psi,$$

so $\mathcal{R}(C^n)$ is also generated by $\{\kappa_i\}$, ψ , $\{\eta_j\}$ and $\{[x_{0,j}]\}$, and $\{[\Delta_{k,l}]\}$. Let $\alpha \in \mathcal{R}(C^n)$ be a monomial in those generators. We would like to show that $\phi_{n,*}(\alpha) \in \mathcal{T}(J)$.

A first step is to separate the variables $\{\kappa_i\}$ and ψ from the rest. Write $\alpha = \beta \cdot \gamma$, with β collecting all factors of $\{\kappa_i\}$ and ψ . Then β is the pull-back of a class $\beta_0 \in \mathcal{R}(S)$ via the structural map $p^n: C^n \rightarrow S$. Since $p^n = \pi \circ \phi_n$ (recall that $\pi: J \rightarrow S$), we find

$$\phi_{n,*}(\alpha) = \phi_{n,*}(p^{n,*}(\beta_0) \cdot \gamma) = \phi_{n,*}(\phi_n^* \pi^*(\beta_0) \cdot \gamma) = \pi^*(\beta_0) \cdot \phi_{n,*}(\gamma).$$

Thanks to the isomorphisms (3.8), we have $\pi^*(\beta_0) \in \mathcal{T}(J)$. So it remains to prove that $\phi_{n,*}(\gamma) \in \mathcal{T}(J)$, or in other words, we may assume that α is a monomial in $\{\eta_j\}$, $\{[x_{0,j}]\}$ and $\{[\Delta_{k,l}]\}$ only.

A second step is to eliminate multiplicities in the variables $\{[\Delta_{k,l}]\}$. Consider for example $[\Delta] = [\Delta_{1,2}] \in \mathcal{R}(C^2)$. Denote by $\Delta: C \rightarrow C^2$ the diagonal map, and by $\text{pr}_2: C^2 \rightarrow C$ the second projection. Then we have by (4.14)

$$[\Delta]^2 = \Delta_* (\Delta^*([\Delta])) = -\Delta_*(K) = -\Delta_*(\Delta^* \text{pr}_2^*(K)) = -[\Delta] \cdot K_2.$$

By pulling back to C^n , we obtain for $1 \leq k < l \leq n$

$$[\Delta_{k,l}]^2 = -[\Delta_{k,l}] \cdot K_l = -[\Delta_{k,l}] \cdot (2\eta_l - 2[x_{0,l}] - \psi).$$

Together with the first step, this allows us to reduce to the case where α is a monomial in $\{\eta_j\}$, $\{[x_{0,j}]\}$ and $\{[\Delta_{k,l}]\}$, with multiplicity at most 1 for each $[\Delta_{k,l}]$.

Further, we may permute the indices of the $\{[\Delta_{k,l}]\}$ factors by applying the identity

$$(4.20) \quad [\Delta_{k,l}] \cdot [\Delta_{l,m}] = [\Delta_{k,m}] \cdot [\Delta_{l,m}].$$

4.2. Tautological rings of the infinite symmetric power

More precisely, if $I = \{i_1, i_2, \dots, i_q\}$ is a subset of $\{1, \dots, n\}$, we define the symbol

$$[\Delta_I] := [\Delta_{i_1, i_2}] \cdot [\Delta_{i_1, i_3}] \cdots [\Delta_{i_1, i_q}].$$

It follows from (4.20) that $[\Delta_I]$ is well-defined. Also we have identities $[\Delta_I] \cdot \eta_{i_1} = [\Delta_I] \cdot \eta_{i_2} = \cdots = [\Delta_I] \cdot \eta_{i_q}$ and $[\Delta_I] \cdot [x_{0, i_1}] = [\Delta_I] \cdot [x_{0, i_2}] = \cdots = [\Delta_I] \cdot [x_{0, i_q}]$. So for $r, s \geq 0$, we can write

$$(\eta^r[x_0]^s)_{\Delta_I} := [\Delta_I] \cdot \eta_{i_1}^r \cdot [x_{0, i_1}]^s.$$

Now combining with the first two steps, we may assume that the $\{[\Delta_{k,l}]\}$ factors of α take the form

$$[\Delta_{I_1}] \cdot [\Delta_{I_2}] \cdots [\Delta_{I_m}],$$

where the I_k are subsets of $\{1, 2, \dots, n\}$ satisfying $I_k \cap I_l = \emptyset$ for $k \neq l$. This means we are reduced to the case where α is of the form

$$\alpha = \prod_{k \in \{1, \dots, m\}} (\eta^{r_k}[x_0]^{s_k})_{\Delta_{I_k}} \cdot \prod_{\substack{j \in \{1, \dots, n\} \\ j \notin \cup_k I_k}} (\eta_j^{u_j}[x_{0,j}]^{v_j}),$$

with I_1, \dots, I_m and $\{j\}$ pairwise disjoint. In this case, the calculation of $\phi_{n,*}(\alpha)$ is rather straightforward: it follows almost from the definitions of $[N]$ and the product $(*)$ that

$$\phi_{n,*}(\alpha) = \prod_{k \in \{1, \dots, m\}}^* [\#I_k]_* \iota_*(\eta^{r_k}[x_0]^{s_k}) * \prod_{\substack{j \in \{1, \dots, n\} \\ j \notin \cup_k I_k}}^* \iota_*(\eta_j^{u_j}[x_{0,j}]^{v_j}),$$

where \prod^* stands for product using $(*)$, and $\#I_k$ is the cardinal of I_k .

Now since $\mathcal{T}(J)$ is stable under $[N]_*$ and $(*)$, the last step is to prove that for all $r, s \geq 0$, we have $\iota_*(\eta^r[x_0]^s) \in \mathcal{T}(J)$. Then by the identity $\eta = \iota^*(\theta)$, we find $\iota_*(\eta^r[x_0]^s) = \theta^r \cdot \iota_*([x_0]^s)$, which further reduces to showing that $\iota_*([x_0]^s) \in \mathcal{T}(J)$. We have $\iota_*([x_0]^0) = [\iota(C)] \in \mathcal{T}(J)$ and $\iota_*([x_0]) = [\phi] \in \mathcal{T}(J)$. When $s \geq 2$, the calculation is similar to (4.15):

$$\iota_*([x_0]^s) = \iota_* x_{0,*} x_0^* ([x_0]^{s-1}) = o_* ((-\phi)^{s-1}) \in \mathcal{T}(J).$$

The proof of the inclusion $\varphi_*(\mathcal{R}_\bullet(C^{[\infty]})) \subset \mathcal{T}_\bullet(J)$ is thus completed. \square

Proof of Theorem 4.14. By Proposition 4.13 (i) and (iv), we know that $L \in \mathcal{R}_\bullet(C^{[\infty]})$, and that $\mathcal{R}_\bullet(C^{[\infty]})$ is stable under $\xi \cap -$. Then the isomorphism (4.19) follows immediately from Theorem 4.4, Lemma 4.15 and Proposition 4.16. By applying the Fourier transform \mathcal{F} and by Proposition 4.13 (iii), we also obtain (4.18). \square

4. A TALE OF TWO TAUTOLOGICAL RINGS (II)

Remark 4.17. Previously, Moonen and Polishchuk considered much bigger tautological rings of $C^{[\infty]}$ and J , for which they obtained similar results as Theorem 4.14 (see [MP10], Corollary 8.6). The significance of our smaller version, is that one can now apply the machinery on the Jacobian side to study enumerative problems on the tautological rings $\mathcal{R}(C^{[n]})$. This will be the main topic of Section 4.3.

Here we propose an alternative (and minimalist) definition of the tautological rings $\mathcal{R}_\bullet(C^{[\infty]})$ and $\mathcal{R}^\bullet(C^{[\infty]})$. It is based on the properties listed in Proposition 4.13.

Definition 4.18. Define $\mathcal{R}'_\bullet(C^{[\infty]})$ and $\mathcal{R}'^\bullet(C^{[\infty]})$ to be the smallest \mathbb{Q} -subalgebras of $\mathrm{CH}_\bullet(C^{[\infty]})$ and $\mathrm{CH}^\bullet(C^{[\infty]})$ respectively, such that

- (i) we have $[C] \in \mathcal{R}'_\bullet(C^{[\infty]})$ and $\xi \in \mathcal{R}'^\bullet(C^{[\infty]})$;
- (ii) the ring $\mathcal{R}'_\bullet(C^{[\infty]})$ (resp. $\mathcal{R}'^\bullet(C^{[\infty]})$) is stable under $[N]_*$ (resp. $[N]^*$), for all $N \geq 0$;
- (iii) the two rings are stable under \mathcal{F} and \mathcal{F}^{-1} ;
- (iv) the ring $\mathcal{R}'_\bullet(C^{[\infty]})$ is stable under cap product with $\mathcal{R}'^\bullet(C^{[\infty]})$.

Note that Condition (iv) is crucial, otherwise $\mathcal{R}'_\bullet(C^{[\infty]})$ and $\mathcal{R}'^\bullet(C^{[\infty]})$ live in two parallel worlds without interactions. That $\xi \in \mathcal{R}'^\bullet(C^{[\infty]})$ can possibly be implied by the other conditions, but we do not yet see a way to prove it.

Corollary 4.19. *We have $\mathcal{R}'_\bullet(C^{[\infty]}) = \mathcal{R}_\bullet(C^{[\infty]})$ and $\mathcal{R}'^\bullet(C^{[\infty]}) = \mathcal{R}^\bullet(C^{[\infty]})$.*

Proof. We prove the first identity, which then implies the second by Fourier duality. Since we assume $\xi \in \mathcal{R}'^\bullet(C^{[\infty]})$, by Definition-Theorem 4.6 we have $L = \mathcal{F}^{-1}(\xi) \in \mathcal{R}'_\bullet(C^{[\infty]})$. Also $\mathcal{R}'_\bullet(C^{[\infty]})$ is stable under $\xi \cap -$. So again by Theorem 4.4 and Lemma 4.15, we have

$$\mathcal{R}'_\bullet(C^{[\infty]}) \simeq \varphi_* (\mathcal{R}'_\bullet(C^{[\infty]})) [t],$$

and it remains to prove that $\varphi_* (\mathcal{R}'_\bullet(C^{[\infty]})) = \mathcal{T}_\bullet(J)$. Then one realizes that $\varphi_* (\mathcal{R}'_\bullet(C^{[\infty]}))$ is, by definition, the smallest \mathbb{Q} -subalgebra of $\mathrm{CH}_\bullet(J)$ that contains $[\iota(C)]$ and that is stable under $[N]_*$ (for all $N \geq 0$), \mathcal{F} and (\cdot) , which is exactly the tautological ring $\mathcal{T}_\bullet(J)$. \square

4.3. Gorenstein properties

In this section, we focus on the case $S = \mathcal{M}_{g,1}$. For $n \geq 1$, denote by $\mathcal{C}_{g,1}^n$ (resp. $\mathcal{C}_{g,1}^{[n]}$) the n -th power (resp. symmetric power) of the universal curve over $\mathcal{M}_{g,1}$, and let $\mathcal{C}_{g,1}^0 = \mathcal{C}_{g,1}^{[0]} = \mathcal{M}_{g,1}$. We also write $\mathcal{C}_{g,1}^{[\infty]}$ for the infinite symmetric power of the universal curve, and $\mathcal{J}_{g,1}$ for the universal Jacobian, both over $\mathcal{M}_{g,1}$. We keep the same notation as in (4.2) for the various maps.

Our goal is to study an analogue of the conjectural Gorenstein property (see Conjecture 3.2 (i)), this time for the tautological rings of $\mathcal{C}_{g,1}^n$ and $\mathcal{C}_{g,1}^{[n]}$. To begin with, we have the results of Looijenga ([Loo95], Theorem) and Faber [Fab97], which locate the expected socle (see also Remark 4.30 below).

Theorem 4.20. *For all $n \geq 0$, we have $\mathcal{R}^i(\mathcal{C}_{g,1}^n) = 0$ for $i > g - 1 + n$, and $\mathcal{R}^{g-1+n}(\mathcal{C}_{g,1}^n) \simeq \mathbb{Q}$. Moreover, the component $\mathcal{R}^{g-1+n}(\mathcal{C}_{g,1}^n)$ is symmetric, i.e. we have*

$$\mathcal{R}^{g-1+n}(\mathcal{C}_{g,1}^n) = \mathcal{R}^{g-1+n}(\mathcal{C}_{g,1}^{[n]}) \simeq \mathbb{Q}. \quad \square$$

We may now ask whether the rings $\mathcal{R}(\mathcal{C}_{g,1}^n)$ and $\mathcal{R}(\mathcal{C}_{g,1}^{[n]})$ satisfy the Gorenstein property. We start with $\mathcal{R}(\mathcal{C}_{g,1}^n)$.

Speculation 4.21. *For all $n \geq 0$ and $0 \leq i \leq g - 1 + n$, the pairing*

$$(4.21) \quad \mathcal{R}^i(\mathcal{C}_{g,1}^n) \times \mathcal{R}^{g-1+n-i}(\mathcal{C}_{g,1}^n) \xrightarrow{\quad} \mathcal{R}^{g-1+n}(\mathcal{C}_{g,1}^n) \simeq \mathbb{Q}$$

is perfect, so that $\mathcal{R}(\mathcal{C}_{g,1}^n)$ is Gorenstein with socle in codimension $g - 1 + n$.

Since the expected socle is symmetric, we may restrict the pairing (4.21) to the symmetric part. It is not difficult to see that the validity of Speculation 4.21 would imply the following.

Speculation 4.22. *For all $n \geq 0$ and $0 \leq i \leq g - 1 + n$, the restriction of the pairing (4.21)*

$$(4.22) \quad \mathcal{R}^i(\mathcal{C}_{g,1}^{[n]}) \times \mathcal{R}^{g-1+n-i}(\mathcal{C}_{g,1}^{[n]}) \xrightarrow{\quad} \mathcal{R}^{g-1+n}(\mathcal{C}_{g,1}^{[n]}) \simeq \mathbb{Q}$$

is perfect, so that $\mathcal{R}(\mathcal{C}_{g,1}^{[n]})$ is Gorenstein with socle in codimension $g - 1 + n$.

A proof of Speculation 4.21 seems difficult in general: when n grows, the ring $\mathcal{R}(\mathcal{C}_{g,1}^n)$ becomes more and more complicated. We can, however, prove Speculation 4.22 for small g and large n (more precisely for $g \leq 7$ and $n \geq 2g - 1$). Our main tools are the link between $\mathcal{R}(\mathcal{C}_{g,1}^{[n]})$ and $\mathcal{T}(\mathcal{J}_{g,1})$ (see Section 4.2), and the machinery developed on the Jacobian side (see Section 3.4).

4. A TALE OF TWO TAUTOLOGICAL RINGS (II)

Jacobian side

We consider a similar Gorenstein property for the tautological ring $\mathcal{T}(\mathcal{J}_{g,1})$. The first step is to locate the expected socle.

Lemma 4.23. *We have $\mathcal{T}^i(\mathcal{J}_{g,1}) = 0$ for $i > 2g - 1$, and $\mathcal{T}^{2g-1}(\mathcal{J}_{g,1}) \simeq \mathbb{Q}$. Moreover, the component $\mathcal{T}^{2g-1}(\mathcal{J}_{g,1})$ has index $j = 2g - 2$ in the Beauville decomposition (2.5), i.e. we have*

$$\mathcal{T}^{2g-1}(\mathcal{J}_{g,1}) = \mathcal{T}_{(2g-2)}^{2g-1}(\mathcal{J}_{g,1}) \simeq \mathbb{Q}.$$

Proof. The surjective map $\phi_g: \mathcal{C}_{g,1}^g \twoheadrightarrow \mathcal{J}_{g,1}$ induces an injective morphism $\phi_g^*: \mathrm{CH}(\mathcal{J}_{g,1}) \hookrightarrow \mathrm{CH}(\mathcal{C}_{g,1}^g)$. By (4.18), we also know that ϕ_g^* restricts to an injective map $\phi_g^*: \mathcal{T}(\mathcal{J}_{g,1}) \hookrightarrow \mathcal{R}(\mathcal{C}_{g,1}^g)$. Then it follows from Theorem 4.20 that $\mathcal{T}^i(\mathcal{J}_{g,1}) = 0$ for $i > 2g - 1$, and that $\mathcal{T}^{2g-1}(\mathcal{J}_{g,1})$ is at most 1-dimensional.

Recall from (3.8) that $\mathcal{T}_{(2g-2)}^{g-1}(\mathcal{J}_{g,1}) \simeq \mathcal{R}^{g-1}(\mathcal{M}_{g,1}) \simeq \mathbb{Q}$. By applying the Fourier transform \mathcal{F} , we obtain

$$\mathcal{T}_{(2g-2)}^{2g-1}(\mathcal{J}_{g,1}) = \mathcal{F}(\mathcal{T}_{(2g-2)}^{g-1}(\mathcal{J}_{g,1})) \simeq \mathbb{Q}.$$

So $\mathcal{T}^{2g-1}(\mathcal{J}_{g,1})$ is indeed 1-dimensional, and is concentrated in $\mathcal{T}_{(2g-2)}^{2g-1}(\mathcal{J}_{g,1})$. \square

With the socle condition verified, we may formulate an analogue of Speculations 4.21 and 4.22 for $\mathcal{T}(\mathcal{J}_{g,1})$.

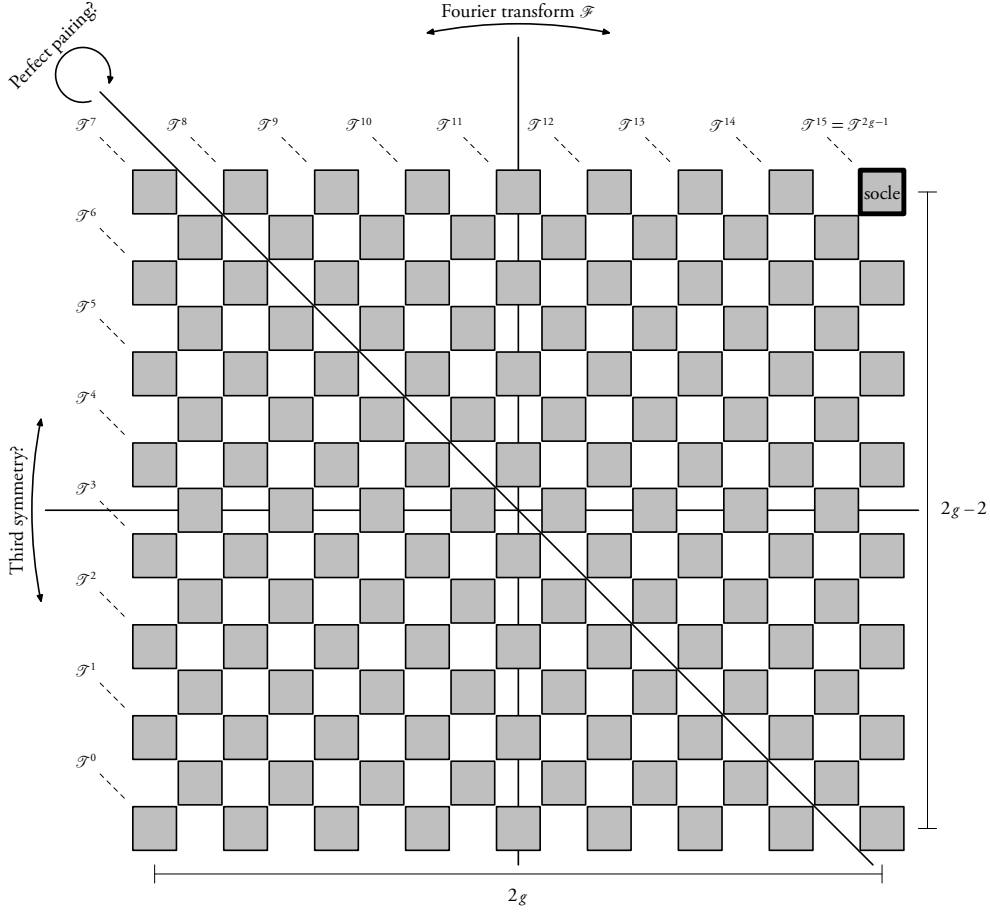
Speculation 4.24. *For all $0 \leq i \leq 2g - 1$, the pairing*

$$(4.23) \quad \mathcal{T}^i(\mathcal{J}_{g,1}) \times \mathcal{T}^{2g-1-i}(\mathcal{J}_{g,1}) \xrightarrow{\quad} \mathcal{T}^{2g-1}(\mathcal{J}_{g,1}) \simeq \mathbb{Q}$$

is perfect, so that $\mathcal{T}(\mathcal{J}_{g,1})$ is Gorenstein with socle in codimension $2g - 1$.

Remark 4.25. The Beauville decomposition provides a nice basis for $\mathcal{T}^i(\mathcal{J}_{g,1})$ with respect to the pairing (4.23). More precisely, recall that $\mathcal{T}^i(\mathcal{J}_{g,1}) = \bigoplus_j \mathcal{T}_{(j)}^i(\mathcal{J}_{g,1})$. Since the socle is in $\mathcal{T}_{(2g-2)}^{g-1}(\mathcal{J}_{g,1})$, the pairing between $\mathcal{T}_{(j)}^i(\mathcal{J}_{g,1})$ and $\mathcal{T}_{(j')}^{2g-1-i}(\mathcal{J}_{g,1})$ is zero unless $j + j' = 2g - 2$. In other words, if we choose for each $\mathcal{T}^i(\mathcal{J}_{g,1})$ a basis from the components $\mathcal{T}_{(j)}^i(\mathcal{J}_{g,1})$, then the pairing matrix of (4.23) becomes block diagonal.

We try to illustrate the pairing using the Dutch house. In figure 4, the socle component is located exactly on the upper right corner. Assuming Speculation 4.24, we would expect a rotational symmetry about the center of the picture. Together with the reflection symmetry about the middle vertical line (given by the Fourier transform), it would then yield a mysterious reflection symmetry about the middle horizontal line. In particular, one should have $\mathcal{T}_{(j)}^i(\mathcal{J}_{g,1}) = 0$ for $j > 2g - 2$, and using the grading in (2.12), a one-to-one correspondence between $\mathcal{T}_{(i,j)}^i(\mathcal{J}_{g,1})$ and $\mathcal{T}_{(i,2g-2-j)}^i(\mathcal{J}_{g,1})$.


 Figure 4. Pairing on $\mathcal{T}(\mathcal{J}_{g,1})$ ($g = 8$).

Once again with the help of Li Ma, we computed the pairing (4.23) for the ring $\tilde{\mathcal{T}}$ defined in (3.12), *i.e.* modulo relations in Construction 3.12. The computer output shows that for $g \leq 7$, we have $\tilde{\mathcal{T}}^{2g-1} \simeq \mathbb{Q}$ and perfect pairings between $\tilde{\mathcal{T}}^i$ and $\tilde{\mathcal{T}}^{2g-1-i}$, for $0 \leq i \leq 2g-1$. Then since the surjective map $\tilde{\mathcal{T}} \rightarrow \mathcal{T}(\mathcal{J}_{g,1})$ in (3.13) is an isomorphism at the socle level, it is in fact an isomorphism. In particular, we can confirm the following.

Theorem 4.26. *Speculation 4.24 is true for $g \leq 7$.* □

For $g = 8$ and several more values of g , however, the relations we find are not sufficient to prove Speculation 4.24.

From the Jacobian to symmetric powers

The main result is a comparison between Speculations 4.22 and 4.24.

