ALGEBRAIC GEOMETRY AND ANALYTIC GEOMETRY

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Introduction

Let X be a projective algebraic variety, defined over the complex numbers. The study of X can be approached from two points of view: the *algebraic* point of view, in which one is interested in the local rings of the points of X, in rational or regular maps from X to other varieties, and the *analytic* (sometimes called "transcendental") point of view in which it is the notion of a holomorphic function on X which plays the principal role. One knows that this second point of view shows itself to be particularly fecund provided X is nonsingular, since this hypothesis allows one to apply all the resources of the theory of Kähler manifolds (harmonic forms, currents, cobordism, etc.).

In many questions, the two points of view lead to essentially equivalent results, although by very different methods. For example, we know that the differential forms which are holomorphic at each point of X are none other than the rational differential forms which are everywhere "of the first kind" (the variety X still assumed to be non-singular); Chow's Thm., which says that every closed analytic subspace of X is an algebraic variety, is another example of the same type.

The principal goal of this paper is to extend this equivalence to *coherent sheaves*; in a precise way, we show that coherent algebraic sheaves and coherent analytic sheaves correspond bijectively and that the correspondence between the two categories of sheaves leaves the cohomology groups invariant (see n° 12 for the statements); we indicate many applications of these results, notably to the comparison between algebraic fiber spaces and analytic fiber spaces.

The first two paragraphs are preliminary. In §1 we recall the definition and the principal properties of "analytic spaces." The definition that we have adopted is the one proposed by H. Cartan in [3], except for the fact that H. Cartan limited himself to *normal* varieties, a useless restriction for our goal; a very similar definition has been used by W.-L. Chow in his works, as yet unpublished, on this subject. In §2, we show how one can equip every algebraic variety X with the structure of an analytic space, and we give many elementary properties of this. The most important thing is without doubt the fact that, if $\mathcal{O}_x$ (resp. $\mathscr{H}_x$) denotes the local ring (resp. the ring of germs of holomorphic functions) of X at the point x, the rings $\mathcal{O}_x$ and $\mathscr{H}_x$ have the same completion, and, as a result, form a "flat couple" in the sense of the Appendix, Def. 4.

§3 contains the proofs of the theorems about coherent sheaves to which we have alluded above. These proofs rest principally both on the theory of coherent algebraic sheaves developed in [18], and on Thms. A and B in [3], exp. XVIII–XIX; in order to be complete, we have reproduced the proofs of these theorems.

§4 is dedicated to applications[1]: invariance of Betti numbers under an automorphism of the field of complex numbers, Chow's Thm., comparison of algebraic and analytic fiber spaces with structure group a given algebraic group. Our results on this last question are still greatly incomplete: of all the semisimple groups, we only know how to deal with the special linear group and the symplectic group.

Finally, we have needed a certain number of results about local rings which are not found explicitly in the literature; we have grouped them into an Appendix.

1. Analytic Spaces

1. **Analytic subsets of affine space.** Let n be an integer ≥ 0, and let $\mathbb{C}^n$ be the n-dimensional vector space over the complex numbers, equipped with the usual topology. If U is a subset of $\mathbb{C}^n$, we say that U is *analytic* if, for each $x \in U$, there are functions $f_1, \dots, f_k$, holomorphic in a neighborhood W of x, such that $U \cap W$ is identical to the set of points $x \in W$ satisfying the equations $f_i(z) = 0$, $i = 1, \dots, k$. The subset U is then locally closed in $\mathbb{C}^n$ (i.e., is the intersection of an open and a closed set), whence locally compact provided one equips it with the topology induced by that of $\mathbb{C}^n$.

We are now going to equip the topological space U with a sheaf. If X is a any space, we denote by $\mathscr{C}(X)$ the sheaf of germs of functions on X with values in $\mathbb{C}$ (cf. [18], n° 32). If $\mathscr{H}$ denotes the sheaf of germs of holomorphic functions on $\mathbb{C}^n$, the sheaf $\mathscr{H}$ is a subsheaf of $\mathscr{C}(\mathbb{C}^n)$. So let x be a point in U; we have a restriction homomorphism

$$\epsilon_x : \mathscr{C}(\mathbb{C}^n)_x \longrightarrow \mathscr{C}(U)_x.$$

The image of $\mathscr{H}_x$ under ϵ_x is a subring $\mathscr{H}_{x,U}$ of $\mathscr{C}(U)_x$; the $\mathscr{H}_{x,U}$ form a subsheaf $\mathscr{H}_U$ of $\mathscr{C}(U)$, which we call the *sheaf of germs of holomorphic functions* on U; it is a sheaf of rings. We denote by $\mathscr{A}_x(U)$ the kernel of $\epsilon_x : \mathscr{H}_x \to \mathscr{H}_{x,U}$; in light of the definition of $\mathscr{H}_{x,U}$, it is the set of $f \in \mathscr{H}_x$ whose restriction to U is zero in a neighborhood of x; we frequently identify $\mathscr{H}_{x,U}$ with the quotient ring $\mathscr{H}_x/\mathscr{A}_x(U)$.

Since we have a topology and a sheaf of functions on U, we can define the notion of a *holomorphic map* (cf. [3], exp. VI as well as [18], n° 32):

Let U and V be two analytic subsets of $\mathbb{C}^r$ and $\mathbb{C}^s$, respectively. A map $\phi : U \to V$ will be called holomorphic if it is continuous, and if $f \in \mathscr{H}_{\phi(x),V}$ implies $f \circ \phi \in \mathscr{H}_{x,U}$. This is the same as saying that the s coordinates of $\phi(x)$, $x \in U$, are holomorphic functions of x, which is to say sections of $\mathscr{H}_U$.

The composition of two holomorphic maps is holomorphic. A bijection $\phi : U \to V$ is called an *analytic isomorphism* (or simply an isomorphism) if ϕ and ϕ^{-1} are holomorphic; this is equivalent to saying that ϕ is a homeomorphism of U onto V which transforms the sheaf $\mathscr{H}_U$ onto the sheaf $\mathscr{H}_V$.

If U and U' are two analytic subsets of $\mathbb{C}^r$ and $\mathbb{C}^{r'}$, the product $U \times U'$ is an analytic subset of $\mathbb{C}^{r+r'}$. The properties stated in [18], n° 33 are valid, by replacing everywhere locally closed subset with analytic subset, and regular map

[1]We have left aside the applications to automorphic functions, for which we refer to [3], exp. XX.

by holomorphic map; in particular, if $\phi : U \to V$ and $\phi' : U' \to V'$ are analytic isomorphisms, then so is

$$\phi \times \phi' : U \times U' \longrightarrow V \times V'.$$

Nevertheless, contrary to the algebraic case, the topology of $U \times U'$ is the product of the topologies of U and U'.

2. **The notion of an analytic space.**

Definition 1. We call an analytic space a set X equipped with a topology and a subsheaf $\mathscr{H}_X$ of the sheaf $\mathscr{C}(X)$, these data being subject to the following axioms:

- **(H1):** There exists an open cover $\{V_i\}$ of X such that each V_i, equipped with the topology and sheaf induced by those of X, is isomorphic to an analytic subset U_i of an affine space, equipped with the topology and sheaf defined in n° 1.
- **(H2):** The topology of X is Hausdorff.

The definitions of n° 1, being of a local nature, extend to analytic spaces. Thus, if X is an analytic space, the sheaf $\mathscr{H}_X$ will be called the sheaf of germs of holomorphic functions on X; if X and Y are two analytic spaces, a map $\phi : X \to Y$ will be called holomorphic if it is continuous and if $f \in \mathscr{H}_{\phi(x),Y}$ implies $f \circ \phi \in \mathscr{H}_{x,X}$; these maps form a family of *morphisms* (in the sense of N. Bourbaki) for the structure of an analytic space.

If V is an open subset of an analytic space X, we will call a *chart* of V any analytic isomorphism of V onto an analytic subspace U of some affine space. The axiom **(H1)** shows that it is possible to cover X by opens possessing charts. If Y is a subset of X, we will say that Y is analytic if, for each chart $\phi : V \to U$, the image $\phi(Y \cap V)$ is an analytic subset of U. If this is the case, then Y is locally closed in X, and can be equipped in a natural way with the structure of an analytic space, said to be *induced* by that of X (cf. [18], n° 35 for the algebraic case). Likewise, let X and X' be two analytic spaces; then there exists on $X \times X'$ a unique analytic structure such that if $\phi : V \to U$ and $\phi' : V' \to U'$ are charts, $\phi \times \phi' : V \times V' \to U \times U'$ is a chart of $V \times V'$; equipped with this structure, $X \times X'$ is called the *product* of the analytic spaces X and X'; one will observe that the topology of $X \times X'$ coincides with the product of the topologies of X and X'.

We leave to the reader the task of transporting the other results of [18], n$^{\text{os}}$ 34–35.

3. **Analytic sheaves.**

The definition of analytic sheaves given in [2], exp. XV extends to the case of an analytic space X: an analytic sheaf $\mathscr{F}$ is simply a sheaf of modules over the sheaf of rings $\mathscr{H}_X$, which is to say a sheaf of $\mathscr{H}_X$-modules (cf. [18], n° 6).

Let Y be a closed analytic subspace of X; for each $x \in X$, let $\mathscr{A}_x(Y)$ be the set of $f \in \mathscr{H}_{x,X}$ whose restriction to Y is zero in a neighborhood of x. The $\mathscr{A}_x(Y)$ form a sheaf of ideals $\mathscr{A}(Y)$ of the sheaf $\mathscr{H}_X$; the sheaf $\mathscr{A}(Y)$ is thus an analytic sheaf. The quotient sheaf $\mathscr{H}_X/\mathscr{A}(Y)$ is zero outside of Y, and its restriction to Y is none other than $\mathscr{H}_Y$, by the very definition of the induced structure; one may thus identify it with $\mathscr{H}_Y$, cf. [18], n° 5.

Proposition 1. *a) The sheaf $\mathscr{H}_X$ is a coherent sheaf of rings ([18], n^o 15).*

b) If Y is a closed analytic subspace of X, the sheaf $\mathscr{A}(Y)$ is a coherent analytic sheaf (i.e., a coherent sheaf of $\mathscr{H}_X$-modules, in the sense of [18], n^o 12).

Proof. In the case where X is open in $\mathbb{C}^n$, these results are due to K. Oka and H. Cartan, cf. [1], Thms. 1 and 2 as well as [2], exp. XV–XVI. The general case reduces immediately to these; indeed, since the question is local, one may assume that X is a closed analytic subset of an open U in $\mathbb{C}^n$; one has $\mathscr{H}_X = \mathscr{H}_U/\mathscr{A}(X)$, and, according to the preceding, $\mathscr{H}_U$ is a coherent sheaf of rings and $\mathscr{A}(X)$ is a coherent sheaf of ideals of $\mathscr{H}_U$; it follows from this that $\mathscr{H}_U$ is coherent, cf. [18], n^o 16. The assertion b) is proven in the same way. □

As other examples of coherent analytic sheaves, we indicate the sheaves of germs of sections of fiber spaces with vector space fibers (cf. n^o 20), and the sheaf of germs of automorphic functions.

4. **A neighborhood of a point in an analytic space.** Let X be an analytic space, x a point of X, and $\mathscr{H}_x$ the ring of germs of holomorphic functions on X at the point x; this ring is an algebra over $\mathbb{C}$, admitting as its unique maximal ideal the ideal $\mathfrak{m}$ formed from the functions f vanishing at x, and the field $\mathscr{H}_x/\mathfrak{m}$ is none other than $\mathbb{C}$; in other words, $\mathscr{H}_x$ is a *local algebra* over $\mathbb{C}$. When $X = \mathbb{C}^n$, the algebra $\mathscr{H}_x$ is none other than the algebra $\mathbb{C}\{z_1, \dots, z_n\}$ of convergent series in n variables; in the general case, $\mathscr{H}_x$ is isomorphic to a quotient algebra $\mathbb{C}\{z_1, \dots, z_n\}/\mathfrak{a}$, since X is locally isomorphic to an analytic subspace of $\mathbb{C}^n$; if follows from this that $\mathscr{H}_x$ is a *Noetherian ring*; it is moreover an *analytic* ring, in the sense of H. Cartan ([3], exp. VII).

Once sees easily that the knowledge of $\mathscr{H}_x$ determines X in a neighborhood of X ([3], *loc. cit.*). In particular, for X to be isomorphic to $\mathbb{C}^n$ in a neighborhood of x, it is necessary and sufficient that the algebra $\mathscr{H}_x$ be isomorphic to $\mathbb{C}\{z_1, \dots, z_n\}$; one sees easily that this condition is equivalent to saying that $\mathscr{H}_x$ is a *regular* local ring of dimension n (for all that concerns local rings, cf. [15]). The point x is then called a *simple* point of dimension n on X; if all the points of X are simple, X is called a *manifold*.

