OVERCONVERGENT THEORY

Contents

The goal of this lecture is to explain the meaning of "overconvergent" and to prove the following theorem of Cherbonnier-Colmez.

Theorem 0.1. *(See* [Ber1, Corollary 25.3]*)*

The functor $V \mapsto \mathrm{D}^\dagger(V)$ induces an equivalence from the category of p-adic representations of Gal_K to the category of étale overconvergent (φ, Γ)-modules over $\mathbf{B}_K^\dagger$.

The main reference is Colmez's paper [Col, Section 4,5,6,7,8,9].

1. Constrcution of Robba rings

Recall that for every $k \geq 0$, there exists a function $w_k : \tilde{\mathbf{A}} \to \mathbb{R} \cup \{+\infty\}$ defined by $w_k(x) = \inf_{0 \leq i \leq k} \nu_{\mathbf{E}}(x_i)$ if $x = \sum_{i \geq 0} p^i [x_i]$ satisfying following properties (See [Ber1, Section 16]).

Fact 1.1. (1) $w_k(x) = +\infty$ if and only if $x \in p^{k+1}\tilde{\mathbf{A}}$;

(2) $w_k(x+y) \geq \inf(w_k(x), w_k(y))$; if $w_k(x) \neq w_k(y)$, it takes " = ";

(3) $w_k(xy) \geq \inf_{i+j \leq k}(w_i(x) + w_j(y))$;

(4) $w_k(\varphi(x)) = p w_k(x)$;

(5) $w_k([\lambda]x) = w_k(x) + \nu_{\mathbf{E}}(\lambda)$ for $\forall \lambda \in \tilde{\mathbf{E}}$;

(6) $w_k(\sigma(x)) = w_k(x)$ for $\forall \sigma \in \mathrm{Gal}_K$.

These functions $\{w_k\}_{k \geq 0}$ define the canonical (or weak) topology on $\tilde{\mathbf{A}}$.

For w_k, we also have the following property.

Lemma 1.2. *If $r > 0$ and if $x = \sum_{n\geq 0} p^n[x_n] \in \tilde{\mathbf{A}}$, then*

(1) $\lim_{k\to+\infty}(\nu_{\mathbf{E}}(x_k) + kr) = +\infty$ *if and only if* $\lim_{k\to+\infty}(w_k(x) + kr) = +\infty$ *and*

(2) *in this case* $\inf_{k\geq 0}(\nu_{\mathbf{E}}(x_k) + kr) = \inf_{k\geq 0}(w_k(x) + kr)$.

Proof. (1) We only prove that $\lim_{k\to+\infty}(\nu_{\mathbf{E}}(x_k)+kr) = +\infty$ implies $\lim_{k\to+\infty}(w_k(x)+kr) = +\infty$; the other direction is obvious since $w_k(x) \leq \nu_{\mathbf{E}}(x_k)$.

We define a function $i : \mathbb{N} \to \mathbb{N}$ by $i(k) = \sup\{n \mid w_k(x) = \nu_{\mathbf{E}}(x_n)\ n \leq k\}$. Then i is an increasing function (because $w_{k+1}(x) \leq w_k(x)$). Clearly, $i(k) \leq k$.

Case I: $\lim_{k\to+\infty} i(k) = +\infty$.

In this case, $w_k(x) + kr = \nu_{\mathbf{E}}(x_{i(k)}) + kr \geq \nu_{\mathbf{E}}(x_{i(k)}) + i(k)r \to +\infty$.

Case II: $\lim_{k\to+\infty} i(k) = n$ for some $n \in \mathbb{N}$.

In this case, there exists an $N \in \mathbb{N}$ such that $i(k) = n$ for all $k \geq N$. In particular, for $k \geq N$, $w_k(x) + kr = \nu_{\mathbf{E}}(x_N) + kr \to +\infty$.

(2) $\inf_{k\geq 0}(\nu_{\mathbf{E}}(x_k) + kr) \geq \inf_{k\geq 0}(w_k(x) + kr) = \inf_{k\geq 0}(\nu_{\mathbf{E}}(x_{i(k)}) + kr) \geq \inf_{k\geq 0}(\nu_{\mathbf{E}}(x_{i(k)}) + i(k)r) \geq \inf_{k\geq 0}(\nu_{\mathbf{E}}(x_k) + kr)$. □

Define

$$\tilde{\mathbf{A}}^{\dagger,r} = \{x \in \tilde{\mathbf{A}} \mid \inf_{k\geq 0}(w_k(x) + \frac{krp}{p-1}) \geq 0 \text{ and } \lim_{k\to+\infty}(w_k(x) + \frac{krp}{p-1}) = +\infty\}$$
$$= \{x \in \tilde{\mathbf{A}} \mid \inf_{k\geq 0}(\nu_{\mathbf{E}}(x_k) + \frac{krp}{p-1}) \geq 0 \text{ and } \lim_{k\to+\infty}(\nu_{\mathbf{E}}(x_k) + \frac{krp}{p-1}) = +\infty\}.$$

Also, we define a function $\nu_r : \tilde{\mathbf{A}}^{\dagger,r} \to \mathbb{R}_{\geq 0}$ by $\nu_r(x) = \inf_{k\geq 0}(w_k(x) + \frac{krp}{p-1})$ for $x \in \tilde{\mathbf{A}}^{\dagger,r}$.

For simplicity, we define $s(r) = \frac{pr}{p-1}$ for $r \geq 0$.

It is straightforward from the definition of $\tilde{\mathbf{A}}^{\dagger,r}$ that for any $r_2 > r_1 > r > 0$,

(1) $\tilde{\mathbf{A}}^{\dagger,r_1} \subset \tilde{\mathbf{A}}^{\dagger,r_2}$ and

(2) $\nu_{r_2}(x) \geq \nu_{r_1}(x)$ for $x \in \tilde{\mathbf{A}}^{\dagger,r}$.

Thus, we can define a function $f_x : \mathbb{R}_{\geq r} \to \mathbb{R}$ by $f_x(t) = \nu_t(x)$.