4. A TALE OF TWO TAUTOLOGICAL RINGS (II)

Theorem 4.27. *We fix $g > 0$. The following three statements are equivalent:*

- (i) *Speculation 4.24 is true;*
- (ii) *Speculation 4.22 is true for some $n \geq 2g - 1$;*
- (iii) *Speculation 4.22 is true for all $n \geq 2g - 1$.*

Proof. Recall that for $n \geq 2g - 1$, the symmetric power $\mathcal{C}_{g,1}^{[n]}$ is a \mathbb{P}^{n-g} -bundle over $\mathcal{J}_{g,1}$. It then follows from (4.5) and (4.18) that we have an isomorphism of \mathbb{Q} -algebras

$$\mathcal{R}(\mathcal{C}_{g,1}^{[n]}) \simeq \mathcal{T}(\mathcal{J}_{g,1})[\xi_n] / \langle P(\xi_n) \rangle,$$

where $P(\xi_n)$ is a polynomial in ξ_n of degree $n - g + 1$ with coefficients in $\mathcal{T}(\mathcal{J}_{g,1})$. In particular, we obtain for the socle components

$$\mathcal{R}^{g-1+n}(\mathcal{C}_{g,1}^{[n]}) \simeq \mathcal{T}^{2g-1}(\mathcal{J}_{g,1}) \cdot \xi_n^{n-g} \simeq \mathbb{Q}.$$

For $0 \leq i \leq g - 1 + n$, we write $\mathcal{R}^i(\mathcal{C}_{g,1}^{[n]}) \simeq \bigoplus_j \mathcal{T}^{i-j}(\mathcal{J}_{g,1}) \cdot \xi_n^j$ with $\max\{0, i - 2g + 1\} \leq j \leq \min\{i, n - g\}$. Then the pairing (4.22) corresponds to

$$\left(\bigoplus_j \mathcal{T}^{i-j}(\mathcal{J}_{g,1}) \cdot \xi_n^j \right) \times \left(\bigoplus_k \mathcal{T}^{g-1+n-i-k}(\mathcal{J}_{g,1}) \cdot \xi_n^k \right) \xrightarrow{\cdot} \mathcal{T}^{2g-1}(\mathcal{J}_{g,1}) \cdot \xi_n^{n-g} \simeq \mathbb{Q}.$$

On the other hand, observe that

$$\left(\mathcal{T}^{i-j}(\mathcal{J}_{g,1}) \cdot \xi_n^j \right) \cdot \left(\mathcal{T}^{g-1+n-i-k}(\mathcal{J}_{g,1}) \cdot \xi_n^k \right) = 0 \text{ if } j + k < n - g.$$

In other words, if we choose a suitable basis for each $\mathcal{R}^i(\mathcal{C}_{g,1}^{[n]})$, then the pairing matrix of (4.22) is block triangular. Moreover, the blocks on the diagonal correspond exactly to the case $j + k = n - g$, *i.e.* the pairing

$$\left(\mathcal{T}^{i-j}(\mathcal{J}_{g,1}) \cdot \xi_n^j \right) \times \left(\mathcal{T}^{2g-1-i+j}(\mathcal{J}_{g,1}) \cdot \xi_n^{n-g-j} \right) \xrightarrow{\cdot} \mathcal{T}^{2g-1}(\mathcal{J}_{g,1}) \cdot \xi_n^{n-g} \simeq \mathbb{Q}.$$

which in turn corresponds to the pairing

$$(4.24) \quad \mathcal{T}^{i-j}(\mathcal{J}_{g,1}) \times \mathcal{T}^{2g-1-i+j}(\mathcal{J}_{g,1}) \xrightarrow{\cdot} \mathcal{T}^{2g-1}(\mathcal{J}_{g,1}) \simeq \mathbb{Q}.$$

In total, saying that (4.22) is perfect for all $0 \leq i \leq g - 1 + n$ ($n \geq 2g - 1$), is equivalent to saying that (4.24) is perfect for all $0 \leq i - j \leq 2g - 1$. The proof of the theorem is thus completed. \square

Remark 4.28. Using the Beauville decomposition (see Remark 4.25), we may choose a basis for the $\mathcal{R}^i(\mathcal{C}_{g,1}^{[n]})$ such that the pairing matrix of (4.22) is block triangular, and that the blocks on the diagonal are further block diagonal matrices.

Then by Theorem 4.26, we obtain an immediate consequence.

Corollary 4.29. *Speculation 4.22 is true for $g \leq 7$ and for all $n \geq 2g - 1$.* \square

This leaves out a finite number of cases for each $g \leq 7$, which could eventually be verified by computer calculations. We also hope to find a theoretical argument for those remaining cases.

We finish this chapter by relating our pointed tautological rings to the classical, unpointed ones. For $n \geq 1$, denote by \mathcal{C}_g^n (resp. $\mathcal{C}_g^{[n]}$) the n -th power (resp. symmetric power) of the universal curve over \mathcal{M}_g , and let $\mathcal{C}_g^0 = \mathcal{C}_g^{[0]} = \mathcal{M}_g$. The classes $\{K_j\}$ and $\{\Delta_{k,l}\}$ in (4.16) live naturally on \mathcal{C}_g^n . The tautological ring $\mathcal{R}(\mathcal{C}_g^n)$ is then defined to be generated by $\{\kappa_i\}$, $\{K_j\}$ and $\{\Delta_{k,l}\}$ (see [Loo95], Section 1), and $\mathcal{R}(\mathcal{C}_g^{[n]})$ is the symmetric part $\mathcal{R}(\mathcal{C}_g^n)^{\mathfrak{S}_n}$.

The connections between the tautological rings are given by the isomorphisms

$$(4.25) \quad \begin{aligned} \mathcal{C}_{g,1}^n &\simeq \mathcal{C}_g^n \times_{\mathcal{M}_g} \mathcal{M}_{g,1} \simeq \mathcal{C}_g^{n+1}, \\ \mathcal{C}_{g,1}^{[n]} &\simeq \mathcal{C}_g^{[n]} \times_{\mathcal{M}_g} \mathcal{M}_{g,1} \simeq \mathcal{C}_g^{[n]} \times_{\mathcal{M}_g} \mathcal{C}_g. \end{aligned}$$

We have a dictionary for the tautological classes: under (4.25), the class $\psi \in \mathcal{R}(\mathcal{C}_{g,1}^n)$ corresponds to $K_{n+1} \in \mathcal{R}(\mathcal{C}_g^{n+1})$, and $[x_{0,j}] \in \mathcal{R}(\mathcal{C}_{g,1}^n)$ corresponds to $[\Delta_{j,n+1}] \in \mathcal{R}(\mathcal{C}_g^{n+1})$. The dictionary gives isomorphisms of \mathbb{Q} -algebras

$$\mathcal{R}(\mathcal{C}_{g,1}^n) \simeq \mathcal{R}(\mathcal{C}_g^{n+1}), \text{ and } \mathcal{R}(\mathcal{C}_{g,1}^{[n]}) \simeq \mathcal{R}(\mathcal{C}_g^{n+1})^{\mathfrak{S}_n},$$

where the symmetric group \mathfrak{S}_n acts on the first n factors of \mathcal{C}_g^{n+1} .

Remark 4.30. We have used this dictionary implicitly when referring to [Loo95] and [Fab97] for the proofs of Proposition 4.10 and Theorem 4.20.

One can also formulate the corresponding Gorenstein property for $\mathcal{R}(\mathcal{C}_g^n)$ and $\mathcal{R}(\mathcal{C}_g^{[n]})$ (note that $\mathcal{R}^{g-2+n}(\mathcal{C}_g^n) = \mathcal{R}^{g-2+n}(\mathcal{C}_g^{[n]}) \simeq \mathbb{Q}$). Since $\mathfrak{S}_{n+1} \supset \mathfrak{S}_n$, we have an inclusion

$$\mathcal{R}(\mathcal{C}_g^{[n+1]}) = \mathcal{R}(\mathcal{C}_g^{n+1})^{\mathfrak{S}_{n+1}} \subset \mathcal{R}(\mathcal{C}_g^{n+1})^{\mathfrak{S}_n} \simeq \mathcal{R}(\mathcal{C}_{g,1}^{[n]}).$$

It is not difficult to see that Corollary 4.29 implies the following.

Corollary 4.31. *For $g \leq 7$ and $n \geq 2g$, the ring $\mathcal{R}(\mathcal{C}_g^{[n]})$ is Gorenstein with socle in codimension $g - 2 + n$.* \square

The difficult task would be to investigate the critical case^{note 6} of $g = 8$, which is beyond the scope of this thesis.

Tautological classes on a Jacobian variety

This chapter specializes to the case of a Jacobian variety. We determine the generic behavior of the tautological ring in codimension 2, and give examples in higher codimensions. As applications, we obtain simple proofs of (i) Green and Griffiths' result on the generic non-vanishing of the Faber-Pandharipande cycle; (ii) Sebastian's result on Voevodsky's smash-nilpotence conjecture for 1-cycles on abelian varieties.

5.1. The tautological ring over a field

From now on, we focus on the case where the base variety S is a field k . The tautological ring of a Jacobian over k (modulo rational equivalence) was studied by Polishchuk [Pol07]. In this section we briefly review his results.

Let C be a smooth projective curve of genus $g > 0$ over k . Denote by (J, θ) the Jacobian of C . By choosing a point $x_0 \in C(k)$, we obtain an embedding $\iota: C \hookrightarrow J$. Recall from Definition 3.4 that the tautological ring $\mathcal{T}(J)$ is the smallest \mathbb{Q} -subalgebra of $(\mathrm{CH}(J), \cdot)$ that contains $[C] = [\iota(C)]$, and that is stable under $[N]^*$ and \mathcal{F} . Following Polishchuk, we define classes

$$\begin{aligned} p_i &:= \mathcal{F}([C]_{(i-1)}) \in \mathrm{CH}_{(i-1)}^i(J), \\ q_i &:= \mathcal{F}(\theta \cdot [C]_{(i)}) \in \mathrm{CH}_{(i)}^i(J). \end{aligned}$$

Note that p_i (resp. q_i) can be non-zero only if $1 \leq i \leq g$ (resp. $0 \leq i \leq g-1$). Also we have $p_1 = -\theta$ and $q_0 = g[J]$. The following result ([Pol07], Theorem 0.2) is a special case of Theorem 3.6.

Theorem 5.1. *The ring $\mathcal{T}(J)$ is generated by the classes $\{p_i\}$ and $\{q_i\}$. Moreover, the operator $f \in \mathfrak{sl}_2$ in (2.17) acts on polynomials in $\{p_i\}$ and $\{q_i\}$ via the following differential operator of degree 2 (again*

5. TAUTOLOGICAL CLASSES ON A JACOBIAN VARIETY

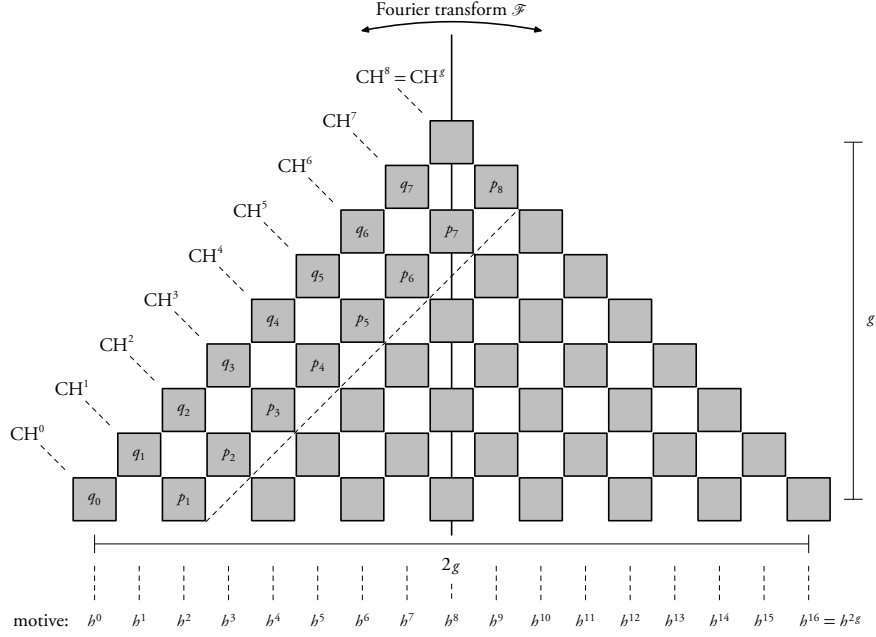


Figure 5. Generators of the tautological ring over k ($g = 8$).

denoted by \mathcal{D}):

$$(5.1) \quad \mathcal{D} = -\frac{1}{2} \sum_{i,j} \binom{i+j}{j} p_{i+j-1} \partial p_i \partial p_j - \sum_{i,j} \binom{i+j-1}{j} q_{i+j-1} \partial q_i \partial p_j + \sum_i q_{i-1} \partial p_i. \quad \square$$

By the Beauville decomposition over k (2.8), the Dutch house in Section 2.3 reduces to its roof. Figure 5 illustrates the case $g = 8$ and locates the classes $\{p_i\}$ and $\{q_i\}$. Here we write $h^i(J) := R^i(J/k)$ for the motivic decomposition. Once again, we have not drawn the components $\mathrm{CH}_{(j)}^i(J)$ with $j < 0$, which are expected to vanish (see Conjecture 2.10 (i)).

Similar to Construction 3.12, one can obtain relations in the tautological ring $\mathcal{T}(J)$ via the \mathfrak{sl}_2 -action: take monomials in $\{p_i\}$ and $\{q_i\}$ that become zero for dimension reasons, and then apply the differential operator \mathcal{D} in (5.1). Using these relations, Polishchuk proved the vanishing of the component on top of the roof ([Pol07], Proposition 4.1 (iii)).

Proposition 5.2. *We have $\mathcal{T}_{(g)}^g(J) = 0$.* □

As is remarked by Polishchuk and Voisin (see [Pol07], Introduction and [Voi13], Section 3), the structure of $\mathcal{T}(J)$ depends highly on the choice of the marked point $x_0 \in C(k)$. One way to avoid this is to follow Beauville [Bea04] and work modulo algebraic equivalence. The quotient ring $\mathcal{T}(J)/\sim_{\mathrm{alg}}$ is then generated by the images of $\{p_i\}$, and relations between them can be obtained

5.2. Codimension 2 and a result of Green and Griffiths

via the \mathfrak{sl}_2 -action. However, the disadvantage of using algebraic equivalence is that one misses many interesting 0-cycles (and more), some of which are also independent of x_0 (see Section 5.2).

There are other sources of relations in both $\mathcal{T}(J)/\sim_{\text{alg}}$ and $\mathcal{T}(J)$, for example via the existence of special linear systems (see [Her07], [GK07], [FH07] and [Moo09], Section 4). However, for the generic curve C (i.e. the generic fiber of $\mathcal{C}_g \rightarrow \mathcal{M}_g$), Polishchuk conjectured that the \mathfrak{sl}_2 -action provides all relations in $\mathcal{T}(J)/\sim_{\text{alg}}$ (see [Pol05], Introduction). Following this idea, we may also raise a similar question.

Question 5.3. *For the generic curve C and generic marked point x_0 (i.e. the generic fiber of $\mathcal{C}_{g,1} \rightarrow \mathcal{M}_{g,1}$), does the \mathfrak{sl}_2 -action provide all relations in $\mathcal{T}(J)$?*

Section 5.2 settles the codimension 2 case. As a consequence, we obtain a simple and characteristic free proof of a result of Green and Griffiths ([GG03], Theorem 2). Further, Section 5.3 gives examples in higher codimensions that extend Proposition 5.2.

Finally, in Section 5.4 we apply results on the tautological ring $\mathcal{T}(J)$ to the study of 1-cycles on an abelian variety. We give a simple proof of a recent result of Sebastian ([Seb12], Theorem 9), which proves Voevodsky's smash-nilpotence conjecture for 1-cycles on abelian varieties.

5.2. Codimension 2 and a result of Green and Griffiths

The goal is to study the structure of the ring $\mathcal{T}(J)$, especially for the generic pointed curve (C, x_0) over $\mathcal{M}_{g,1}$. To begin with, for any smooth projective curve C over a field k with $x_0 \in C(k)$, we have

$$\mathcal{T}^0(J) = \mathbb{Q} \cdot [J], \quad \mathcal{T}_{(0)}^1(J) = \mathbb{Q} \cdot p_1, \quad \text{and} \quad \mathcal{T}_{(1)}^1(J) = \mathbb{Q} \cdot q_1.$$

We know that $p_1 = -\theta \neq 0$ when the genus $g > 0$. For the class q_1 , we have $q_1 = 0$ for $g = 1$, and in general Polishchuk showed that $q_1 = 0$ if and only if $K = (2g - 2)[x_0]$ (here K is the canonical divisor class of C ; see [Pol07], Section 1). So when $g \geq 2$, we have $q_1 \neq 0$ as long as we avoid those so-called *subcanonical points*, i.e. points x_0 satisfying $K = (2g - 2)[x_0]$.

In this section we consider the codimension 2 part $\mathcal{T}^2(J)$ for curves of genus $g \geq 2$. First we have the Beauville decomposition

$$\mathcal{T}^2(J) = \mathcal{T}_{(0)}^2(J) \oplus \mathcal{T}_{(1)}^2(J) \oplus \mathcal{T}_{(2)}^2(J).$$

The easiest component is $\mathcal{T}_{(0)}^2(J)$: we have $\mathcal{T}_{(0)}^2(J) = \mathbb{Q} \cdot p_1^2$, and $p_1^2 \neq 0$ for $g \geq 2$ since $\theta = -p_1$ is ample. For the component $\mathcal{T}_{(1)}^2(J)$, we know that it is spanned by $q_1 p_1$ and p_2 . When $g = 2$, we have $p_1 p_2 \in \mathcal{T}_{(1)}^3(J) = 0$ for dimension reasons. Then a relation in $\mathcal{T}_{(1)}^2(J)$ is given by applying the differential operator \mathcal{D} in (5.1)

$$\mathcal{D}(p_1 p_2) = q_1 p_1 - p_2 = 0.$$

5. TAUTOLOGICAL CLASSES ON A JACOBIAN VARIETY

In this case $\mathcal{T}_{(1)}^2(J)$ is 1-dimensional if x_0 is not a Weierstrass point, and zero otherwise. When $g \geq 3$, a classical theorem of Ceresa shows that p_2 is not algebraically equivalent to zero for the generic curve C over \mathcal{M}_g ([Cer83], Theorem 3.1). It then follows that for the generic pointed curve (C, x_0) over $\mathcal{M}_{g,1}$ with $g \geq 3$, the classes $q_1 p_1$ and p_2 are linearly independent in $\mathcal{T}_{(1)}^2(J)$.

We now treat the remaining component $\mathcal{T}_{(2)}^2(J)$, which is spanned by the classes q_1^2 and q_2 . By Proposition 5.2 we have $\mathcal{T}_{(2)}^2(J) = 0$ for $g = 2$ ($q_2 = 0$ and one more relation given by $\mathcal{D}(q_1 p_2) = 0$). When $g = 3$, since $p_2^2 \in \mathcal{T}_{(2)}^4(J) = 0$, we again obtain a relation

$$(5.2) \quad \mathcal{D}^2(p_2^2) = \mathcal{D}(-6p_3 + 2q_1 p_2) = 2(q_1^2 - 4q_2) = 0.$$

We can show that for the generic pointed curve (C, x_0) over $\mathcal{M}_{3,1}$, all relations in $\mathcal{T}_{(2)}^2(J)$ are given by (5.2), *i.e.* $\mathcal{T}_{(2)}^2(J)$ is 1-dimensional (essentially the same proof as that of Theorem 5.4).

When $g \geq 4$, there are no more relations coming from the \mathfrak{sl}_2 -action. The following result confirms our expectation.

Theorem 5.4. *For the generic pointed curve (C, x_0) over $\mathcal{M}_{g,1}$ with $g \geq 4$, the classes q_1^2 and q_2 are linearly independent in $\mathcal{T}_{(2)}^2(J)$.*

The difficulty of proving Theorem 5.4 lies in the fact that classes in $\mathrm{CH}_{(2)}^2(J)$ are Abel-Jacobi trivial. This means those classes cannot be detected by conventional invariants. When $k = \mathbb{C}$, there exist more sophisticated Hodge-theoretic invariants (see for example [GG03], Appendix), but they are in general difficult to compute.

Our situation is more specific: the Jacobian J is an abelian variety or, in the relative setting, an abelian scheme. In this case, the cycle class map as described in (2.11) provides a simple invariant for detecting non-trivial cycles on the generic Jacobian. Further, we apply a degeneration argument due to Fakhruddin [Fak96], which takes full advantage of the boundary of $\mathcal{M}_{g,1}$. The rest of the proof is just some elementary computation.

Remark 5.5. When the base field k is uncountable, the statement in Theorem 5.4 remains true if one replaces *generic* by the term *very general*, which means away from a countable union of Zariski-closed proper subsets of $\mathcal{M}_{g,1}$.

In fact, all data (*i.e.* $\mathcal{M}_{g,1}$, J , q_1^2 and q_2) are defined over the prime field $k_0 = \mathbb{Q}$ or \mathbb{F}_p . Then for any class $\alpha \in \mathbb{Q} \cdot q_1^2 + \mathbb{Q} \cdot q_2$, if α is non-zero over the generic point $\eta \in \mathcal{M}_{g,1}/k_0$, by base change it is also non-zero over any point $s \in \mathcal{M}_{g,1}(k)$ that maps to η . In other words, we have $\alpha \neq 0$ over any point $s \in \mathcal{M}_{g,1}(k)$ that does not lie in a subvariety of $\mathcal{M}_{g,1}$ defined over k_0 . Since k_0 is countable, there are only countably many such varieties.

On the other hand, Theorem 5.4 is only expected to hold for a very general curve: according to the Bloch-Beilinson conjecture (see [Jann94], Remark 4.12 (c)), both q_1^2 and q_2 should vanish if the curve is defined over $\overline{\mathbb{Q}}$ or $\overline{\mathbb{F}_p(t)}$.

Proof of Theorem 5.4

Note that Theorem 5.4 is of geometric nature: if the statement is true over the base field k , then it is automatically true over any base field $k' \subset k$. Therefore we may assume k to be algebraically closed.

Let $\pi: A \rightarrow S$ be a principally polarized abelian scheme with S a smooth connected variety over k (the principal polarization is not important here). Consider a class $\alpha \in \mathrm{CH}_{(j)}^i(A)$. Recall from (2.11) that the cycle class $\mathrm{cl}(\alpha)$ lands in $H^j(S, R^{2i-j}\pi_*\mathbb{Q}_\ell(i))$. Denote by A_η the generic fiber of A/S , and by $\alpha_\eta \in \mathrm{CH}_{(j)}^i(A_\eta)$ the restriction of α to A_η . Suppose $\alpha_\eta = 0$. Then by the *spreading out* procedure, there exists a non-empty open subset $U \subset S$ such that $\alpha_U = 0$ in $\mathrm{CH}_{(j)}^i(A_U)$, where $A_U := A \times_S U$ and $\alpha_U := \alpha|_{A_U}$. Combining with (2.11), we have the following implication.

Proposition 5.6. *If $\alpha_\eta = 0$, then there exists a non-empty open subset $U \subset S$ such that*

$$\mathrm{cl}(\alpha_U) = 0 \text{ in } H^j(U, R^{2i-j}\pi_*\mathbb{Q}_\ell(i)). \quad \square$$

Now consider the diagram (2.15) with $S = \mathcal{M}_{g,1}$. An important feature is that (2.15) can be extended to $\mathcal{M}_{g,1}^{\mathrm{ct}}$, i.e. the moduli stack of stable pointed curves of genus g and of compact type. The precise procedure of the extension is documented in [GZ12], Section 6. Roughly speaking, when a curve of compact type has several irreducible components, its Jacobian becomes a product of Jacobians. Then the embedding ι is obtained by taking the product of the embeddings in the factors. Again the fact that $\mathcal{M}_{g,1}^{\mathrm{ct}}$ is a stack plays no role in the discussion: since we work with \mathbb{Q} -coefficients, for our purpose it is equivalent to pass to a finite cover of the moduli stack that is an honest variety (see [ACV03], Theorem 7.6.4 for the existence of such a cover).

The universal Jacobian over $\mathcal{M}_{g,1}^{\mathrm{ct}}$ is still a principally polarized abelian scheme. The classes q_1 and q_2 are defined over the whole $\mathcal{M}_{g,1}^{\mathrm{ct}}$ (previously known as $p_{1,1}$ and $p_{2,2}$ over $\mathcal{M}_{g,1}$; see Section 3.2). Let $g \geq 4$. In view of Proposition 5.6, we would like to show that for all non-zero linear combinations $\alpha = r q_1^2 + s q_2$, and for all non-empty open subsets $U \subset \mathcal{M}_{g,1}^{\mathrm{ct}}$, we have

$$\mathrm{cl}(\alpha_U) \neq 0 \text{ in } H^2(U, R^2\pi_*\mathbb{Q}_\ell(2)).$$

Using the following lemma by Fakhruddin ([Fak96], Lemma 4.1), we can reduce the proof of this to a calculation on the boundary of $\mathcal{M}_{g,1}^{\mathrm{ct}}$.

Lemma 5.7. *Let X, S be smooth connected varieties over k and $\pi: X \rightarrow S$ be a smooth proper map. Consider a class $h \in H^m(X, \mathbb{Q}_\ell(n))$. Suppose there exists a non-empty subvariety $T \subset S$ such that for all non-empty open subsets $V \subset T$, we have $h_V \neq 0$, where $h_V := h|_{X_V}$. Then for all non-empty open subsets $U \subset S$, we have $h_U \neq 0$. \square*

Therefore to prove Theorem 5.4, it suffices to construct families of *test curves* over the boundary of $\mathcal{M}_{g,1}^{\mathrm{ct}}$, and to show that for each $\alpha = r q_1^2 + s q_2$ with $(r, s) \neq (0, 0) \in \mathbb{Q}^2$, we can find a family

5. TAUTOLOGICAL CLASSES ON A JACOBIAN VARIETY

such that the class $\text{cl}(\alpha)$ does not vanish over any non-empty open subset of the base variety. We shall construct two families of stable curves over the same base T . For simplicity, we begin with the case $g = 4$, while the argument for the general case is almost identical (see the end of the proof).

Construction 5.8. Take two smooth curves C_1 and C_2 of genus 2 over k , with Jacobians (J_1, θ_1) and (J_2, θ_2) . Let x (resp. y) be a varying point on C_1 (resp. C_2), and c be a fixed point on C_2 . We construct the first family of stable curves by joining x with y and using c as the marked point, and then the second family by joining x with c and using y as the marked point, as is shown in Figure 6.



Figure 6. Two families of test curves ($g = 4$).

With x and y varying, both families have the same base variety $T := C_1 \times (C_2 \setminus \{c\})$. We denote them by $\mathcal{C} \rightarrow T$ and $\mathcal{C}' \rightarrow T$ respectively. Observe that \mathcal{C} and \mathcal{C}' have also the same relative Jacobian $\mathcal{J} := J_1 \times J_2 \times T$, a constant abelian scheme over T via the last projection.