We return to the general case; the ring $\mathscr{H}_x$ having no nonzero nilpotent elements, it follows that (cf. [16], Chap. IV, §2) one has:

$$0 = \bigcap \mathfrak{p}_i,$$

the $\mathfrak{p}_i$ denoting the minimal prime ideals of $\mathscr{H}_x$. If one denotes by X_i the irreducible components of X at x, one has $\mathfrak{p}_i = \mathscr{A}_x(X_i)$ and $\mathscr{H}_x/\mathfrak{p}_i = \mathscr{H}_{x,X_i}$. This essentially carries the local study of X to that of the X_i; for example, the *dimension* (analytic—i.e., half of the topological dimension) of X at x is the supremum of the dimensions of the X_i. One observes that this dimension *coincides* with the dimension (in the sense of Krull) of the local ring $\mathscr{H}_x$; indeed, it suffices to verify this when X is irreducible at x, i.e., when $\mathscr{H}_x$ is an integral domain; in this case, if one denotes by r the analytic dimension of X at x, one knows (cf. [14], §4 as well as [3], exp. VIII) that $\mathscr{H}_x$ is a finite extension of $\mathbb{C}\{z_1, \dots, z_r\}$; since $\mathbb{C}\{z_1, \dots, z_r\}$ has as its completion the algebra of formal power series $\mathbb{C}[\![z_1, \dots, z_r]\!]$, its dimension is r, and thus it is also so for $\mathscr{H}_x$, according to [15], p. 18, which proves our assertion.

2. The Analytic Space Associated to an Algebraic Variety

In what follows, we are going to consider algebraic varieties over the field $\mathbb{C}$. Such a variety will be equipped with with two topologies: the "usual" topology, and the "Zariski" topology. In order to avoid confusion, we will place the letter Z in front of notions relative to the latter topology; for example, "Z-open" will signify "open in the Zariski topology."

5. **Definition of the analytic space associated to an algebraic variety.** The possibility of equipping every algebraic variety with the structure of an analytic space results from the following lemma:

Lemma 1. *a) The Z-topology of $\mathbb{C}^n$ is less fine than the usual topology.*

b) Every Z-locally closed subset of $\mathbb{C}^n$ is analytic.

c) If U and U' are two Z-locally closed subsets of $\mathbb{C}^n$ and $\mathbb{C}^{n'}$, and if $f : U \to U'$ is a regular map, then f is holomorphic.

d) In the hypotheses of c), if one supposes moreover that f is a biregular isomorphism, then it is also an analytic isomorphism.

Proof. By definition, a Z-closed subset of $\mathbb{C}^n$ is defined as the zero set of a certain number of polynomials; since a polynomial is continuous (resp. holomorphic) in the usual topology, one deduces from this a) (resp. b)). In order to prove c), one may assume that $U' = \mathbb{C}$; one must then show that every regular function on U is holomorphic, which results again from the fact that a polynomial is a holomorphic function. Finally, d) is an immediate consequence of c), applied to f^{-1}. □

Now let X be an algebraic variety over the field $\mathbb{C}$ (in the sense of [18], n° 34, so not necessarily irreducible). Let V be a Z-open subset of X, possessing an (algebraic) chart

$$\phi : V \longrightarrow U$$

onto a Z-locally closed subset U of an affine space. According to Lemma 1, b), U can be equipped with the structure of an analytic space.

Proposition 2. *There exists on X a unique structure of an analytic space such that, for every chart $\phi : V \to U$, the Z-open set V is open, and ϕ is an analytic isomorphism of V (equipped with the analytic structure induced by that of X) onto U (equipped with the analytic structure defined in n° 1).*

(More briefly: every algebraic chart must be an analytic chart).

Proof. The uniqueness is clear, since one can cover X by Z-open sets V possessing charts. In order to prove existence, let $\phi : V \to U$ be a chart, and transport to V the analytic structure of U by way of ϕ^{-1}. If $\phi' : V' \to U'$ is another chart, the analytic structures induced on $V \cap V'$ by V and V' are the same, by virtue of Lemma 1, d); moreover, $V \cap V'$ is open at once in V and in V', by virtue of Lemma 1, a). By gluing, one obtains also on X a topology and a sheaf $\mathscr{H}_X$ which visibly satisfies the axiom **(H1)**. In order to satisfy **(H2)**, we use the axiom **(VA2′)** of [18], n° 34; with the notation of this axiom, the graphs T_{ij} of the identification relations between two U_i and U_j are Z-closed in $U_i \times U_j$, thus *a fortiori* closed, which surely shows that X is Hausdorff. □

Remark. One can give a direct definition of the analytic structure on X, without passing to the charts $\phi : V \to U$. One defines the topology as the least fine in which the regular functions on the Z-open subsets of X are still continuous, and one defines $\mathscr{H}_{x,X}$ to be the analytic subring of $\mathscr{C}(X)_x$ generated by $\mathcal{O}_{x,X}$ (in the sense of [3], exp. VIII). We leave to the reader the task of verifying the equivalence of these two definitions.

In what follows, we denote by X^h the set X equipped with the structure of an analytic space which was just defined. The topology of X^h is *finer* than the topology of X; since X^h can be covered by a finite number of opens possessing charts, X^h is a locally compact *σ-compact*[2] space.

The following properties result immediately from the definition of X^h:

If X and Y are two algebraic varieties, one has $(X \times Y)^h = X^h \times Y^h$. If Y is a Z-locally closed subset of X, then Y^h is an analytic subset of X^h; moreover, the analytic structure of Y^h coincides with the analytic structure induced on Y by X^h. Finally, if $f : X \to Y$ is a regular map of an algebraic variety X to an algebraic variety Y, f is also a holomorphic map from X^h to Y^h.

6. **Relations between the local ring at a point and the ring of holomorphic functions at that point.** Let X be an algebraic variety, and let x be a point of X. We intend to compare the local ring $\mathcal{O}_x$ of regular functions on X at the point x with the local ring $\mathscr{H}_x$ of holomorphic functions on X^h in a neighborhood of x.

Since every regular function is holomorphic, every function $f \in \mathcal{O}_x$ defines a germ of a holomorphic function at x, which we denote by $\theta(f)$. The map $\theta : \mathcal{O}_x \to \mathscr{H}_x$ is a homomorphism, and maps the maximal ideal of $\mathcal{O}_x$ into that of $\mathscr{H}_x$; it extends then by continuity to a homomorphism $\hat{\theta} : \hat{\mathcal{O}}_x \to \hat{\mathscr{H}}_x$ of the completion of $\mathcal{O}_x$ into that of $\mathscr{H}_x$ (cf. Appendix, n° 24).

Proposition 3. *The homomorphism* $\hat{\theta} : \hat{\mathcal{O}}_x \to \hat{\mathscr{H}}_x$ *is bijective.*

We will prove the preceding proposition at the same time as another result:

Let Y be a Z-locally closed subset of X, and let $\mathscr{I}_x(Y)$ (or $\mathscr{I}_x(Y, X)$ if one wants to specify X) be the ideal of $\mathcal{O}_x$ formed from functions f whose restriction to Y is zero in a Z-neighborhood of x (cf. [18], n° 39). The image of $\mathscr{I}_x(Y)$ under θ is clearly contained in the ideal $\mathscr{A}_x(Y)$ of $\mathscr{H}_x$ defined in n° 3.

Proposition 4. *The ideal* $\mathscr{A}_x(Y)$ *is generated by* $\theta\big(\mathscr{I}_x(Y)\big)$.

Proof. We will first prove Props. 3 and 4 in the particular case where X is the affine space $\mathbb{C}^n$. Prop. 3 is then trivial, for $\hat{\mathcal{O}}_x$ and $\hat{\mathscr{H}}_x$ are none other than the algebra $\mathbb{C}[\![z_1, \dots, z_n]\!]$ of formal power series in n indeterminates. We pass to Prop. 4; let $\mathfrak{a}$ be the ideal of $\mathscr{H}_x$ generated by $\mathscr{I}_x(Y)$ (the ring $\mathcal{O}_x$ being identified with a subring of $\mathscr{H}_x$ by way of θ). Each ideal of $\mathscr{H}_x$ defines the germ of an analytic subset of X

[2] In fact, X^h is easily seen to be second countable (hence *a fortiori* σ-compact, as it is locally compact) [translator's note].

at x, cf. [1], n° 3 or [3], exp. VI, p. 6; it is clear that the germ defined by $\mathfrak{a}$ is none other than Y. So let f be an element of $\mathscr{A}_x(Y)$; by virtue of the "zero theorem" (which is valid for the ideals of $\mathscr{H}_x$, cf. [14], p. 278, as well as [2], exp. XIV, p. 3, and [3], exp. VIII, p. 9) there exists an integer $r \geq 0$ such that $f^r \in \mathfrak{a}$. *A fortiori*, one has

$$f^r \in \mathfrak{a}\hat{\mathscr{H}}_x = \mathscr{I}_x(Y)\hat{\mathscr{H}}_x = \mathscr{I}_x(Y)\hat{\mathcal{O}}_x.$$

But the ideal $\mathscr{I}_x(Y)$ is an intersection of prime ideals, which correspond to the irreducible components of Y passing through x. According to a theorem of Chevalley (cf. [15], p. 40 as well as [17], p. 67), the same is then true of the ideal $\mathscr{I}_x\hat{\mathcal{O}}_x$, and the relation $f^r \in \mathscr{I}_x\hat{\mathcal{O}}_x$ thus implies $f \in \mathscr{I}_x\hat{\mathcal{O}}_x$. Since $\mathscr{H}_x$ is a Noetherian local ring, one has $\mathfrak{a}\hat{\mathscr{H}}_x \cap \mathscr{H}_x = \mathfrak{a}$ (cf. [16], Chap. IV, or Appendix, Prop. 27); one then has $f \in \mathfrak{a}$, which proves Prop. 4 in the case under consideration.

We pass to the general case. The question being local, one may assume that X is a subvariety of an affine space which we denote by U. By definition, one has:

$$\mathcal{O}_x = \mathcal{O}_{x,U}/\mathscr{I}_x(X,U) \quad \text{and} \quad \mathscr{H}_x = \mathscr{H}_{x,U}/\mathscr{A}_x(X,U).$$

The map $\theta : \mathcal{O}_x \to \mathscr{H}_x$ is obtained by passage to the quotient from the map $\theta : \mathcal{O}_{x,U} \to \mathscr{H}_{x,U}$, and, according to the preceding, we know that $\hat{\theta} : \hat{\mathcal{O}}_{x,U} \to \hat{\mathscr{H}}_{x,U}$ is bijective, and that $\mathscr{A}_x(X,U) = \theta\big(\mathscr{I}_x(X,U)\big)\mathscr{H}_{x,U}$. Prop. 3 results from this immediately, by applying Prop. 29 of the Appendix. As for Prop. 4, it results from the fact that $\mathscr{A}_x(Y)$ is the canonical image of the ideal $\mathscr{A}_x(Y,U)$, which is generated by $\theta\big(\mathscr{I}_x(Y,U)\big)$ according to the preceding. □

Prop. 3 shows in particular that $\theta : \mathcal{O}_x \to \mathscr{H}_x$ is *injective*, which permits us to identify $\mathcal{O}_x$ with the subring $\theta(\mathcal{O}_x)$ of $\mathscr{H}_x$. Taking into account this identification, one has:

Corollary 1. *The pair of rings $(\mathcal{O}_x, \mathscr{H}_x)$ is a flat couple (in the sense of the Appendix, Def. 4).*

Proof. This follows immediately from Prop. 3 and Prop. 28 of the Appendix. □

Corollary 2. *The rings $\mathcal{O}_x$ and $\mathscr{H}_x$ have the same dimension.*

Proof. Indeed, one knows that the dimension of a Noetherian local ring is equal to that of its completion (cf. [15], p. 26). □

Taking into account some results stated in n° 4, one obtains the following result (where we assume X to be irreducible in order to simplify the statement):

Corollary 3. *If X is an irreducible algebraic variety of dimension r, the analytic space X^h is of analytic dimension r at each of its points.*

7. Relations between the usual topology and the Zariski topology of an algebraic variety.

Proposition 5. *Let X be an algebraic variety, and U a subset of X. If U is Z-open and Z-dense in X, then U is dense in X.*

Proof. Let Y be the complement of U in X; it is a Z-closed subset of X. Let x be a point of X; if x does not belong to the closure of U, one has $Y = X$ in a neighborhood of x, whence $\mathscr{A}_x(Y) = 0$, with the notation of n° 6. Since $\mathscr{A}_x(Y)$ contains $\theta\big(\mathscr{I}_x(Y)\big)$, and since θ is injective (Prop. 3), one then has that $\mathscr{I}_x(Y) = 0$, which means that $Y = X$ in a Z-neighborhood of x, contrary to the hypothesis that U is Z-dense in X, QED. □

Remark. One sees easily that Prop. 5 is *equivalent* to the fact that $\theta : \mathcal{O}_x \to \mathscr{H}_x$ is injective, a much more elementary fact than Prop. 3, and which one can, for example, prove by reduction to the case of a curve.

We are now going to give two simple applications of Prop. 5.

Proposition 6. *In order for an algebraic variety X to be complete, it is necessary and sufficient that it be compact.*

Proof. We first recall a result of Chow (cf. [7], as well as [19], n° 4): for each algebraic variety X, there exists a projective variety Y, a subset U of Y, Z-open and Z-dense in Y, and a surjective regular map $f : U \to X$ whose graph T is Z-closed in $X \times Y$. One has $U = Y$ if an only if X is complete.