Proposition 1.3 (Newton Polygon of x)**.** *Assume $r > 0$ and $x = \sum_{n\geq 0}[x_n]p^n \in \tilde{\mathbf{A}}^{\dagger,r}$.*

(1) *The function $f_x : \mathbb{R}_{\geq r} \to \mathbb{R}_{\geq 0}$ is an increasing, piecewise linear, concave continuous function. All slopes of f_x belong to $\frac{p}{p-1}\mathbb{Z}_{\geq 0}$ and f_x has finitely many slopes and cusps.*

(2) *Let $\partial_l f_x$* (resp.$\partial_r f_x$) *be the left (resp. right) derivation of f_x. Then $\frac{p-1}{p}\partial_l f_x(t)$ (*resp.$\frac{p-1}{p}\partial_r f_x(t)$*) is the maximal (resp. minimal) integer N satisfying $v_t(x) =$*

$\nu_{\mathbf{E}}(x_N)+\frac{tpN}{p-1}$. *As a consequence,* $f_x(t)$ *is derivable at* $t=t_0>r$ *if and only if there exists exactly one* $=k\geq 0$ *such that* $\nu_{t_0}(x)=\nu_{\mathbf{E}}(x_k)+ks(t_0)$ *and* $k=\frac{p-1}{p}f'_x(t_0)$.

(3) *If* $x_0\neq 0$*, then there exists an* $r_0\geq r$ *such that for any* $t\geq r_0$*,* $f_x(t)=\nu_{\mathbf{E}}(x_0)$*. In particular, the last slope of* f_x *is* 0.

Proof. By definition of f_x, it is increasing.

For $r_0\geq r$, because $x\in\tilde{\mathbf{A}}^{\dagger,r_0}$, the set

$$\Omega_x:=\{i\in\mathbb{N}\mid f_x(r_0)(=\nu_{r_0}(x))=\nu_{\mathbf{E}}(x_i)+is(r_0)\}$$

is finite. Thus, we can write $\Omega_x=\{n_1<n_2<\cdots<n_k\}$.

Since $\lim_{m\to+\infty}\nu_{\mathbf{E}}(x_m)+ms(r_0)=+\infty$, there exists an $M>f_x(r_0)$ such that for any $n\notin\Omega_x$, $\nu_{\mathbf{E}}(x_n)+ns(r_0)\geq M$. Therefore, for any $r'\approx r_0$ (of course, we require $r'\geq r$), $f_x(r')=\inf_{1\leq i\leq k}\nu_{\mathbf{E}}(x_{n_i})+n_is(r')=f_x(r_0)+\inf_{1\leq i\leq k}(\frac{pn_i}{p-1}(r'-r_0))$.

When $(r\leq)r'<r_0$, $f_x(r')=f_x(r_0)+\frac{pn_k}{p-1}(r'-r_0)$.

When $r'\geq r_0$, $f_x(r')=f_x(r_0)+\frac{pn_1}{p-1}(r'-r_0)$.

This shows (1) and (2).

To prove (3), we notice that for every $r'>r$

$$\nu_{r'}(x)=\inf(\nu_{\mathbf{E}}(x_0),\inf_{i\geq 1}(\nu_{\mathbf{E}}(x_i)+is(r'))).$$

The second term

$$\inf_{i\geq 1}(\nu_{\mathbf{E}}(x_i)+is(r'))\geq\nu_r(x)+s(r')-s(r).$$

Thus, for $r'\gg r$, we have $f_x(r')=\nu_{\mathbf{E}}(x_0)$. This completes the proof. □

Lemma 1.4. *Assume* $r>0$.

(1) $\tilde{\mathbf{A}}^{\dagger,r}$ *is a sub-ring of* $\tilde{\mathbf{A}}$ *which is stable under the action of* $\mathrm{Gal}_{\mathbb{Q}_p}$.

(2) $\varphi:\tilde{\mathbf{A}}^{\dagger,r}\to\tilde{\mathbf{A}}^{\dagger,pr}$ *is a bijection.*

Proof. Put $s=s(r)$.

(1) If $x,y\in\tilde{\mathbf{A}}^{\dagger,r}$, by Fact 1.1 (2), (3), we have that

$$w_k(x+y)+sk\geq\inf(w_k(x)+sk,w_k(y)+sk)$$

and that

$$w_k(xy)+sk\geq\inf_{i+j\leq k}(w_i(x)+w_j(y))+sk\geq\inf_{i+j\leq k}(w_i(x)+is)+(w_j(y)+js).$$

In particular, both of $x+y$ and xy belong to $\tilde{\mathbf{A}}^{\dagger,r}$. Also, we prove that $\nu_r(xy)\geq\nu_r(x)+\nu_r(y)$ and that $\nu_r(x+y)\geq\inf(\nu_r(x),\nu_r(y))$ which takes equality when $\nu_r(x)\neq\nu_r(y)$.

(2) If $x=\sum_{n\geq 0}p^n[x_n]\in\tilde{\mathbf{A}}$, then $\varphi(x)=\sum_{n\geq 0}p^n[x_n^p]$. From

$$w_k(\varphi(x))+ks(pr)=pw_k(x)+ks(pr)=p(w_k(x)+ks(r)),$$

we see that $x\in\tilde{\mathbf{A}}^{\dagger,r}$ if and only if $\varphi(x)\in\tilde{\mathbf{A}}^{\dagger,pr}$ and in this situation $\nu_{pr}(\varphi(x))=p\nu_r(x)$. □

The next Lemma shows that ν_r is a norm on $\tilde{\mathbf{A}}^{\dagger,r}$.

Lemma 1.5. *Assume $r>0$. Let $x=\sum_{n\geq 0}p^n[x_n], y=\sum_{n\geq 0}p^n[y_n]\in\tilde{\mathbf{A}}^{\dagger,r}$ and $\alpha\in\tilde{\mathbf{E}}$.*

(1) $\nu_r(x)=+\infty$ *if and only if* $x=0$;
(2) $\nu_r(x+y)\geq\inf(\nu_r(x),\nu_r(y))$;
(3) $\nu_r(xy)=\nu_r(x)+\nu_r(y)$;
(4) $\nu_{pr}(\varphi(x))=p\nu_r(x)$;
(5) $\nu_r(px)=\nu_r(x)+s(r)$ *and* $\nu_r([\alpha]x)=\nu_{\mathbf{E}}(\alpha)+\nu_r(x)$;
(6) $\nu_r(\sigma(x))=\nu_r(x)$ *for all* $\sigma\in\mathrm{Gal}_{\mathbb{Q}_p}$.