Consider the embeddings $\mathcal{C} \hookrightarrow \mathcal{J}$ with respect to c , and $\mathcal{C}' \hookrightarrow \mathcal{J}$ with respect to y . An important fact is that both embeddings naturally extend over $C_1 \times C_2 \supset T$. More precisely, we have

$$\begin{aligned} \psi_1: C_1 \times C_1 \times C_2 &\hookrightarrow J_1 \times J_2 \times C_1 \times C_2 \text{ given by } (z, x, y) \mapsto (\mathcal{O}_{C_1}(z - x), \mathcal{O}_{C_2}(y - c), x, y), \\ \psi_2: C_2 \times C_1 \times C_2 &\hookrightarrow J_1 \times J_2 \times C_1 \times C_2 \text{ given by } (w, x, y) \mapsto (0, \mathcal{O}_{C_2}(w - c), x, y), \\ \psi'_1: C_1 \times C_1 \times C_2 &\hookrightarrow J_1 \times J_2 \times C_1 \times C_2 \text{ given by } (z, x, y) \mapsto (\mathcal{O}_{C_1}(z - x), \mathcal{O}_{C_2}(c - y), x, y), \\ \psi'_2: C_2 \times C_1 \times C_2 &\hookrightarrow J_1 \times J_2 \times C_1 \times C_2 \text{ given by } (w, x, y) \mapsto (0, \mathcal{O}_{C_2}(w - y), x, y). \end{aligned}$$

We take $\overline{T} := C_1 \times C_2$ as the base variety and view the other varieties as \overline{T} -schemes through the projections onto the last two factors. We also write $\overline{\mathcal{J}} := J_1 \times J_2 \times \overline{T}$. The divisor θ corresponding to $\overline{\mathcal{J}} \rightarrow \overline{T}$ is $\theta = \theta_1 \times [J_2] \times [\overline{T}] + [J_1] \times \theta_2 \times [\overline{T}]$.

Let $\overline{\mathcal{C}} \subset \overline{\mathcal{J}}$ be the union of the images of ψ_1 and ψ_2 ; similarly, let $\overline{\mathcal{C}}' \subset \overline{\mathcal{J}}$ be the union of the images of ψ'_1 and ψ'_2 . Then the restriction of $\overline{\mathcal{C}}$ (resp. $\overline{\mathcal{C}}'$) over T is exactly \mathcal{C} (resp. \mathcal{C}'). Define

$$\overline{q}_i := \mathcal{F}(\theta \cdot [\overline{\mathcal{C}}]_{(i)}) \in \text{CH}_{(i)}^i(\overline{\mathcal{J}}), \text{ and } \overline{q}'_i := \mathcal{F}(\theta \cdot [\overline{\mathcal{C}}']_{(i)}) \in \text{CH}_{(i)}^i(\overline{\mathcal{J}}).$$

Again, the restriction of \overline{q}_i (resp. \overline{q}'_i) over T is exactly the class q_i of \mathcal{C} (resp. \mathcal{C}').

As $\overline{\mathcal{J}}$ is a constant abelian scheme over \overline{T} , we have a Künneth decomposition

$$H^m(\overline{\mathcal{J}}) = \bigoplus_{a_1+b_1+a_2+b_2=m} H^{a_1}(J_1) \otimes H^{b_1}(C_1) \otimes H^{a_2}(J_2) \otimes H^{b_2}(C_2).$$

5.2. Codimension 2 and a result of Green and Griffiths

Here, and in what follows, we omit the coefficients of the cohomology groups. Also on the right hand side we have sorted the factors in the order J_1 - C_1 - J_2 - C_2 , as this turns out to be convenient in our calculations. Given a class $h \in H^m(\overline{\mathcal{J}})$, we denote by $h^{[a_1, b_1, a_2, b_2]}$ its Künneth component in the indicated degrees.

In this case, the cycle class map (2.11) takes the form

$$(5.3) \quad \text{cl}: \text{CH}_{(j)}^i(\overline{\mathcal{J}}) \rightarrow \bigoplus_{\substack{a_1+a_2=2i-j \\ b_1+b_2=j}} H^{a_1}(J_1) \otimes H^{b_1}(C_1) \otimes H^{a_2}(J_2) \otimes H^{b_2}(C_2).$$

So if α is one of the classes $\overline{q}_1^2, \overline{q}_2, \overline{q}_1'^2, \overline{q}_2' \in \text{CH}_{(2)}^2(\overline{\mathcal{J}})$, we can only have $\text{cl}(\alpha)^{[a_1, b_1, a_2, b_2]} \neq 0$ if $a_1 + a_2 = 2$ and $b_1 + b_2 = 2$. Moreover, we remark that $H^2(C_1)$ (resp. $H^2(C_2)$) is supported on a point of C_1 (resp. C_2). As we should like to have the cycle classes after restriction to open subsets $V \subset T \subset \overline{T}$, the only interesting components are $\text{cl}(\alpha)^{[a_1, 1, a_2, 1]}$ with $a_1 + a_2 = 2$ (in fact, we will see in the proof of Proposition 5.9 that for $\text{cl}(\alpha)^{[a_1, 1, a_2, 1]}$ to be non-zero, we also have $a_1 = a_2 = 1$).

The following elementary calculation is the key point in the proof of Theorem 5.4.

Proposition 5.9. *There exist non-zero classes*

$$\begin{aligned} h_1 &\in H^1(J_1) \otimes H^1(C_1) \otimes H^1(J_2) \otimes H^1(C_2) \\ h_2, h_4 &\in H^0(J_1) \otimes H^0(C_1) \otimes H^1(J_2) \otimes H^1(C_2) \\ h_3 &\in H^1(J_1) \otimes H^1(C_1) \otimes H^0(J_2) \otimes H^0(C_2) \end{aligned}$$

such that

$$\begin{aligned} \text{cl}(\overline{q}_2)^{[1, 1, 1, 1]} &= h_1, & \text{cl}(\overline{q}_1^2)^{[1, 1, 1, 1]} &= 2h_2 \cup h_3, \\ \text{cl}(\overline{q}_2')^{[1, 1, 1, 1]} &= -h_1, & \text{cl}(\overline{q}_1'^2)^{[1, 1, 1, 1]} &= -2h_2 \cup h_3 + 2h_3 \cup h_4. \end{aligned}$$

Moreover, the classes $h_2 \cup h_3$ and $h_3 \cup h_4$ are also non-zero.

Proof. The proof is just a careful analysis of the embeddings $\phi_1, \phi_2, \phi_1', \phi_2'$. We first calculate the relevant Künneth components of $\text{cl}([\overline{\mathcal{C}}]_{(i)})$ and $\text{cl}([\overline{\mathcal{C}}']_{(i)})$. Then by intersecting with $\text{cl}(\theta)$ and applying \mathcal{F} in cohomology, we obtain the relevant components of $\text{cl}(\overline{q}_i)$ and $\text{cl}(\overline{q}_i')$.

We start with the cycle classes of $[\overline{\mathcal{C}}]_{(1)}$ and $[\overline{\mathcal{C}}]_{(2)}$. Observe that the image of ϕ_2 only gives a class in $H^4(J_1) \otimes H^0(C_1) \otimes H^2(J_2) \otimes H^0(C_2)$, which by (5.3), does not contribute to either $[\overline{\mathcal{C}}]_{(1)}$ or $[\overline{\mathcal{C}}]_{(2)}$. Regarding ϕ_1 , we may view it as the product of

$$\begin{aligned} \phi_3: C_1 \times C_1 &\hookrightarrow J_1 \times C_1 & \phi_4: C_2 &\hookrightarrow J_2 \times C_2 \\ (z, x) &\mapsto (\mathcal{O}_{C_1}(z - x), x), & y &\mapsto (\mathcal{O}_{C_2}(y - c), y). \end{aligned}$$

5. TAUTOLOGICAL CLASSES ON A JACOBIAN VARIETY

The class of $\text{Im}(\psi_3)$ has components in $H^2(J_1) \otimes H^0(C_1)$, $H^1(J_1) \otimes H^1(C_1)$ and $H^0(J_1) \otimes H^2(C_1)$. The third component is irrelevant due to the appearance of $H^2(C_1)$. We claim that the other two components are both non-zero. For the first, we regard $J_1 \times C_1$ as a constant family over C_1 . Then $C_1 \times C_1$ is fiberwise an ample divisor, which gives a non-zero class in $H^2(J_1) \otimes H^0(C_1)$. For the component in $H^1(J_1) \otimes H^1(C_1)$, we consider

$$\begin{array}{ccccccc} C_1 \times C_1 & \xrightarrow{\text{id} \times \Delta} & C_1 \times C_1 \times C_1 & \xrightarrow{\sigma_2 \times \text{id}} & C_1^{[2]} \times C_1 & \xrightarrow{\varphi_2 \times \text{id}} & J_1 \times C_1 \\ (z, x) & \mapsto & (z, x, x) & \mapsto & ((z, x), x) & \mapsto & (\mathcal{O}_{C_1}(z + x - 2x), x). \end{array}$$

The class of the diagonal in $C_1 \times C_1$ has a component in $H^1(C_1) \otimes H^1(C_1)$ which, viewed as a correspondence, gives the identity $H^1(C_1) \xrightarrow{\sim} H^1(C_1)$. It follows that the class of $\text{Im}(\text{id} \times \Delta)$ has a non-zero component in $H^0(C_1) \otimes H^1(C_1) \otimes H^1(C_1)$. Moreover, we have isomorphisms

$$\sigma_{2,*}: H^0(C_1) \otimes H^1(C_1) \xrightarrow{\sim} H^1(C_1^{[2]}), \text{ and } \varphi_{2,*}: H^1(C_1^{[2]}) \xrightarrow{\sim} H^1(J_1),$$

the latter due to the fact that $C_1^{[2]}$ is obtained by blowing up a point in J_1 . Therefore $\text{Im}(\psi_3)$ as a correspondence gives an isomorphism $H^1(J_1) \xrightarrow{\sim} H^1(C_1)$, which implies a non-zero component in $H^1(J_1) \otimes H^1(C_1)$.

Similarly, the class of $\text{Im}(\psi_4)$ has non-zero components in $H^4(J_2) \otimes H^0(C_2)$ and $H^3(J_2) \otimes H^1(C_2)$. Now we collect all non-zero contributions to the classes of $[\overline{\mathcal{C}}]_{(1)}$ and $[\overline{\mathcal{C}}]_{(2)}$ that do not involve either $H^2(C_1)$ or $H^2(C_2)$. For $[\overline{\mathcal{C}}]_{(2)}$, there is only one non-zero class

$$h_1^0 \in H^1(J_1) \otimes H^1(C_1) \otimes H^3(J_2) \otimes H^1(C_2).$$

By intersecting with $\text{cl}(\theta)$ and applying \mathcal{F} , we obtain a non-zero class

$$h_1 \in H^1(J_1) \otimes H^1(C_1) \otimes H^1(J_2) \otimes H^1(C_2),$$

For $[\overline{\mathcal{C}}]_{(1)}$, there are two non-zero classes

$$h_2^0 \in H^2(J_1) \otimes H^0(C_1) \otimes H^3(J_2) \otimes H^1(C_2), \text{ and } h_3^0 \in H^1(J_1) \otimes H^1(C_1) \otimes H^4(J_2) \otimes H^0(C_2).$$

Again by intersecting with $\text{cl}(\theta)$ and applying \mathcal{F} , we obtain non-zero classes

$$h_2 \in H^0(J_1) \otimes H^0(C_1) \otimes H^1(J_2) \otimes H^1(C_2), \text{ and } h_3 \in H^1(J_1) \otimes H^1(C_1) \otimes H^0(J_2) \otimes H^0(C_2).$$

Then we have $\text{cl}(\overline{q}_2)^{[1,1,1,1]} = h_1$ and $\text{cl}(\overline{q}_1^2)^{[1,1,1,1]} = h_2 \cup h_3 + h_3 \cup h_2 = 2h_2 \cup h_3$.

For the cohomology classes of \overline{q}_2' and $\overline{q}_1'^2$, we remark that the embedding ψ_1' differs from ψ_1 only by an action of $[-1]$ on the J_2 factor. As a consequence, by repeating the same procedure we

5.2. Codimension 2 and a result of Green and Griffiths

obtain classes $h'_1 = -h_1$, $h'_2 = -h_2$ and $h'_3 = h_3$, so that $2h'_2 \cup h'_3 = -2h_2 \cup h_3$. However, this time the embedding ψ'_2 makes an additional contribution. The class of $\text{Im}(\psi'_2)$ has a non-zero component

$$h_4^0 \in H^4(J_1) \otimes H^0(C_1) \otimes H^1(J_2) \otimes H^1(C_2),$$

which belongs to the class of $[\overline{\mathcal{C}}']_{(1)}$. By intersecting with $\text{cl}(\theta)$ and applying \mathcal{F} , we get a non-zero class

$$h_4 \in H^0(J_1) \otimes H^0(C_1) \otimes H^1(J_2) \otimes H^1(C_2).$$

It follows that $\text{cl}(\overline{q}_2')^{[1,1,1,1]} = -h_1$ and $\text{cl}(\overline{q}_1^2)^{[1,1,1,1]} = -2h_2 \cup h_3 + 2h_3 \cup h_4$.

Finally, since the 0-th cohomology groups $H^0(C_i)$ and $H^0(J_i)$ are generated by the unit of the ring structures, we see that both $h_2 \cup h_3$ and $h_3 \cup h_4$ are non-zero. \square

As $h_1 \neq 0$ and $h_3 \cup h_4 \neq 0$, it follows from Proposition 5.9 that for any $(r, s) \neq (0, 0) \in \mathbb{Q}^2$, at least one of the classes $\text{cl}(r\overline{q}_1^2 + s\overline{q}_2)^{[1,1,1,1]}$ and $\text{cl}(r\overline{q}_1^2 + s\overline{q}_2')^{[1,1,1,1]}$ is non-zero in $H^1(C_1) \otimes H^1(J_1) \otimes H^1(C_2) \otimes H^1(J_2)$.

It remains to ensure that this non-zero cohomology class does not vanish when restricted to non-empty open subsets of $\overline{T} = C_1 \times C_2$, *i.e.* that it is not supported on a divisor of $C_1 \times C_2$. We can achieve this by imposing additional assumptions on C_1 and C_2 . In positive characteristic, we choose C_1 to be ordinary and C_2 supersingular. Over $\overline{\mathbb{Q}}$, and hence for any $k = \overline{k}$ of characteristic 0, we take C_1 and C_2 such that J_1 and J_2 are both simple, and such that $\text{End}(J_1) = \mathbb{Z}$ and J_2 is of CM type (see [CF96], Chapters 14 and 15 for explicit examples). In both situations we have $\text{Hom}(J_1, J_2) = 0$, which implies that there is no non-zero divisor class in $H^1(C_1) \otimes H^1(C_2)$. This completes the proof of Theorem 5.4 for $g = 4$.

When $g > 4$, we may attach to both families a constant curve C_0 of genus $g - 4$ via a fixed point $c' \in C_0$, and use another fixed point $c'' \in C_0$ as the marked point, as is shown in Figure 7. We repeat the same procedure, and the proof is exactly the same. Alternatively, one can follow the degeneration argument of Ceresa (see [Cer83], Section 3) and reduce to the genus 4 case. \square



Figure 7. Two families of test curves ($g > 4$).

5. TAUTOLOGICAL CLASSES ON A JACOBIAN VARIETY

Consequence: a result of Green and Griffiths

Let C be a smooth projective curve of genus g over k . Denote by $K \in \mathrm{CH}^1(C)$ the canonical divisor class of C . Faber and Pandharipande introduced the 0-cycle (class)

$$Z := K \times K - (2g - 2)K_\Delta \in \mathrm{CH}^2(C \times C),$$

where K_Δ is the divisor K on the diagonal $\Delta \subset C \times C$. The cycle Z is of degree 0 and lies in the kernel of the Albanese map.

It is easy to see that $Z = 0$ for $g = 0, 1$, or 2 . Faber and Pandharipande showed that it is also the case when $g = 3$, using the fact that curves of genus 3 are either hyperelliptic or plane curves (see Proposition 5.10 (iii) for a unified proof). They asked if Z vanishes in general.

The question over $k = \mathbb{C}$ was answered in the negative by Green and Griffiths using lengthy computations of infinitesimal invariants ([GG03], Theorem 2; see Corollary 5.11). We give a new proof of this result in arbitrary characteristic, showing that it is an immediate consequence of Theorem 5.4.

The idea is the following^{note 7}: observe that $Z \in \mathcal{R}(C \times C)^{\mathfrak{S}_2} \simeq \mathcal{R}(C^{[2]})$, where $C^{[2]}$ is the second symmetric power of C and the tautological rings $\mathcal{R}(C \times C)$ and $\mathcal{R}(C^{[2]})$ are defined in Section 4.2. Then by the isomorphism (4.18), we should be able to express Z as the pull-back of a certain class $W \in \mathcal{T}(J)$ (possibly also the class $\xi_2 \in \mathrm{CH}^1(C \times C)^{\mathfrak{S}_2}$ defined in (4.4)) via the map $\phi_2: C \times C \rightarrow J$ with respect to x_0 . Since Z is Abel-Jacobi trivial, we should look for W in $\mathcal{T}_{(2)}^2(J)$.

Proposition 5.10.

- (i) Define $W := 2(q_1^2 - (2g - 2)q_2) \in \mathcal{T}_{(2)}^2(J)$. Then $Z = \phi_2^*(W)$.
- (ii) We have $Z = 0$ if and only if $W = 0$. In particular, whether W vanishes or not is independent of the point x_0 .
- (iii) If $g = 3$, then $Z = 0$ in $\mathrm{CH}^2(C \times C)$.

Proof. Statement (i) is obtained by an explicit calculation. The essential ingredients are the pull-back of θ via $\iota: C \hookrightarrow J$, and the pull-back of ℓ via $\iota \times \phi_2: C \times (C \times C) \rightarrow J \times J$. Write $\eta = \iota^*(\theta)$ and $\ell_{1,2} = (\iota \times \phi_2)^*(\ell)$. Then we have by (2.19) and (4.13)

$$\eta = \frac{1}{2}K + [x_0]$$

and

$$\ell_{1,2} = [\Delta_1] + [\Delta_2] - 2\mathrm{pr}_1^*([x_0]) - \mathrm{pr}_2^*([x_0 \times C] + [C \times x_0]),$$

where $\Delta_1 = \{(x, x, y) : x, y \in C\}$ and $\Delta_2 = \{(x, y, x) : x, y \in C\}$, and $\mathrm{pr}_1: C \times (C \times C) \rightarrow C$, $\mathrm{pr}_2: C \times (C \times C) \rightarrow C \times C$ are the projections.

5.2. Codimension 2 and a result of Green and Griffiths

We chase through the following cartesian squares

$$\begin{array}{ccccc}
 C \times (C \times C) & \xrightarrow{\iota \times \text{id}_{C \times C}} & J \times (C \times C) & \xrightarrow{\text{pr}_2} & C \times C \\
 \text{id}_C \times \phi_2 \downarrow & & \downarrow \text{id}_J \times \phi_2 & & \downarrow \phi_2 \\
 C \times J & \xrightarrow{\iota \times \text{id}_J} & J \times J & \xrightarrow{\text{pr}_2} & J \\
 \text{pr}_1 \downarrow & & \downarrow \text{pr}_1 & & \\
 C & \xrightarrow{\iota} & J & &
 \end{array}$$

and we find

$$\begin{aligned}
 \phi_2^*(\mathcal{F}(\theta \cdot [C])) &= \text{pr}_{2,*}(\text{pr}_1^*(\eta) \cdot \exp(\ell_{1,2})) \\
 &= \text{pr}_{2,*}\left(\text{pr}_1^*\left(\frac{1}{2}K + [x_0]\right) \cdot \exp([\Delta_1] + [\Delta_2] - 2\text{pr}_1^*([x_0]))\right) \cdot \exp(-[x_0 \times C] - [C \times x_0]) \\
 &= \text{pr}_{2,*}\left(\text{pr}_1^*\left(\left(\frac{1}{2}K + [x_0]\right) \cdot \exp(-2[x_0])\right) \cdot \exp([\Delta_1] + [\Delta_2])\right) \cdot \exp(-[x_0 \times C] - [C \times x_0]) \\
 &= \text{pr}_{2,*}\left(\text{pr}_1^*\left(\frac{1}{2}K + [x_0]\right) \cdot \exp([\Delta_1] + [\Delta_2])\right) \cdot \exp(-[x_0 \times C] - [C \times x_0]).
 \end{aligned}$$

Then by expanding the exponentials while keeping track of the codimension, we get

$$\begin{aligned}
 \phi_2^*(q_1) &= \text{pr}_{2,*}\left(\text{pr}_1^*\left(\frac{1}{2}K + [x_0]\right) \cdot ([\Delta_1] + [\Delta_2])\right) \\
 &\quad - \text{pr}_{2,*}\text{pr}_1^*\left(\frac{1}{2}K + [x_0]\right) \cdot ([x_0 \times C] + [C \times x_0]) \\
 &= \frac{1}{2}(K \times [C] + [C] \times K) - (g-1)([x_0 \times C] + [C \times x_0])
 \end{aligned}$$

and

$$\begin{aligned}
 \phi_2^*(q_2) &= \text{pr}_{2,*}\left(\text{pr}_1^*\left(\frac{1}{2}K + [x_0]\right) \cdot \frac{1}{2}([\Delta_1] + [\Delta_2])^2\right) \\
 &\quad - \text{pr}_{2,*}\left(\text{pr}_1^*\left(\frac{1}{2}K + [x_0]\right) \cdot ([\Delta_1] + [\Delta_2])\right) \cdot ([x_0 \times C] + [C \times x_0]) \\
 &\quad + \text{pr}_{2,*}\text{pr}_1^*\left(\frac{1}{2}K + [x_0]\right) \cdot \frac{1}{2}([x_0 \times C] + [C \times x_0])^2
 \end{aligned}$$

5. TAUTOLOGICAL CLASSES ON A JACOBIAN VARIETY

$$\begin{aligned}
&= \text{pr}_{2,*} \left(\text{pr}_1^* \left(\frac{1}{2}K + [x_0] \right) \cdot ([\Delta_1] \cdot [\Delta_2]) \right) - \frac{1}{2}(K \times [x_0] + [x_0] \times K) + (g-2)[x_0 \times x_0] \\
&= \frac{1}{2}K_\Delta - \frac{1}{2}(K \times [x_0] + [x_0] \times K) + (g-1)[x_0 \times x_0].
\end{aligned}$$

Hence

$$\phi_2^*(q_1^2) = \frac{1}{2}K \times K - (g-1)(K \times [x_0] + [x_0] \times K) + 2(g-1)^2[x_0 \times x_0],$$

and we obtain $\phi_2^*(2(q_1^2 - (2g-2)q_2)) = K \times K - (2g-2)K_\Delta$.

For (ii), we calculate the push-forward $\phi_{2,*}(Z)$. The algorithm is given by the proof of Proposition 4.16: we have

$$\begin{aligned}
Z &= K \times K - (2g-2)K_\Delta \\
&= 4 \left(\frac{1}{2}K + [x_0] \right) \times \left(\frac{1}{2}K + [x_0] \right) - 4 \left(\frac{1}{2}K + [x_0] \right) \times [x_0] - 4[x_0] \times \left(\frac{1}{2}K + [x_0] \right) \\
&\quad - (4g-4) \left(\frac{1}{2}K + [x_0] \right)_\Delta + 4g[x_0 \times x_0],
\end{aligned}$$

so that

$$\begin{aligned}
\phi_{2,*}(Z) &= 4(\theta \cdot [C]) * (\theta \cdot [C]) - 8\theta \cdot [C] - (4g-4)[2]_*(\theta \cdot [C]) + 4g[\theta] \\
&= 0(\theta \cdot [C]_{(0)}) + 0(\theta \cdot [C]_{(1)}) + 4(\theta \cdot [C]_{(1)}) * (\theta \cdot [C]_{(1)}) - (8g-8)(\theta \cdot [C]_{(2)}) \\
&\quad + (\text{terms in } \oplus_{j \geq 3} \mathcal{T}_{(j)}^g(J)) \\
&= 4((\theta \cdot [C]_{(1)}) * (\theta \cdot [C]_{(1)}) - (2g-2)(\theta \cdot [C]_{(2)})) + (\text{terms in } \oplus_{j \geq 3} \mathcal{T}_{(j)}^g(J)).
\end{aligned}$$

Observe that the Fourier dual of $(\theta \cdot [C]_{(1)}) * (\theta \cdot [C]_{(1)}) - (2g-2)(\theta \cdot [C]_{(2)})$ is exactly W . Therefore if $Z = 0$, then every component of $\phi_{2,*}(Z)$ in the Beauville decomposition is zero, and hence $W = 0$.

Statement (iii) follows directly from (i) and (5.2). \square

By applying Theorem 5.4 and Proposition 5.10, we obtain the result of Green and Griffiths over an arbitrary base field k .

Corollary 5.11. *For the generic curve C over \mathcal{M}_g with $g \geq 4$, we have $Z \neq 0$. Also if k is uncountable, the same statement holds for a very general curve C over \mathcal{M}_g with $g \geq 4$.* \square

5.3. Examples in higher codimensions

In this section we assume $k = \mathbb{C}$, as we shall use a Hodge-theoretic argument. The goal is to prove the following result, which extends Proposition 5.2.