This being the case, we first suppose that X is complete; one then has $X = f(Y)$, and, since every projective variety is compact in the usual topology, one concludes that X is compact. Conversely, suppose X is compact; then T, which is closed in $X \times Y$, is likewise compact; thus U is closed in Y, and Prop. 5 shows that $U = Y$, which completes the proof. □

The following lemma is essentially due to Chevalley:

Lemma 2. *Let $f : X \to Y$ be a regular map of an algebraic variety X to an algebraic variety Y, and suppose that $f(X)$ is Z-dense in Y. Then there exists a subset $U \subseteq f(X)$ which is Z-open and Z-dense in Y.*

Proof. Provided that X and Y are irreducible, this result is well know, cf. [4], exp. 3 or [17], p. 15, for example. We are going to reduce the general case to this: let X_i, $i \in I$, be the irreducible components of X, and let Y_i be the Z-closure of $f(X_i)$ in Y; the Y_i are irreducible, and one has $Y = \bigcup Y_i$; there is then $J \subseteq I$ such that the Y_j, $j \in J$, are the irreducible components of Y. According to the result recalled at the beginning, for each $j \in J$, there exist a subset $U_j \subseteq f(X_j)$ which is Z-open and Z-dense in Y_j; even if it means restricting U_j, one can moreover assume that U_j does not meet any of the Y_k, $k \in J$, $k \neq j$. Then by setting $U = \bigcup_{j \in J} U_j$, one obtains a subset of Y which satisfies all the required properties. □

Proposition 7. *If $f : X \to Y$ is a regular map of an algebraic variety X to an algebraic variety Y, the closure and the Z-closure of $f(X)$ coincide.*

Proof. Let T be the Z-closure of $f(X)$ in Y. By applying Lemma 2 to $f : X \to T$, one sees that there exists a subset $U \subseteq f(X)$ which is Z-open and Z-dense in T. According to Prop. 5, U is then dense in T, and the same is *a fortiori* true of $f(X)$; this shows that T is contained in the closure of $f(X)$; since the opposite inclusion is clear, this completes the proof. □

8. **An analytic criterion for regularity.** One knows that every regular map is holomorphic. The following proposition (which we will add to moreover in n° 19) indicates in which case the converse is true.

Proposition 8. *Let X and Y be two algebraic varieties, and let $f : X \to Y$ be a holomorphic map from X to Y. If the graph T of f is a Z-locally closed subset (i.e., an algebraic subvariety) of $X \times Y$, the map f is regular.*

Proof. Let $p = \mathrm{pr}_X$ be the canonical projection of T onto the first factor X of $X \times Y$; the map p is regular, bijective, and its inverse map is the map $x \mapsto \big(x, f(x)\big)$, which is holomorphic by hypothesis; thus p is an analytic isomorphism, and we have reduced to showing that p is a biregular isomorphism (since one has $f = \mathrm{pr}_Y \circ p^{-1}$). This results from the following proposition: □

Proposition 9. *Let T and X be two algebraic varieties, and let $p : T \to X$ be a bijective regular map. If p is an analytic isomorphism from T onto X, then it is also a biregular isomorphism.*

Proof. We show first that p is a homeomorphism for the Zariski topologies of T and X. Let F be a Z-closed subset of T; since p is an analytic isomorphism, it is *a fortiori* a homeomorphism, and $p(F)$ is closed in X. By applying Prop. 7 to $p : F \to X$, one concludes that $p(F)$ is Z-closed in X, which proves our assertion.

It remains for us to show that p transforms the sheaf $\mathcal{O}_X$ of local rings of X into the sheaf $\mathcal{O}_T$ of local rings of T. More precisely, if t is a point of T, and if $x = p(t)$, the map p defines a homomorphism

$$p^* : \mathcal{O}_{x,X} \longrightarrow \mathcal{O}_{t,T},$$

and we need to prove that p^* is bijective[3].

From the fact that p is a Z-homeomorphism, p^* is injective, which allows us to identify $\mathcal{O}_{x,X}$ with a subring of $\mathcal{O}_{t,T}$. In order to simplify the exposition, we put $A = \mathcal{O}_{x,X}$, $A' = \mathcal{O}_{t,T}$, so that one has $A \subseteq A'$. Likewise, we denote by B (resp. B') the ring $\mathscr{H}_{x,X}$ (resp. $\mathscr{H}_{t,T}$), and we consider A and A' as included respectively into B and B' (which is allowed, by virtue of Prop. 3). The hypothesis that p is an analytic isomorphism means that $B = B'$.

Let X_i be the irreducible components of X passing through x; each X_i determines a prime ideal $\mathfrak{p}_i = \mathscr{I}_x(X_i)$ of the ring A, and the local quotient $A_i = A/\mathfrak{p}_i$ is none other than the local ring of x on X_i; the quotient field of A_i, call it K_i, is thus none other than the field of rational functions on the irreducible variety X_i. The ideals $\mathfrak{p}_i$ are clearly the minimal primes of the ring A, and one has $0 = \bigcap \mathfrak{p}_i$. The set S of elements of A which do not belong to any of the $\mathfrak{p}_i$ is multiplicative (it is easy to see that it is the set of regular elements of A). The total ring of fractions A_S is equal to the direct product of the fields K_i (cf. Lemma 3 hereafter).

Let $T_i = p^{-1}(X_i)$; since p is a Z-homeomorphism, the T_i are the irreducible components of T passing through t, and define the prime ideals $\mathfrak{p}'_i$ of A'; we put again $A'_i = A'/\mathfrak{p}'_i$, and we denote by K'_i the field of fraction of A'_i; the total ring of

[3]The proof which follows was communicated to me by P. Samuel.

fractions $A'_{S'}$ is equal to the direct product of the K'_i. We note that $\mathfrak{p}'_i \cap A = \mathfrak{p}_i$, whence $A_i \subseteq A'_i$, $K_i \subseteq K'_i$, and $A_S \subseteq A'_{S'}$.

We are first going to show that $K_i = K'_i$, said differently that p defines a "birational" correspondence between T_i and X_i; since $p : T_i \to X_i$ is a Z-homeomorphism, T_i and X_i have the same dimension, and the fields K_i and K'_i have the same transcendence degree over $\mathbb{C}$. If one puts then $n_i = [K'_i : K_i]$, one knows[4] that there exists a nonempty Z-open subset U_i of X_i such that the inverse image of every point of U_i is made up of exactly n_i points of T_i. Since p is bijective, this shows that $n_i = 1$, and one has of course $K_i = K'_i$.

Since A_S (resp. $A'_{S'}$) is the direct product of the K_i (resp. the K'_i), it follows that $A_S = A'_{S'}$. So let $f' \in A'$; according to what precedes, one has $f' \in A_S$, i.e., there exist $g \in A$ and $s \in S$ such that $g = sf'$. One then has $g \in sA'$, whence $g \in sB'$, which is to say $g \in sB$. But, according to Cor. 1 of Prop. 3, the couple (A, B) is a flat couple, and one thus has $sB \cap A = sA$, cf. Appendix, n° 22. One then gets $g \in sA$, i.e., there exists $f \in A$ such that $g = sf$, where again $s(f - f') = 0$, and, since s is not a zerodivisor in A'; this implies $f = f'$, which is to say $A = A'$, QED. □

We have used in the course of the proof the following result, which we are now going to prove:

Lemma 3. *Let A be a commutative ring, in which the ideal* 0 *is the intersection of a finite number of distinct minimal prime ideals $\mathfrak{p}_i$; let K_i be the field of fractions of $A/\mathfrak{p}_i$, and let S be the set of those elements of A which do not belong to any of the $\mathfrak{p}_i$. The ring of fractions A_S is then isomorphic to the direct product of the K_i.*

Proof. One knows that the prime ideals of A_S correspond bijectively to those prime ideals of A which do not meet S (cf. [16], Chap. VI, §3, to which we return for all that concerns rings of fractions). It follows that, if one puts $\mathfrak{m}_i = \mathfrak{p}_i A_S$, the $\mathfrak{m}_i$ are the only prime ideals of A_S; in particular, they are maximal ideals, clearly distinct, since $\mathfrak{m}_i \cap A = \mathfrak{p}_i$ ([16], *loc. cit.*). Moreover, the field $A_S/\mathfrak{m}_i$ is generated by $A/\mathfrak{p}_i$, and thus coincides with K_i. It remains to show that the canonical homomorphism

$$\phi : A_S \longrightarrow \prod A_S/\mathfrak{m}_i = \prod K_i$$

is bijective.

In the first place, the relation $\bigcap \mathfrak{p}_i = 0$ implies $\bigcap \mathfrak{m}_i = 0$, which shows that ϕ is injective. We denote then by $\mathfrak{b}_i$ the product (in the ring A_S) of the ideals $\mathfrak{m}_j$, $j \neq i$, and we put $\mathfrak{b} = \sum \mathfrak{b}_i$. The ideal $\mathfrak{b}$ is not contained in any of the $\mathfrak{m}_i$, and so is identical to A_S, and there exist elements $x_i \in \mathfrak{b}_i$ such that $\sum x_i = 1$. One has:

$$x_i \equiv 1 \pmod{\mathfrak{m}_i} \quad \text{and} \quad x_i \equiv 0 \pmod{\mathfrak{m}_j}, \quad j \neq i,$$

which shows that $\phi(A_S)$ contains the elements $(1, 0, \ldots, 0), \ldots, (0, \ldots, 0, 1)$ of $\prod K_i$. Since these elements generate the A_S-module $\prod K_i$, this indeed shows that ϕ is bijective, and completes the proof. □

[4] This is a classical result, and easy to prove, about correspondences. One will find in [17], p. 16 a slightly weaker result, but sufficient for the application for which we are using it.

3. The Correspondence Between Algebraic Sheaves and Coherent Analytic Sheaves

9. **The analytic sheaf associated to an algebraic sheaf.** Let X be an algebraic variety, and let X^h be the analytic space which is associated to it by the process in n° 5. If $\mathscr{F}$ is any sheaf on X, we will equip the set $\mathscr{F}$ with a new topology which makes it into a sheaf on X^h; this topology is defined in the following manner: if $\pi : \mathscr{F} \to X$ denotes the projection from $\mathscr{F}$ to X, one injects $\mathscr{F}$ into $X^h \times \mathscr{F}$ by the map $f \mapsto (\pi(f), f)$, and the topology in question is that induced on $\mathscr{F}$ by that of $X^h \times \mathscr{F}$. One verifies immediately that one has equipped the set $\mathscr{F}$ with the structure of a sheaf on X^h, a sheaf which we denote by $\mathscr{F}'$. For each $x \in X$, one then has $\mathscr{F}'_x = \mathscr{F}_x$; the sheaves $\mathscr{F}$ and $\mathscr{F}'$ only differ in their topologies ($\mathscr{F}'$ is nothing more than the *inverse image* of $\mathscr{F}$ under the continuous map $X^h \to X$).

What precedes applies in particular to the sheaf $\mathcal{O}$ of local rings of X; Prop. 3 of n° 6 allows us to identify the sheaf $\mathcal{O}'$ so obtained with a subsheaf of the sheaf $\mathscr{H}$ of germs of holomorphic functions on X^h.

Definition 2. Let $\mathscr{F}$ be an algebraic sheaf on X. One calls the analytic sheaf associated to $\mathscr{F}$ the sheaf $\mathscr{F}^h$ on X^h defined by the formula:

$$\mathscr{F}^h = \mathscr{F}' \otimes \mathscr{H},$$

the tensor product being taken over the sheaf of rings $\mathcal{O}'$.

(Said differently, $\mathscr{F}^h$ comes from $\mathscr{F}'$ by extension of the ring of operators to $\mathscr{H}$.)

The sheaf $\mathscr{F}^h$ is a sheaf of $\mathscr{H}$-modules, which is to say an *analytic* sheaf; the injection $\mathcal{O}' \to \mathscr{H}$ defines a canonical homomorphism $\alpha : \mathscr{F}' \to \mathscr{F}^h$.

Every algebraic homomorphism (which is to say $\mathcal{O}$-linear)

$$\phi : \mathscr{F} \longrightarrow \mathscr{G}$$

defines, by extension of the ring of operators, an analytic homomorphism

$$\phi^h : \mathscr{F}^h \longrightarrow \mathscr{G}^h.$$

Thus $\mathscr{F}^h$ is a *covariant functor* of $\mathscr{F}$.

Proposition 10. *a) The functor $\mathscr{F}^h$ is an exact functor.*

b) For every algebraic sheaf $\mathscr{F}$, the homomorphism $\alpha : \mathscr{F}' \to \mathscr{F}^h$ is injective.

c) If $\mathscr{F}$ is a coherent algebraic sheaf, $\mathscr{F}^h$ is a coherent analytic sheaf.

Proof. If $\mathscr{F}_1 \to \mathscr{F}_2 \to \mathscr{F}_3$ is an exact sequence of algebraic sheaves, it is clear that the sequence $\mathscr{F}'_1 \to \mathscr{F}'_2 \to \mathscr{F}'_3$ is too, and also the sequence

$$\mathscr{F}'_1 \otimes \mathscr{H} \longrightarrow \mathscr{F}'_2 \otimes \mathscr{H} \longrightarrow \mathscr{F}'_3 \otimes \mathscr{H},$$

according to Cor. 1 to Prop. 3, which proves a). The assertion b) is likewise a result of the same corollary.