Proof. We have established (2) and (4) in the proof of Lemma 1.4. From the definition of ν_r that $\nu_r(x)=\inf_{n\geq 0}(\nu_{\mathbf{E}}(x_n)+ns(r))$, (1), (5) and (6) are easy to check. We only prove (3) here.

Recall we have proved (3)′ which says $\nu_r(xy)\geq\nu_r(x)+\nu_r(y)$ in the proof of Lemma 1.4.

By Proposition 1.3, except finitely many $r'\geq r$, there exists a unique n and a unique m such that $\nu_{r'}(x)=\nu_{\mathbf{E}}(x_n)+ns(r')<\nu_{r'}(x-[x_n]p^n)$ and that $\nu_{r'}(y)=\nu_{\mathbf{E}}(y_m)+ms(r')<\nu_{r'}(y-[y_m]p^m)$. Considering

$$xy=[x_ny_m]p^{n+m}+(x-[x_n]p^n)y+[x_n]p^n(y-[y_m]p^m),$$

by (3)′, $\nu_{r'}$ takes values at the last two terms strictly bigger than $\nu_r(x)+\nu_r(y)$. By (2), $\nu_{r'}(xy)=\nu_{\mathbf{E}}(x_ny_m)+(n+m)s(r')=\nu_{r'}(x)+\nu_{r'}(y)$. In other words, $f_{xy}(r')=f_x(r')+f_y(r')$ for all but finitely many $r'\in\mathbb{R}_{\geq r}$. By continuities, $f_{xy}(t)=f_x(t)+f_y(t)$ for all $t\geq r$. In particular, $\nu_r(xy)=\nu_r(x)+\nu_r(y)$. □

Remark 1.1. By Lemma 1.5, we see that (1) $f_{xy}=f_x+f_y$; (2) $\frac{1}{p}f_{\varphi(x)}(p\bullet)=f_x(\bullet)$; (3) $f_{px}(\bullet)=f_x(\bullet)+s(\bullet)$; (4) $f_{[\alpha]x}(\bullet)=f_x(\bullet)+\nu_{\mathbf{E}}(\alpha)$.

Define $\tilde{\mathbf{B}}^{\dagger,r}=\tilde{\mathbf{A}}^{\dagger,r}[\frac{1}{p}]$. By using Lemma 1.5 (5), one can extend ν_r to a norm on $\tilde{\mathbf{B}}^{\dagger,r}$ such that Proposition 1.3 and Lemma 1.5 are still true for elements in $\tilde{\mathbf{B}}^{\dagger,r}$.

We remark that $\tilde{\mathbf{A}}^{\dagger,r}$ is not the ring of integers in $(\tilde{\mathbf{B}}^{\dagger,r},\nu_r)$ (for example, $r=\frac{p-1}{p}$, then $\nu_r(\frac{[\tilde{p}]}{p})=0$). However, $\tilde{\mathbf{A}}^{\dagger,r}$ is the ring of integers in $(\tilde{\mathbf{B}}^{\dagger,r}\cap\tilde{\mathbf{A}},\nu_r)$.

If $x=\sum_{n\gg-\infty}[x_n]p^n\in\tilde{\mathbf{B}}^+$, we can define $\nu_0(x)=\inf_k\nu_{\mathbf{E}}(x_k)$. Then the above properties are still true except that it happens that f_x has infinitely many slopes and cusps (in a neighborhood of 0).

Fact 1.6. For every $0\neq\alpha\in\tilde{\mathbf{E}}$ and $r>0$, $[\alpha]\in\tilde{\mathbf{B}}^{\dagger,r}$, therefore it is a unit. This is because there exists an $N\geq 0$ such that $p^N[\alpha]\in\tilde{\mathbf{A}}^{\dagger,r}$.

Proposition 1.7. *The topology on* $\tilde{\mathbf{A}}^{\dagger,r}$ *is separated and completed.*

Proof. The separateness follows from Lemma 1.5 (1). We remain check the completeness.

Let $\{x_i\}_{i\geq 0}$ be a sequence converging to 0. Then $\nu_r(x_i)\to+\infty$ while $i\to+\infty$. Therefore, for a fixed $k\geq 0$, $w_k(x_i)\geq\nu_r(x_i)-ks(r)\to+\infty$. In other words, the sequence $\{x_i\}_{i\geq 0}$ converges to 0 in $\tilde{\mathbf{A}}$ under the canonical topology. Put $x=\sum_{i\geq 0}x_i$. Then $w_k(x)+ks(r)\geq\inf_i w_k(x_i)+ks(r)\geq 0$ for all $k\geq 0$.

We need to check that $x\in\tilde{\mathbf{A}}^{\dagger,r}$.

For any given $M>0$, there exists an $N\in\mathbb{N}$ such that for every $i\geq N$, $\nu_r(x_i)\geq M$. In particular, $w_k(x_i)+ks(r)>M$ for all $k\geq 0$ and $i\geq N$. There exists an $K>N$ such that for every $i\leq N$ and $k\geq K$, $w_k(x_i)+ks(r)>M$. Therefore, for every $k\geq K$,

$$w_k(x)+ks(r)\geq\inf(\inf_{i\leq N}(w_k(x_i)+ks(r)),\inf_{i\geq N}(w_k(x_i)+ks(r)))>M$$

Thus, $x\in\tilde{\mathbf{A}}^{\dagger,r}$. □

Lemma 1.8. *Assume* $r>0$.

(1) *The action of* $\mathrm{Gal}_{\mathbb{Q}_p}$ *on* $\tilde{\mathbf{A}}^{\dagger,r}$ *is continuous*
(2) *The map* $\varphi:\tilde{\mathbf{A}}^{\dagger,r}\to\tilde{\mathbf{A}}^{\dagger,pr}$ *is an homeomorphism.*

Proof. The (2) follows from Lemma 1.5 (4). By Lemma 1.5 (6), it remains to prove that for a given $x=\sum_{n\geq 0}[x_n]p^n\in\tilde{\mathbf{A}}^{\dagger,r}$, the function $\mathrm{Gal}_{\mathbb{Q}_p}\to\mathbb{R}$ by mapping $\sigma\mapsto\nu_r(\sigma(x))$ is continuous. It suffices to check that $\lim_{\sigma\to 1}\nu_r(\sigma(x)-x)=+\infty$.