5.3. Examples in higher codimensions

Proposition 5.12. *For a very general pointed curve (C, x_0) over $\mathcal{M}_{g,1}$, we have*

$$(5.4) \quad \mathcal{T}_{(g-1)}^{g-1}(J) \simeq \mathcal{T}_{(g-1)}^g(J) \simeq \mathbb{Q}.$$

The first isomorphism in (5.4) is given by the Fourier transform \mathcal{F} . The proof of the second isomorphism consists of two steps. First we use relations coming from the \mathfrak{sl}_2 -action to show that $\mathcal{T}_{(g-1)}^{g-1}(J)$ is at most 1-dimensional (valid over any base field k). Then we apply a similar degeneration argument as in Theorem 5.4 to prove non-triviality (over $k = \mathbb{C}$).

Lemma 5.13. *For any smooth projective pointed curve (C, x_0) over a field, the class q_{g-1} generates $\mathcal{T}_{(g-1)}^{g-1}(J)$. Hence $\mathcal{T}_{(g-1)}^{g-1}(J)$ is at most 1-dimensional.*

Proof. The argument is essentially due to Polishchuk (see [Pol07], Proposition 4.1). By Theorem 5.1, the component $\mathcal{T}_{(g-1)}^{g-1}(J)$ is spanned by monomials

$$(5.5) \quad q_{i_1} q_{i_2} \cdots q_{i_m} \text{ with } i_1 + i_2 + \cdots + i_m = g - 1.$$

We would like to show that all such monomials are proportional to q_{g-1} , and we proceed by induction on the number of factors m . The assertion is trivial for $m = 1$. Now suppose all monomials in (5.5) with $1 \leq m < m_0$ are proportional to q_{g-1} . Then to each $q_{i_1} q_{i_2} \cdots q_{i_{m_0}}$ with $i_1 + i_2 + \cdots + i_{m_0} = g - 1$, we associate the monomial

$$p_{i_1+1} p_{i_2+1} \cdots p_{i_{m_0}+1} \in \mathcal{T}_{(g-1)}^{g-1+m_0}(J).$$

Since $m_0 > 1$, we have $\mathcal{T}_{(g-1)}^{g-1+m_0}(J) = 0$ for dimension reasons. By applying m_0 times the differential operator \mathcal{D} in (5.1), we obtain

$$\mathcal{D}^{m_0}(p_{i_1+1} p_{i_2+1} \cdots p_{i_{m_0}+1}) = 0 \text{ in } \mathcal{T}_{(g-1)}^{g-1}(J).$$

But if we analyze the explicit expression of \mathcal{D} , two of the three terms in (5.1) have the effect of merging two factors into one, while only the third term $\sum_i q_{i-1} \partial p_i$ keeps the same number of factors. So in $\mathcal{D}^{m_0}(p_{i_1+1} p_{i_2+1} \cdots p_{i_{m_0}+1})$ we find a positive multiple of $q_{i_1} q_{i_2} \cdots q_{i_{m_0}}$, and all other terms have the number of factors strictly smaller than m_0 . Those terms are proportional to q_{g-1} by the induction hypothesis, and so is $q_{i_1} q_{i_2} \cdots q_{i_{m_0}}$. \square

Proposition 5.14. *Let (C, x_0) be a very general pointed curve over $\mathcal{M}_{g,1}/\mathbb{C}$. Then we have $p_i \neq 0$ for $1 \leq i \leq g$, and $q_i \neq 0$ for $0 \leq i \leq g - 1$.*

This means all the classes $\{p_i\}$ and $\{q_i\}$ depicted in Figure 5 are non-trivial on a very general pointed curve.

5. TAUTOLOGICAL CLASSES ON A JACOBIAN VARIETY

Proof. The strategy is exactly the same as in the codimension 2 case (see Theorem 5.4). For simplicity, we only prove non-triviality for p_g and q_{g-1} . The general case can be obtained either by attaching constant components to the family shown in Figure 8 and repeating the same procedure, or by applying Ceresa's degeneration argument (see [Cer83], Section 3).

Moreover, since $p_1 p_g \in \mathcal{T}_{(g-1)}^{g+1}(J) = 0$, we have $\mathcal{D}(p_1 p_g) = q_{g-1} p_1 - p_g = 0$. Then we see that $p_g = e(q_{g-1})$ and $q_{g-1} = f(p_g)$, where $e, f \in \mathfrak{sl}_2$. So it suffices to prove non-triviality for p_g , or for the class $[C]_{(g-1)}$ by Fourier duality.

The statement is trivial for $g = 1$, so we assume $g \geq 2$. In view of Remark 5.5, Proposition 5.6 and Lemma 5.7, it suffices to construct a family of test curves \mathcal{C} over the boundary of $\mathcal{M}_{g,1}^{\text{ct}}$, and to show that the cycle class of $[\mathcal{C}]_{(g-1)}$ does not vanish over any non-empty open subset of the base variety.

The following is analogous to the first family in Construction 5.8. Take a complex smooth curve C of genus 2, with J its Jacobian. Also take $g - 2$ complex elliptic curves E_1, \dots, E_{g-2} , and denote by o_1, \dots, o_{g-2} their respective zeros. Let x be a varying point of C , and y_1, \dots, y_{g-2} be varying points on E_1, \dots, E_{g-2} respectively. The family is obtained by joining x with y_1 , and o_i with y_{i+1} for $1 \leq i \leq g - 3$. Finally o_{g-2} serves as the marked point, as is shown in Figure 8.

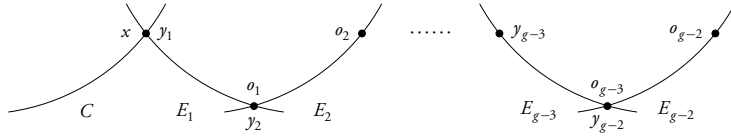


Figure 8. A family of test curves.

The base variety is then

$$T = C \times (E_1 \setminus \{o_1\}) \times \cdots \times (E_{g-2} \setminus \{o_{g-2}\}).$$

Write $\mathcal{C} \rightarrow T$ for this family, whose relative Jacobian is simply $\mathcal{J} = J \times E_1 \times \cdots \times E_{g-2} \times T$ over the last factor T . Similar to the codimension 2 case, the embedding $\mathcal{C} \hookrightarrow \mathcal{J}$ with respect to o_{g-2} naturally extends over $\overline{T} = C \times E_1 \times \cdots \times E_{g-2}$. More precisely, we have

$$\begin{aligned} \psi_C : C \times C \times E_1 \times \cdots \times E_{g-2} &\hookrightarrow J \times E_1 \times \cdots \times E_{g-2} \times C \times E_1 \times \cdots \times E_{g-2} \\ (z, x, y_1, \dots, y_{g-2}) &\mapsto (\mathcal{O}_C(z - x), y_1, \dots, y_{g-2}, x, y_1, \dots, y_{g-2}), \end{aligned}$$

and for $1 \leq i \leq g - 2$

$$\begin{aligned} \psi_{E_i} : E_i \times C \times E_1 \times \cdots \times E_{g-2} &\hookrightarrow J \times E_1 \times \cdots \times E_{g-2} \times C \times E_1 \times \cdots \times E_{g-2} \\ (w_i, x, y_1, \dots, y_{g-2}) &\mapsto (0, 0, \dots, 0, w_i, y_{i+1}, \dots, y_{g-2}, x, y_1, \dots, y_{g-2}). \end{aligned}$$

5.4. Application: smash-nilpotent 1-cycles on an abelian variety

Write $\overline{\mathcal{J}} = J \times E_1 \times \cdots \times E_{g-2} \times \overline{T}$. Let $\overline{\mathcal{C}} \subset \overline{\mathcal{J}}$ be the union of the images of $\psi_C, \psi_{E_1}, \dots, \psi_{E_{g-2}}$, whose restriction over T is exactly $\mathcal{C} \subset \mathcal{J}$.

Our goal is to calculate the cycle class of $[\overline{\mathcal{C}}]_{(g-1)} \in \mathrm{CH}_{(g-1)}^{g-1}(\overline{\mathcal{J}})$ using the Künneth formula (we omit the coefficients of the cohomology groups)

$$H^m(\overline{\mathcal{J}}) = \bigoplus_{\sum_{i=0}^{g-2} a_i + b_i = m} H^{a_0}(J) \otimes H^{b_0}(C) \otimes H^{a_1}(E_1) \otimes H^{b_1}(E_1) \otimes \cdots \otimes H^{a_{g-2}}(E_{g-2}) \otimes H^{b_{g-2}}(E_{g-2}).$$

Here the cycle class map (2.11) takes the form

$$\mathrm{cl}: \mathrm{CH}_{(j)}^i(\overline{\mathcal{J}}) \rightarrow \bigoplus_{\substack{\sum_{i=0}^{g-2} a_i = 2i-j \\ \sum_{i=0}^{g-2} b_i = j}} H^{a_0}(J) \otimes H^{b_0}(C) \otimes H^{a_1}(E_1) \otimes H^{b_1}(E_1) \otimes \cdots \otimes H^{a_{g-2}}(E_{g-2}) \otimes H^{b_{g-2}}(E_{g-2}).$$

The computation is somewhat tedious but elementary (essentially the same as in Proposition 5.9). In the end the only relevant component (*i.e.* non-zero, and not easily killed by restricting to non-empty open subsets $V \subset \overline{T}$) of $\mathrm{cl}([\overline{\mathcal{C}}]_{(g-1)})$ is given by the embedding ψ_C . We call it h and we have

$$(5.6) \quad h \in H^1(J) \otimes H^1(C) \otimes H^1(E_1) \otimes H^1(E_1) \otimes \cdots \otimes H^1(E_{g-2}) \otimes H^1(E_{g-2}),$$

i.e. all the a_i and b_i are equal to 1.

It remains to ensure that the class h in (5.6) does not vanish when restricted to non-empty open subsets of \overline{T} , *i.e.* that it is not supported on a divisor of \overline{T} . We have the following observation: by construction h is a tensor product of classes in

$$H^1(J) \otimes H^1(C), \quad H^1(E_1) \otimes H^1(E_1), \quad \dots, \quad H^1(E_{g-2}) \otimes H^1(E_{g-2}).$$

So its factor in $H^1(C) \otimes H^1(E_1) \otimes \cdots \otimes H^1(E_{g-2}) \subset H^{g-1}(\overline{T})$ has maximal Hodge level, which cannot be supported on a divisor of \overline{T} . The proof is thus completed. \square

Finally, Proposition 5.12 is obtained by combining Lemma 5.13 and Proposition 5.14. \square

5.4. Application: smash-nilpotent 1-cycles on an abelian variety

The subject of this section is independent of the two previous sections. We discuss Voevodsky's smash-nilpotence conjecture, and explain how we can apply our knowledge of tautological classes to a special case of the conjecture.

Let k be a field and X be an object in \mathcal{V}_k . Voevodsky [Voe95] introduced the following notion.

Definition 5.15. A class $\alpha \in \mathrm{CH}(X)$ is called *smash-nilpotent* if there exists an integer $N > 0$ such that

$$\underbrace{\alpha \times \alpha \times \cdots \times \alpha}_N = 0 \quad \text{in } \mathrm{CH}(X^N).$$

5. TAUTOLOGICAL CLASSES ON A JACOBIAN VARIETY

The set of smash-nilpotent classes forms an ideal in $\mathrm{CH}(X)$ and is stable under correspondences. It is easy to see that smash-nilpotent classes are homologically equivalent to zero, and hence numerically equivalent to zero. Voevodsky conjectured the converse ([Voe95], Conjecture 4.2).

Conjecture 5.16. *If $\alpha \in \mathrm{CH}_{\mathrm{num}}(X)$, i.e. α is numerically equivalent to zero, then α is smash-nilpotent.*

One can view Conjecture 5.16 as a stronger version of the standard conjecture $\sim_{\mathrm{hom}} = \sim_{\mathrm{num}}$. Also it fits into the Bloch-Beilinson-Murre (BBM) framework: notably the BBM conjecture plus the standard conjecture implies Conjecture 5.16. We refer to [And05], Sections 2.6 and 2.7 for more details.

The first known case of Conjecture 5.16 was given by Voevodsky ([Voe95], Corollary 3.2), and independently by Voisin ([Voi96], Lemma 2.3).

Theorem 5.17. *Conjecture 5.16 is true for $\alpha \in \mathrm{CH}_{\mathrm{alg}}(X)$, i.e. α algebraically equivalent to zero. \square*

A related (and weaker; see [And05], Théorème 3.33) conjecture is the finite dimensionality conjecture of Kimura-O’Sullivan. Let $M \in \mathcal{M}_k$ be a Chow motive over k (see Section 2.2). Recall that we have the n -th symmetric (resp. wedge) product $S^n(M)$ (resp. $\wedge^n(M)$) (see Examples 2.13 (iii)). Kimura and independently O’Sullivan introduced the following notion and conjecture ([Kim05], Definition 3.7 and Conjecture 7.1).

Definition 5.18.

- (i) The motive M is called *even* (resp. *odd*) if there exists an integer $N > 0$ such that $\wedge^N(M) = 0$ (resp. $S^N(M) = 0$).
- (ii) The motive M is called *finite-dimensional* if M can be written as a direct sum $M^+ \oplus M^-$ with M^+ even and M^- odd.

Conjecture 5.19. *All motives $M \in \mathcal{M}_k$ are finite-dimensional.*

So far Conjecture 5.19 is only known for the subcategory of \mathcal{M}_k generated by the motives of curves (or abelian varieties) ([Kim05], Theorem 4.2). In the case of an abelian variety A , if we write $h(A) = \bigoplus_i h^i(A)$ for the motivic decomposition (see Theorem 2.14; here $h^i(A) = R^i(A/k)$), then $h^i(A)$ is even (resp. odd) for i even (resp. odd).

On the other hand, using properties of finite-dimensional motives, Kahn and Sebastian obtained a second case of Conjecture 5.16 ([KS09], Proposition 1).

Theorem 5.20. *Conjecture 5.16 is true for $\alpha \in \mathrm{CH}(M)$ with M odd. \square*

More recently, based on Theorems 5.17 and 5.20, Sebastian proved the following case ([Seb12], Theorem 9).

5.4. Application: smash-nilpotent 1-cycles on an abelian variety

Theorem 5.21. *Conjecture 5.16 is true for 1-cycles on an abelian variety. In other words, if A is an abelian variety of dimension g and $\alpha \in \mathrm{CH}_{1,\mathrm{num}}(A) = \mathrm{CH}_{\mathrm{num}}^{g-1}(A)$, then α is smash-nilpotent.*

Remark 5.22. The standard conjecture $\sim_{\mathrm{hom}} = \sim_{\mathrm{num}}$ is known for 1-cycles on abelian varieties. In fact, since Conjecture 2.10 (i) and (ii) are true for $\mathrm{CH}^{g-1}(A)$, we have $\mathrm{CH}_{\mathrm{num}}^{g-1}(A) = \bigoplus_{j \geq 1} \mathrm{CH}_{(j)}^{g-1}(A)$.

Here we present a simple proof of Theorem 5.21 using tautological classes on Jacobian varieties. Our starting point will also be Theorems 5.17 and 5.20. Let (C, x_0) be a smooth projective pointed curve of genus g over k , with (J, θ) its Jacobian. The first step is to prove smash-nilpotence for tautological 1-cycles.

Lemma 5.23. *For $j \geq 1$, the class $[C]_{(j)} \in \mathcal{T}_{(j)}^{g-1}(J)$ is smash-nilpotent. As a consequence, all classes in $\mathcal{T}_{\mathrm{num}}(J) = \bigoplus_{j \geq 1} \mathcal{T}_{(j)}(J)$ are smash-nilpotent.*

Proof. By Fourier duality, it is equivalent to prove that p_i is smash-nilpotent for all $i \geq 2$. First of all, we know that $p_2 \in \mathrm{CH}(h^3(J))$ (see Figure 5). Since $h^3(J)$ is odd, Theorem 5.20 implies that p_2 is smash-nilpotent.

The following argument is essentially in [Her07], Lemma 4 and [Pol08], Proposition 2.5 (iii). For $i \geq 3$, apply the differential operator \mathcal{D} in (5.1) to the class $p_2 p_{i-1}$, and we get

$$(5.7) \quad \mathcal{D}(p_2 p_{i-1}) = -\frac{i(i+1)}{2} p_i + q_1 p_{i-1} + q_{i-1} p_2.$$

Since p_2 is smash-nilpotent, the identity (5.7) shows that if p_{i-1} is smash-nilpotent, then so is p_i . Hence by induction we obtain smash-nilpotence for all p_i with $i \geq 2$. \square

The second step is a simple observation that all 1-cycles on an abelian variety are tautological in some sense.

Proof of Theorem 5.21. Let $\alpha \in \mathrm{CH}_{\mathrm{num}}^{g-1}(A)$. By Remark 5.22, we have $\alpha \in \bigoplus_{j \geq 1} \mathrm{CH}_{(j)}^{g-1}(A)$. Choose a representative of α

$$\alpha = \sum_{i=1}^m n_i [C_i],$$

where the $C_i \subset A$ are (possibly singular) irreducible curves. By Theorem 5.17, we may assume after translation that all C_i pass through $o \in A$.

Take the normalization $\sigma_i: \tilde{C}_i \rightarrow C_i$ and write $f_i: \tilde{C}_i \rightarrow A$. By functoriality, if we denote by \tilde{J}_i the Jacobian of \tilde{C}_i , then the map f_i factors through

$$\begin{array}{ccc} \tilde{C}_i & \xrightarrow{f_i} & A \\ \searrow \iota_i & & \nearrow \phi_i \\ & \tilde{J}_i & \end{array}$$

5. TAUTOLOGICAL CLASSES ON A JACOBIAN VARIETY

where $\iota_i: \tilde{C}_i \hookrightarrow \tilde{J}_i$ is the embedding with respect to $x_{0,i} \in \tilde{C}_i(k)$ satisfying $f_i(x_{0,i}) = o$.

Back to the class α . We have

$$\alpha = \sum_{i=1}^m n_i [C_i] = \sum_{i=1}^m n_i f_{i,*}([\tilde{C}_i]) = \sum_{i=1}^m n_i \phi_{i,*}([\iota_i(\tilde{C}_i)])$$

Apply the Beauville decomposition to $[\iota_i(\tilde{C}_i)]$, and we get

$$(5.8) \quad \alpha = \sum_{i=1}^m n_i \phi_{i,*}([\iota_i(\tilde{C}_i)]_{(0)}) + \sum_{i=1}^m \sum_{j \geq 1} n_i \phi_{i,*}([\iota_i(\tilde{C}_i)]_{(j)})$$

Now since $\alpha \in \bigoplus_{j \geq 1} \text{CH}_{(j)}^{g-1}(A)$, the first term on the right-hand side of (5.8) is zero, and the second term is smash-nilpotent by Lemma 5.23. \square

From Theorem 5.21, it is not difficult to deduce a slightly stronger statement ([Seb12], Theorem 6).

Corollary 5.24. *Conjecture 5.16 is true for 1-cycles on a product of curves.*

The statement follows essentially from the observation below.

Lemma 5.25. *Let X be an object in \mathcal{V}_k , and $\pi: P \rightarrow X$ be a projective bundle of relative dimension n . Then if Conjecture 5.16 is true for 1-cycles on X , it is also true for 1-cycles on P .*

Proof. Write $P = \mathbb{P}(V)$ for some vector bundle $V \rightarrow X$. Denote by $\xi \in \text{CH}^1(P)$ the first Chern class of the line bundle $\mathcal{O}_{\mathbb{P}(V)}(1)$. Take a class $\alpha \in \text{CH}_1(P)$. It is classical that α is uniquely expressible as

$$\alpha = \xi^n \cdot \pi^*(\beta) + \xi^{n-1} \cdot \pi^*(\gamma),$$

with $\beta \in \text{CH}_1(X)$ and $\gamma \in \text{CH}_0(X)$ (see [Ful98], Theorem 3.3). Also $\alpha \in \text{CH}_{1,\text{num}}(P)$ if and only if $\beta \in \text{CH}_{1,\text{num}}(X)$ and $\gamma \in \text{CH}_{0,\text{num}}(X)$. In this case, since $\text{CH}_{0,\text{num}}(X) = \text{CH}_{0,\text{alg}}(X)$, we know from Theorem 5.17 that γ is smash-nilpotent. Then if β is smash-nilpotent, so is α . \square

Proof of Corollary 5.24. For $1 \leq i \leq m$, let C_i be a smooth projective curve of genus g_i over k , together with a point $x_{0,i} \in C_i(k)$. We would like to prove Conjecture 5.16 for 1-cycles on $C_1 \times \cdots \times C_m$. Denote by J_i the Jacobian of C_i . Take $n_i \geq 2g_i - 1$ and consider the product of symmetric powers $C_1^{[n_1]} \times \cdots \times C_m^{[n_m]}$. Using the points $x_{0,i}$ we have maps

$$C_1 \times \cdots \times C_m \xrightarrow{\iota} C_1^{[n_1]} \times \cdots \times C_m^{[n_m]} \twoheadrightarrow J_1 \times \cdots \times J_m.$$

We know that $C_1^{[n_1]} \times \cdots \times C_m^{[n_m]}$ can be written as an iteration of projective bundles over $J_1 \times \cdots \times J_m$. Then by Theorem 5.21 and Lemma 5.25, we obtain Conjecture 5.16 for 1-cycles on $C_1^{[n_1]} \times \cdots \times C_m^{[n_m]}$.

5.4. Application: smash-nilpotent 1-cycles on an abelian variety

Finally, the passage from $C_1^{[n_1]} \times \cdots \times C_m^{[n_m]}$ to $C_1 \times \cdots \times C_m$ is given by a correspondence

$$\Gamma \in \text{Corr}(C_1^{[n_1]} \times \cdots \times C_m^{[n_m]}, C_1 \times \cdots \times C_m)$$

satisfying $\Gamma_* \circ \iota_* = \text{id}$ (see Remark 4.2). If $\alpha \in \text{CH}_{1,\text{num}}(C_1 \times \cdots \times C_m)$, then we have $\alpha = \Gamma_*(\iota_*(\alpha))$ and that $\iota_*(\alpha) \in \text{CH}_{1,\text{num}}(C_1^{[n_1]} \times \cdots \times C_m^{[n_m]})$ is smash-nilpotent. Since smash-nilpotence is stable under correspondences, we see that α is also smash-nilpotent. \square

Further, an object $X \in \mathcal{V}_k$ is called *dominated* by $Y \in \mathcal{V}_k$ if there exists a surjective morphism $f: Y \twoheadrightarrow X$.

Corollary 5.26. *Conjecture 5.16 is true for 1-cycles on $X \in \mathcal{V}_k$ that is dominated by a product of curves.*

Proof. Let $f: Y = C_1 \times \cdots \times C_m \twoheadrightarrow X$ be a surjective map of relative dimension $n = m - \dim(X)$. Consider a divisor $l \in \text{CH}^1(Y)$ that is ample relative to f . We have $f_*(l^n) = N[X]$ for some $N > 0$. Then for $\alpha \in \text{CH}_{1,\text{num}}(X)$, by applying the projection formula we find $f_*(l^n \cdot f^*(\alpha)) = N\alpha$. We know by Corollary 5.24 that $l^n \cdot f^*(\alpha) \in \text{CH}_{1,\text{num}}(Y)$ is smash-nilpotent, and so is α . \square

We refer to Schoen's paper [Sch96] for more discussions on the set of varieties that are dominated by products of curves.

Tautological classes in higher Griffiths groups

We consider tautological classes modulo equivalences induced by S. Saito's filtration on Chow groups. For the Jacobian of a very general complex pointed curve, we detect non-trivial classes in Saito's higher Griffiths groups. This improves a result of Ikeda.

6.1. Motivation

In this chapter we work over $k = \mathbb{C}$. Let (C, x_0) be a smooth projective pointed curve of genus $g > 0$ over \mathbb{C} , with J its Jacobian. Using the embedding $\iota: C \hookrightarrow J$ with respect to x_0 , we obtain classes $[C]_{(j)} \in \mathrm{CH}_{(j)}^{g-1}(J)$ for $j \geq 0$.

We would like to study these classes (or more generally, classes in the tautological ring $\mathcal{T}(J)$) modulo other equivalences, *e.g.* algebraic equivalence. The history began with Ceresa's theorem, stating that $[C]_{(1)}$ is not algebraically equivalent to zero for a very general curve C over \mathcal{M}_g with $g \geq 3$ ([Cer83], Theorem 3.1; the statement makes sense since the class $[C]_{(j)}$ modulo algebraic equivalence is independent of the point x_0). Later Fakhruddin proved the same non-triviality statement for $[C]_{(2)}$ and $g \geq 11$ ([Fak96], Corollary 4.6), and Beauville raised the question for higher components $[C]_{(j)}$ (see [Bea04], Section 5.5).

The general picture has become clearer since Polishchuk's work on $\mathcal{T}(J)/\sim_{\mathrm{alg}}$ (see [Pol05], Introduction). In particular, we have precise expectations for the smallest $g = g(j)$ such that $[C]_{(j)} \not\sim_{\mathrm{alg}} 0$ when C is very general and of genus g (see [Moo09], Corollary 2.9). The same bound can be obtained via Colombo and van Geemen's gonality consideration (see [CG93], Theorem 1.3 and [Voi13], Conjecture 0.4).

Conjecture 6.1. *For a very general curve C over \mathcal{M}_g with $g \geq 2j + 1$, the class $[C]_{(j)}$ is not algebraically equivalent to zero.*

6. TAUTOLOGICAL CLASSES IN HIGHER GRIFFITHS GROUPS

What makes Conjecture 6.1 difficult is again the fact that $[C]_{(j)}$ is Abel-Jacobi trivial for $j \geq 2$. So one needs a more sophisticated yet computable invariant which detects cycles modulo algebraic equivalence. Recently such an invariant has been constructed by Voisin (see [Voi13], Section 1.2). She carried out computations for a family of plane curves studied by Ikeda [Ike03]. In particular, she was able to improve Fakhruddin's result for $[C]_{(2)}$ to $g \geq 6$ ([Voi13], Corollary 0.7), while the optimal bound should be $g \geq 5$ according to Conjecture 6.1.