In order to prove c), we first remark that one has $\mathcal{O}^h = \mathscr{H}$; so if $\mathscr{F}$ is coherent algebraic, and if x is a point of X, one can find an exact sequence:

$$\mathcal{O}^q \longrightarrow \mathcal{O}^p \longrightarrow \mathscr{F} \longrightarrow 0,$$

valid on a Z-neighborhood U of x. According to a), one gets an exact sequence

$$\mathscr{H}^q \longrightarrow \mathscr{H}^q \longrightarrow \mathscr{F}^h \longrightarrow 0,$$

valid over U. Since U is a neighborhood of x, and since the sheaf of rings $\mathscr{H}$ is coherent (Prop. 1, n° 3), this indeed shows that $\mathscr{F}^h$ is coherent ([18], n° 15). □

The preceding proposition shows in particular that, if $\mathscr{I}$ is a sheaf of ideals of $\mathcal{O}$, the sheaf $\mathscr{I}^h$ is none other than the sheaf of ideals of $\mathscr{H}$ generated by the elements of $\mathscr{I}$.

10. **Extension of a sheaf.** Let Y be a Z-closed subvariety of the algebraic variety X, and let $\mathscr{F}$ be a coherent algebraic sheaf on Y. If one denotes by $\mathscr{F}^X$ the sheaf obtained by extending $\mathscr{F}$ by 0 on $X \smallsetminus Y$ (cf. [18], n° 5), one knows that $\mathscr{F}^X$ is a coherent algebraic sheaf on X, and the sheaf $(\mathscr{F}^X)^h$ is well-defined; it is a coherent analytic sheaf on X^h. But on the other hand, the sheaf $\mathscr{F}^h$ is a coherent analytic sheaf on Y^h, that one can extend to 0 on $X^h \smallsetminus Y^h$, thus obtaining a new sheaf $(\mathscr{F}^h)^X$. One has:

Proposition 11. *The sheaves $(\mathscr{F}^h)^X$ and $(\mathscr{F}^X)^h$ are canonically isomorphic.*

Proof. The two sheaves in question are zero outside of Y^h; thus it will suffice for us to show that their restrictions to Y^h are isomorphic.

Let x be a point of Y. We put, in order to simplify notation:

$$A = \mathcal{O}_{x,X}, \quad A' = \mathcal{O}_{x,Y}, \quad B = \mathscr{H}_{x,X}, \quad B' = \mathscr{H}_{x,Y}, \quad E = \mathscr{F}_x.$$

One then has

$$(\mathscr{F}^h)^X_x = E \otimes_{A'} B' \quad \text{and} \quad (\mathscr{F}^X)^h_x = E \otimes_A B.$$

The ring A' is the quotient of A by an ideal $\mathfrak{a}$, and, according to Prop. 4 of n° 6, one has $B' = B/\mathfrak{a}B = B \otimes_A A'$. By virtue of the associativity of the tensor product, one then obtains an isomorphism:

$$\theta_x : E \otimes_{A'} B' = E \otimes_{A'} AA' \otimes_A B \longrightarrow E \otimes_A B,$$

which varies continuously with x, as one easily sees; the proposition follows from this. □

One can express Prop. 11 by saying that the functor $\mathscr{F}^h$ is *compatible with the usual identification of $\mathscr{F}$ with $\mathscr{F}^X$.*

11. **Homomorphisms induced on cohomology.** The notation being the same as in n° 9, let X be an algebraic variety, $\mathscr{F}$ an algebraic sheaf on X, and $\mathscr{F}^h$ the analytic sheaf associated to $\mathscr{F}$. If U is a Z-open subset of X, and if s is a section of $\mathscr{F}$ over U, one can consider s as a section s' of $\mathscr{F}'$ over the open U^h of X^h, and $\alpha(s') = s' \otimes 1$ is a section of $\mathscr{F}^h = \mathscr{F}' \otimes \mathscr{H}$ over U^h. The map $s \mapsto \alpha(s')$ is a homomorphism

$$\epsilon : \Gamma(U, \mathscr{F}) \longrightarrow \Gamma(U^h, \mathscr{F}^h).$$

Now let $\mathfrak{U} = \{U_i\}$ be a finite Z-open covering of X; the U_i^h form a finite open covering of X^h, which we denote by $\mathfrak{U}^h$. For all systems of indices $i_0, \ldots, i_q$, one has, according to the preceding, a canonical homomorphism

$$\epsilon : \Gamma(U_{i_0} \cap \cdots \cap U_{i_q}, \mathscr{F}) \longrightarrow \Gamma(U_{i_0}^h \cap \cdots U_{i_q}^h, \mathscr{F}^h),$$

and hence a homomorphism

$$\epsilon : C(\mathfrak{U}, \mathscr{F}) \longrightarrow C(\mathfrak{U}^h, \mathscr{F}^h),$$

with the notation of [18], n° 18.

This homomorphism commutes with the coboundary d, and so defines, by passage to cohomology, homomorphisms:

$$\epsilon : H^q(\mathfrak{U}, \mathscr{F}) \longrightarrow H^q(\mathfrak{U}^h, \mathscr{F}^h).$$

Finally, by passage to the inductive limit over $\mathfrak{U}$, one obtains *the homomorphisms induced on the cohomology groups*

$$\epsilon : H^q(X, \mathscr{F}) \longrightarrow H^q(X^h, \mathscr{F}^h).$$

These homomorphisms enjoy the usual functorial properties; they commute with the homomorphisms $\phi : \mathscr{F} \to \mathscr{G}$; if one has an exact sequence of algebraic sheaves:

$$0 \longrightarrow \mathscr{A} \longrightarrow \mathscr{B} \longrightarrow \mathscr{C} \longrightarrow 0,$$

where the sheaf $\mathscr{A}$ is *coherent*, the diagram:

$$\begin{array}{ccc} H^q(X, \mathscr{C}) & \xrightarrow{\delta} & H^{q+1}(X, \mathscr{A}) \\ \downarrow{\scriptstyle\epsilon} & & \downarrow{\scriptstyle\epsilon} \\ H^q(X^h, \mathscr{C}^h) & \xrightarrow[\delta]{} & H^{q+1}(X^h, \mathscr{A}^h) \end{array}$$

is commutative: this is seen, for example, by taking for coverings $\mathfrak{U}$ coverings by open affines (cf. [18]).

12. **Projective varieties. Statements of the theorems.** Suppose that X is a *projective variety*, which is to say a Z-closed subvariety of a projective space $\mathbb{P}_r(\mathbb{C})$. One then has the following theorems, which we will prove in the paragraphs to come:

Theorem 1. *For every coherent algebraic sheaf $\mathscr{F}$ on X, and for every integer $q \geq 0$, the homomorphism*

$$\epsilon : H^q(X, \mathscr{F}) \longrightarrow H^q(X^h, \mathscr{F}^h),$$

defined in n° 11, is bijective.

For $q = 0$, one obtains in particular an isomorphism of $\Gamma(X, \mathscr{F})$ with $\Gamma(X^h, \mathscr{F}^h)$.

Theorem 2. *If $\mathscr{F}$ and $\mathscr{G}$ are two coherent algebraic sheaves on X, every analytic homomorphism of $\mathscr{F}^h$ into $\mathscr{G}^h$ comes from a unique algebraic homomorphism of $\mathscr{F}$ into $\mathscr{G}$.*

Theorem 3. *For every coherent analytic sheaf $\mathscr{M}$ on X^h, there exists a coherent algebraic sheaf $\mathscr{F}$ on X such that $\mathscr{F}^h$ is isomorphic to $\mathscr{M}$. Moreover, this property determines $\mathscr{F}$ in a unique fashion, up to isomorphism.*

Remarks. 1. These three theorems signify that the theory of coherent analytic sheaves on X^h essentially coincides with that of coherent algebraic sheaves on X. Of course, they are due to the fact that X is a *projective* variety, and so are inaccurate for an affine variety.

2. One can factor ϵ as:

$$H^q(X, \mathscr{F}) \longrightarrow H^q(X^h, \mathscr{F}') \longrightarrow H^q(X^h, \mathscr{F}^h).$$

Thm. 1 leads one to ask whether $H^q(X, \mathscr{F}) \to H^q(X^h, \mathscr{F}')$ is bijective. The response is negative; indeed, if this homomorphism were bijective for every coherent algebraic sheaf $\mathscr{F}$, it would also be the case for the constant sheaf $K = \mathbb{C}(X)$ of rational functions on X (supposed irreducible), since this sheaf is the union of coherent sheaves (compare with [19], §2); but one has $H^q(X, K) = 0$ for $q > 0$, while on the other hand $H^q(X^h, K)$ is a K-vector space of dimension equal to the qth Betti number of X^h.

13. **Proof of Theorem 1.** Suppose that X is embedded in the projective space $\mathbb{P}_r(\mathbb{C})$; if we identify $\mathscr{F}$ with the sheaf obtained by extending it by 0 outside of X, one knows ([18], n° 26) that one has:

$$H^q(X, \mathscr{F}) = H^q(\mathbb{P}_r(\mathbb{C}), \mathscr{F}) \quad \text{and} \quad H^q(X^h, \mathscr{F}^h) = H^q(\mathbb{P}_r(\mathbb{C})^h, \mathscr{F}^h),$$

the notation $\mathscr{F}^h$ being justified by Prop. 11. One sees then that it suffices to prove that

$$\epsilon : H^q(\mathbb{P}_r(\mathbb{C}), \mathscr{F}) \longrightarrow H^q(\mathbb{P}_r(\mathbb{C})^h, \mathscr{F}^h)$$

is bijective, in other words, we reduce to the case where $X = \mathbb{P}_r(\mathbb{C})$.

We first establish two lemmas:

Lemma 4. *Thm. 1 is true for the sheaf* $\mathcal{O}$.

Proof. For $q = 0$, $H^0(X, \mathcal{O})$ and $H^0(X^h, \mathcal{O}^h)$ are both reduced to the constants. For $q > 0$, one knows that $H^q(X, \mathcal{O}) = 0$, cf. [18], n° 85, Prop. 8; on the other hand, according to Dolbeault's Thm. (cf. [8]), $H^q(X^h, \mathcal{O}^h)$ is isomorphic to the cohomology of type $(0, q)$ of the projective space X, and so is reduced to 0, QED[5]. □

Lemma 5. *Thm. 1 is true for the sheaf* $\mathcal{O}(n)$.

(For the definition of $\mathcal{O}(n)$, cf. [18], n° 54, as well as n° 16 hereafter.)

Proof. We argue by induction on $r = \dim X$, the case $r = 0$ being trivial. Let t be a linear form, not identically zero, on the homogeneous coordinates $t_0, \dots, t_r$, and let E be the hyperplane defined by the equation $t = 0$. One has an exact sequence:

$$0 \longrightarrow \mathcal{O}(-1) \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}_E \longrightarrow 0,$$

where $\mathcal{O} \to \mathcal{O}_E$ is the restriction homomorphism, so that $\mathcal{O}(-1) \to \mathcal{O}$ is multiplication by t (cf. [18], n° 81). From this, one gets an exact sequence, valid for all $n \in \mathbb{Z}$:

$$0 \longrightarrow \mathcal{O}(n-1) \longrightarrow \mathcal{O}(n) \longrightarrow \mathcal{O}_E(n) \longrightarrow 0.$$

[5]One can also *directly* calculate $H^q(X, \mathcal{O})$ by using the open covering of X defined in n° 16, as well as some developments in Laurent series (J. Frenkel, not published). One thus avoids all recourse to the theory of Kähler manifolds.

According to n° 11, one has a commutative diagram:

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & H^q(X,\mathcal{O}(n-1)) & \longrightarrow & H^q(X,\mathcal{O}(n)) & \longrightarrow & H^q(X,\mathcal{O}_E(n)) & \longrightarrow & H^{q+1}(X,\mathcal{O}(n-1)) & \longrightarrow \cdots \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\cdots \longrightarrow & H^q(X^h,\mathcal{O}(n-1)^h) & \longrightarrow & H^q(X^h,\mathcal{O}(n)^h) & \longrightarrow & H^q(X^h,\mathcal{O}_E(n)^h) & \longrightarrow & H^{q+1}(X^h,\mathcal{O}(n-1)^h) & \longrightarrow \cdots
\end{array}$$

Given the inductive hypothesis, the homomorphism

$$\epsilon : H^q\big(E,\mathcal{O}_E(n)\big) \longrightarrow H^q\big(E^h,\mathcal{O}_E(n)^h\big)$$

is bijective for all $q \geq 0$ and all $n \in \mathbb{Z}$. By applying the Five Lemma, one then sees that, if Thm. 1 is true for $\mathcal{O}(n)$, it is true for $\mathcal{O}(n-1)$, and conversely. Since it is true for $n = 0$ by Lemma 4, it is thus true for all n. □

We can now proceed to the proof of Thm. 1. We will reason by descending induction on q, the theorem being trivial for $q > 2r$, since then $H^q(X,\mathscr{F})$ and $H^q(X^h,\mathscr{F}^h)$ are both zero. According to [18], n° 55, Cor. to Thm. 1, there exists an exact sequence of coherent algebraic sheaves:

$$0 \longrightarrow \mathscr{R} \longrightarrow \mathscr{L} \longrightarrow \mathscr{F} \longrightarrow 0,$$

where $\mathscr{L}$ is a direct sum of sheaves isomorphic to $\mathcal{O}(n)$; given Lemma 5, Thm. 1 is true for the sheaf $\mathscr{L}$.