By Fact 1.1 (2), (6), for every $k\geq 0$, $w_k(\sigma(x)-x)+ks(r)\geq w_k(x)+ks(r)$. Thus, for any given $M>0$, there is an $N>0$ such that for every $k\geq N$, $w_k(\sigma(x)-x)+ks(r)\geq w_k(x)+ks(r)>M$. For $k\leq N$, since $\mathrm{Gal}_{\mathbb{Q}_p}$ acts on $\tilde{\mathbf{A}}$ continuously, there exists an open subgroup $H\leq\mathrm{Gal}_{\mathbb{Q}_p}$ such that for every $\sigma\in H$, $w_k(\sigma(x)-x)+$

$ks(r) > M$. Therefore, for any $\sigma \in H$, we have $\inf_k(w_k(\sigma(x)-x)+ks(r)) > M$; that is $\nu_r(\sigma(x)-x) > M$. This proves the lemma. □

Recall $H_K = \mathrm{Gal}_{K(\zeta_{p^\infty})}$. Thus, we can define $\tilde{\mathbf{B}}_K^{\dagger,r} = (\tilde{\mathbf{B}}^{\dagger,r})^{H_K}$. Also, we can define $\mathbf{B}^{\dagger,r} = \tilde{\mathbf{B}}^{\dagger,r} \cap \mathbf{B}$ as well as $\mathbf{B}_K^{\dagger,r} = (\mathbf{B}^{\dagger,r})^{H_K}$. Similarly, the meaning of $\tilde{\mathbf{A}}_K^{\dagger,r}$, $\mathbf{A}^{\dagger,r}$ and $\mathbf{A}_K^{\dagger,r}$ are clear.

Define $\tilde{\mathbf{B}}^\dagger = \bigcup_{r>0} \tilde{\mathbf{B}}^{\dagger,r}$. Then the meaning of $\mathbf{B}^\dagger$ and $\mathbf{B}_K^\dagger$ are clear as well.

Proposition 1.9. *The ring $\tilde{\mathbf{B}}^\dagger$ is a field. As a consequence, all of $\tilde{\mathbf{B}}_K^\dagger$, $\mathbf{B}^\dagger$ and $\mathbf{B}_K^\dagger$ are fields.*

To prove this Proposition, we need to study units of $\tilde{\mathbf{A}}^{\dagger,r}$.

Lemma 1.10. *Let $x = \sum_{n\geq 0} p^n[x_n] \in \tilde{\mathbf{A}}^{\dagger,r}$. Then x is a unit if and only if for all $k \geq 1$, $0 = \nu_r(x) = \nu_{\mathbf{E}}(x_0) < \nu_{\mathbf{E}}(x_k) + ks(r)$.*

Proof. Assume that for all $k \geq 1$, we have $0 = \nu_r(x) = \nu_{\mathbf{E}}(x_0) < \nu_{\mathbf{E}}(x_k) + ks(r)$. Since $x_0 \in \tilde{\mathbf{E}}^+$, $[x_0]$ is a unit in $\tilde{\mathbf{A}}^{\dagger,r}$. Thus, we may assume that $x_0 = 1$ by using $[x_0]^{-1}x$ instead of x. In this case, $x = 1-x'$ for some $x' \in \tilde{\mathbf{A}}^{\dagger,r}$ satisfying $\nu_r(x') > 0$. Then $\sum_{n\geq 0}(x')^n$ converges in $\tilde{\mathbf{A}}^{\dagger,r}$ and is the inverse of $x = 1-x'$.

Conversely, if x is a unit of $\tilde{\mathbf{A}}^{\dagger,r}$ with the inverse $y = \sum_{n\geq 0} p^n[y_n]$. Since $xy = 1$, modulo p, we must have $x_0y_0 = 1$. Moreover, because $\nu_r(x), \nu_r(y) \geq 0$, it follows from

$$0 = \nu_r(1) = \nu_r(xy) = \nu_r(x) + \nu_r(y)$$

that $\nu_r(x) = 0 = \nu_r(y)$. On the other hand, since $\nu_{\mathbf{E}}(x_0) \geq \nu_r(x) = 0$, $x_0 \in \tilde{\mathbf{E}}^+$. For the same reason $y_0 \in \tilde{\mathbf{E}}^+$. Thus, $\nu_{\mathbf{E}}(x_0) = \nu_{\mathbf{E}}(y_0) = 0$.

It remains to show that $\inf_{k\geq 1}(\nu_{\mathbf{E}}(x_k) + ks(r)) > 0$ (equivalently, $\nu_r(x-[x_0]) > 0$). We may assume that $x_0 = y_0 = 1$. Otherwise, assume $\nu_r(x-1) = 0$, then we claim that $\nu_r(y-1) = 0$. In fact, if $-z = y-1$ satisfying $\nu_r(z) > 0$, then $\nu_r(x-1) = \nu_r(\sum_{n\geq 1} z^n) > 0$, which is impossible. Now, let n_0 (resp m_0) be the largest integer such that $\nu_{\mathbf{E}}(x_{n_0}) + n_0 s(r) = \nu_r(x-1)$ (resp. $\nu_{\mathbf{E}}(y_{m_0}) + m_0 s(r) = \nu_r(y-1)$). Since

$$1 = xy \equiv \sum_{n+m<n_0+m_0} [x_n y_m]p^m + [\sum_{n+m=n_0+m_0} x_n y_m]p^{n_0+m_0} + p^{n_0+m_0+1}z$$

for some $z \in \tilde{\mathbf{A}}$, by the addition law for Witt vectors, there exists an element

$$S(\ldots, x_{ij}, \ldots) \in \mathbb{F}_p[x_{ij}^{\frac{1}{p^{n_0+m_0-i-j}}} \mid i+j < n_0+m_0]$$

which is homogenous of degree 1 (putting $\deg x_{ij} = 1$) such that

$$0 = \sum_{n+m=n_0+m_0} x_n y_m + S(\dots, x_i y_j, \dots).$$

By the choice of (n_0, m_0), for every $(n, m) \neq (n_0, m_0)$ satisfying $n + m \leq n_0 + m_0$, $\nu_{\mathbf{E}}(x_n y_m) + (n_0 + m_0)s(r) > 0$. As a consequence,