In a series of papers ([Sai96], [Sai00], [Sai00b] and [Sai02]), S. Saito introduced a related theme. The idea is based on a closer look at the definition of algebraic equivalence: for $X \in \mathcal{V}_{\mathbb{C}}$ we have

$$(6.1) \quad \mathrm{CH}_{\mathrm{alg}}^i(X) = \sum_{Y, \Gamma} \mathrm{Im}(\Gamma_* : \mathrm{CH}_{0, \mathrm{hom}}(Y) \rightarrow \mathrm{CH}^i(X)),$$

where the sum ranges over all $Y \in \mathcal{V}_{\mathbb{C}}$ and all $\Gamma \in \mathrm{CH}^i(Y \times X)$. A key observation is that $\mathrm{CH}_{0, \mathrm{hom}}(Y)$ represents the first term of a filtration $F^\bullet \mathrm{CH}_0(Y)$ conjectured by Bloch-Beilinson-Murre (BBM). More generally, the BBM conjecture predicts for all $X \in \mathcal{V}_{\mathbb{C}}$ a descending filtration $F^\bullet \mathrm{CH}^i(X)$ on $\mathrm{CH}^i(X)$, called the Bloch-Beilinson (BB) filtration. It should be preserved by correspondences, satisfy $F^0 \mathrm{CH}^i(X) = \mathrm{CH}^i(X)$ and $F^{i+1} \mathrm{CH}^i(X) = 0$, and for all $0 \leq j \leq i$ the graded piece

$$\mathrm{Gr}_F^j \mathrm{CH}^i(X) = F^j \mathrm{CH}^i(X) / F^{j+1} \mathrm{CH}^i(X)$$

should in some sense be controlled by the cohomology group $H^{2i-j}(X, \mathbb{Q})$ (see [Mur93], [Jann94] and [Voi02], Chapitre 23 for more details). Further, one expects $F^1 \mathrm{CH}^i(X) = \mathrm{CH}_{\mathrm{hom}}^i(X)$. It then follows that we can rewrite (6.1) as

$$\mathrm{CH}_{\mathrm{alg}}^i(X) = \sum_{Y, \Gamma} \mathrm{Im}(\Gamma_* : F^1 \mathrm{CH}_0(Y) \rightarrow \mathrm{CH}^i(X)).$$

In theory, if we replace $F^1 \mathrm{CH}_0(Y)$ by deeper terms $F^j \mathrm{CH}_0(Y)$ of the BB filtration, we obtain a family of equivalences that refine algebraic equivalence.

In practice we use Saito's candidate of the BB filtration to make statements unconditional. From now on $F^\bullet \mathrm{CH}^i(X)$ stands for Saito's filtration on $\mathrm{CH}^i(X)$. The precise definition and properties of $F^\bullet \mathrm{CH}^i(X)$ will be given in Section 6.2. Define

$$Z_0 F^j \mathrm{CH}^i(X) = \sum_{Y, \Gamma} \mathrm{Im}(\Gamma_* : F^j \mathrm{CH}_0(Y) \rightarrow \mathrm{CH}^i(X)),$$

where the sum is taken in the same way as (6.1). Then we have $Z_0 F^j \mathrm{CH}^i(X) \subset F^j \mathrm{CH}^i(X)$, and $Z_0 F^1 \mathrm{CH}^i(X) = \mathrm{CH}_{\mathrm{alg}}^i(X)$. As mentioned, each of the terms $Z_0 F^j$ can be interpreted as a refined algebraic equivalence.

Also recall the Griffiths group $\mathrm{Griff}^i(X) = \mathrm{CH}_{\mathrm{hom}}^i(X) / \mathrm{CH}_{\mathrm{alg}}^i(X)$. In the context of the filtration $F^\bullet \mathrm{CH}^i(X)$, Saito introduced the higher Griffiths group

$$\mathrm{Griff}^{i,j}(X) = F^j \mathrm{CH}^i(X) / (Z_0 F^j \mathrm{CH}^i(X) + F^{j+1} \mathrm{CH}^i(X)).$$

Note that $\text{Griff}^{i,1}(X)$ is the quotient of $\text{Griff}^i(X)$ by $F^2 \text{CH}^i(X)$.

Similar to Conjecture 6.1, one may now ask the question of detecting non-trivial classes in higher Griffiths groups. The question was partially answered by Ikeda, who proved non-triviality for a specific family of plane curves ([Ike03], Theorem 1). In this chapter we improve Ikeda's result in the very general case (modulo a slight difference in the definition of Saito's filtration), with sharp bounds on the genus. To give an example, we can prove the following statement (see Theorem 6.13 for the full result).

Theorem 6.2. *For a very general pointed curve (C, x_0) over $\mathcal{M}_{g,1}$ with $g \geq j + 2$ and $j \geq 1$, the class $[C]_{(j)}$ is not zero in $\text{Griff}^{g-1,j}(X)$.*

A key ingredient in the proof is that the Mumford-Tate group (or more precisely the Hodge group) of a very general curve is the full symplectic group. Using this fact, we find a simple invariant for the higher Griffiths groups of a very general Jacobian. The invariant is easily computable via Fakhruddin's degeneration argument (already seen in the proof of Theorem 5.4).

6.2. Saito's filtration and higher Griffiths groups

In this section we briefly review (a version of) Saito's filtration on Chow groups, and discuss its basic properties. The filtration induces a family of equivalence relations, for which one can define the notion of higher Griffiths groups.

We begin with the definition in the relative setting. Let S be a smooth connected variety over \mathbb{C} , and let $\pi: X \rightarrow S$ be an object in \mathcal{V}_S .

Definition 6.3. We define a descending filtration $F_S^\bullet \text{CH}^i(X)$, indexed by the non-negative integers, in the following inductive way.

- (i) We set $F_S^0 \text{CH}^i(X) = \text{CH}^i(X)$ for all $i \geq 0$.
- (ii) Suppose that $F_S^j \text{CH}^{i'}(Y)$ is defined for every Y/S in \mathcal{V}_S and every $i' \geq 0$. Then we let

$$F_S^{j+1} \text{CH}^i(X) := \sum_{Y, \Gamma} \text{Im}(\Gamma_*: F_S^j \text{CH}^{i'}(Y) \rightarrow \text{CH}^i(X)),$$

where the sum ranges over all $\phi: Y \rightarrow S$ in \mathcal{V}_S and all correspondences $\Gamma \in \text{Corr}_S^{i-i'}(Y, X)$, with the condition that

$$(6.2) \quad \Gamma_*: R^{2i'-j} \phi_* \mathbb{Q} \rightarrow R^{2i-j} \pi_* \mathbb{Q}$$

is the zero map.

6. TAUTOLOGICAL CLASSES IN HIGHER GRIFFITHS GROUPS

Note that when $S = \mathbb{C}$, the map (6.2) becomes $\Gamma_*: H^{2i'-j}(Y, \mathbb{Q}) \rightarrow H^{2i-j}(X, \mathbb{Q})$. Also in this case we drop the subscript S and write simply $F^\bullet \mathrm{CH}^i(X)$.

Remark 6.4. When $S = \mathbb{C}$, the filtration above is denoted by $F_B^\bullet \mathrm{CH}^i(X)$ in [Sai00], Section 1. It differs slightly from the one adopted by Ikeda [Ike03], which is denoted by $F_H^\bullet \mathrm{CH}^i(X)$ in [Sai96], Section 9. Both filtrations are candidates of the Bloch-Beilinson filtration, and conjecturally^{note 8} they coincide. Here our choice of filtration is based on the compatibility with the (classical) Leray filtration in cohomology (see Lemma 6.6).

We list a few properties of $F_S^\bullet \mathrm{CH}^i(X)$, which can be found in [Sai02], Sections 0 and 2.

Proposition 6.5.

- (i) For $j \geq 0$, the terms F_S^j are preserved by correspondences. In particular we have $F_S^{j+1} \mathrm{CH}^i(X) \subset F_S^j \mathrm{CH}^i(X)$, meaning that $F_S^\bullet \mathrm{CH}^i(X)$ is indeed a descending filtration.
- (ii) The filtration $F_S^\bullet \mathrm{CH}^i(X)$ is also preserved under base change: if $f: S' \rightarrow S$ is a morphism of smooth connected varieties over \mathbb{C} , then for all $i, j \geq 0$ we have

$$f^*(F_S^j \mathrm{CH}^i(X)) \subset F_{S'}^j \mathrm{CH}^i(X \times_S S').$$

- (iii) Let $\alpha \in \mathrm{CH}^i(X)$. For $s \in S$, denote by α_s the restriction of α to the fiber X_s . By (ii) we know that $\alpha \in F_S^j \mathrm{CH}^i(X)$ implies $\alpha_s \in F^j \mathrm{CH}^i(X_s)$ for all $s \in S$. Conversely, if $\alpha_s \in F^j \mathrm{CH}^i(X_s)$ for all $s \in S$ (or equivalently^{note 9}, for a very general $s \in S$), then there exists a non-empty open subset $U \subset S$ such that $\alpha_U \in F_U^j \mathrm{CH}^i(X_U)$ (here $X_U = X \times_S U$ and $\alpha_U = \alpha|_{X_U}$).
- (iv) When $S = \mathbb{C}$, the first term $F^1 \mathrm{CH}^i(X)$ is $\mathrm{CH}_{\mathrm{hom}}^i(X)$. The second term $F^2 \mathrm{CH}^i(X)$ is contained in the kernel of the Abel-Jacobi map $\mathrm{aj}: \mathrm{CH}_{\mathrm{hom}}^i(X) \rightarrow J^i(X) \otimes_{\mathbb{Z}} \mathbb{Q}$, and conjecturally they coincide. Further, we have

$$F^2 \mathrm{CH}^i(X) \cap \mathrm{CH}_{\mathrm{alg}}^i(X) = \mathrm{Ker}(\mathrm{aj}) \cap \mathrm{CH}_{\mathrm{alg}}^i(X).$$

In particular, we know that $F^2 \mathrm{CH}_0(X) = \mathrm{Ker}(\mathrm{alb}: \mathrm{CH}_{0,\mathrm{hom}}(X) \rightarrow \mathrm{Alb}(X) \times_{\mathbb{Z}} \mathbb{Q})$, where $\mathrm{CH}_0(X)$ is the Chow group of 0-cycles on X and alb is the Albanese map. \square

Recall that in cohomology, the Leray spectral sequence

$$E_2^{p,q} = H^p(S, R^q \pi_* \mathbb{Q}) \Rightarrow H^{p+q}(X, \mathbb{Q})$$

degenerates at E_2 ([Del68], Proposition 2.1). Denote by $L_S^\bullet H^{p+q}(X, \mathbb{Q})$ the Leray filtration on $H^{p+q}(X, \mathbb{Q})$ relative to S . Then we have

$$H^p(S, R^q \pi_* \mathbb{Q}) \simeq \mathrm{Gr}_{L_S}^p H^{p+q}(X, \mathbb{Q}).$$

6.2. Saito's filtration and higher Griffiths groups

Also recall the cycle class map $\text{cl}: \text{CH}^i(X) \rightarrow H^{2i}(X, \mathbb{Q})$ (for simplicity we omit the Tate twists). The following compatibility property is essential in the discussion.

Lemma 6.6. *The cycle class map cl preserves the filtrations on both sides, i.e. we have for all $i, j \geq 0$*

$$\text{cl}(F_S^j \text{CH}^i(X)) \subset L_S^j H^{2i}(X, \mathbb{Q}).$$

Proof. We proceed by induction on j . The statement is trivial for $j = 0$. Suppose for all $Y/S \in \mathcal{V}_S$ and $i' \geq 0$ we have

$$\text{cl}(F_S^j \text{CH}^{i'}(Y)) \subset L_S^j H^{2i'}(Y, \mathbb{Q}).$$

Let $\alpha \in F_S^{j+1} \text{CH}^i(X)$. By Definition 6.3 we may assume $\alpha = \Gamma_*(\beta)$, with $\beta \in F_S^j \text{CH}^{i'}(Y)$ for some $\phi: Y \rightarrow S$ in \mathcal{V}_S , and $\Gamma \in \text{Corr}^{i-i'}(Y, X)$ such that

$$\Gamma_*: R^{2i'-j} \phi_* \mathbb{Q} \rightarrow R^{2i-j} \pi_* \mathbb{Q}$$

is the zero map. By the induction assumption, we have the following commutative diagram.

$$\begin{array}{ccc} F_S^j \text{CH}^{i'}(Y) & \xrightarrow{\Gamma_*} & F_S^{j+1} \text{CH}^i(X) \\ \text{cl} \downarrow & & \downarrow \text{cl} \\ L_S^j H^{2i'}(Y, \mathbb{Q}) & \xrightarrow{\Gamma_*} & L_S^j H^{2i}(X, \mathbb{Q}) \\ \text{pr} \downarrow & & \downarrow \text{pr} \\ \text{Gr}_{L_S}^j H^{2i'}(Y, \mathbb{Q}) & \xrightarrow{\Gamma_*} & \text{Gr}_{L_S}^j H^{2i}(X, \mathbb{Q}) \\ \simeq \downarrow & & \downarrow \simeq \\ H^j(S, R^{2i'-j} \phi_* \mathbb{Q}) & \xrightarrow{\Gamma_*} & H^j(S, R^{2i-j} \pi_* \mathbb{Q}) \end{array}$$

Here the arrows pr are projections to the graded pieces. The condition on Γ implies that the last horizontal arrow is zero, so that

$$\text{pr}(\text{cl}(\alpha)) = \text{pr}(\text{cl}(\Gamma_*(\beta))) = \Gamma_*(\text{pr}(\text{cl}(\beta))) = 0 \in \text{Gr}_{L_S}^j H^{2i}(X, \mathbb{Q}).$$

Hence $\text{cl}(\alpha) \in L_S^{j+1} H^{2i}(X, \mathbb{Q})$. □

We are mostly interested in the case of abelian schemes. Let A/S be a principally polarized abelian scheme over S (again the principal polarization is not important here). The Beauville decomposition provides another filtration on $\text{CH}^i(A)$, namely

$$\tilde{F}_S^j \text{CH}^i(A) := \bigoplus_{j' \geq j} \text{CH}_{(j')}^i(A).$$

6. TAUTOLOGICAL CLASSES IN HIGHER GRIFFITHS GROUPS

At least when $S = \mathbb{C}$, the two filtrations $F^j \text{CH}^i(A)$ and $\tilde{F}^j \text{CH}^i(A)$ are believed to coincide (since they are both candidates of the Bloch-Beilinson filtration). Ikeda proved the following inclusion ([Ike03], Lemma 3.1).

Lemma 6.7. *For all $i, j \geq 0$, we have*

$$(6.3) \quad \bigoplus_{j' \geq j} \text{CH}_{(j')}^i(A) \subset F_S^j \text{CH}^i(A). \quad \square$$

Induced equivalence relations

Using the filtration $F_S^\bullet \text{CH}^i(X)$, we define a family of equivalences on $\text{CH}^i(X)$ which in some sense generalize algebraic equivalence. For simplicity we restrict to the case $S = \mathbb{C}$, and we refer to [Sai02], Section 5 for the definition in the relative setting. If X is an object in $\mathcal{V}_{\mathbb{C}}$, denote by $\text{CH}_r(X)$ the Chow group of dimension r cycles on X .

Definition 6.8. For $i, j, r \geq 0$, define subgroups of $F^j \text{CH}^i(X)$

$$Z_r F^j \text{CH}^i(X) := \sum_{Y, \Gamma} \text{Im}(\Gamma_* : F^j \text{CH}_r(Y) \rightarrow \text{CH}^i(X)),$$

where the sum ranges over all $Y \in \mathcal{V}_{\mathbb{C}}$ and all correspondences $\Gamma \in \text{CH}^{i+r}(Y \times X)$.

In particular, we have $Z_0 F^1 \text{CH}^i(X) = \text{CH}_{\text{alg}}^i(X)$. Similar to algebraic equivalence, one may regard $Z_r F^j \text{CH}^i(X)$ as an equivalence on $\text{CH}^i(X)$ by setting for $\alpha, \beta \in \text{CH}^i(X)$

$$\alpha \sim_{Z_r F^j} \beta \text{ if and only if } \alpha - \beta \in Z_r F^j \text{CH}^i(X).$$

The terms $Z_r F^j$ are preserved by correspondences. Also we have by definition

$$Z_r F^{j+1} \text{CH}^i(X) \subset Z_r F^j \text{CH}^i(X),$$

so $Z_r F^\bullet \text{CH}^i(X)$ is a descending filtration. Moreover, to every pair $\beta \in F^j \text{CH}_r(Y)$ and $\Gamma \in \text{CH}^{i+r}(Y \times X)$ one can associate $\beta \times \mathbb{P}^1 \in F^j \text{CH}_{r+1}(Y \times \mathbb{P}^1)$ and $\Gamma \times \text{pt} \in \text{CH}^{i+r+1}((Y \times \mathbb{P}^1) \times X)$. This implies

$$Z_r F^j \text{CH}^i(X) \subset Z_{r+1} F^j \text{CH}^i(X),$$

meaning that $Z_\bullet F^j \text{CH}^i(X)$ is an ascending filtration. Finally if X is connected and of dimension n , we have

$$(6.4) \quad Z_{n-i} F^j \text{CH}^i(X) = F^j \text{CH}^i(X).$$

6.3. Detecting non-trivial classes in higher Griffiths groups

To summarize, we have the following diagram of filtrations.

$$\begin{array}{ccccccc}
\mathrm{CH}^i(X) \supset Z_0 F^1 \mathrm{CH}^i(X) \supset Z_0 F^2 \mathrm{CH}^i(X) \supset \cdots \supset Z_0 F^j \mathrm{CH}^i(X) \supset \cdots \\
\parallel \qquad \qquad \cap \qquad \qquad \cap \qquad \qquad \cap \\
\mathrm{CH}^i(X) \supset Z_1 F^1 \mathrm{CH}^i(X) \supset Z_1 F^2 \mathrm{CH}^i(X) \supset \cdots \supset Z_1 F^j \mathrm{CH}^i(X) \supset \cdots \\
\parallel \qquad \qquad \cap \qquad \qquad \cap \qquad \qquad \cap \\
\mathrm{CH}^i(X) \supset Z_2 F^1 \mathrm{CH}^i(X) \supset Z_2 F^2 \mathrm{CH}^i(X) \supset \cdots \supset Z_2 F^j \mathrm{CH}^i(X) \supset \cdots \\
\parallel \qquad \qquad \cap \qquad \qquad \cap \qquad \qquad \cap \\
\vdots \qquad \qquad \vdots \qquad \qquad \vdots \qquad \qquad \ddots \qquad \qquad \vdots \qquad \qquad \ddots \\
\parallel \qquad \qquad \cap \qquad \qquad \cap \qquad \qquad \cap \\
\mathrm{CH}^i(X) \supset Z_r F^1 \mathrm{CH}^i(X) \supset Z_r F^2 \mathrm{CH}^i(X) \supset \cdots \supset Z_r F^j \mathrm{CH}^i(X) \supset \cdots \\
\parallel \qquad \qquad \cap \qquad \qquad \cap \qquad \qquad \cap \\
\vdots \qquad \qquad \vdots \qquad \qquad \vdots \qquad \qquad \ddots \qquad \qquad \vdots \qquad \qquad \ddots \\
\parallel \qquad \qquad \cap \qquad \qquad \cap \qquad \qquad \cap \\
\mathrm{CH}^i(X) \supset Z_{n-i} F^1 \mathrm{CH}^i(X) \supset Z_{n-i} F^2 \mathrm{CH}^i(X) \supset \cdots \supset Z_{n-i} F^j \mathrm{CH}^i(X) \supset \cdots \\
\parallel \qquad \qquad \parallel \qquad \qquad \parallel \qquad \qquad \parallel \\
\mathrm{CH}^i(X) \supset F^1 \mathrm{CH}^i(X) \supset F^2 \mathrm{CH}^i(X) \supset \cdots \supset F^j \mathrm{CH}^i(X) \supset \cdots
\end{array}$$

Here the column $Z_\bullet F^1 \mathrm{CH}^i(X)$ is known as Nori's filtration (see [Nor93], Section 5), with one end $Z_0 F^1 \mathrm{CH}^i(X) = \mathrm{CH}_{\mathrm{alg}}^i(X)$ and the other $Z_{n-i} F^1 \mathrm{CH}^i(X) = F^1 \mathrm{CH}^i(X) = \mathrm{CH}_{\mathrm{hom}}^i(X)$.

Definition 6.9. For $i, j \geq 0$, define the *higher Griffiths groups*

$$\mathrm{Griff}^{i,j}(X) := F^j \mathrm{CH}^i(X) / (Z_0 F^j \mathrm{CH}^i(X) + F^{j+1} \mathrm{CH}^i(X)).$$

Note that $\mathrm{Griff}^{i,1}(X)$ is not the classical Griffiths group $\mathrm{Griff}^i(X) = \mathrm{CH}_{\mathrm{hom}}^i(X) / \mathrm{CH}_{\mathrm{alg}}^i(X)$, but its quotient by $F^2 \mathrm{CH}^i(X)$.

6.3. Detecting non-trivial classes in higher Griffiths groups

We work in the setting of the diagram (2.15), with $S = \mathcal{M}_{g,1}$ over \mathbb{C} . Let $s \in \mathcal{M}_{g,1}$ be a very general point, *i.e.* a point outside a countable union of Zariski-closed proper subsets. Denote by $(C_s, x_{0,s})$ the pointed curve over s , and by J_s the Jacobian of C_s . We refer to Section 5.1 for the definition of the tautological ring $\mathcal{T}(J_s)$.

Our goal is to detect non-trivial tautological classes (*e.g.* $[C_s]_{(j)} \in \mathcal{T}_{(j)}^{g-1}(J_s)$) in the higher Griffiths groups $\mathrm{Griff}^{i,j}(J_s)$. To begin with, we introduce an invariant for $\mathrm{Griff}^{i,j}(J_s)$, which can be viewed as an analogue of the one used in Proposition 5.6 (for detecting classes modulo rational equivalence).

6. TAUTOLOGICAL CLASSES IN HIGHER GRIFFITHS GROUPS

Consider the universal Jacobian $\pi: J \rightarrow \mathcal{M}_{g,1}$ and the cycle class map

$$\text{cl}: \text{CH}_{(j)}^i(J) \rightarrow H^j(\mathcal{M}_{g,1}, R^{2i-j}\pi_*\mathbb{Q}).$$

Recall from Corollary 2.17 that there is the Lefschetz decomposition

$$R^{2i-j}\pi_*\mathbb{Q} = \bigoplus_{k=\max\{0, 2i-j-g\}}^{\lfloor (2i-j)/2 \rfloor} (R\pi_*(u))^k R_{\text{prim}}^{2i-j-2k}\pi_*\mathbb{Q},$$

where $u = \text{cl}(-\theta) \in H^2(J, \mathbb{Q})$ and $R\pi_*(u): R^i\pi_*\mathbb{Q} \rightarrow R^{i+2}\pi_*\mathbb{Q}$ is the induced map. For simplicity, when $2i-j > g$ we write $R_{\text{prim}}^{2i-j}\pi_*\mathbb{Q}$ for the sheaf

$$(R\pi_*(u))^k R_{\text{prim}}^{2i-j-2k}\pi_*\mathbb{Q} \text{ with } k = 2i-j-g.$$

Proposition 6.10. *Let g, i, j, r be non-negative integers satisfying*

$$(6.5) \quad g > i + r, \text{ and } i > j + r.$$

Consider a class $\alpha \in \text{CH}_{(j)}^i(J)$, with $\pi: J \rightarrow \mathcal{M}_{g,1}$ the universal Jacobian. Suppose for a very general point $s \in \mathcal{M}_{g,1}$, the restriction α_s lies in

$$Z_r F^j \text{CH}^i(J_s) + F^{j+1} \text{CH}^i(J_s).$$

Then there exists a non-empty open subset $U \subset \mathcal{M}_{g,1}$, such that the component of $\text{cl}(\alpha_U)$ in

$$H^j(U, R_{\text{prim}}^{2i-j}\pi_*\mathbb{Q})$$

is zero (here $\alpha_U = \alpha|_{J_U}$ with $J_U = J \times_{\mathcal{M}_{g,1}} U$).