One has a commutative diagram:

$$\begin{array}{ccccccccc}
H^q(X,\mathscr{R}) & \longrightarrow & H^q(X,\mathscr{L}) & \longrightarrow & H^q(X,\mathscr{F}) & \longrightarrow & H^{q+1}(X,\mathscr{R}) & \longrightarrow & H^{q+1}(X,\mathscr{L}) \\
\downarrow{\scriptstyle \epsilon_1} & & \downarrow{\scriptstyle \epsilon_2} & & \downarrow{\scriptstyle \epsilon_3} & & \downarrow{\scriptstyle \epsilon_4} & & \downarrow{\scriptstyle \epsilon_5} \\
H^q(X^h,\mathscr{R}^h) & \longrightarrow & H^q(X^h,\mathscr{L}^h) & \longrightarrow & H^q(X^h,\mathscr{F}^h) & \longrightarrow & H^{q+1}(X^h,\mathscr{R}^h) & \longrightarrow & H^{q+1}(X^h,\mathscr{L}^h)
\end{array}$$

In this diagram, the homomorphisms ϵ_4 and ϵ_5 are bijective, according to the inductive hypothesis; according to what we have just said, the same is true of ϵ_2. The Five Lemma shows then that ϵ_3 is surjective. This result, being valid for every coherent algebraic sheaf $\mathscr{F}$ applies in particular to $\mathscr{R}$, which shows that ϵ_1 is surjective. A new application of the Five Lemma shows then that ϵ_3 is bijective, which completes the proof.

14. **Proof of Theorem 2.** Let $\mathscr{A} = \mathrm{Hom}(\mathscr{F},\mathscr{G})$ be the sheaf of germs of homomorphisms of $\mathscr{F}$ into $\mathscr{G}$ (cf. [18], n^os 11 and 14). An element $f \in \mathscr{A}_x$ is the germ of a homomorphism of $\mathscr{F}$ into $\mathscr{G}$ in a neighborhood of x, and so defines a germ of a homomorphism f^h of the analytic sheaf $\mathscr{F}^h$ into the sheaf $\mathscr{G}^h$; the map $f \mapsto f^h$ is an $\mathcal{O}'$-linear homomorphism of the sheaf $\mathscr{A}'$ defined by $\mathscr{A}$ (cf. n° 9) into the sheaf $\mathscr{B} = \mathrm{Hom}(\mathscr{F}^h,\mathscr{G}^h)$; this homomorphism extends by linearity to a homomorphism

$$\iota : \mathscr{A}^h \longrightarrow \mathscr{B}.$$

Lemma 6. *The homomorphism $\iota : \mathscr{A}^h \to \mathscr{B}$ is bijective.*

Proof. Let $x \in X$. Since $\mathscr{F}$ is coherent, one has, according to [18], n° 14:

$$\mathscr{A}_x = \mathrm{Hom}(\mathscr{F}_x,\mathscr{G}_x), \quad \text{whence} \quad \mathscr{A}^h_x = \mathrm{Hom}(\mathscr{F}_x,\mathscr{G}_x) \otimes \mathscr{H}_x,$$

the functors $\otimes$ and Hom being applied over the ring $\mathcal{O}_x$.

Since $\mathscr{F}^h$ is coherent, one likewise has:

$$\mathscr{B}_x = \mathrm{Hom}(\mathscr{F}_x \otimes \mathscr{H}_x, \mathscr{G}_x \otimes \mathscr{H}_x),$$

the functor $\otimes$ being taken over $\mathcal{O}_x$, and the functor Hom over $\mathscr{H}_x$.

Everything reduces to seeing that the homomorphism

$$\iota_x : \mathrm{Hom}(\mathscr{F}_x, \mathscr{G}_x) \otimes \mathscr{H}_x \longrightarrow \mathrm{Hom}(\mathscr{F}_x \otimes \mathscr{H}_x, \mathscr{G}_x \otimes \mathscr{H}_x)$$

is bijective, which results from the fact that the couple $(\mathcal{O}_x, \mathscr{H}_x)$ is flat and from Prop. 24 of the Appendix. □

We now prove Thm. 2. Let us consider the homomorphisms

$$H^0(X, \mathscr{A}) \xrightarrow{\epsilon} H^0(X^h, \mathscr{A}^h) \xrightarrow{\iota} H^0(X^h, \mathscr{B}).$$

An element of $H^0(X^h, \mathscr{A})$ $\big($resp. of $H^0(X^h, \mathscr{B})\big)$ is none other than a homomorphism from $\mathscr{F}$ into $\mathscr{G}$ (resp. of $\mathscr{F}^h$ into $\mathscr{G}^h$). Moreover, if $f \in H^0(X, \mathscr{A})$, one has $\iota \circ \epsilon(f) = f^h$, by very definition of ι. Thm. 2 thus comes again to affirm that $\iota \circ \epsilon$ is bijective. But ϵ is bijective according to Thm. 1 (which is applicable because $\mathscr{A}$ is coherent, according to [18], n° 14), and ι is bijective according to Lemma 6, QED.

15. **Proof of Theorem 3. Preliminaries.** The *uniqueness* of the sheaf $\mathscr{F}$ results from Thm. 2. Indeed, if $\mathscr{F}$ and $\mathscr{G}$ are two coherent algebraic sheaves on X responding to the question, there exists by hypothesis an isomorphism $g : \mathscr{F}^h \to \mathscr{G}^h$. According to Thm. 2, there thus exists a homomorphism $f : \mathscr{F} \to \mathscr{G}$ such that $g = f^h$. If one denotes by $\mathscr{A}$ and $\mathscr{B}$ the kernel and cokernel of f, one has an exact sequence:

$$0 \longrightarrow \mathscr{A} \longrightarrow \mathscr{F} \xrightarrow{f} \mathscr{G} \longrightarrow \mathscr{B} \longrightarrow 0,$$

whence, according to Prop. 10 a), an exact sequence:

$$0 \longrightarrow \mathscr{A}^h \longrightarrow \mathscr{F}^h \xrightarrow{g} \mathscr{G}^h \longrightarrow \mathscr{B}^h \longrightarrow 0.$$

Since g is bijective, this implies that $\mathscr{A}^h = \mathscr{B}^h = 0$, whence, according to Prop. 10 b), $\mathscr{A} = \mathscr{B} = 0$, which indeed shows that f is bijective.

It remains to prove the *existence* of $\mathscr{F}$. I claim that one may limit oneself to the case where X *is a projective space* $\mathbb{P}_r(\mathbb{C})$. Indeed, let Y be an algebraic subvariety of $X = \mathbb{P}_r(\mathbb{C})$, and let $\mathscr{M}$ be a coherent analytic sheaf on Y^h. The sheaf $\mathscr{M}^X$ obtained by extending $\mathscr{M}$ by 0 outside of Y^h is a coherent analytic sheaf on X^h. If one assumes Thm. 3 proven for the space X, there then exists a coherent algebraic sheaf $\mathscr{G}$ on X such that $\mathscr{G}^h$ is isomorphic to $\mathscr{M}^X$. Let $\mathscr{I} = \mathscr{I}(Y)$ be the coherent sheaf of ideals defined by the subvariety Y. If $f \in \mathscr{I}_x$, multiplication by f is an endomorphism ϕ of $\mathscr{G}_x$; the endomorphism ϕ^h of $\mathscr{G}^h_x = \mathscr{M}^h_x$ is reduced to 0, since $\mathscr{M}$ is a coherent analytic sheaf on Y^h; the same is thus true of ϕ by Prop. 10 b). Hence, one has $\mathscr{I} \cdot \mathscr{G} = 0$, which means that there exists a coherent algebraic sheaf $\mathscr{F}$ on Y such that $\mathscr{G} = \mathscr{F}^X$ ([18], n° 39, Prop. 3). According to Prop. 11, $(\mathscr{F}^h)^X$ is isomorphic to $(\mathscr{F}^X)^h = \mathscr{G}^h$, which is isomorphic to $\mathscr{M}^X$. By restriction to Y, one sees that $\mathscr{F}^h$ is isomorphic to $\mathscr{M}$, which proves our assertion.

16. **Proof of Theorem 3. The sheaves $\mathscr{M}(n)$.** Given the preceding n°, we suppose that $X = \mathbb{P}_r(\mathbb{C})$, and reason by induction on r, the case $r = 0$ being trivial.

For all $n \in \mathbb{Z}$, we first define a new analytic sheaf, the sheaf $\mathscr{M}(n)$:

Let $t_0, \ldots, t_r$ be a system of homogeneous coordinates on X, and let U_i be the open set formed from the points where $t_i \neq 0$; we denote by $\mathscr{M}_i$ the restriction of the sheaf $\mathscr{M}$ to U_i; multiplication by t_j^n/t_i^n is an isomorphism of $\mathscr{M}_i$ with $\mathscr{M}_j$, defined over $U_i \cap U_j$. The sheaf $\mathscr{M}(n)$ is thus defined by regluing the sheaves $\mathscr{M}_i$ by way of the preceding isomorphisms (cf. [18], n° 54, where the same construction is applied to algebraic sheaves). The sheaf $\mathscr{M}(n)$ is locally isomorphic to $\mathscr{M}$, and thus coherent because $\mathscr{M}$ is; one has a canonical isomorphism $\mathscr{M}(n) = \mathscr{M} \otimes \mathscr{H}(n)$, the tensor product being over $\mathscr{H}$. If $\mathscr{F}$ is an algebraic sheaf, one has $\mathscr{F}^h(n) = \mathscr{F}(n)^h$.

Lemma 7. *Let E be a hyperplane of $\mathbb{P}_r(\mathbb{C})$, and let $\mathscr{A}$ be a coherent analytic sheaf on E. One has $H^q\big(E^h, \mathscr{A}(n)\big) = 0$ for $q > 0$ and sufficiently large n.*

(This is the "Theorem B" of [3], exp. XVIII.)

Proof. By virtue of the inductive hypothesis, there exists a coherent algebraic sheaf $\mathscr{F}$ on E such that $\mathscr{A} = \mathscr{F}^h$, whence $\mathscr{A}(n) = \mathscr{F}(n)^h$; according to Thm. 1, $H^q\big(E^h, \mathscr{A}(n)\big)$ is isomorphic to $H^q\big(E, \mathscr{F}(n)\big)$, and Lemma 7 then results from Prop. 7 of [18], n° 65. □

Lemma 8. *Let $\mathscr{M}$ be a coherent analytic sheaf on $X = \mathbb{P}_r(\mathbb{C})$. There exists an integer $n(\mathscr{M})$ such that, for all $n \geq n(\mathscr{M})$, and for all $x \in X$, the $\mathscr{H}_x$-module $\mathscr{M}(n)_x$ is generated by the elements of $H^0\big(X^h, \mathscr{M}(n)\big)$.*

(This is the "Theorem A" of [3], exp. XVIII.)

Proof. We first remark that, if $H^0\big(X^h, \mathscr{M}(n)\big)$ generates $\mathscr{M}(n)_x$, the same property holds for all $m \geq n$. Indeed, let k be an index such that $x \in U_k$; for each i, let θ_i be the endomorphism of $\mathscr{M}_i$ given by multiplication by $(t_k/t_i)^{m-n}$; the θ_i commute with the identifications which define respectively $\mathscr{M}(n)$ and $\mathscr{M}(m)$, and so give rise to a homomorphism $\theta : \mathscr{M}(n) \to \mathscr{M}(m)$; since θ is an isomorphism over U_k, our assertion follows.

We also remark that if $H^0\big(X^h, \mathscr{M}(n)\big)$ generates $\mathscr{M}(n)_x$, it also generates $\mathscr{M}(n)_y$ for y near enough to x, according to [18], n° 12.

These two remarks, together with the compactness of X^h, bring us to proving the following statement:

For all $x \in X$, there exists an integer n, depending on x and $\mathscr{M}$, such that $H^0\big(X^h, \mathscr{M}(n)\big)$ generates $\mathscr{M}(n)_x$.

We choose a hyperplane E passing through x with homogeneous equation $t = 0$. If $\mathscr{A}(E)$ denotes the sheaf of ideals defined by E (cf. n° 3), one has an exact sequence:

$$0 \longrightarrow \mathscr{A}(E) \longrightarrow \mathscr{H} \longrightarrow \mathscr{H}_E \longrightarrow 0.$$

Moreover, the sheaf $\mathscr{A}(E)$ is isomorphic to $\mathscr{H}(-1)$, the isomorphism $\mathscr{H}(-1) \to \mathscr{A}(E)$ being defined by multiplication by t (cf. the proof of Lemma 5).