$$\nu_{\mathbf{E}}(S(\dots, x_i y_i, \dots)) + (n_0 + m_0)s(r) > 0.$$

This implies that

$$0 = \nu_{\mathbf{E}}(x_{n_0} y_{m_0}) + (n_0+m_0)s(r) = \nu_{\mathbf{E}}(\sum_{n+m=n_0+m_0, n\neq n_0} x_n y_m + S(\dots, x_i y_j, \dots)) + (n_0+m_0)s(r)$$

$$\geq \inf(\inf_{n+m=n_0+m_0, n\neq n_0} (\nu_{\mathbf{E}}(x_n y_m)), \nu_{\mathbf{E}}(S(\dots, x_i y_j, \dots))) + (n_0 + m_0)s(r) > 0.$$

A contradiction! We complete the proof. □

Corollary 1.11. *$x = \sum_{n\geq 0} p^n [x_n] \in \tilde{\mathbf{A}}^{\dagger,r}$ is a unit if and only if the set of slopes of f_x is exact $\{0\}$ and 0 is the only integer satisfying $0 = \nu_r(x) = \nu_{\mathbf{E}}(x_k) + ks(r)$.*

Corollary 1.12. *For $x = \sum_{n\geq 0} p^n [x_n] \in \tilde{\mathbf{A}}^{\dagger,r}$ such that $[x_0] \neq 0$, there is an $r_0 > r$ such that $\frac{x}{[x_0]}$ is a unit in $\tilde{\mathbf{A}}^{\dagger,r_0}$.*

Proof. Since $x_0 \neq 0$, by Proposition1.3 (3), we can choose $r_1 \geq r$ such that for all $t \geq r_1$, $f_x(t) = \nu_{\mathbf{E}}(x_0)$. Put $y = \frac{x}{[x_0]}$, since $f_x(t) = f_y(t) + \nu_{\mathbf{E}}(x_0)$, we see that $y \in \tilde{\mathbf{A}}^{\dagger,r_1}$ and $f_y(t) = 0$ for $t \geq r_1$. Using Proposition1.3 (2), for any $r_2 > r_1$, 0 is the only integer satisfying $\nu_{r_2}(y) = \nu_{\mathbf{E}}(y_k) + ks(r_2)$. Thus if we fix an $r_0 > r_1$, then y is a unit in $\tilde{\mathbf{A}}^{\dagger,r_0}$. □

Example 1.13. For $r \geq 1$, $\frac{\pi}{[\bar{\pi}]}$ is a unit in $\tilde{\mathbf{A}}^{\dagger,r}$.

In fact, $\pi = [\epsilon] - 1 = \sum_{n\geq 0} p^n [x_n]$. Then $x_0 = \bar{\pi} = \epsilon - 1$ and for $k \geq 1$, x_k is a polynomial in $\epsilon^{\frac{1}{p^k}} - 1$ of degree p^k with no constant term. Thus, $\nu_{\mathbf{E}}(x_k) \geq \nu_{\mathbf{E}}(\epsilon^{\frac{1}{p^k}} - 1) = \frac{1}{p^{k-1}(p-1)}$. For $r \geq 1$,

$$\nu_{\mathbf{E}}(x_k) + ks(r) - \nu_{\mathbf{E}}(\bar{\pi}) \geq \frac{1}{p^{k-1}(p-1)} + \frac{p}{p-1}(kr - 1) > 0.$$

Thus, by Lemma 1.10, $\frac{\pi}{[\bar{\pi}]}$ is a unit in $\tilde{\mathbf{A}}^{\dagger,r}$.

(In fact, $r > \frac{p-1}{p}$ is enough.)

Proof. (of Proposition 1.9)

For any given $x \in \tilde{\mathbf{B}}^{\dagger}$, since p is invertible, by definition of $\tilde{\mathbf{B}}^{\dagger}$, we may assume $x \in \tilde{\mathbf{A}}^{\dagger,r}$ for some $r > 0$.

We claim that for any $z \in \tilde{\mathbf{A}}^{\dagger,r} \cap p\tilde{\mathbf{A}}$, there exists a $0 \neq \alpha \in \tilde{\mathbf{E}}$ such that $\frac{[\alpha]}{p} z \in \tilde{\mathbf{A}}^{\dagger,r}$.

In fact, put $y = \frac{z}{p}$. Then $w_k(y) = w_{k+1}(z)$. Choose $0 \neq \alpha$ satisfying $\nu_{\mathbf{E}}(\alpha) > s(r)$, then

$$w_k([\alpha]y) + ks(r) \geq w_{k+1}(z) + (k+1)s(r).$$

This implies that $[\alpha]y \in \tilde{\mathbf{A}}^{\dagger,r}$.

Now, by Fact 1.6, for any $0 \neq \alpha \in \tilde{\mathbf{E}}$, $[\alpha]$ is invertible in $\tilde{\mathbf{B}}^{\dagger,r}$. We may assume $x = \sum_{n\geq 0}[x_n]p^n \in \tilde{\mathbf{A}}^{\dagger,r}$ such that $x_0 \neq 0$ for some $r > 0$. By Corollary 1.12, there exists an $r_0 > r$ such that $\frac{x}{[x_0]}$ is a unit in $\tilde{\mathbf{A}}^{\dagger,r_0}$. It follows that x is invertible in $\tilde{\mathbf{B}}^{\dagger,r_0}$.

This completes the proof. □

Now, let V be a p-adic representation of Gal_K, then $\mathrm{D}^\dagger(V) := (V \otimes \mathbf{B}^\dagger)^{H_K}$ is a vector space of dimension $\leq \dim(V)$ over $\mathbf{B}_K^\dagger$.