Remark 6.11. The two numerical conditions in (6.5) are necessary. If either of the two is not satisfied, then the assumption on the class α_s always holds (while the conclusion of Proposition 6.10 may not be true). In fact, we know that α_s lies in $\text{CH}_{(j)}^i(J_s)$. If $g \leq i + r$, by (6.4) and (6.3) we have

$$\text{CH}_{(j)}^i(J_s) \subset F^j \text{CH}^i(J_s) = Z_r F^j \text{CH}^i(J_s).$$

Similarly if $i \leq j + r$, by applying the Fourier transform \mathcal{F} we find

$$\text{CH}_{(j)}^i(J_s) = \mathcal{F}(\text{CH}_{(j)}^{g-i+j}(J_s)) \subset Z_r F^j \text{CH}^i(J_s).$$

Proof of Proposition 6.10. Write $\alpha_s = \alpha_{1,s} + \alpha_{2,s}$, with $\alpha_{1,s} \in Z_r F^j \text{CH}^i(J_s)$ and $\alpha_{2,s} \in F^{j+1} \text{CH}^i(J_s)$. Using the fact that both $Z_r F^j \text{CH}^i(J_s)$ and $F^{j+1} \text{CH}^i(J_s)$ are stable under correspondences, we may assume $\alpha_{1,s}, \alpha_{2,s} \in \text{CH}_{(j)}^i(J_s)$ by taking Beauville components. Since s is very general, by the spreading

6.3. Detecting non-trivial classes in higher Griffiths groups

out procedure we can find a non-empty open subset $U \subset \mathcal{M}_{g,1}$, and classes $\alpha_{1,U}, \alpha_{2,U} \in \text{CH}^i(J_U)$ whose restrictions over s are $\alpha_{1,s}$ and $\alpha_{2,s}$ respectively. Again by taking Beauville components, we may assume $\alpha_{1,U}, \alpha_{2,U} \in \text{CH}_{(j)}^i(J_U)$. This means we are reduced to the proof of two separate cases $\alpha_s \in Z_r F^j \text{CH}^i(J_s)$ or $\alpha_s \in F^{j+1} \text{CH}^i(J_s)$.

If $\alpha_s \in F^{j+1} \text{CH}^i(J_s)$, by Proposition 6.5 (iii) and Lemma 6.6 there exists a non-empty open subset $U \subset \mathcal{M}_{g,1}$ such that $\text{cl}(\alpha_U) = 0$. In particular, the primitive component of $\text{cl}(\alpha_U)$ is zero.

Suppose $\alpha_s \in Z_r F^j \text{CH}^i(J_s)$. By Definition 6.8, we may assume $\alpha_s = \Gamma_{s,*}(\beta_s)$, with $\beta_s \in F^j \text{CH}_r(Y_s)$ for some $Y_s \in \mathcal{V}_{\mathbb{C}}$, and $\Gamma_s \in \text{CH}^{i+r}(Y_s \times J_s)$. Without loss of generality, we take Y_s to be connected and of dimension n . Now since s is very general, by the spreading out procedure we may also assume there exists a non-empty open subset $U \subset \mathcal{M}_{g,1}$ containing s , such that Y_s is the fiber over s of some $\phi: Y_U \rightarrow U$ in \mathcal{V}_U (with connected fibers), that β_s (resp. Γ_s) is the restriction over s of some $\beta_U \in \text{CH}^{n-r}(Y_U)$ (resp. $\Gamma_U \in \text{CH}^{i+r}(Y_U \times_U J_U)$), and that $\alpha_U = \Gamma_{U,*}(\beta_U)$. Further, we know from Proposition 6.5 (iii) that by possibly shrinking U , we can take $\beta_U \in F_U^j \text{CH}^{n-r}(Y_U)$.

The cycle class $\text{cl}(\alpha_U)$ lies in $H^j(U, R^{2i-j} \pi_* \mathbb{Q})$, which is canonically isomorphic to

$$\text{Gr}_{L_U}^j H^{2i}(J_U, \mathbb{Q}) = L_U^j H^{2i}(J_U, \mathbb{Q}) / L_U^{j+1} H^{2i}(J_U, \mathbb{Q}).$$

On the other hand, by Lemma 6.6 we have $\text{cl}(\beta_U) \in L_U^j H^{2n-2r}(Y_U, \mathbb{Q})$. Then since the Leray filtration satisfies $L_U^p \cdot L_U^q \subset L_U^{p+q}$, we see that $\text{cl}(\alpha_U)$ depends only on the class of $\text{cl}(\Gamma_U)$ in

$$\text{Gr}_{L_U}^0 H^{2i+2r}(Y_U \times_U J_U, \mathbb{Q}) = L_U^0 H^{2i+2r}(Y_U \times_U J_U, \mathbb{Q}) / L_U^1 H^{2i+2r}(Y_U \times_U J_U, \mathbb{Q}),$$

which is isomorphic to

$$(6.6) \quad H^0(U, R^{2i+2r}(\phi \times_U \pi)_* \mathbb{Q}).$$

Recall the Künneth formula

$$R^{2i+2r}(\phi \times_U \pi)_* \mathbb{Q} = \bigoplus_{p+q=2i+2r} R^p \phi_* \mathbb{Q} \otimes R^q \pi_* \mathbb{Q}.$$

Consider $Y_U \times_U J_U$ as an abelian scheme over Y_U , endowed with the action of $[N]$, *i.e.* the multiplication by N (for $N \in \mathbb{Z}$). By comparing the action of $[N]^*$, we find that the only relevant component in (6.6) (for obtaining $\text{cl}(\alpha_U)$) is

$$(6.7) \quad H^0(U, R^{2r+j} \phi_* \mathbb{Q} \otimes R^{2i-j} \pi_* \mathbb{Q}).$$

By the theorem of the fixed part, the latter is further isomorphic to

$$(6.8) \quad (H^{2r+j}(Y_s, \mathbb{Q}) \otimes H^{2i-j}(J_s, \mathbb{Q}))^{\pi_1(U,s)}.$$

Denote by h the corresponding class of $\text{cl}(\Gamma_U)$ in (6.8). Then h is a Hodge class of type $(i+r, i+r)$. We now apply the two numerical conditions in (6.5), and we distinguish two cases.

6. TAUTOLOGICAL CLASSES IN HIGHER GRIFFITHS GROUPS

(i) Case $2i - j \leq g$. We have a Hodge decomposition

$$H^{2i-j}(J_s, \mathbb{C}) = H^{2i-j,0}(J_s) \oplus H^{2i-j-1,1}(J_s) \oplus \cdots \oplus H^{1,2i-j-1}(J_s) \oplus H^{0,2i-j}(J_s),$$

with $H^{2i-j,0}(J_s) \neq 0$. The numerical conditions force $g \geq 2i - j > i + r$.

(ii) Case $2i - j \geq g$. This time the Hodge decomposition becomes

$$H^{2i-j}(J_s, \mathbb{C}) = H^{g,2i-j-g}(J_s) \oplus H^{g-1,2i-j-g+1}(J_s) \oplus \cdots \oplus H^{2i-j-g+1,g-1}(J_s) \oplus H^{2i-j-g,g}(J_s),$$

with $H^{g,2i-j-g}(J_s) \neq 0$. The numerical conditions force $2i - j \geq g > i + r$.

To summarize, the numerical conditions ensure that h does not reach the maximal Hodge level in the factor $H^{2i-j}(J_s, \mathbb{C})$. In other words, if we denote by \mathbb{V} the largest sub-variation of Hodge structures (VHS) of $R^{2i-j}\pi_*\mathbb{Q}$ of Hodge coniveau at least 1, the corresponding class of $\text{cl}(\Gamma_U)$ in (6.7) lies in

$$\text{Im}(H^0(U, R^{2r+j}\phi_*\mathbb{Q} \otimes \mathbb{V}) \rightarrow H^0(U, R^{2r+j}\phi_*\mathbb{Q} \otimes R^{2i-j}\pi_*\mathbb{Q})).$$

As a result, we have

$$(6.9) \quad \text{cl}(\alpha_U) \in \text{Im}(H^j(U, \mathbb{V}) \rightarrow H^j(U, R^{2i-j}\pi_*\mathbb{Q})).$$

Now the crucial step is an argument using Mumford-Tate groups. We refer to [Moo99] for the general theory. As before $s \in U$ is a very general point. Write $H := H^1(J_s, \mathbb{Q})$. It is well known that the Hodge group (also called special Mumford-Tate group) of H is the full symplectic group $\text{Sp}(H)$ (for example, we know that the mapping class group is surjective to the symplectic group). Then by the classical representation theory of symplectic groups (see [FulH91], Section 17.2), the fact that

$$\mathbb{V}_s \subset H^{2i-j}(J_s, \mathbb{Q}) = \wedge^{2i-j}(H)$$

is not of maximal Hodge level implies that \mathbb{V}_s is included in the non-primitive part of $H^{2i-j}(J_s, \mathbb{Q})$. It follows that \mathbb{V} and $R_{\text{prim}}^{2i-j}\pi_*\mathbb{Q}$ are, as VHS's and also as local systems, direct summands of $R^{2i-j}\pi_*\mathbb{Q}$ with trivial intersection (in fact we have $R^{2i-j}\pi_*\mathbb{Q} = R_{\text{prim}}^{2i-j}\pi_*\mathbb{Q} \oplus \mathbb{V}$ by the definition of \mathbb{V}). By (6.9), we conclude that the component of $\text{cl}(\alpha_U)$ in $H^j(U, R_{\text{prim}}^{2i-j}\pi_*\mathbb{Q})$ is zero. \square

Remark 6.12. The above proof only requires that the Hodge group of a very general fiber is the full symplectic group. So whenever this condition is satisfied, Proposition 6.10 stays valid. This is the case even over the hyperelliptic locus (see [AC79], Théorème 1).

With the invariant in hand, we detect non-trivial tautological classes in higher Griffiths groups. To be coherent with the literature, we state the result in terms of the curve class $[C] \in \text{CH}^{g-1}(J)$ and the Pontryagin product $(*)$. Let $m \geq 1$ and $j_1, \dots, j_m \geq 1$. Write $j = j_1 + \cdots + j_m$, and consider

$$([C]^{*m})_{(j_1, \dots, j_m)} := [C]_{(j_1)} * \cdots * [C]_{(j_m)} \in \text{CH}_{(j)}^{g-m}(J).$$

6.3. Detecting non-trivial classes in higher Griffiths groups

Let $r \geq 0$. The two numerical conditions in (6.5) for a class in $\mathrm{CH}_{(j)}^{g-m}(J)$ read

$$(6.10) \quad m \geq r + 1, \text{ and } g \geq m + j + r + 1.$$

Theorem 6.13. *Let $g, m, j_1, \dots, j_m, j, r$ be as above satisfying the conditions in (6.10). Then if $s \in \mathcal{M}_{g,1}$ is a very general point, the class $([C_s]^{*m})_{(j_1, \dots, j_m)}$ does not lie in*

$$Z_r F^j \mathrm{CH}^{g-m}(J_s) + F^{j+1} \mathrm{CH}^{g-m}(J_s).$$

In particular for $r = 0$ and $g \geq m + j + 1$, we have

$$([C_s]^{*m})_{(j_1, \dots, j_m)} \neq 0 \text{ in } \mathrm{Griff}^{g-m, j}(J_s).$$

It follows from Remark 6.11 that for $m \geq r+1$ fixed, the bound on the genus g is sharp. The proof of Theorem 6.13 is similar to those of Theorem 5.4 and Proposition 5.14: according to Lemma 5.7 and Proposition 6.10, it suffices to construct a family of test curves over the boundary of $\mathcal{M}_{g,1}^{\mathrm{ct}}$, and to show that the relevant primitive cohomology class does not vanish over any non-empty open subset of the base variety.

Proof of Theorem 6.13. For simplicity, we only prove the case $g = m + j + r + 1$: the general case follows either from the same construction (see Figure 9) with more constant components, or from Ceresa's degeneration argument (see [Cer83], Section 3).

To construct a family of test curves for $g = m + j + r + 1$, we need the following ingredients:

- (i) a collection of m complex smooth curves C_1, \dots, C_m of genus 2, with varying points $x_i \in C_i$ for $1 \leq i \leq m$, and fixed points $c_i \in C_i$ for $2 \leq i \leq m$;
- (ii) a collection of $j - m$ complex elliptic curves E_1, \dots, E_{j-m} (by definition $j \geq m$), with varying points $y_i \in E_i$ and zeros $o_i \in E_i$, for $1 \leq i \leq j - m$;
- (iii) another collection of $r + 1$ complex elliptic curves E'_1, \dots, E'_{r+1} , with fixed points $b_i \in E'_i$ and zeros $o'_i \in E'_i$ such that $b_i \neq o'_i$, for $1 \leq i \leq r + 1$.

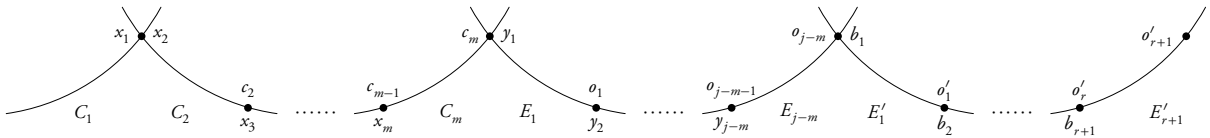


Figure 9. A family of test curves ($g = m + j + r + 1$).

As is shown in Figure 9, the family is obtained by joining x_1 with x_2 , c_i with x_{i+1} (for $2 \leq i \leq m-1$), c_m with y_1 , o_i with y_{i+1} (for $1 \leq i \leq j-m-1$), o_{j-m} with b_1 , and o'_i with b_{i+1} (for $1 \leq i \leq r$).

6. TAUTOLOGICAL CLASSES IN HIGHER GRIFFITHS GROUPS

Finally we let o'_{r+1} serve as the marked point. There are some extreme cases: we join x_1 with y_1 if $j > m = 1$, c_m with b_1 if $j = m > 1$, and x_1 with b_1 if $j = m = 1$. We verify that the genus of the resulting curves is $2m + (j - m) + (r + 1) = m + j + r + 1$.

Write $\mathcal{C} \rightarrow T$ for this family, where

$$T = C_1 \times (C_2 \setminus \{c_2\}) \times \cdots \times (C_m \setminus \{c_m\}) \times (E_1 \setminus \{o_1\}) \times \cdots \times (E_{j-m} \setminus \{o_{j-m}\}).$$

For $1 \leq i \leq m$, denote by (J_i, θ_i) the Jacobian of C_i . Then the relative Jacobian of $\mathcal{C} \rightarrow T$ is

$$\mathcal{J} = J_1 \times \cdots \times J_m \times E_1 \times \cdots \times E_{j-m} \times E'_1 \times \cdots \times E'_{r+1} \times T$$

over the last factor T . An important observation is that the embedding $\mathcal{C} \hookrightarrow \mathcal{J}$ with respect to o'_{r+1} can be extended over

$$\overline{T} = C_1 \times \cdots \times C_m \times E_1 \times \cdots \times E_{j-m}.$$

More precisely, write $\overline{\mathcal{J}} = J_1 \times \cdots \times J_m \times E_1 \times \cdots \times E_{j-m} \times E'_1 \times \cdots \times E'_{r+1} \times \overline{T}$. We have embeddings

$$\begin{aligned} \psi_{C_i} : C_i \times \overline{T} &\hookrightarrow \overline{\mathcal{J}}, \text{ for } 1 \leq i \leq m, \\ \psi_{E_i} : E_i \times \overline{T} &\hookrightarrow \overline{\mathcal{J}}, \text{ for } 1 \leq i \leq j - m, \\ \psi_{E'_i} : E'_i \times \overline{T} &\hookrightarrow \overline{\mathcal{J}}, \text{ for } 1 \leq i \leq r + 1, \end{aligned}$$

and if we denote by $\overline{\mathcal{C}} \subset \overline{\mathcal{J}}$ the union of the $\text{Im}(\psi_{C_i})$, $\text{Im}(\psi_{E_i})$ and $\text{Im}(\psi_{E'_i})$, then the restriction of $\overline{\mathcal{C}}$ over T is exactly \mathcal{C} .

The formulae for these embeddings are a bit too long to write down, but are essentially the same as those in the proof of Proposition 5.14. For example ψ_{C_1} is given by

$$\begin{aligned} &(z, x_1, \dots, x_m, y_1, \dots, y_{j-m}) \mapsto \\ &(\mathcal{O}_{C_1}(z - x_1), \mathcal{O}_{C_2}(x_2 - c_2), \dots, \mathcal{O}_{C_m}(x_m - c_m), y_1, \dots, y_{j-m}, b_1, \dots, b_{r+1}, x_1, \dots, x_m, y_1, \dots, y_{j-m}). \end{aligned}$$

More generally, all the embeddings ψ_{C_i} , ψ_{E_i} and $\psi_{E'_i}$ can be written as products of the following maps, which we shall refer to as basic building blocks:

- (i) $C_1 \times C_1 \rightarrow J_1 \times C_1$, given by $(z, x_1) \mapsto (\mathcal{O}_{C_1}(z - x_1), x_1)$;
- (ii) (for $2 \leq k \leq m$) $C_k \times C_k \rightarrow J_k \times C_k$, given by $(z, x_k) \mapsto (\mathcal{O}_{C_k}(z - c_k), x_k)$;
- (iii) (for $2 \leq k \leq m$) $C_k \rightarrow J_k \times C_k$, given by $x_k \mapsto (\mathcal{O}_{C_k}(x_k - c_k), x_k)$;
- (iv) (for $1 \leq k \leq m$) $C_k \rightarrow J_k \times C_k$, given by $x_k \mapsto (0, x_k)$;
- (v) (for $1 \leq k \leq j - m$) $E_k \times E_k \rightarrow E_k \times E_k$, given by $(z, y_k) \mapsto (z, y_k)$;

6.3. Detecting non-trivial classes in higher Griffiths groups

- (vi) (for $1 \leq k \leq j-m$) $E_k \rightarrow E_k \times E_k$, given by $y_k \mapsto (y_k, y_k)$;
- (vii) (for $1 \leq k \leq j-m$) $E_k \rightarrow E_k \times E_k$, given by $y_k \mapsto (0, y_k)$;
- (viii) (for $1 \leq k \leq r+1$) $E'_k \rightarrow E'_k$, given by $z \mapsto z$;
- (ix) (for $1 \leq k \leq r+1$) $\text{pt} \rightarrow E'_k$, given by $z \mapsto b_k$;
- (x) (for $1 \leq k \leq r+1$) $\text{pt} \rightarrow E'_k$, given by $z \mapsto 0$.

The example of ψ_{C_1} is then the product of (i), (iii), (vi) and (ix). Note that the building blocks involve only one index k at a time, *i.e.* there is no interaction between different curves. For future reference, when looking at an embedding ψ_{C_i} , ψ_{E_i} or $\psi_{E'_i}$, we shall call its building block with target in $J_k \times C_k$ (resp. $E_k \times E_k$ and E'_k) the J_k - C_k (resp. E_k - E_k and E'_k) factor of the embedding.

We use the Künneth formula to calculate the cycle class of

$$([\overline{\mathcal{C}}]^{*m})_{(j_1, \dots, j_m)} = [\overline{\mathcal{C}}]_{(j_1)} * \dots * [\overline{\mathcal{C}}]_{(j_m)} \in \text{CH}_{(j)}^{j+r+1}(\overline{\mathcal{J}}),$$

which lies in

$$H^{j+2r+2}(J_1 \times \dots \times J_m \times E_1 \times \dots \times E_{j-m} \times E'_1 \times \dots \times E'_{r+1}) \otimes H^j(C_1 \times \dots \times C_m \times E_1 \times \dots \times E_{j-m}).$$

Here, and in what follows, we omit the coefficients of the cohomology groups. We know that the only relevant Künneth component (*i.e.* not killed by restricting to non-empty open subsets $V \subset \overline{T}$) of $H^j(\overline{T}) = H^j(C_1 \times \dots \times C_m \times E_1 \times \dots \times E_{j-m})$ is

$$(6.11) \quad H^1(C_1) \otimes \dots \otimes H^1(C_m) \otimes H^1(E_1) \otimes \dots \otimes H^1(E_{j-m}).$$

Then we have two immediate observations. First, the images of the embeddings $\psi_{E'_i}$ do not contribute to the class of $([\overline{\mathcal{C}}]^{*m})_{(j_1, \dots, j_m)}$. In fact, since $\psi_{E'_i}$ is a section of the projection $\overline{\mathcal{J}} \rightarrow E'_i \times \overline{T}$, we have

$$[\text{Im}(\psi_{E'_i})] = [\text{Im}(\psi_{E'_i})]_{(0)}.$$

Second, the images of ψ_{E_i} make no essential contribution either. This is because the E_i - E_i factor of ψ_{E_i} is the identity $\text{id}: E_i \times E_i \rightarrow E_i \times E_i$, which only gives a class in $H^0(E_i) \otimes H^0(E_i)$. On the other hand, the E_i - E_i factor of other embeddings is either the diagonal map $\Delta: E_i \rightarrow E_i \times E_i$, which gives classes in $H^2(E_i) \otimes H^0(E_i)$, $H^1(E_i) \otimes H^1(E_i)$ and $H^0(E_i) \otimes H^2(E_i)$, or the map $(0, \text{id}): E_i \rightarrow E_i \times E_i$, which gives a class in $H^2(E_i) \otimes H^0(E_i)$. Among them, only $H^2(E_i) \otimes H^0(E_i)$ has non-zero Pontryagin product with $H^0(E_i) \otimes H^0(E_i)$, but the product does not lead to a factor in (6.11).

It follows that one only need to consider the images of ψ_{C_i} , or more precisely the cycle class of

$$(6.12) \quad [\text{Im}(\psi_{C_1})]_{(j_1)} * \dots * [\text{Im}(\psi_{C_m})]_{(j_m)} + \text{permutations}.$$

6. TAUTOLOGICAL CLASSES IN HIGHER GRIFFITHS GROUPS

The computation is just careful bookkeeping (essentially the same as in Proposition 5.9): we collect all non-zero cohomology classes given by the J_k - C_k (resp. E_k - E_k and E'_k) factors, do tensor products, and sum all terms up. Here with the same building blocks, we always get the same class for each of the factors. Then the sum of non-zero classes is again non-zero, meaning that there is no cancellation effect. In the end we obtain only one relevant component (*i.e.* non-zero, and not easily killed by restricting to non-empty open subsets $V \subset \overline{T}$) of the class of (6.12). We denote it by h and it lies in

$$(6.13) \quad \begin{aligned} & H^1(J_1) \otimes \cdots \otimes H^1(J_m) \otimes H^1(E_1) \otimes \cdots \otimes H^1(E_{j-m}) \otimes H^2(E'_1) \otimes \cdots \otimes H^2(E'_{r+1}) \\ & \otimes H^1(C_1) \otimes \cdots \otimes H^1(C_m) \otimes H^1(E_1) \otimes \cdots \otimes H^1(E_{j-m}). \end{aligned}$$

We now look at the component of h in

$$H_{\text{prim}}^{j+2r+2}(J_1 \times \cdots \times J_m \times E_1 \times \cdots \times E_{j-m} \times E'_1 \times \cdots \times E'_{r+1}) \otimes H^j(C_1 \times \cdots \times C_m \times E_1 \times \cdots \times E_{j-m}).$$

Here for the abelian scheme $\overline{\mathcal{J}} \rightarrow \overline{T}$, the associated divisor class $\theta \in \text{CH}_{(0)}^1(\overline{\mathcal{J}})$ is

$$\theta = \text{pr}_{J_1}^*(\theta_1) + \cdots + \text{pr}_{J_m}^*(\theta_m) + \text{pr}_{E_1}^*([o_1]) + \cdots + \text{pr}_{E_{j-m}}^*([o_{j-m}]) + \text{pr}_{E'_1}^*([o'_1]) + \cdots + \text{pr}_{E'_{r+1}}^*([o'_{r+1}]),$$

where $\text{pr}_{J_i}: \overline{\mathcal{J}} \rightarrow J_i$, $\text{pr}_{E_i}: \overline{\mathcal{J}} \rightarrow E_i$ and $\text{pr}_{E'_i}: \overline{\mathcal{J}} \rightarrow E'_i$ are the projections. Then it follows immediately from the form of (6.13) that $h \neq \text{cl}(-\theta) \cup h'$, *i.e.* the primitive part of h is non-zero.

Finally it remains to ensure that the primitive part of h does not vanish when restricted to non-empty open subsets of \overline{T} , *i.e.* that it is not supported on a divisor of \overline{T} . The observation is the same as in the proof of Proposition 5.14: by construction h is a tensor product of classes in

$$\begin{aligned} & H^1(J_1) \otimes H^1(C_1), \dots, H^1(J_m) \otimes H^1(C_m), H^1(E_1) \otimes H^1(E_1), \dots, H^1(E_{j-m}) \otimes H^1(E_{j-m}), \\ & H^2(E'_1), \dots, H^2(E'_{r+1}). \end{aligned}$$

So its factor in $H^1(C_1) \otimes \cdots \otimes H^1(C_m) \otimes H^1(E_1) \otimes \cdots \otimes H^1(E_{j-m}) \subset H^j(\overline{T})$ has maximal Hodge level. The same goes for the primitive part of h , which then cannot be supported on a divisor of \overline{T} . In view of Lemma 5.7 and Proposition 6.10, the proof is now completed. \square

We end this chapter by a remark and a conjecture. Let (C, x_0) be a very general pointed curve over $\mathcal{M}_{g,1}$, with J its Jacobian. From Theorem 6.13 we know that

$$[C]_{(j)} \notin Z_0 F^j \text{CH}^{g-1}(J) \text{ for } g \geq j+2.$$

Moreover, by Conjecture 6.1 we expect that

$$[C]_{(j)} \notin Z_0 F^1 \text{CH}^{g-1}(J) \text{ for } g \geq 2j+1.$$

Observe that there are exactly $(2j+1) - (j+2) = j-1$ values of g to fill the gap between the terms $Z_0 F^j \text{CH}^{g-1}(J)$ and $Z_0 F^1 \text{CH}^{g-1}(J)$. So we make the following guess.