By tensoring with $\mathscr{M}$, we get an exact sequence:

$$\mathscr{M} \otimes \mathscr{A}(E) \longrightarrow \mathscr{M} \longrightarrow \mathscr{M} \otimes \mathscr{H}_E \longrightarrow 0.$$

We denote by $\mathscr{B}$ the sheaf $\mathscr{M} \otimes \mathscr{H}_E$, and we denote by $\mathscr{C}$ the kernel of the homomorphism $\mathscr{M} \otimes \mathscr{A} \to \mathscr{M}$ $\big($one has $\mathscr{C} = \mathrm{Tor}_1(\mathscr{M}, \mathscr{H}_E)\big)$; from the fact that $\mathscr{A}(E)$ is isomorphic to $\mathscr{H}(-1)$, the sheaf $\mathscr{M} \otimes \mathscr{A}(E)$ is isomorphic to $\mathscr{M}(-1)$, and one thus obtains an exact sequence:

$$0 \longrightarrow \mathscr{C} \longrightarrow \mathscr{M}(-1) \longrightarrow \mathscr{M} \longrightarrow \mathscr{B} \longrightarrow 0. \tag{1}$$

By applying the functor $\mathscr{M}(n)$ to the exact sequence (1), one obtains a new exact sequence:

$$0 \longrightarrow \mathscr{C}(n) \longrightarrow \mathscr{M}(n-1) \longrightarrow \mathscr{M}(n) \longrightarrow \mathscr{B}(n) \longrightarrow 0. \tag{2}$$

Let $\mathscr{P}_n$ be the kernel of the homomorphism $\mathscr{M}(n) \to \mathscr{B}(n)$; the sequence (2) decomposes into two exact sequences:

$$0 \longrightarrow \mathscr{C}(n) \longrightarrow \mathscr{M}(n-1) \longrightarrow \mathscr{P}_n \longrightarrow 0, \tag{3}$$

$$0 \longrightarrow \mathscr{P}_n \longrightarrow \mathscr{M}(n) \longrightarrow \mathscr{B}(n) \longrightarrow 0, \tag{4}$$

which, each in turn, give rise to exact sequences in cohomology:

$$H^1\big(X^h, \mathscr{M}(n-1)\big) \longrightarrow H^1(X^h, \mathscr{P}_n) \longrightarrow H^2\big(X^h, \mathscr{C}(n)\big) \tag{5}$$

and

$$H^1(X^h, \mathscr{P}_n) \longrightarrow H^1\big(X^h, \mathscr{M}(n)\big) \longrightarrow H^1\big(X^h, \mathscr{B}(n)\big). \tag{6}$$

According to the definition of $\mathscr{B}$ and $\mathscr{C}$, one has $\mathscr{A}(E) \cdot \mathscr{B} = 0$ and $\mathscr{A}(E) \cdot \mathscr{C} = 0$, which shows that $\mathscr{B}$ and $\mathscr{C}$ are coherent analytic sheaves on the hyperplane E. Thus, applying Lemma 7, one sees that there exists an integer n_0 such that one has, for each $n \geq n_0$, $H^1\big(X^h, \mathscr{B}(n)\big) = 0$ an $H^2\big(X^h, \mathscr{C}(n)\big) = 0$. The exact sequences (5) and (6) thus give the inequalities:

$$\dim H^1\big(X^h, \mathscr{M}(n-1)\big) \geq \dim H^1(X^h, \mathscr{P}_n) \geq \dim H^1\big(X^h, \mathscr{M}(n)\big). \tag{7}$$

These dimensions are *finite*, according to [5] (see also [3], exp. XVII). It follows from this that $\dim H^1\big(X^h, \mathscr{M}(n)\big)$ is a *decreasing* function of n, for $n \geq n_0$; thus there exists an integer $n_1 \geq n_0$ such that the function $\dim H^1\big(X^h, \mathscr{M}(n)\big)$ is *constant* for $n \geq n_1$. One has then:

$$\dim H^1\big(X^h, \mathscr{M}(n)\big) = \dim H^1(X^h, \mathscr{P}_n) = \dim H^1\big(X^h, \mathscr{M}(n)\big) \quad \text{if } n > n_1. \tag{8}$$

Since $n_1 \geq n_0$, one has $H^1\big(X^h, \mathscr{B}(n)\big) = 0$, and the exact sequence (6) shows that $H^1(X^h, \mathscr{P}_n) \to H^1\big(X^h, \mathscr{M}(n)\big)$ is surjective; but, according to (8), these two vector spaces have the same dimension; the homomorphism in question is thus injective, and the exact sequence in cohomology associated to the exact sequence (4) shows that[6]:

$$H^0\big(X^h, \mathscr{M}(n)\big) \longrightarrow H^0\big(X^h, \mathscr{B}(n)\big) \quad \textit{is surjective for } n > n_1. \tag{9}$$

We now choose a integer $n > n_1$ such that $H^0\big(X^h, \mathscr{B}(n)\big)$ generates $\mathscr{B}(n)_x$; this is possible, for $\mathscr{B}$, being a coherent analytic sheaf on E, is of the form $\mathscr{G}^h$,

[6]One recalls the process used by Kodaira-Spencer to prove the theorem of Lefschetz (cf. [12]).

whence $H^0\big(X^h, \mathscr{B}(n)\big) = H^0\big(X^h, \mathscr{G}(n)\big)$, according to Thm. 1, and one knows that $H^0\big(X^h, \mathscr{G}(n)\big)$ generates $\mathscr{G}(n)_x$ for n big enough, cf. [18], n° 55, Thm. 1.

This being the case, I say that every integer n works. Indeed, we put, to simplify the notation, $A = \mathscr{H}_x$, $M = \mathscr{M}(n)_x$, $\mathfrak{p} = \mathscr{A}_x(E)$, and let N be the A-submodule of M generated by $H^0\big(X^h, \mathscr{M}(n)\big)$. One has that $\mathscr{B}(n)_x = \mathscr{M}(n)_x \otimes \mathscr{H}_{x,E} = M \otimes_A A/\mathfrak{p} = M/\mathfrak{p}M$; on the other hand, it results from the preceding that the canonical image of N in $M/\mathfrak{p}M$ generates $M/\mathfrak{p}M$. This can be written as $M = N + \mathfrak{p}M$, whence, *a fortiori*, $M = N + \mathfrak{m}M$ ($\mathfrak{m}$ denoting the maximal ideal of the local ring A), which indeed implies $M = N$ (Appendix, Prop. 24, Cor.), and completes the proof of Lemma 8. □

17. **End of the proof of Theorem 3.** Let $\mathscr{M}$ always be a coherent analytic sheaf on $X = \mathbb{P}_r(\mathbb{C})$. By virtue of Lemma 8, there exists an integer n such that $\mathscr{M}(n)$ is isomorphic to a quotient sheaf of a sheaf $\mathscr{H}^p$, and $\mathscr{M}$ is thus isomorphic to a quotient of $\mathscr{H}(-n)^p$. If we denote by $\mathscr{L}_0$ the coherent algebraic sheaf $\mathcal{O}(-n)^p$, one sees then that one has an exact sequence:

$$0 \longrightarrow \mathscr{R} \longrightarrow \mathscr{L}_0^h \longrightarrow \mathscr{M} \longrightarrow 0,$$

where $\mathscr{R}$ is a coherent analytic sheaf.

Applying the same reasoning to the sheaf $\mathscr{R}$, one constructs a coherent algebraic sheaf $\mathscr{L}_1$ and a surjective analytic homomorphism $\mathscr{L}_1^h \to \mathscr{R}$. Whence an exact sequence:

$$\mathscr{L}_1^h \xrightarrow{g} \mathscr{L}_0^h \longrightarrow \mathscr{M} \longrightarrow 0.$$

According to Thm. 2, there exists a homomorphism $f : \mathscr{L}_1 \to \mathscr{L}_0$ such that $g = f^h$. If one denotes by $\mathscr{F}$ the cokernel of f, one has an exact sequence:

$$\mathscr{L}_1 \xrightarrow{f} \mathscr{L}_0 \longrightarrow \mathscr{F} \longrightarrow 0,$$

whence (Prop. 10) a new exact sequence:

$$\mathscr{L}_1^h \xrightarrow{g} \mathscr{L}_0^h \longrightarrow \mathscr{F}^h \longrightarrow 0,$$

which shows indeed that $\mathscr{M}$ is isomorphic to $\mathscr{F}^h$, which completes the proof of Thm. 3.

4. Applications

18. **The algebraic nature of Betti numbers.** Let σ be an automorphism of the field $\mathbb{C}$; if x is a point of $\mathbb{P}_r(\mathbb{C})$, with homogeneous coordinates $[t_0, \dots, t_r]$, we denote by x^σ the point with homogeneous coordinates $[t_0^\sigma, \dots, t_r^\sigma]$; thus, σ defines an automorphism of $\mathbb{P}_r(\mathbb{C})$.

If X is a Z-closed algebraic subvariety of $\mathbb{P}_r(\mathbb{C})$, its transform X^σ by σ is again a Z-closed algebraic subvariety of $\mathbb{P}_r(\mathbb{C})$; if X is nonsingular, the same is true of X^σ (because of the Jacobian criterion, for example).

Proposition 12. *If X is nonsingular, then the Betti numbers of X and X^σ are the same.*

Proof. Let $b_n(X)$ be the nth Betti number of X, and let $\Omega^p(X)^h$ be the sheaf of germs of degree p holomorphic differentials on X. We put:

$$h^{p,q}(X) = \dim H^q\big(X^h, \Omega^p(X)^h\big).$$

According to Dolbeault's Thm. (cf. [18]), one has:

$$b_n(X) = \sum_{p+q=n} h^{p,q}(X),$$

and likewise:

$$b_n(X^\sigma) = \sum_{p+q=n} h^{p,q}(X^\sigma).$$

But, according to Thm. 1, one has $h^{p,q}(X) = \dim H^q\big(X, \Omega^p(X)\big)$, where this time $\Omega^p(X)$ denotes the coherent algebraic sheaf of degree p regular differential forms on X, and likewise $h^{p,q}(X^\sigma) = \dim H^q\big(X^\sigma, \Omega^p(X^\sigma)\big)$. Moreover, if ω is a regular differential form on a Z-open subset U of X, the form ω^σ is regular on the Z-open subset U^σ of X^σ; one concludes from this that for each Z-open cover $\mathfrak{U}$ of X, σ defines a semilinear isomorphism of $C\big(\mathfrak{U}, \Omega^p(X)\big)$ onto $C\big(\mathfrak{U}^\sigma, \Omega^p(X^\sigma)\big)$, thus of $H^q\big(\mathfrak{U}, \Omega^p(X)\big)$ onto $H^q\big(\mathfrak{U}^\sigma, \Omega^p(X^\sigma)\big)$, thus also of $H^q\big(X, \Omega^p(X)\big)$ onto $H^q\big(X^\sigma, \Omega^p(X^\sigma)\big)$, and one has indeed that $h^{p,q}(X) = h^{p,q}(X^\sigma)$, which proves the proposition. □

Prop. 12 implies the following result, conjectured by A. Weil:

Corollary. *Let V be a nonsingular projective variety defined over an algebraic number field K. The complex varieties X obtained from V by embedding K into $\mathbb{C}$ have Betti numbers independent of the choice of embedding.*

Proof. Indeed, one knows that two embeddings of K into $\mathbb{C}$ only differ by an automorphism of $\mathbb{C}$. □

Remark. I ignore whether the varieties X and X^σ are always homeomorphic[7]; at any rate, the example of a curve of genus 1 shows already that they are not always analytically isomorphic.

19. **Chow's Theorem.** This is the following result (cf. [6]):

Proposition 13. *Every closed analytic subset of projective space is algebraic.*

Proof. We show how this proposition results from Thm. 3. Let X be a projective space, and let Y be a closed analytic subset of X^h. According to a theorem of H. Cartan, cited above (n° 3, Prop. 1), the sheaf $\mathscr{H}_Y = \mathscr{H}_X/\mathscr{A}(Y)$ is a coherent analytic sheaf on X^h; thus there exists (Thm. 3) a coherent algebraic sheaf $\mathscr{F}$ on X such that $\mathscr{H}_Y = \mathscr{F}^h$. According to Prop. 10 b), the support of $\mathscr{F}^h$ is equal to that of $\mathscr{F}$ (recalling, cf. [18], n° 81, that the support of $\mathscr{F}$ is the set of $x \in X$ such that $\mathscr{F}_x \neq 0$), thus is Z-closed, since $\mathscr{F}$ is coherent. Since $\mathscr{F}^h = \mathscr{H}_Y$, this means that Y is Z-closed, QED. □

[7] They aren't: in the paper *Exemples de variétés projectives conjugées non homéomorphes*, C.R. Acad. Sci. Paris **258** (1964), 4194–4196, Serre constructs a nonsingular projective surface X such that the varieties X and X^σ are not homeomorphic [translator's note].

We now indicate some simple applications of Chow's Thm.:

Proposition 14. *If X is an algebraic variety, every compact analytic subset X' of X is algebraic.*

Proof. We reestablish the notation used in the proof of Prop. 6: let Y be a projective variety, U a subset of Y, Z-open and Z-dense in Y, and $f : U \to X$ a surjective regular map whose graph T is Z-closed in $X \times Y$. Let $T' = T \cap (X' \times Y)$; since X' and Y are compact, and since T is closed, T' is compact; this is therefore also the case for the projection Y' of T' onto the factor Y. On the other hand, $Y' = f^{-1}(X')$, which shows that Y' is an analytic subset of U, and hence of Y; Chow's Thm. then shows that Y' is a Z-closed subset of Y. By applying Prop. 7 to $f : Y' \to X$, one concludes that $X' = f(Y')$ is Z-closed in X, QED. □

Proposition 15. *Every holomorphic map f of a compact algebraic variety X into an algebraic variety Y is regular.*

Proof. Let T be the graph of f in $X \times Y$. Since f is holomorphic, T is a compact analytic subset of $X \times Y$; Prop. 14 shows then that T is algebraic, whence the fact that f is regular, according to Prop. 8. □

Corollary. *Each compact analytic space possesses at most one structure of an algebraic variety.*

20. **Algebraic fiber spaces and analytic fiber spaces.** Let G be an algebraic group and X an algebraic variety. The germs of regular maps from X into G form a sheaf of groups, in general not abelian, which we denote by $\mathscr{G}$.