We say V is *overconvergent* if $\dim_{\mathbf{B}_K^\dagger}(\mathrm{D}^\dagger(V)) = \dim(V)$. Equivalently, V is overconvergent if and only if

$$\mathrm{D}^\dagger(V) \otimes_{\mathbf{B}_K^\dagger} \mathbf{B}^\dagger \simeq V \otimes \mathbf{B}^\dagger.$$

Since φ acts on $\mathbf{B}^\dagger$, $\mathrm{D}^\dagger(V)$ is a (φ, Γ_K)-module over $\mathbf{B}_K^\dagger$. A (φ, Γ_K)-module $\mathrm{D}^\dagger$ over $\mathbf{B}_k^\dagger$ is *étale* if $\mathrm{D}^\dagger \otimes_{\mathbf{B}_K^\dagger} \mathbf{B}_K$ is an étale (φ, Γ_K)-module over $\mathbf{B}_K$.

In the rest of this section, we shall show that $\mathbf{B}_K^{\dagger,r}$ is a ring consisting of Laurent series on some annulus for suitable r.

We fix some notations.

K: a finite extension of $\mathbb{Q}_p$;

F: the maximal unramified subfield of K_∞.

k_F: the residue field of F.

By previous talks, we have $\mathbf{E}_F = k_F[[\bar{\pi}]][\bar{\pi}^{-1}]$ with ring of integers $\mathbf{E}_F^+ = k_F[[\bar{\pi}]]$. $\mathbf{A}_F^+ = \mathcal{O}_F[[\pi]]$, $\mathbf{A}_F = \mathcal{O}_F\widehat{[[\pi]][\pi^{-1}]}$ and $\mathbf{B}_F = \mathbf{A}_F[\frac{1}{p}]$.

Also, $\mathbf{E}_K/\mathbf{E}_F$ is totally ramified with index $e_K = e(K_\infty/\mathbb{Q}_{p,\infty})$ and $\mathbf{E}_F/\mathbf{E}_{\mathbb{Q}_p}$ is unramified of degree $f_K = f(K_\infty/\mathbb{Q}_{p,\infty})$. Put $d_K = e_K f_K$, then

$$d_K = [\mathbf{B}_K : \mathbf{B}_{\mathbb{Q}_p}] = [\mathbf{E}_K : \mathbf{E}_{\mathbb{Q}_p}] = [K_\infty : \mathbb{Q}_{p,\infty}]$$

Let $\bar{\pi}_K$ be a uniformizer of $\mathbf{E}_K$ and let $\bar{P}_K$ be the minimal polynomial of $\bar{\pi}_K$ over $\mathbf{E}_F^+$. We choose a lifting $P_K \in \mathbf{A}_F^+[T]$ of $\bar{P}_K$. By Hensel's Lemma, there exists a unique $\pi_K \in \mathbf{A}_K$ with reduction $\bar{\pi}_K$ modulo p satisfying $P_K(\pi_K) = 0$.

Let $\mathcal{D}_K$ be the relative differential of $\mathbf{E}_K$ over $\mathbf{E}_F$. Then $\nu_{\mathbf{E}}(\mathcal{D}_K) = \nu_{\mathbf{E}}(\bar{P}'_K(\bar{\pi}_K))$.

Lemma 1.14. *For every $k \geq 1$, $w_k(\pi_K) \geq -(2k-1)\nu_{\mathbf{E}}(\mathcal{D}_K)$.*

Proof. (See [Col, Lemma 6.4])

If $\pi_K = \sum_{i \geq 0} [x_i] p^i$, then we need to show that for every $k \geq 1$, $w_k(\pi_K) \geq -(2k-1)\nu_{\mathbf{E}}(\mathcal{D}_K)$. Put $z_k = \sum_{i=0}^k p^i [x_i]$. Then $P_K(z_k) \in p^{k+1}\mathbf{A}_K$.

Firstly, assume $k = 1$. Because $P_K \in \mathbf{A}_F^+[T]$, if $P_K([\bar{\pi}_K]) = p[u] + p^2 v$, then $u \in \mathbf{E}^+$. Therefore, we have

$$0 \equiv P_K(z_1) = P_K([\bar{\pi}_K] + p[x_1]) \equiv P_K([\bar{\pi}_K]) + P'_K([\bar{\pi}_K])[x_1]p \equiv [u + \bar{P}'_K(\bar{\pi}_K)x_1]p \mod p^2\mathbf{A}_K.$$

Thus, $\nu_{\mathbf{E}}(x_1) \geq -\nu_{\mathbf{E}}(\bar{P}'_K(\bar{\pi}_K))$ as desired.

For general case, we do induction on k. By inductive hypothesis, for every $n \geq k+1$, we have

$$\begin{aligned} w_n(P_K(z_k)) &\geq \inf_{1 \leq i_j \leq k, i_1 + \cdots + i_r \leq n} -(2i_j - 1)\nu_{\mathbf{E}}(\bar{P}'_K(\bar{\pi}_K)) \\ &= \inf_{1 \leq i_j \leq k, i_1 + \cdots + i_r = n} -(2i_j - 1)\nu_{\mathbf{E}}(\bar{P}'_K(\bar{\pi}_K)) \\ &= -\sup_{1 \leq i_j \leq k, i_1 + \cdots + i_r = n} (2n - r)\nu_{\mathbf{E}}(\bar{P}'_K(\bar{\pi}_K)) \\ &\geq -(2n-2)\nu_{\mathbf{E}}(\bar{P}'_K(\bar{\pi}_K)) \qquad (\because n \geq k+1 \therefore r \geq 2). \end{aligned}$$

In particular, we get $w_{k+1}(P_K(z_k)) \geq -2k\nu_{\mathbf{E}}(\bar{P}_K(\bar{\pi}_K))$. In other words, if we write $P_K(z_k) = p^{k+1}[y_{k+1}] + p^{k+2}v$, then $\nu_{\mathbf{E}}(y_{k+1}) \geq -2k\nu_{\mathbf{E}}(\bar{P}_K(\bar{\pi}_K))$. Therefore, we have

$$\begin{aligned} 0 \equiv P_K(z_{k+1}) = P_K(z_k + p^{k+1}[x_{k+1}]) &\equiv P_K(z_k) + P'_K(z_k)[x_{k+1}]p^{k+1} \\ \equiv [y_{k+1}]p^{k+1} + P'_K([\bar{\pi}_K])[x_{k+1}]p^{k+1} &\equiv [y_{k+1} + \bar{P}'_K(\bar{\pi}_K)x_{k+1}]p^{k+1} \mod p^{k+2}. \end{aligned}$$

Thus, $\nu_{\mathbf{E}}(x_{k+1}) \geq -(2k+1)\nu_{\mathbf{E}}(\bar{P}_K(\bar{\pi}_K))$ as desired. □

Corollary 1.15. *For every $k \geq 1$, $w_k(P'_K(\pi_K)) \geq -(2k-1)\nu_{\mathbf{E}}(\bar{P}'_K(\bar{\pi}_K))$.*

Proof. The proof is similar to the proof of general case in Lemma 1.14.