6.3. Detecting non-trivial classes in higher Griffiths groups

Conjecture 6.14. *For (C, x_0) very general over $\mathcal{M}_{g,1}$ with $g \geq 2j + 2 - k$ and $1 \leq k \leq j$, we have*

$$[C]_{(j)} \notin Z_0 F^k \text{CH}^{g-1}(J).$$

Using Polishchuk's \mathfrak{sl}_2 -action on $\mathcal{T}(J)$ (see Section 5.1), we show that the bound on g is sharp. Recall the classes $p_i = \mathcal{F}([C]_{(i-1)})$ and $q_i = \mathcal{F}(\theta \cdot [C]_{(i)})$. The following result generalizes [Pol05], Corollary 0.2 (which corresponds to the case $k = 1$).

Proposition 6.15. *Let (C, x_0) be a smooth projective pointed curve of genus g over \mathbb{C} . Then for $g \leq 2i - 1 - k$ and $1 \leq k \leq i - 1$, we have*

$$\mathcal{T}_{(i-1)}^i(J) \subset Z_0 F^k \text{CH}^i(J).$$

In particular $p_i \in Z_0 F^k \text{CH}^i(J)$.

Proof. The argument is essentially the same as in Lemma 5.13. By Theorem 5.1, the component $\mathcal{T}_{(i-1)}^i(J)$ is spanned by monomials

$$(6.14) \quad q_{a_1} \cdots q_{a_m} p_b \quad \text{with } a_1 + \cdots + a_m + b = i.$$

To show that all such monomials belong to $Z_0 F^k \text{CH}^i(J)$, we proceed by induction on b . For $b \leq i - k$, i.e. $a_1 + \cdots + a_m = i - b \geq k$, we have by (6.4) and (6.3)

$$q_{a_1} \cdots q_{a_m} \in \text{CH}_{(i-b)}^{i-b}(J) = \mathcal{F}(\text{CH}_{(i-b)}^g(J)) \subset Z_0 F^{i-b} \text{CH}^{i-b}(J) \subset Z_0 F^k \text{CH}^{i-b}(J),$$

so that $q_{a_1} \cdots q_{a_m} p_b \in Z_0 F^k \text{CH}^i(J)$.

Now suppose all monomials in (6.14) with $i - k \leq b < b_0$ are in $Z_0 F^k \text{CH}^i(J)$. Then for each $q_{a_1} \cdots q_{a_m} p_{b_0}$ with $a_1 + \cdots + a_m = i - b_0$, we look for a monomial

$$(6.15) \quad q_{a_1} \cdots q_{a_m} p_{c_1} \cdots p_{c_{g-i+1}} \in \text{CH}_{(i-1)}^g(J) \subset Z_0 F^{i-1} \text{CH}^g(J) \subset Z_0 F^k \text{CH}^g(J),$$

with $c_1 + \cdots + c_{g-i+1} = g - i + b_0$ and $c_1, \dots, c_{g-i+1} \geq 2$. The existence of such monomials is guaranteed by our assumption: in fact, since $g \leq 2i - 1 - k$ and $b_0 \geq i - k + 1$, we have

$$g - i + b_0 \geq 2(g - i + 1),$$

meaning that one can always divide $g - i + b_0$ into $g - i + 1$ parts, with all parts greater than or equal to 2. We apply $g - i$ times the differential operator \mathcal{D} in (5.1) to (6.15), and we obtain

$$(6.16) \quad \mathcal{D}^{g-i}(q_{a_1} \cdots q_{a_m} p_{c_1} \cdots p_{c_{g-i+1}}) \in Z_0 F^k \text{CH}^i(J).$$

Then by analyzing the explicit expression of \mathcal{D} , we find in (6.16) a non-zero multiple of $q_{a_1} \cdots q_{a_m} p_{b_0}$ (with sign $(-1)^{g-i}$), plus multiples of monomials $q_{a'_1} \cdots q_{a'_{m'}} p_{b'}$ with $b' < b_0$. Those monomials are in $Z_0 F^k \text{CH}^i(J)$ by the induction assumption, and so is $q_{a_1} \cdots q_{a_m} p_{b_0}$.

In particular, the last stage of the induction shows that $p_i \in Z_0 F^k \text{CH}^i(J)$. □

6. TAUTOLOGICAL CLASSES IN HIGHER GRIFFITHS GROUPS

g	$[C]_{(1)}$		$[C]_{(2)}$		$[C]_{(3)}$		$[C]_{(4)}$		$[C]_{(5)}$	
3	$\notin Z_0 F^1$	C								
4	$\notin Z_0 F^1$	C	$\notin Z_0 F^2$ $\in Z_0 F^1$	P						
5	$\notin Z_0 F^1$	C	$\notin Z_0 F^1$?	$\notin Z_0 F^3$ $\in Z_0 F^2$					
6	$\notin Z_0 F^1$	C	$\notin Z_0 F^1$	V	$\notin Z_0 F^2$ $\in Z_0 F^1$? P	$\notin Z_0 F^4$ $\in Z_0 F^3$			
7	$\notin Z_0 F^1$	C	$\notin Z_0 F^1$	V	$\notin Z_0 F^1$?	$\notin Z_0 F^3$ $\in Z_0 F^2$?	$\notin Z_0 F^5$ $\in Z_0 F^4$	

Table 2. Results and questions in genus $g \leq 7$.

By applying \mathcal{F}^{-1} to p_{j+1} , we obtain that $[C]_{(j)} \in Z_0 F^k \text{CH}^{g-1}(J)$ for $g \leq 2j + 1 - k$ and $1 \leq k \leq j$.

In Table 2, we summarize some of our findings in low genus cases. Statements marked with C, P and V are known to Ceresa, Polishchuk and Voisin respectively. Those without marks are proven in this section, while those with a question mark remain open.

Notes

2. Preliminaries

1. (page 8) The polarization λ provides $L_\lambda \in \text{Pic}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$ defined in (2.1). Conversely, to every $L \in \text{Pic}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$ that is relatively ample, symmetric and trivialized along the zero section, we may associate a symmetric quasi-isogeny $\varphi_L \in \text{Hom}(A, A^t) \otimes_{\mathbb{Z}} \mathbb{Q}$. It sends a point $x \in A_s$ for some $s \in S$ to the class $\varphi_L(x) := L_s \otimes T_x^*(L_s)^{-1}$, where T_x is the translation of A_s by x . In total we have $\varphi_{L_\lambda} = \lambda \otimes_{\mathbb{Z}} \mathbb{Q}$.

There is a classical sign issue in the definition of φ_L , hence of the polarization λ (see [Pol03], Section 17.3). After some struggle we decide to choose the somewhat unconventional definition (which is also adopted by Beauville). The reason is well explained in [Bea10], Section 1.6, and we dare to repeat it here.

Most people follow Mumford's formula $\phi_L(x) = -\varphi_L(x) = T_x^*(L) \otimes L^{-1}$ (see [Mum70], Section 6). Then for an elliptic curve E defined over k , and for $L = \mathcal{O}_E(o)$, we have $\phi_L(x) = \mathcal{O}_E(o - x)$. More generally, consider a smooth projective curve C over k with its Jacobian $J := \text{Pic}^0(C)$. By choosing a point $x_0 \in C(k)$ we obtain an embedding $\iota: C \rightarrow J$, defined on points by $\iota(x) := \mathcal{O}_C(x - x_0)$. On the other hand, we have the Albanese map $\text{alb}: C \rightarrow J^t$ also with respect to x_0 . We would very much like to identify the two maps ι and alb . However, the identification is given by φ_L and not $\phi_L = -\varphi_L$ (here L is the line bundle associated to a symmetric theta divisor).

Here we also followed [GMxx], Chapter 11 in the definition of a polarization, as it turned out to be the right approach. The point is that one should regard the polarization as an isogeny, rather than a line bundle or a divisor class. In fact, the line bundle or divisor class with \mathbb{Z} -coefficients may not even exist in the relative setting (see [GH13], Section 1), and with \mathbb{Q} -coefficients it is difficult to recover the corresponding isogeny.

2. (page 12) Regarding Part (ii) of Conjecture 2.10, Grothendieck's standard conjectures predict that numerical and homological equivalences coincide (see [Kle94], Section 5). This is known for

abelian varieties in characteristic zero, but only for codimension $i = 0, 1, g - 1$ and g in positive characteristic (see [Mil02] for more details).

Conjecture 2.10 is a concrete special case of a more general conjecture, the so-called Bloch-Beilinson-Murre (BBM) conjecture. We refer to [Mur93], [Jann94] and [Voi02], Chapitre 23 for the precise statements. Roughly speaking, the BBM conjecture predicts for all varieties $X \in \mathcal{V}_k$ a descending filtration $F^\bullet \mathrm{CH}^i(X)$ on $\mathrm{CH}^i(X)$, with $F^0 \mathrm{CH}^i(X) = \mathrm{CH}^i(X)$ and $F^{i+1} \mathrm{CH}^i(X) = 0$, such that the terms F^j are stable under correspondences. Also for all $0 \leq j \leq i$, the graded piece

$$\mathrm{Gr}_F^j \mathrm{CH}^i(X) = F^j \mathrm{CH}^i(X) / F^{j+1} \mathrm{CH}^i(X)$$

should in some sense be controlled by the cohomology group $H^{2i-j}(X)$. Further, one expects that $H^{2i}(X)$ controls $\mathrm{Gr}_F^0 \mathrm{CH}^i(X)$ via the cycle class map, and that $H^{2i-1}(X)$ controls $\mathrm{Gr}_F^1 \mathrm{CH}^i(X)$ via the Abel-Jacobi map.

In the case of an abelian variety A , the Beauville decomposition provides a candidate for the conjectural filtration, namely

$$F^j \mathrm{CH}^i(A) := \bigoplus_{j' \geq j} \mathrm{CH}_{(j')}^i(A).$$

Conjecture 2.10 states exactly what is needed for this candidate to satisfy all requirements set by the BBM conjecture.

3. A tale of two tautological rings (I)

3. (page 26) We refer to [Fab99], Conjecture 1, [FP00], Section 0.5, and [Pan02], Conjecture 1 for various versions of the Faber conjectures. These conjectures concern the tautological ring of the following moduli spaces:

- (i) moduli of stable n -pointed curves of genus g , denoted by $\overline{\mathcal{M}}_{g,n}$;
- (ii) moduli of stable n -pointed curves of genus g and of compact type, denoted by $\mathcal{M}_{g,n}^{\mathrm{ct}}$;
- (iii) moduli of stable n -pointed curves of genus g with rational tails, denoted by $\mathcal{M}_{g,n}^{\mathrm{rt}}$;
- (iv) moduli of smooth curves of genus g , denoted by \mathcal{M}_g .

See also Faber's notes [Fab13] for the current status of these conjectures, and [PT12], Corollary 2.5 for a first counterexample in Case (i) (for $\overline{\mathcal{M}}_{2,n}$).

Our $\mathcal{M}_{g,1}$ version (Conjecture 3.2) is the closest to the original \mathcal{M}_g version. Although it is never stated explicitly in the literature, one can relate it to Case (iii) via the isomorphism(s)

$$\mathcal{M}_{g,1} \simeq \mathcal{M}_{g,1}^{\mathrm{rt}} (\simeq \mathcal{C}_g).$$

More generally, for $n \geq 1$ there are surjective maps $\mathcal{M}_{g,n}^{\text{rt}} \twoheadrightarrow \mathcal{C}_g^n$, where \mathcal{C}_g^n is the n -th power of the universal curve over \mathcal{M}_g . One can also formulate a version of the conjectures for \mathcal{C}_g^n (see Speculation 4.21). It is believed that the versions for $\mathcal{M}_{g,n}^{\text{rt}}$ and \mathcal{C}_g^n are equivalent, but to the best of our knowledge there is no written proof of this.

4. (page 38) Consider the component $\tilde{\mathcal{R}}^{g-1}$, where the expected socle is situated. It is spanned by the image of $\text{Mon}_{(0,2g-2)}$, and the goal is to relate all elements in $\text{Mon}_{(0,2g-2)}$ to a single element, namely $x_{0,2g-2}$. So far we have been able to associate a relation to every element in $\text{Mon}_{(0,2g-2)}$ other than $x_{0,2g-2}$. By defining a partial order on $\text{Mon}_{(0,2g-2)}$, we can show that the relation matrix is block triangular. What is combinatorially difficult is to prove that all diagonal blocks are of maximal rank.

The construction goes as follows: first remark that there is a correspondence between $\text{Mon}_{(0,2g-2)}$ and the set of partitions of all integers k such that $0 \leq k \leq g-1$. In fact, to every such partition $\lambda = (i_1, \dots, i_m)$ with $i_1 + \dots + i_m = k$, we associate the element $y^{g-1-k} x_{0,2i_1} \cdots x_{0,2i_m}$ in $\text{Mon}_{(0,2g-2)}$. In particular, the cardinal $\#\text{Mon}_{(0,2g-2)}$ is $p(0) + \dots + p(g-1)$, where $p(-)$ stands for the partition function. Then observe that to every partition λ above, we may associate a partition λ' of $2g-2$

$$\lambda = (i_1, \dots, i_m) \mapsto \lambda' = (2i_1 + g - 1 - k, 2i_2, \dots, 2i_m, \underbrace{1, \dots, 1}_{g-1-k}).$$

The partial order on $\text{Mon}_{(0,2g-2)}$ is defined as the first part of λ' , i.e. $2i_1 + g - 1 - k$. Using the elements $\{x_{i+2,i}\}$ located to the right of the diagonal $i = j$ in the Dutch house, we define the monomial

$$M_{\lambda'} = x_{2i_1+g-1-k+2, 2i_1+g-1-k} x_{2i_2+2, 2i_2} \cdots x_{2i_m+2, 2i_m} x_{3,1}^{g-1-k},$$

which belongs to $\text{Mon}_{(4g-4+2m-2k, 2g-2)}$. We see that for all λ except $\lambda = (g-1)$, we have $4g-4+2m-2k > 2g$, so that $M_{\lambda'} = 0$ in $\tilde{\mathcal{R}}$. Then to every element in $\text{Mon}_{(0,2g-2)}$ other than $x_{0,2g-2}$, we associate a relation $F^{2g-2+m-k}(M_{\lambda'}) = 0$ in $\tilde{\mathcal{R}}^{g-1}$. By the explicit expression of F (3.11), we can prove that $F^{2g-2+m-k}(M_{\lambda'})$ only contains terms of order greater than or equal to the first part of λ' .

For $\tilde{\mathcal{R}}^i$ with $i > g-1$, the situation is similar: this time there are no more new generators $x_{0,2i}$, and every element in $\text{Mon}_{(0,2i)}$ corresponds to a relation. The question is still to show that the relation matrix is of maximal rank. Here we should say that, difficult combinatorics aside, we do get a feeling why the socle should lie in codimension $g-1$, and why beyond $g-1$ everything should vanish.

4. A tale of two tautological rings (II)

5. (page 50) This notion of tautological rings can be extended to more general settings. In fact, for any object $X \in \mathcal{V}_S$, we have the same tautological maps

$$T: X^m \rightarrow X^n, \text{ with } m \geq 1 \text{ and } n \geq 0,$$

such that each factor of T is a projection. Then for $n \geq 0$, we define the system of tautological rings of X^n to be the collection of smallest \mathbb{Q} -subalgebras $\{\mathcal{R}(X^n) \subset \mathrm{CH}(X^n)\}$, such that

- (i) the ring $\mathcal{R}(X)$ contains a (usually finite) set A of geometrically constructed classes;
- (ii) the system is stable under pull-backs and push-forwards via all tautological maps T .

Over a field (*i.e.* $S = k$), this notion is studied by O’Sullivan in [O’S10]. Also an interesting case is when X is a $K3$ surface. Here we take A to be a finite set that spans $\mathrm{Pic}(X) \times_{\mathbb{Z}} \mathbb{Q}$. In this case, Voisin conjectured that for all $n \geq 1$, the restriction of the cycle class map $\mathrm{cl}: \mathrm{CH}(X^n) \rightarrow H(X^n)$ to $\mathcal{R}(X^n)$ is injective ([Voi08], Conjecture 1.6). When $n = 1$, this is the well-known result of Beauville and Voisin ([BV04], Theorem 1). The conjecture turns out to be rather strong: for example it implies that the motive of X is finitely dimensional in the sense of Kimura-O’Sullivan (see Conjecture 5.19).

For any variety $X \in \mathcal{V}_k$, one may further generalize this notion of tautological rings by including a finite set $A \in \mathcal{R}(X^m)$ for some $m > 1$. O’Sullivan showed that the finite dimensionality of $\mathcal{R}(X^n)$ for all n is roughly equivalent to the Kimura-O’Sullivan finite dimensionality of the motive of X ([O’S10], Theorem 1.1).

6. (page 61) Recently a new set of tautological relations for various moduli spaces (including \mathcal{C}_g^n) has been conjectured by Pixton ([Pix12], Conjecture 1), and proven by Pandharipande-Pixton-Zvonkine in cohomology ([PPZ13], Corollary 2), and by Janda in the Chow ring ([Jan13], Proposition 1). Data have been collected regarding the discrepancies between Pixton’s relations and the conjectural Gorenstein property, for \mathcal{C}_g^n with many values of g and n . Our computation seems to be coherent with those data.

Corollary 4.31 shows that when $g \leq 7$, the Gorenstein property for the symmetric powers $\mathcal{C}_g^{[n]}$ holds for n arbitrarily large. It then appears that after $g = 24$ for $n = 0$ and $g = 20$ for $n = 1$, the number $g = 8$ is the ultimate critical number for any large n (at least if one restricts to the symmetric powers). It is certainly one of the most interesting cases for those who want to prove or disprove the various Gorenstein type properties.

On the other hand, following the first counterexample given by Petersen and Tommasi (for $\overline{\mathcal{M}}_{2,n}$; see [PT12], Corollary 2.5), one tends to think that the tautological rings behave worse as n increases. So it is somewhat surprising to get positive results such as Corollary 4.31 for arbitrarily large n . Previously, similar positive results were obtained only for the moduli spaces $\overline{\mathcal{M}}_{0,n}$ ([Kee92], Section 2), $\mathcal{M}_{1,n}^{\mathrm{ct}}$ ([Tav11], Theorem 0.1), $\overline{\mathcal{M}}_{1,n}$ ([Pet12], Section 1) and $\mathcal{M}_{2,n}^{\mathrm{rt}}$ ([Tav11b], Theorem 0.1).

Finally, our approach sheds some light on the importance of the universal Jacobian. One might hope that the nice structures and powerful machinery on the Jacobian side would help resolving further problems on the moduli and curve sides.

5. Tautological classes on a Jacobian variety

7. (page 72) The following remark is based on a discussion with Johan Commelin. Similar to the Faber-Pandharipande cycle, there is a geometrically constructed 1-cycle on $C \times C \times C$ introduced by Gross and Schoen (see [GS95], Section 0). It comes in two versions, one with respect to a point $x_0 \in C(k)$ and the other canonical. Denote by Δ the diagonal $C \subset C \times C$ and by Δ_{sm} the small diagonal $C \subset C \times C \times C$. We then define

$$\begin{aligned}\Delta_{x_0} &= [\Delta_{\text{sm}}] - ([\Delta \times x_0] + \text{permutations}) + ([x_0 \times x_0 \times C] + \text{permutations}), \\ \Delta_K &= [\Delta_{\text{sm}}] - \frac{1}{2g-2}([\Delta] \times K + \text{permutations}) + \frac{1}{(2g-2)^2}(K \times K \times [C] + \text{permutations}).\end{aligned}$$

We have $\Delta_{x_0}, \Delta_K \in \text{CH}^2(C \times C \times C)^{\mathfrak{S}_3}$, and they are both homologically trivial.

Similar to Proposition 5.10, we can express Δ_{x_0} and Δ_K as the pull-back of certain classes in $\mathcal{T}(J)$ via the map $\phi_3: C \times C \times C \rightarrow J$ with respect to x_0 , together with the class $\xi_3 = [x_0 \times C \times C] + \text{permutations} \in \text{CH}^1(C \times C \times C)^{\mathfrak{S}_3}$. After some tedious but elementary computations, we obtain

$$\Delta_{x_0} = \phi_3^*(p_2) + 2\phi_3^*(q_2) + \phi_3^*(q_1) \cdot \xi_3,$$

and

$$\Delta_K = \phi_3^*(p_2) - \frac{2}{2g-2}\phi_3^*(q_1 p_1) + \frac{4g}{2g-2}\phi_3^*(q_2) - \frac{2(2g-3)}{(2g-2)^2}\phi_3^*(q_1^2).$$

If we work modulo algebraic equivalence, since q_1 and q_2 are algebraically trivial, only the p_2 terms on the right-hand side above will survive. In fact, by a similar argument as in Proposition 5.10 (ii), one can show that Δ_{x_0} (resp. Δ_K) is algebraically trivial if and only if p_2 is algebraically trivial. The latter is also equivalent to the Ceresa cycle $[C] - [-1]^*[C]$ being algebraically trivial.

6. Tautological classes in higher Griffiths groups

8. (page 86) When $S = \mathbb{C}$, the filtration $F_H^\bullet \text{CH}^i(X)$ is defined by replacing the vanishing of

$$\Gamma_*: H^{2i'-j}(Y, \mathbb{Q}) \rightarrow H^{2i-j}(X, \mathbb{Q})$$

by the condition that

$$\Gamma_*(H^{2i'-j}(Y, \mathbb{Q})) \subset N^{i-j+1} H^{2i-j}(X, \mathbb{Q}),$$

where $N^k H^m(X, \mathbb{Q})$ is the largest \mathbb{Q} -sub-Hodge structure of $H^m(X, \mathbb{Q})$ such that $N^k H^m(X, \mathbb{C}) \subset F^k H^m(X, \mathbb{C})$. Besides $F_B^\bullet \text{CH}^i(X)$ and $F_H^\bullet \text{CH}^i(X)$, there is yet a third version of Saito's filtration, denoted by $F_{BM}^\bullet \text{CH}^i(X)$ in [Sai00], Section 1. It replaces the condition above by

$$\Gamma_*(H^{2i'-j}(Y, \mathbb{Q})) \subset \tilde{N}^{i-j+1} H^{2i-j}(X, \mathbb{Q}),$$

where

$$\tilde{N}^k H^m(X, \mathbb{Q}) = \sum_{\substack{Y \hookrightarrow X \\ \text{codim}(Y) \geq k}} \text{Ker}(H^m(X, \mathbb{Q}) \rightarrow H^m(X - Y, \mathbb{Q})).$$

The three filtrations $F_{BM}^\bullet \text{CH}^i(X)$, $F_B^\bullet \text{CH}^i(X)$ and $F_H^\bullet \text{CH}^i(X)$ are studied in [Sai96], [Sai00] and [Sai00b] respectively for different purposes. By definition we have $F_B^j \text{CH}^i(X) = F_{BM}^j \text{CH}^i(X) = F_H^j \text{CH}^i(X)$ for $j \leq 2$, and in general

$$F_B^j \text{CH}^i(X) \subset F_{BM}^j \text{CH}^i(X) \subset F_H^j \text{CH}^i(X).$$

Further, the generalized Hodge conjecture predicts $\tilde{N}^\bullet = N^\bullet$, and thus $F_{BM}^\bullet \text{CH}^i(X) = F_H^\bullet \text{CH}^i(X)$. Saito proved that by assuming the standard conjecture $\sim_{\text{hom}} = \sim_{\text{num}}$, we also have $F_B^\bullet \text{CH}^i(X) = F_{BM}^\bullet \text{CH}^i(X)$ ([Sai00], Theorem 1.1).

9. (page 86) In fact, the locus $\{s \in S : \alpha_s \in F^j \text{CH}^i(X_s)\}$ is a countable union of Zariski-closed subsets of S . One way to see this is to use relative Hilbert schemes (or Chow varieties). Since objects in \mathcal{V}_S are projective over S , one can parametrize all data in Definition 6.3 by countably many projective schemes over S (note that the cohomological condition in each inductive step is Zariski-closed). The more precise procedure is documented in [Voi02], Section 22.2 (see also [Voi12b], Sections 0.1 and 2.1).

Similar to Remark 5.5, there is also a more general argument without using projectivity. First remark that the data $\pi: X \rightarrow S$ and α can be defined over a finitely generated subfield $k \subset \mathbb{C}$. Denote by η the generic point of S/k . We can extend Definition 6.3 to more general settings (over an arbitrary field) using ℓ -adic cohomology. In particular, we obtain the filtration $F^\bullet \text{CH}^i(X_\eta)$ over η . Then if $\alpha_\eta \in F^j \text{CH}^i(X_\eta)$, by specialization we have $\alpha_s \in F^j \text{CH}^i(X_s)$ for all $s \in S(\mathbb{C})$. On the other hand if $\alpha_\eta \notin F^j \text{CH}^i(X_\eta)$, by base change we have $\alpha_s \notin F^j \text{CH}^i(X_s)$ for any $s \in S(\mathbb{C})$ that maps to η , or equivalently, for any $s \in S(\mathbb{C})$ that does not lie in a subvariety of S defined over k . Since k is finitely generated and hence countable, there are only countably many such varieties. This argument can also be found in [Ike03], proof of Proposition 2.10.