One knows that if $\mathscr{A}$ is a sheaf of groups, one can define the *group* $H^0(X, \mathscr{A})$ and the *set* $H^1(X, \mathscr{A})$: cf. [9] as well as [10], Chap. V, for example. In particular, $H^1(X, \mathscr{G})$ is defined; the elements of this set are none other than the *classes of principal algebraic fiber spaces* with base X and structure group G (in the sense defined by A. Weil, cf. [20]). For example, the elements of $H^1(X, \mathcal{O}_X)$ are the classes of fiber spaces with group the additive group $\mathbb{C}$.

Likewise, if $\mathscr{G}^h$ is the sheaf of germs of holomorphic maps of X into G, the elements of $H^1(X^h, \mathscr{G}^h)$ are none other than the classes of *analytic* fiber spaces with base X and group G. Every algebraic fiber space E defines an analytic fiber space E^h, whence a map

$$\epsilon : H^1(X, \mathscr{G}) \longrightarrow H^1(X^h, \mathscr{G}^h)$$

analogous to the map defined in n° 11.

Proposition 16. *If X is compact, the map ϵ is injective.*

Proof. Let E and E' be two principal algebraic fiber spaces with base X and structure group G. Prop. 16 means that if E and E' are analytically isomorphic, then they are also algebraically isomorphic. In fact, we are going to prove a result which is a little more precise, namely that every analytic isomorphism $\phi : E \to E'$ is an algebraic isomorphism (which is to say, regular).

The space $E \times E'$ is a principal algebraic fiber space with base $X \times X$ and structure group $G \times G$; we denote by (E, E') its pullback under the diagonal map

$X \to X \times X$: this is the "fiber product" of E and E'. We make $G \times G$ act on G by the formula:

$$(g, g') \cdot h = ghg'^{-1}.$$

Let T be the fiber space associated to the principal fiber space (E, E') allowing for fiber type the group G, equipped with the preceding operations[8]. One sees immediately that the sections of T correspond bijectively to the isomorphisms of E onto E'; in particular, the isomorphism ϕ corresponds to an analytic section s of T. By applying Prop. 15 to $s : X \to T$, one sees that s is regular, which shows that ϕ is regular and proves the proposition. □

Suppose now that X is a *projective variety*. One can ask whether $\epsilon : H^1(X, \mathscr{G}) \to H^1(X^h, \mathscr{G}^h)$ is bijective, said differently (given Prop. 16), whether every analytic fiber space is algebraic. This is clearly untrue if one does not impose any conditions on G, as is shown by the case where G is an abelian variety (or a finite group); in the following propositions, we are going to indicate a certain number of groups G for which this holds.

Proposition 17. *If G is the additive group $\mathbb{C}$, the map ϵ is bijective.*

Proof. Indeed, one has then $\mathscr{G} = \mathcal{O}$ and $\mathscr{G}^h = \mathcal{O}^h$, and the proposition is a special case of Thm. 1. □

Proposition 18. *If G is the general linear group $GL_n(\mathbb{C})$, the map ϵ is bijective.*

Proof. To every principal fiber space with structure group $GL_n(\mathbb{C})$ is associated a fiber space with vector space fibers, of fiber type $\mathbb{C}^n$, which characterizes it. Given the correspondence between fiber spaces with vector space fibers and locally free sheaves (cf. [18], n° 41, for example), one can thus reduce to proving the following statement:

If $\mathscr{M}$ is a coherent analytic sheaf on X^h which is locally isomorphic to $\mathscr{H}^n$, there exists a coherent algebraic sheaf $\mathscr{F}$ on X which is locally isomorphic to $\mathcal{O}^n$ such that $\mathscr{F}^h$ is isomorphic to $\mathscr{M}$.

According to Thm. 3, there exists a coherent algebraic sheaf $\mathscr{F}$ on X satisfying the second condition. For each $x \in X$, the $\mathscr{H}_x$-module $\mathscr{F}^h = \mathscr{F}_x \otimes \mathscr{H}_x$ is thus isomorphic to $\mathscr{H}_x^n$; by applying Prop. 30 of the Appendix to the rings $A = \mathcal{O}_x$, $A' = \mathscr{H}_x$ and to the module $E = \mathscr{F}_x$, one concludes that $\mathscr{F}_x$ is isomorphic to $\mathcal{O}_x^n$; since $\mathscr{F}$ is coherent, this shows that $\mathscr{F}$ is locally isomorphic to $\mathcal{O}^n$, and completes the proof. □

Remarks. 1. For $n = 1$, $GL_n(\mathbb{C})$ coincides with the multiplicative group $\mathbb{C}^*$; if one assumes that X is a *normal* variety, the group $H^1(X, \mathscr{G})$ coincides with the group of divisor classes locally linearly equivalent to zero (cf. [20], §3) and Prop. 18 shows that every analytic fiber space with base X and structure group $\mathbb{C}^*$ comes from such a divisor. Provided X is *nonsingular*, this result has been obtained by Kodaira-Spencer [12]; in this case, it is essentially equivalent anyway to the theorem of Lefschetz on the existence of divisors with a given homology class.

[8] I.e., T is the fiber space with base X, fiber G, structure group $G \times G$, and transition functions the same as those for (E, E') [translator's note].

2. Prop. 18 permits one to understand other results of Kodaira on arbitrary projective varieties (allowing singularities); this is notably the case with Thms. 7 and 8 in [11]. We will not discuss these.

Now let G be an algebraic group, and H an algebraic subgroup of G; one knows (cf. [13], for example) that the homogeneous space G/H can be equipped with the structure of an algebraic variety, the quotient of that of G. The group H acts on G by right translation; we assume that these operations define on G the structure of a *principal algebraic fiber space* with base G/H and structure group H, where, which amounts to the same thing, we assume that there exists a *rational section* $G/H \to G$ (which is not always the case, as we will see further on). Under this hypothesis, one has the following result, which was communicated to me, as well as its proof, by A. Grothendieck:

Proposition 19. *Let X be a compact algebraic variety, and let P be a principal analytic fiber space with structure group H and base X. In order for P to be algebraic, it is necessary and sufficient that the same be true of the principal fiber space $P \times_H G$ induced by P by extending the structure group H to G.*

Proof. The necessity is clear. In order to prove the sufficiency, we suppose that $P \times_H G$ is algebraic. This means that there exists an algebraic principal fiber space P_0 with structure group G and an analytic isomorphism $h : P_0 \to P \times_H G$. We consider the fiber space E (resp. E_0) associated to $P \times_H G$ (resp. to P_0) with fiber type G/H on which G operates by translations. One has:

$$E_0 = P_0 \times_G G/H \quad \text{and} \quad E = (P \times_H G) \times_G G/H = P \times_H G/H.$$

The analytic isomorphism h defines an analytic isomorphism $f : E_0 \to E$. But the fiber space $E = P \times_H G/H$ possesses a canonical section s, since the group H leaves invariant the point of G/H corresponding to the neutral element of G. The isomorphism f transforms s into a section $s_0 = f^{-1} \circ s$ of E_0; the section s_0 is holomorphic, thus regular, according to Prop. 15.

On the other hand, since G operates on P_0, the same is true of H, and P_0/H is none other than E_0; more precisely, P_0 is a principal algebraic fiber space with structure group G and base E_0: this is easily verified by reasoning locally, using the hypothesis that G is a principal algebraic fiber space with structure group H and base G/H. So let $P_1 = s_0^{-1}(P_0)$ be the pullback of P_0 under the map $s_0 : X \to E_0$; the fiber space P_1 is a principal algebraic fiber space with base X and structure group H. We are going to show that P_1 is analytically isomorphic to P, which will prove the proposition.

The relation $s_0 = f^{-1} \circ s$, together with the fact that f is an analytic isomorphism, shows that $P_1 = s_0^{-1}(P_0)$ is analytically isomorphic to the pullback of $P \times_H G$ (considered as a principal fiber space with structure group H) under the map $s : X \to E$. But this last pullback is none other than P, as is shown by the following commutative diagram:

$$\begin{array}{ccc} P & \longrightarrow & P \times_H G \\ \big\downarrow & & \big\downarrow \\ X & \xrightarrow[s]{} & E = P \times_H G/H. \end{array}$$

This completes the proof. □

By combining Props. 18 and 19 one obtains:

Proposition 20. *Let G be an algebraic subgroup of the group $GL_n(\mathbb{C})$ satisfying the following condition:*

(R): *There exists a rational section $GL_n(\mathbb{C})/G \to GL_n(\mathbb{C})$.*

Then, for every projective variety X, the map:

$$\epsilon : H^1(X, \mathscr{G}) \longrightarrow H^1(X^h, \mathscr{G}^h)$$

is bijective.

Examples. The condition **(R)** is satisfied in the following cases:

a) if G is *solvable*, by virtue of a theorem of Rosenlicht, [13];

b) if $G = SL_n(\mathbb{C})$, the rational section being evident in this case;

c) if $G = Sp_n(\mathbb{C})$, $n = 2m$; in this case, the homogeneous space GL_n/G is the space of nondegenerate alternating forms $\sum_{i<j} u_{ij} x_i \wedge x_j$, and the condition **(R)** results from the fact that the *generic* $\sum_{i<j} u_{ij} x_i \wedge x_j$ can be reduced to the canonical form $\sum_{i=1}^{m} x_{2i-1} \wedge x_{2i}$ by a linear change of variables with coefficients in the field $\mathbb{C}(u_{ij})$.

These last two examples lead one to conjecture that the condition **(R)** is satisfied each time that G is a *semisimple simply-connected* group.

On the contrary, one can show that the special orthogonal group $G = O_n^+(\mathbb{C})$ does not satisfy the condition **(R)** provided that $n \geq 3$. I ignore whether, in this case, the map $\epsilon : H^1(X, \mathscr{G}) \to H^1(X^h, \mathscr{G}^h)$ is bijective.

Appendix

All the rings considered below are assumed *commutative* and *with unity*; all the modules over these rings are assumed *unitary*.

21. **Flat modules.**

Definition 3. Let B be an A-module. One says that B is A-flat (or flat) if, for every exact sequence of A-modules

$$E \longrightarrow F \longrightarrow G,$$

the sequence

$$E \otimes_A B \longrightarrow F \otimes_A B \longrightarrow G \otimes_A B$$

is exact.

In view of the definition of the Tor functors, the preceding condition is equivalent to saying that $\mathrm{Tor}_1^A(B, Q) = 0$ for every A-module Q; since Tor commutes with inductive limits, one can reduce to the case of modules Q of finite type, and likewise (thanks to the exact sequence of the Tor) to modules Q with one generator; thus for B to be A-flat, it is necessary and sufficient that $\mathrm{Tor}_1^A(B, A/\mathfrak{a}) = 0$ for every ideal $\mathfrak{a}$ of A, in other words that the canonical homomorphism $\mathfrak{a} \otimes_A B \to B$ is injective.

Examples. 1. If A is a PID, it follows that "B is A-flat" is equivalent to "B is torsion free."

2. If S is a multiplicative subset of a ring A, the ring of fractions A_S is A-flat, according to [18], n° 48, Lemma 1.

Let A and B be two rings, and let $\theta : A \to B$ a homomorphism from A to B; this homomorphism makes B into an A-module. If E and F are two A-modules, $E \otimes_A B$ and $F \otimes_A B$ are equipped with the structure of B-modules; moreover, if $f : E \to F$ is a homomorphism, $f \otimes 1$ is a B-homomorphism from $E \otimes_A B$ to $F \otimes_A B$; one obtains in this way a canonical A-linear map:

$$\mathrm{Hom}_A(E, F) \longrightarrow \mathrm{Hom}_B(E \otimes_A B, F \otimes_A B),$$

which extends by linearity to a B-linear map:

$$\iota : \mathrm{Hom}_A(E, F) \otimes_A B \longrightarrow \mathrm{Hom}_B(E \otimes_A B, F \otimes_A B).$$

Proposition 21. *The homomorphism ι defined above is bijective provided A is a Noetherian ring, E is a finite A-module, and B is A-flat.*

Proof. For a fixed module F, we set:

$$T(E) = \mathrm{Hom}_A(E, F) \otimes_A B \quad \text{and} \quad T'(E) = \mathrm{Hom}_B(E \otimes_A B, F \otimes_A B),$$

so that ι is a homomorphism of the functor $T(E)$ to the functor $T'(E)$.

For $E = A$, one has $T(E) = T'(E) = F \otimes_A B$ and ι is bijective; the same is true provided E is a free module of finite type.