Let $P_K(T) = a_d T^d + a_{d-1} T^{d_1} + \cdots + a_0 \in \mathbf{A}_F^+[T]$ with $a_d = 1$. Then for every $n \geq 1$,

$$P'_K(\pi_K) \equiv \sum_{i=0}^{d-1} (i+1) a_{i+1} (\sum_{j=0}^{n} p^j [x_j])^i \mod p^{n+1}.$$

Because $a_k \in \mathbf{A}_F^+$, by Lemma 1.14, we see that

$$\begin{aligned} w_n(P'_K(\pi_K)) &\geq \inf_{1\leq i_j\leq n, i_1+\cdots+i_r\leq n} -(2i_j-1)\nu_{\mathbf{E}}(\bar{P}_K(\bar{\pi}_K)) \\ &\geq \inf_{1\leq i_j\leq n, i_1+\cdots+i_r\leq n} -(2n-r)\nu_{\mathbf{E}}(\bar{P}_K(\bar{\pi}_K)) \geq -(2n-1)\nu_{\mathbf{E}}(\bar{P}_K(\bar{\pi}_K)) \end{aligned}$$

as expected. ☐

Define

$$r_K = \begin{cases} \frac{2\nu_{\mathbf{E}}(\mathcal{D}_K)(p-1)}{p}, & \text{if } \mathbf{E}_K/\mathbf{E}_{\mathbb{Q}_p} \text{ramified} \\ \frac{p-1}{p}, & \text{if } \mathbf{E}_K/\mathbf{E}_{\mathbb{Q}_p} \text{unramified} \end{cases}$$

Lemma 1.16. *For $r > r_K$, $\pi_K \in \mathbf{A}_K^{\dagger,r}$. Moreover, we have*

(1) $\frac{\pi_K}{[\bar{\pi}_K]}$ *is a unit in* $\in \mathbf{A}_K^{\dagger,r}$;
(2) $\frac{P'_K(\pi_K)}{[\bar{P}'_K(\bar{\pi}_K)]}$ *is a unit in* $\in \mathbf{A}_K^{\dagger,r}$.

Proof. By Lemma 1.14, for any $k \geq 1$, $w_k(\pi_K) + ks(r_K) \geq \nu_{\mathbf{E}}(\bar{P}'_K(\bar{\pi}_K))$. Thus, for any $r > r_K$ and $k \geq 1$, $w_k(\pi_K) + ks(r) \geq \nu_{\mathbf{E}}(\bar{P}'_K(\bar{\pi}_K)) + ks(r - r_K)$. Thus, $\pi_K \in \mathbf{A}_K^{\dagger,r}$ and by Lemma 1.10 $\frac{\pi_K}{[\bar{\pi}_K]}$ is a unit. The proof of (2) is similar. ☐

Now, we put $f_i = \pi_K^{i-1}$ for $1 \leq i \leq e_K$. Then $\{f_i\}_{1\leq i\leq e_K}$ is a basis of $\mathbf{A}_K$ over $\mathbf{A}_F$. Let $\{f_i^*\}_{1\leq i\leq e_K}$ be the dual basis of $\mathbf{A}_K$ over $\mathbf{A}_F$ with respect to the perfect pair

$$(-,-) : \mathbf{A}_K \times \mathbf{A}_K \to \mathbf{A}_F, \quad (x,y) \mapsto \mathrm{Tr}_{\mathbf{B}_K/\mathbf{B}_F}(xy).$$

Lemma 1.17. *For $1 \leq i \leq e_K$, $P'_K(\pi_K)f_i^* \in \mathbf{A}_F^+[\pi_K]$.*

Proof. By [Ser, III.§6.Lemma 2], we see that

$$\mathrm{Tr}\left(\frac{\pi_K^j}{P'_K(\pi_K)}\right) = \begin{cases} 0, & 0 \leq j \leq e_K - 2 \\ 1, & j = e_K - 1 \end{cases}$$

Since for all $i \geq 0$, π_K^i is a linear combination of π_K^j for $0 \leq j \leq e_K - 1$, we see that f_i^* is of form $\frac{Q_i(\pi_K)}{P'_K(\pi_K)}$ for some monic polynomial $Q_i \in \mathbf{A}_F^+[T]$. This shows the lemma. ☐

Corollary 1.18. *For $r > r_K$, $\mathbf{B}_K^{\dagger,r}$ is a free module over $\mathbf{B}_{\mathbb{Q}_p}^{\dagger,r}$ of rank d_K. As a consequence, $[\mathbf{B}_K^\dagger : \mathbf{B}_{\mathbb{Q}_p}^\dagger] = d_K$.*

Proof. By Lemma1.16 (2), $f_i^* \in \mathbf{B}_K^{\dagger,r}$ for $r > r_K$. (In fact, $[\bar{P}'_K(\bar{\pi}_K)]f_i^* \in \mathbf{A}_K^{\dagger,r}$.) Thus, for any $x \in \mathbf{B}_K^{\dagger,r}$, x can be uniquely written as

$$x = \sum_{j=0}^{e_K-1} \mathrm{Tr}(x\pi_K^j)f_j^*.$$

Therefore, $\{f_j^*\}_{1\leq j\leq e_K}$ is a basis of $\mathbf{B}_K^{\dagger,r}$ over $\mathbf{B}_F^{\dagger,r}$.

For $K=F$, this follows from the fact $\mathbf{B}_F=\mathbf{B}_{\mathbb{Q}_p}\otimes_{\mathbb{Z}_p}\mathcal{O}_F$. □

Recall we have proved following Lemma in previous talks.

Lemma 1.19. *For any $x\in\mathbf{A}_K$, x can be uniquely written as*

$$x=\sum_{n\in\mathbb{Z}}a_n\pi_K^n,\quad a_n\in\mathcal{O}_F$$

satisfying $\lim_{n\to-\infty}a_n=0$.