Bibliography

- [ACV03] ABRAMOVICH, D., A. CORTI, and A. VISTOLI. “Twisted bundles and admissible covers.” Special issue in honor of Steven L. Kleiman. *Communications in Algebra* 31 (2003), no. 8, 3547–3618. (page 67)
- [A’C79] A’CAMPO, N. “Tresses, monodromie et le groupe symplectique.” *Commentarii Mathematici Helvetici* 54 (1979), no. 2, 318–327. (page 92)
- [And04] ANDRÉ, Y. *Une introduction aux motifs (motifs purs, motifs mixtes, périodes)*. Panoramas et Synthèses, 17. Société Mathématique de France, Paris, 2004. xii+261 pp. (page 14)
- [And05] ANDRÉ, Y. “Motifs de dimension finie (d’après S.-I. Kimura, P. O’Sullivan...).” Séminaire Bourbaki, Vol. 2003/2004. *Astérisque* no. 299 (2005), Exposé no. 929, viii, 115–145. (pages 15 and 78)
- [Bea83] BEAUVILLE, A. “Quelques remarques sur la transformation de Fourier dans l’anneau de Chow d’une variété abélienne.” *Algebraic geometry* (Tokyo/Kyoto, 1982), 238–260, Lecture Notes in Mathematics, 1016, Springer, Berlin, 1983. (pages 9 and 12)
- [Bea86] BEAUVILLE, A. “Sur l’anneau de Chow d’une variété abélienne.” *Mathematische Annalen* 273 (1986), no. 4, 647–651. (pages 9 and 12)
- [Bea04] BEAUVILLE, A. “Algebraic cycles on Jacobian varieties.” *Compositio Mathematica* 140 (2004), no. 3, 683–688. (pages 11, 27, 42, 64, and 83)
- [Bea10] BEAUVILLE, A. “The action of SL_2 on abelian varieties.” *Journal of the Ramanujan Mathematical Society* 25 (2010), no. 3, 253–263. (pages 10, 11, and 99)
- [BV04] BEAUVILLE, A. and C. VOISIN. “On the Chow ring of a $K3$ surface.” *Journal of Algebraic Geometry* 13 (2004), no. 3, 417–426. (page 102)

BIBLIOGRAPHY

- [Ber11] BERGVALL, O. “Relations in the tautological ring of the universal curve.” Master thesis, KTH Stockholm, 2011. (page 39)
- [Bol12] BOLDSSEN, S. “Improved homological stability for the mapping class group with integral or twisted coefficients.” *Mathematische Zeitschrift* 270 (2012), no. 1-2, 297–329. (page 27)
- [CF96] CASSELS, J. and E. FLYNN. *Prolegomena to a middlebrow arithmetic of curves of genus 2*. London Mathematical Society Lecture Note Series, 230. Cambridge University Press, Cambridge, 1996. xiv+219 pp. (page 71)
- [Cer83] CERESA, G. “ C is not algebraically equivalent to C^- in its Jacobian.” *Annals of Mathematics* (2) 117 (1983), no. 2, 285–291. (pages 42, 66, 71, 76, 83, and 93)
- [Cha10] CHARLES, F. “On the zero locus of normal functions and the étale Abel-Jacobi map.” *International Mathematics Research Notices* (2010), no. 12, 2283–2304. (page 12)
- [CG93] COLOMBO, E. and B. VAN GEEMEN. “Note on curves in a Jacobian.” *Compositio Mathematica* 88 (1993), no. 3, 333–353. (page 83)
- [Del68] DELIGNE, P. “Théorème de Lefschetz et critères de dégénérescence de suites spectrales.” *Publications Mathématiques de l’IHÉS* 35 (1968), 107–126. (pages 16 and 86)
- [DM91] DENINGER, C. and J. MURRE. “Motivic decomposition of abelian schemes and the Fourier transform.” *Journal für die Reine und Angewandte Mathematik* 422 (1991), 201–219. (pages 9, 10, 13, 14, and 16)
- [Fab97] FABER, C. “A non-vanishing result for the tautological ring of \mathcal{M}_g .” Preprint, 1997. arXiv:math/9711219. (pages 26, 57, and 61)
- [Fab99] FABER, C. “A conjectural description of the tautological ring of the moduli space of curves.” *Moduli of curves and abelian varieties*, 109–129, Aspects of Mathematics, E33, Vieweg, Braunschweig, 1999. (pages 26, 27, 40, and 100)
- [Fab13] FABER, C. “Tautological algebras of moduli spaces of curves.” *Moduli spaces of Riemann surfaces*, 197–220, IAS/Park City Mathematics Series, 20, American Mathematical Society, Providence, RI, 2013. (pages 26, 41, 42, and 100)
- [FP00] FABER, C. and R. PANDHARIPANDE. “Logarithmic series and Hodge integrals in the tautological ring.” With an appendix by Don Zagier. *Michigan Mathematical Journal* 48 (2000), 215–252. (pages 26 and 100)

- [FP05] FABER, C. and R. PANDHARIPANDE. “Relative maps and tautological classes.” *Journal of the European Mathematical Society* 7 (2005), no. 1, 13–49. (page 49)
- [FP13] FABER, C. and R. PANDHARIPANDE. “Tautological and non-tautological cohomology of the moduli space of curves.” *Handbook of moduli*, Vol. I, 293–330, Advanced Lectures in Mathematics, 24, International Press, Boston, 2013. (page 26)
- [Fak96] FAKHRUDDIN, N. “Algebraic cycles on generic abelian varieties.” *Compositio Mathematica* 100 (1996), no. 1, 101–119. (pages 42, 66, 67, and 83)
- [Fu10] FU, B. “Remarks on hard Lefschetz conjectures on Chow groups.” *Science China. Mathematics* 53 (2010), no. 1, 105–114. (page 12)
- [FH07] FU, B. and F. HERBAUT. “On the tautological ring of a Jacobian modulo rational equivalence.” *Geometriae Dedicata* 129 (2007), 145–153. (pages 27 and 65)
- [Ful98] FULTON, W. *Intersection theory*. Second edition. *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics*, 2. Springer-Verlag, Berlin, 1998. xiv+470 pp. (pages 44 and 80)
- [FulH91] FULTON, W. and J. HARRIS. *Representation theory: a first course*. Graduate Texts in Mathematics, 129. Readings in Mathematics. Springer-Verlag, New York, 1991. xvi+551 pp. (page 92)
- [GK07] VAN DER GEER, G. and A. KOUVIDAKIS. “Cycle relations on Jacobian varieties.” With an appendix by Don Zagier. *Compositio Mathematica* 143 (2007), no. 4, 900–908. (pages 27 and 65)
- [GMxx] VAN DER GEER, G. and B. MOONEN. *Abelian varieties*. Book project. (page 99)
- [GG03] GREEN, M. and P. GRIFFITHS. “An interesting 0-cycle.” *Duke Mathematical Journal* 119 (2003), no. 2, 261–313. (pages 65, 66, and 72)
- [GS95] GROSS, B. and C. SCHOEN. “The modified diagonal cycle on the triple product of a pointed curve.” *Annales de l’Institut Fourier* 45 (1995), no. 3, 649–679. (page 103)
- [GH13] GRUSHEVSKY, S. and K. HULEK. “Geometry of theta divisors — a survey.” *A celebration of algebraic geometry*, 361–390, Clay Mathematics Proceedings, 18, American Mathematical Society, Providence, RI, 2013. (page 99)

BIBLIOGRAPHY

- [GZ12] GRUSHEVSKY, S. and D. ZAKHAROV. “The zero section of the universal semiabelian variety, and the double ramification cycle.” Preprint, 2012. [arXiv:1206.3534](#). To appear in *Duke Mathematical Journal*. (page 67)
- [HR01] HAIN, R. and D. REED. “Geometric proofs of some results of Morita.” *Journal of Algebraic Geometry* 10 (2001), no. 2, 199–217. (page 32)
- [Her07] HERBAUT, F. “Algebraic cycles on the Jacobian of a curve with a linear system of given dimension.” *Compositio Mathematica* 143 (2007), no. 4, 883–899. (pages 27, 65, and 79)
- [Ike03] IKEDA, A. “Algebraic cycles and infinitesimal invariants on Jacobian varieties.” *Journal of Algebraic Geometry* 12 (2003), no. 3, 573–603. (pages 42, 84, 85, 86, 88, and 104)
- [Ion05] IONEL, E.-N. “Relations in the tautological ring of \mathcal{M}_g .” *Duke Mathematical Journal* 129 (2005), no. 1, 157–186. (page 26)
- [Jan13] JANDA, F. “Tautological relations in moduli spaces of weighted pointed curves.” Preprint, 2013. [arXiv:1306.6580](#). (page 102)
- [Jann94] JANNSSEN, U. “Motivic sheaves and filtrations on Chow groups.” *Motives* (Seattle, WA, 1991), 245–302, Proceedings of Symposia in Pure Mathematics, 55, Part 1, American Mathematical Society, Providence, RI, 1994. (pages 66, 84, and 100)
- [KS09] KAHN, B. and R. SEBASTIAN. “Smash-nilpotent cycles on abelian 3-folds.” *Mathematical Research Letters* 16 (2009), no. 6, 1007–1010. (page 78)
- [Kee92] KEEL, S. “Intersection theory of moduli space of stable n -pointed curves of genus zero.” *Transactions of the American Mathematical Society* 330 (1992), no. 2, 545–574. (page 102)
- [Kim05] KIMURA, S.-I. “Chow groups are finite dimensional, in some sense.” *Mathematische Annalen* 331 (2005), no. 1, 173–201. (pages 14 and 78)
- [KV96] KIMURA, S.-I. and A. VISTOLI. “Chow rings of infinite symmetric products.” *Duke Mathematical Journal* 85 (1996), no. 2, 411–430. (pages 12, 44, 45, 48, and 49)
- [Kle94] KLEIMAN, S. “The standard conjectures.” *Motives* (Seattle, WA, 1991), 3–20, Proceedings of Symposia in Pure Mathematics, 55, Part 1, American Mathematical Society, Providence, RI, 1994. (pages 12 and 99)
- [Kün93] KÜNNEMANN, K. “A Lefschetz decomposition for Chow motives of abelian schemes.” *Inventiones Mathematicae* 113 (1993), no. 1, 85–102. (pages 10, 12, and 14)

- [LX12] LIU, K. and H. XU. “Computing top intersections in the tautological ring of \mathcal{M}_g .” *Mathematische Zeitschrift* 270 (2012), no. 3-4, 819–837. (pages 39 and 40)
- [Loo95] LOOIJENGA, E. “On the tautological ring of \mathcal{M}_g .” *Inventiones Mathematicae* 121 (1995), no. 2, 411–419. (pages 26, 50, 57, and 61)
- [Loo96] LOOIJENGA, E. “Stable cohomology of the mapping class group with symplectic coefficients and of the universal Abel-Jacobi map.” *Journal of Algebraic Geometry* 5 (1996), no. 1, 135–150. (page 27)
- [Mil02] MILNE, J. “Polarizations and Grothendieck’s standard conjectures.” *Annals of Mathematics* (2) 155 (2002), no. 2, 599–610. (page 100)
- [Moo99] MOONEN, B. “Notes on Mumford-Tate groups.” Lecture notes at the Centre Emile Borel, 1999. (page 92)
- [Moo09] MOONEN, B. “Relations between tautological cycles on Jacobians.” *Commentarii Mathematici Helvetici* 84 (2009), no. 3, 471–502. (pages 18, 27, 34, 42, 65, and 83)
- [Moo11] MOONEN, B. “On the Chow motive of an abelian scheme with non-trivial endomorphisms.” Preprint, 2011. arXiv:1110.4264. To appear in *Journal für die Reine und Angewandte Mathematik*. (page 12)
- [MP10] MOONEN, B. and A. POLISHCHUK. “Algebraic cycles on the relative symmetric powers and on the relative Jacobian of a family of curves. II.” *Journal of the Institute of Mathematics of Jussieu* 9 (2010), no. 4, 799–846. (pages 44, 46, 47, 48, and 56)
- [Mum70] MUMFORD, D. *Abelian varieties*. Tata Institute of Fundamental Research Studies in Mathematics, no. 5. Oxford University Press, London, 1970. viii+242 pp. (page 99)
- [Mum75] MUMFORD, D. *Curves and their Jacobians*. The University of Michigan Press, Ann Arbor, Michigan, 1975. vi+104 pp. (page 21)
- [Mum83] MUMFORD, D. “Towards an enumerative geometry of the moduli space of curves.” *Arithmetic and geometry*, Vol. II, 271–328, Progress in Mathematics, 36, Birkhäuser Boston, Boston, MA, 1983. (page 26)
- [Mur93] MURRE, J. “On a conjectural filtration on the Chow groups of an algebraic variety. I. The general conjectures and some examples.” *Indagationes Mathematicae* (New Series) 4 (1993), no. 2, 177–188. (pages 84 and 100)

BIBLIOGRAPHY

- [Nor93] NORI, M. “Algebraic cycles and Hodge-theoretic connectivity.” *Inventiones Mathematicae* 111 (1993), no. 2, 349–373. (page 89)
- [O’S10] O’SULLIVAN, P. “A finiteness theorem for algebraic cycles.” Preprint, 2010. arXiv:1003.4789. (page 102)
- [Pan02] PANDHARIPANDE, R. “Three questions in Gromov-Witten theory.” *Proceedings of the International Congress of Mathematicians*, Vol. II (Beijing, 2002), 503–512, Higher Education Press, Beijing, 2002. (pages 26 and 100)
- [PP13] PANDHARIPANDE, R. and A. PIXTON. “Relations in the tautological ring of the moduli space of curves.” Preprint, 2013. arXiv:1301.4561. (pages 41 and 42)
- [PPZ13] PANDHARIPANDE, R., A. PIXTON, and D. ZVONKINE. “Relations on $\overline{\mathcal{M}}_{g,n}$ via 3-spin structures.” Preprint, 2013. arXiv:1303.1043. (page 102)
- [Pet12] PETERSEN, D. “The structure of the tautological ring in genus one.” Preprint, 2012. arXiv:1205.1586. To appear in *Duke Mathematical Journal*. (page 102)
- [PT12] PETERSEN, D. and O. TOMMASI. “The Gorenstein conjecture fails for the tautological ring of $\overline{\mathcal{M}}_{2,n}$.” Preprint, 2012. arXiv:1210.5761. To appear in *Inventiones Mathematicae*. (pages 100 and 102)
- [Pix12] PIXTON, A. “Conjectural relations in the tautological ring of $\overline{\mathcal{M}}_{g,n}$.” Preprint, 2012. arXiv:1207.1918. (page 102)
- [Pol03] POLISHCHUK, A. *Abelian varieties, theta functions and the Fourier transform*. Cambridge Tracts in Mathematics, 153. Cambridge University Press, Cambridge, 2003. xvi+292 pp. (page 99)
- [Pol05] POLISHCHUK, A. “Universal algebraic equivalences between tautological cycles on Jacobians of curves.” *Mathematische Zeitschrift* 251 (2005), no. 4, 875–897. (pages 27, 35, 42, 65, 83, and 97)
- [Pol07] POLISHCHUK, A. “Lie symmetries of the Chow group of a Jacobian and the tautological subring.” *Journal of Algebraic Geometry* 16 (2007), no. 3, 459–476. (pages 10, 27, 31, 63, 64, 65, and 75)
- [Pol07b] POLISHCHUK, A. “Algebraic cycles on the relative symmetric powers and on the relative Jacobian of a family of curves. I.” *Selecta Mathematica* (New Series) 13 (2007), no. 3, 531–569. (pages 21, 22, 27, and 30)

- [Pol08] POLISHCHUK, A. “Fourier-stable subrings in the Chow rings of abelian varieties.” *Mathematical Research Letters* 15 (2008), no. 4, 705–714. (pages 11 and 79)
- [Sai96] SAITO, S. “Motives and filtrations on Chow groups.” *Inventiones Mathematicae* 125 (1996), no. 1, 149–196. (pages 84, 86, and 104)
- [Sai00] SAITO, S. “Motives and filtrations on Chow groups. II.” *The arithmetic and geometry of algebraic cycles* (Banff, AB, 1998), 321–346, NATO Science Series C: Mathematical and Physical Sciences, 548, Kluwer Academic Publishers, Dordrecht, 2000. (pages 84, 86, 103, and 104)
- [Sai00b] SAITO, S. “Motives, algebraic cycles and Hodge theory.” *The arithmetic and geometry of algebraic cycles* (Banff, AB, 1998), 235–253, CRM Proceedings & Lecture Notes, 24, American Mathematical Society, Providence, RI, 2000. (pages 84 and 104)
- [Sai02] SAITO, S. “Higher normal functions and Griffiths groups.” *Journal of Algebraic Geometry* 11 (2002), 161–201. (pages 84, 86, and 88)
- [Sch96] SCHOEN, C. “Varieties dominated by product varieties.” *International Journal of Mathematics* 7 (1996), no. 4, 541–571. (page 81)
- [Schw63] SCHWARZENBERGER, R. “Jacobians and symmetric products.” *Illinois Journal of Mathematics* 7 (1963), 257–268. (page 46)
- [Seb12] SEBASTIAN, R. “Smash nilpotent cycles on products of curves.” Preprint, 2012. arXiv:1207.7344. To appear in *Compositio Mathematica*. (pages 65, 78, and 80)
- [Sou84] SOULÉ, C. “Groupes de Chow et K-théorie de variétés sur un corps fini.” *Mathematische Annalen* 268 (1984), no. 3, 317–345. (page 12)
- [Tav11] TAVAKOL, M. “The tautological ring of $\mathcal{M}_{1,n}^{\text{ct}}$.” *Annales de l’Institut Fourier* 61 (2011), no. 7, 2751–2779. (page 102)
- [Tav11b] TAVAKOL, M. “The tautological ring of the moduli space $\mathcal{M}_{2,n}^{\text{rt}}$.” Preprint, 2011. arXiv:1101.5242. To appear in *International Mathematics Research Notices*. (page 102)
- [Voe95] VOEVODSKY, V. “A nilpotence theorem for cycles algebraically equivalent to zero.” *International Mathematics Research Notices* (1995), no. 4, 187–198. (pages 77 and 78)
- [Voi96] VOISIN, C. “Remarks on zero-cycles of self-products of varieties.” *Moduli of vector bundles* (Sanda, 1994; Kyoto, 1994), 265–285, Lecture Notes in Pure and Applied Mathematics, 179, Dekker, New York, 1996. (page 78)

BIBLIOGRAPHY

- [Voi02] VOISIN, C. *Théorie de Hodge et géométrie algébrique complexe*. Cours Spécialisés, 10. Société Mathématique de France, Paris, 2002. viii+595 pp. (pages 84, 100, and 104)
- [Voi08] VOISIN, C. “On the Chow ring of certain algebraic hyper-Kähler manifolds.” *Pure and Applied Mathematics Quarterly* 4 (2008), no. 3, part 2, 613–649. (page 102)
- [Voi12] VOISIN, C. “Chow rings and decomposition theorems for families of $K3$ surfaces and Calabi-Yau hypersurfaces.” *Geometry & Topology* 16 (2012), no. 1, 433–473. (pages 16 and 17)
- [Voi12b] VOISIN, C. *Chow rings, decomposition of the diagonal and the topology of families*. Notes for the Hermann Weyl Lectures, 2012. To appear in *Annals of Mathematics Studies*, Princeton University Press, Princeton, NJ. (page 104)
- [Voi13] VOISIN, C. “Infinitesimal invariants for cycles modulo algebraic equivalence and 1-cycles on Jacobians.” Preprint, 2013. arXiv:1304.4095. To appear in *Algebraic Geometry*. (pages 42, 64, 83, and 84)

Index

- Abel-Jacobi map, 12, 66, 84, 86
- Abel-Jacobi map for curves of compact type, 67
- addition map, 7, 44
- Albanese map, 72, 86

- Beauville conjecture, 12, 18
- Beauville decomposition, 9, 11, 17
- Bloch-Beilinson conjecture, 66
- Bloch-Beilinson filtration, 84, 100
- Bloch-Beilinson-Murre (BBM) conjecture, 78, 84, 100
- Brill-Noether theory, 42

- cap product, 45
- Ceresa cycle, 66, 83, 103
- Chow cohomology, 45
- Chow homology, 44
- Chow motive, 13
- Chow variety, 104
- CM type, 71
- coniveau, 92
- correspondence, 13
- cycle class map, 12, 17, 67, 87, 90

- Dutch house, 18, 28, 37, 58, 64

- Faber conjectures, 26, 100
- Faber-Pandharipande cycle, 72
- Fakhruddin's lemma, 67
- finite-dimensional motive, 78
- Fourier transform, 9, 48

- generalized Hodge conjecture, 104
- generic, 5, 66
- Gorenstein property, 26, 57, 58
- Gross-Schoen cycle, 103

- higher Griffiths group, 89
- Hilbert scheme, 104
- Hodge level, 77, 92, 96

- infinite symmetric power, 44
- infinitesimal invariant, 72, 84

- Kimura-O'Sullivan conjecture, 78

- ℓ -adic realization, 16
- Lefschetz decomposition, 10
- Lefschetz motive, 14
- Leray spectral sequence, 16, 86
- lower correspondence, 49

- moduli of curves, 26, 67, 100
- moduli stack, 26, 67

INDEX

- Morita's identity, 32
- motivic decomposition, 15
- motivic Lefschetz decomposition, 15
- motivic relation, 42
- multiplication by N , 7, 44
- multiplicative decomposition, 16
- Nori's filtration, 89
- Poincaré line bundle, 7
- polarization, 8, 99
- Pontryagin product, 8, 45
- Riemann-Roch theorem, 44
- S. Saito's filtration, 85, 103
- \mathfrak{sl}_2 -action, 11, 21, 35, 64
- smash-nilpotence, 77
- socle, 26, 57, 58
- spreading out, 67, 91
- standard conjectures, 78, 99, 104
- subcanonical point, 65
- supersingular curve, 71
- Tate motive, 14
- Tate twist, 16
- tautological class, 25, 27, 50, 51
- tautological cohomology, 51
- tautological homology, 51
- tautological map, 49, 101
- tautological relation, 35
- tautological ring, 25, 27, 50, 102
- theorem of the fixed part, 91
- theorem of the square, 12, 22
- theta characteristic, 23
- theta divisor, 23
- upper correspondence, 48
- variety dominated by a product of curves, 81
- very general, 5, 66
- Voevodsky conjecture, 78
- Voisin's conjecture on products of $K3$ surfaces, 102
- Weil cohomology, 12
- Weil conjectures, 12

Samenvatting

Tautologische klassen zijn meetkundig geconstrueerde klassen van algebraïsche cykels. De meetkunde en de enumeratieve eigenschappen van dergelijke klassen zijn bijzonder interessant.

Het eerste deel van dit proefschrift verenigt twee klassieke noties van tautologische klassen: de ene op de modulieruimte van krommen (in de zin van Mumford, Faber, *etc.*), en de andere op de Jacobiaan van een kromme (in de zin van Beauville, Polishchuk, *etc.*). In navolging van Polishchuk, construeren we relaties tussen tautologische klassen gebruikmakend van motivische structuren van de Jacobiaan. Met deze relaties verkrijgen we diverse gevolgen van de bekende vermoedens van Faber.

Het tweede deel is gewijd aan het detecteren van tautologische klassen die niet verdwijnen op de generieke Jacobiaan. Gebruikmakend van een degeneratie-argument van Fakhruddin ontwikkelen we een invariant in deze context. We detecteren niet-triviale klassen in de Chowgroepen en in de hogere Griffithsgroepen in de zin van S. Saito. In het bijzonder krijgen we een nieuw bewijs van een stelling van Green en Griffiths, alsook een verbetering van een resultaat van Ikeda.

Résumé

Les classes tautologiques sont des classes de cycles algébriques construites de façon géométrique. La géométrie et les propriétés énumératives autour de ces classes sont particulièrement intéressantes.

La première partie de cette thèse unifie deux notions classiques de classes tautologiques : l'une sur l'espace de modules des courbes (d'après Mumford, Faber, *etc.*), et l'autre sur la jacobienne d'une courbe (d'après Beauville, Polishchuk, *etc.*). Suivant Polishchuk, on construit des relations entre les classes tautologiques en utilisant les structures motiviques de la jacobienne. Avec ces relations, on obtient diverses conséquences sur les célèbres conjectures de Faber.

La deuxième partie est consacrée à la détection des classes tautologiques qui ne s'annulent pas sur la jacobienne générique. En utilisant un argument de dégénération dû à Fakhruddin, on développe un invariant simple dans ce contexte. On détecte des classes non-triviales dans les groupes de Chow et dans les groupes de Griffiths supérieurs au sens de S. Saito. En particulier, on obtient une nouvelle preuve d'un théorème de Green et Griffiths, ainsi qu'une amélioration d'un résultat d'Ikeda.

Curriculum vitae

Qizheng Yin was born on 18 July 1986 in Hebei, China. He grew up in Beijing and spent the best time of his childhood in Beijing Jingshan School, where he became interested in mathematics. After three years of undergraduate studies in Peking University, he participated in the Sélection internationale of the École normale supérieure in 2007. He was selected and went on studying in Paris.

The three years in Paris broadened his mathematical knowledge and vision. In 2009 Qizheng finished his master thesis under the supervision of Claire Voisin, who kindly brought him to the wonderful world of algebraic geometry. In 2010 he continued to study under Ben Moonen, first in Amsterdam and later in Nijmegen. The elegance, precision and distinguished style of his new advisor inspired him to complete the present work. Qizheng also received a postdoctoral position from ETH Zürich. Starting from January 2014, he will have the great opportunity to work with Rahul Pandharipande.

In 2013 Qizheng married his long-time girlfriend Zhiyue Zhou. Having lived in several cities in the world, they are both looking forward to a new journey in Zürich. Qizheng is also a part-time street photographer and has so far sold one photo in his career.

QIZHENG YIN

**Tautological Cycles
on Curves and Jacobians**



2013