But the ring A is Noetherian, and E is of finite type; thus there exists an exact sequence:

$$L_1 \longrightarrow L_0 \longrightarrow E \longrightarrow 0,$$

where L_0 and L_1 are free modules of finite type. We consider the commutative diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & T(E) & \longrightarrow & T(L_0) & \longrightarrow & T(L_1) \\ & & \downarrow{\scriptstyle \iota} & & \downarrow{\scriptstyle \iota_0} & & \downarrow{\scriptstyle \iota_1} \\ 0 & \longrightarrow & T'(E) & \longrightarrow & T'(L_0) & \longrightarrow & T'(L_1) \end{array}$$

The first line of this sequence is exact from the fact that B is A-flat; the second line is too from general properties of the functors $\otimes$ and Hom. Since we know that ι_0 and ι_1 are bijective, it follows that ι is bijective, QED. □

22. **Flat couples.**

Definition 4. Let A be a ring, and let B a ring containing A. One says that the couple (A, B) is flat if the A-module B/A is A-flat.

One has:

Proposition 22. *For a couple (A, B) to be flat, it is necessary and sufficient that B be A-flat, and that one of the following properties is satisfied:*

a) (resp. a′)) For every A-module (resp. for each A-module of finite type) E, the homomorphism $E \to E \otimes_A B$ is injective.

a″) For every ideal $\mathfrak{a}$ of A, one has $\mathfrak{a}B \cap A = \mathfrak{a}$.

Proof. If E is any A-module, the exact sequence:

$$0 \longrightarrow A \longrightarrow B \longrightarrow B/A \longrightarrow 0,$$

gives rise to the exact sequence:

$$\mathrm{Tor}_1^A(A, E) \longrightarrow \mathrm{Tor}_1^A(B, E) \longrightarrow \mathrm{Tor}_1^A(B/A, E) \longrightarrow A \otimes_A E \longrightarrow B \otimes_A E.$$

Given that $A \otimes_A E = E$ and $\mathrm{Tor}_1^A(A, E) = 0$, one obtains the new exact sequence:

$$0 \longrightarrow \mathrm{Tor}_1^A(B, E) \longrightarrow \mathrm{Tor}_1^A(B/A, E) \longrightarrow E \longrightarrow E \otimes_A B.$$

One sees then that for $\mathrm{Tor}_1^A(B/A, E)$ to be 0, it is necessary and sufficient that the same be true of $\mathrm{Tor}_1^A(B, E)$ and that the homomorphism $E \to E \otimes_A B$ be injective; the proposition results immediately from that (noting that the property a″) reduces to saying that the homomorphism $A/\mathfrak{a} \to A/\mathfrak{a} \otimes_A B$ is injective). □

Proposition 23. *Let $A \subseteq B \subseteq C$ be three rings. If the couples (A, C) and (B, C) are flat, then so is the couple (A, B).*

Proof. We first show that B is A-flat, which is to say that if one has an exact sequence of A-modules:

$$0 \longrightarrow E \longrightarrow F,$$

the sequence: $0 \to E \otimes_A B \to F \otimes_A B$ is then exact.

Let N be the kernel of the homomorphism $E \otimes_A B \to F \otimes_A B$; since C is B-flat, one has an exact sequence:

$$0 \longrightarrow N \otimes_B C \longrightarrow (E \otimes_A B) \otimes_B C \longrightarrow (F \otimes_A B) \otimes_B C.$$

But, according to the associativity of the tensor product, $(E \otimes_A B) \otimes_B C$ is identified with $E \otimes_A C$, and likewise $(F \otimes_A B) \otimes_B C$ is identified with $F \otimes_A C$. Moreover, C being A-flat, the homomorphism $E \otimes_A C \to F \otimes_A C$ is injective. It follows that $N \otimes_B C = 0$, and, upon applying Prop. 22 to the couple (B, C), one sees that $N = 0$, which completes the proof that B is A-flat.

On the other hand, if E is any A-homomorphism, the composite homomorphism $E \to E \otimes_A B \to E \otimes_A C$ is injective (since the couple (A, C) is flat), and the same is true *a fortiori* of $E \to E \otimes_A B$; this shows that the couple (A, B) satisfies all the hypotheses of Prop. 22, QED. □

Remark. Analogous reasoning shows that if (A, B) and (B, C) are flat, then so is (A, C). On the contrary, it can happen that (A, B) and (A, C) are flat without (B, C) being so.

23. **Modules over a local ring.** In this paragraph, we denote by A a *local Noetherian ring*[9], *with maximal ideal* $\mathfrak{m}$.

Proposition 24. *If a finite A-module E satisfies the relation $E = \mathfrak{m}E$, then one has $E = 0$.*

(Cf. [16], p. 138 or [4], exp. I, for example.)

Proof. Suppose $E \neq 0$, and let $e_1, \dots, e_n$ be a system of generators of E having the smallest possible number of elements. Since $e_n \in \mathfrak{m}E$, one has $e_n = x_1e_1 + \cdots + x_ne_n$, with $x_i \in \mathfrak{m}$, whence

$$(1 - x_n)e_n = x_1e_1 + \cdots + x_{n-1}e_{n-1};$$

since $1 - x_n$ is invertible in A, this shows that the $e_1, \dots, e_{n-1}$ generate E, which is a contradiction to the hypothesis made on n. □

Corollary 4. *Let E be a finite A-module. If a submodule F of E satisfies the relation $E = F + \mathfrak{m}E$, one has $E = F$.*

Proof. Indeed, this relation means that $E/F = \mathfrak{m}(E/F)$. □

We equip A-module E with the $\mathfrak{m}$-*adic topology* in which the submodules $\mathfrak{m}^nE$ form a base of neighborhoods of 0 (cf. [16], p. 153).

Proposition 25. *Let E be a finite A-module. Then:*

a) The topology induced on a submodule F of E by the $\mathfrak{m}$-adic topology of E coincides with the $\mathfrak{m}$-adic topology of F.

b) Every submodule of E is closed in the $\mathfrak{m}$-adic topology of E (and, in particular, E is totally disconnected).

(Cf. [16], *loc. cit.*, as well as [3], exp. VIII bis).

Proof. We recall briefly the proof of this proposition. One begins by proving a), which can be done either by using the theory of primary decomposition (Krull, cf. [16]), or by establishing the existence of an integer r such that one has

$$F \cap \mathfrak{m}^nE = \mathfrak{m}^{n-r}(F \cap \mathfrak{m}^rE) \quad \text{for } n \geq r$$

(Artin-Rees, cf. [4], exp. 2).

One shows then that E is totally disconnected: by applying a) to the submodule F, the closure of 0 in E, one sees that $F = \mathfrak{m}F$, whence $F = 0$, according to Prop. 24. By applying this result to the quotient modules, one deduces b). □

Let E still be a finite A-module, and let $\hat{E}$ and $\hat{A}$ be the completions of E and A with respect to the $\mathfrak{m}$-adic topology. The bilinear map $A \times E \to E$ extends by

[9] In fact, all the results proved in these two paragraphs are true as stated for any *Zariski ring* (cf. [16], p. 157).

continuity to a map $\hat{A} \times \hat{E} \to \hat{E}$ which makes $\hat{E}$ into a $\hat{A}$-module. The canonical injection of E into $\hat{E}$ thus extends by linearity to a homomorphism

$$\epsilon : E \otimes_A \hat{A} \to \hat{E}.$$

Proposition 26. *For every finite A-module E, the homomorphism ϵ defined above is bijective.*

Proof. Let $0 \to R \to L \to E \to 0$ be an exact sequence of A-modules, L being a free module of finite type. From the fact that A is Noetherian, R is of finite type; on the other hand, Prop. 25 shows that the $\mathfrak{m}$-adic topology of R is induced by that of L, and it is clear that the topology of E is the quotient of that of L; as these topologies are *metrizable*, one gets an exact seqence:

$$0 \longrightarrow \hat{R} \longrightarrow \hat{L} \longrightarrow \hat{E} \longrightarrow 0.$$

We then consider the commutative diagram:

$$\begin{array}{ccccccc}
R \otimes_A \hat{A} & \longrightarrow & L \otimes_A \hat{A} & \longrightarrow & E \otimes_A \hat{A} & \longrightarrow & 0 \\
\downarrow{\scriptstyle \epsilon''} & & \downarrow{\scriptstyle \epsilon'} & & \downarrow{\scriptstyle \epsilon} & & \\
\hat{R} & \longrightarrow & \hat{L} & \longrightarrow & \hat{E} & \longrightarrow & 0.
\end{array}$$

The two rows of this diagram are exact, and, on the other hand, it is clear that ϵ' is bijective. One concludes that ϵ is surjective (otherwise said, one has $\hat{E} = \hat{A} \cdot E$, cf. [16], p. 153, Lemma 1). This result, being proven for every finite A-module, applies in particular to R, which shows that ϵ'' is surjective, and, by applying the Five Lemma, one concludes that ϵ is bijective, QED. □

24. **Flatness properties of local rings.** All the local rings considered below are assumed *Noetherian.*

Proposition 27. *Let A be a local ring, and let $\hat{A}$ be its completion. The couple $(A, \hat{A})$ is flat.*

Proof. First off, $\hat{A}$ is A-flat. Indeed, it suffices to show that if $E \to F$ is injective, then so is $E \otimes_A \hat{A} \to F \otimes_A \hat{A}$, and one can also assume that E and F are of finite type. In this case, Prop. 26 shows that $E \otimes_A \hat{A}$ is identified with $\hat{E}$, and likewise $F \otimes_A \hat{A}$ is identified with $\hat{F}$, and our assertion results then from the obvious fact that $\hat{E}$ injects into $\hat{F}$.

Likewise, the fact that $E \to \hat{E}$ is injective if E is of finite type shows that the couple $(A, \hat{A})$ satisfies the property a$'$) of Prop. 22, and so is indeed a flat couple. □

Now let A and B be two local rings, and let θ be a homomorphism from A into B. We assume that θ maps the maximal ideal of A into the maximal ideal of B. Then θ is continuous, and extends by continuity to a homomorphism $\hat{\theta} : \hat{A} \to \hat{B}$.

Proposition 28. *Suppose that $\hat{\theta} : \hat{A} \to \hat{B}$ is bijective, and identify A with a subring of B by way of θ. The couple (A, B) is then a flat couple.*

Proof. One has $A \subseteq B \subseteq \hat{B} = \hat{A}$, and the couples $(A, \hat{A})$ and $(B, \hat{B})$ are flat, according to the preceding proposition. Prop. 23 shows then that (A, B) is a flat couple. □

Proposition 29. *Let A and B be two local rings, let $\mathfrak{a}$ be an ideal of A, and let θ be a homomorphism from A into B. If θ satisfies the hypothesis of Prop. 28, then so does the homomorphism from $A/\mathfrak{a}$ into $B/\theta(\mathfrak{a})B$ defined by θ (which shows that the couple $(A/\mathfrak{a}, B/\theta(\mathfrak{a})B)$ is a flat couple).*

Proof. According to Prop. 26, the completion of $A/\mathfrak{a}$ is $\hat{A}/\mathfrak{a}\hat{A}$, and, likewise, that of $B/\theta(\mathfrak{a})B$ is $\hat{B}/\theta(\mathfrak{a})\hat{B}$, whence the result. □

Proposition 30. *Let A and A' be two local rings, let θ be a homomorphism of A into A' satisfying the hypothesis of Prop. 28, and let E be a finite A-module. If the A'-module $E' = E \otimes_A A'$ is isomorphic to A'^n, then E is isomorphic to A^n.*

Proof. We identify A with a subring of A' by way of θ. If $\mathfrak{m}$ and $\mathfrak{m}'$ are the maximal ideals of A and A', one then has $\mathfrak{m} \subseteq \mathfrak{m}'$; on the other hand, since $\mathfrak{m}'$ is a neighborhood of 0 in A', and since A is dense in A', one has $A' = \mathfrak{m}' + A$, which shows that $A/\mathfrak{m} = A'/\mathfrak{m}'$,whence $E/\mathfrak{m}E = E'/\mathfrak{m}'E'$. Since the A'-module E is a free module of rank n, the same is true of the $A'/\mathfrak{m}'$-module $E'/\mathfrak{m}'E'$. One concludes that it is possible to choose n elements $e_1, \ldots, e_n$ in E whose images in $E/\mathfrak{m}E$ form a basis of $E/\mathfrak{m}E$, considered as a vector space over $A/\mathfrak{m}$. The elements e_i define a homomorphism $f : A^n \to E$ which is surjective by virtue of the corollary to Prop. 24. We are going to show that f is injective, which will prove the proposition.

Let N be the kernel of f. From the fact that the couple (A, A') is flat (Prop. 28), the exact sequence

$$0 \longrightarrow N \longrightarrow A^n \xrightarrow{f} E \longrightarrow 0$$

gives rise to the exact sequence

$$0 \longrightarrow N' \longrightarrow A'^n \xrightarrow{f'} E' \longrightarrow 0.$$

Since the module E' is free, N' is a direct summand in A'^n, and one has an exact sequence

$$0 \longrightarrow N'/\mathfrak{m}'N' \longrightarrow A'^n/\mathfrak{m}'A'^n \longrightarrow E'/\mathfrak{m}'E' \longrightarrow 0.$$

But, by the same construction, f' defines a bijection of $A'^n/\mathfrak{m}'A'^n$ onto $E'/\mathfrak{m}'E'$. It follows that $N'/\mathfrak{m}'N' = 0$, whence $N' = 0$ (Prop. 24), whence $N = 0$ since the couple (A, A') is flat, QED. □

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