Proof. Recall $\mathbf{E}_K=k_F[[\bar{\pi}_K]][\bar{\pi}_K^{-1}]$. We define a section $s:\mathbf{E}_K\to\mathbf{A}_K$ of natural projection $\mathbf{A}_K\to\mathbf{E}_K$ by

$$s(\sum_{n\gg-\infty}\bar{b}_n\bar{\pi}_K^n)=\sum_{n\gg-\infty}[\bar{b}_n]\pi_K^n.$$

For $x\in\mathbf{A}_K$, put $x_0:=x$. Define $x_{n+1}=\frac{x-s(\bar{x}_n)}{p}$ inductively. Then we have

$$x=\sum_{n\geq0}p^ns(\bar{x}_n).$$

The uniqueness is trivial by construction. □

Lemma 1.20. *Assume $r>r_K$ For $\bar{x}\in\mathbf{E}_K$, then $s(\bar{x})\in\mathbf{A}_K^{\dagger,r}[\frac{1}{[\bar{\pi}]}]$. In this case, $\nu_r(s(\bar{x}))=\nu_{\mathbf{E}}(\bar{x})$.*

Proof. Because $\mathbf{E}_K$ is a free module over $\mathbf{E}_F$ of rank e_K with a set of basis $\{\bar{\pi}_K^j\}_{0\leq j\leq e_K-1}$, it suffices to check that $s(\bar{\pi}_K)\in\mathbf{A}_K^{\dagger,r}$. This follows from Lemma1.16.

If $\bar{x}=\sum_{n\geq n_0}a_n\bar{\pi}_K^n$ ($a_n\in k_F$) with $0\neq a_{n_0}$, then $\nu_{\mathbf{E}}(\bar{x})=n_0\nu_{\mathbf{E}}(\bar{\pi}_K)$. However, for $n\geq n_0$, $\nu_r([a_n]\pi_K^n)=\nu_r([a_n\bar{\pi}_K^n])=n\nu_{\mathbf{E}}(\bar{\pi}_K)$. Thus, $\nu_r(s(\bar{x}))=\nu_{\mathbf{E}}(\bar{x})$. □

Lemma 1.21. *If $x\in\mathbf{A}_K$ and if $k\geq0$, then*

$$w_k(\frac{x-s(\bar{x})}{p})\geq\inf(w_{k+1}(x),w_0(x)-(k+1)s(r_K)).$$

Proof. Replacing x by a multiplication of x by a power of $[\bar{\pi}_K]$, we may assume $\bar{x}\in\mathbf{E}_K^{+,\times}$; that is $\nu_{\mathbf{E}}(\bar{x})=0$. Since

$$w_k(\frac{x-s(\bar{x})}{p})=w_{k+1}(x-s(\bar{x}))\geq\inf(w_{k+1}(x),w_{k+1}(s(\bar{x}))),$$

it suffices to check that $w_{k+1}(s(\bar{x}))\geq-(k+1)s(r_K)$.

For $n\geq0$, by Fact1.1 and Lemma1.14,

$$w_{k+1}(\pi_K^n)\geq\inf_{i_1+\cdots+i_n=k+1}(w_{i_1}(\pi_K)+\cdots+w_{i_n}(\pi_K))$$

$$\geq \inf_{i_1+\cdots+i_n=k+1}((2i_1-1)\nu_{\mathbf{E}}(\mathcal{D}_K)+\cdots+(2i_n-1)\nu_{\mathbf{E}}(\mathcal{D}_K))$$

$$\geq -(2k+2-n)\nu_{\mathbf{E}}(\mathcal{D}_K) \geq -(k+1)(2\nu_{\mathbf{E}}(\mathcal{D}_K)) = -(k+1)s(r_K).$$

Because we have assumed $\bar{x} \in \mathbf{E}_K^{+,\times}$, $w_{k+1}(s(\bar{x})) \geq -(k+1)s(r_K)$ by definition of s. □

For $x \in \mathbf{A}_K$, we define $x_0 = x$ and define $x_{n+1} = \frac{x_n - s(\bar{x}_n)}{p}$ inductively.

Corollary 1.22. *If $n \geq 0$, then $\nu_{\mathbf{E}}(\bar{x}_n) \geq \inf_{0\leq i\leq n}(w_i(x) - (n-i)s(r_K))$.*

Proof. For $n = 0$, the result is trivial. So we assume $n \geq 1$.

We prove that for every $k \geq 0$, for $n \geq 1$,

$$w_k(x_n) \geq \inf(w_{k+n}(x), \inf_{0\leq i\leq n-1}(w_i(x) - (k+n-i)s(r_K))).$$

The result is the case for $k = 0$.

We give the proof by induction on n. For $n = 1$, this is the result of Lemma1.21. By Lemma1.21 again,

$$w_k(x_{n+1}) \geq \inf(w_{k+1}(x_n), w_0(x_n) - (k+1)s(r_K)).$$

By inductive hypothesis (on n), $w_0(x_n) \geq w_n(x)$ and

$$w_{k+1}(x_n) \geq \inf(w_{k+1+n}(x), \inf_{0\leq i\leq n-1}(w_i(x) - (k+1+n-i)s(r_K))).$$

Combining these inequalities, we prove the desired result for $n+1$. □

Let $\mathcal{A}_F^r$ be the ring of Laurent series $f(T) = \sum_{n\in\mathbb{Z}} a_n T^n$ with $a_n \in \mathcal{O}_F$ such that $\nu_p(a_n) + nr \geq 0$ and that $\lim_{n\to-\infty} \nu_p(a_n) + nr = +\infty$. If $f \in \mathcal{A}_F^r$, we define $\omega_r(f) = \inf_n(\nu_p(a_n) + nr)$. Then it can be checked that ω_r is a valuation on $\mathcal{A}_F^r$. The $\mathcal{A}_F^r$ can be viewed as the ring of analytic functions on annulus $\{0 < \nu_p(T) \leq r\}$ which are bounded by 1 with coefficients in $\mathcal{O}_F$. Let $\mathcal{B}_F^r = \mathcal{A}_F^r[\frac{1}{p}]$, which is the ring of bounded analytic functions on annulus $\{0 < \nu_p(T) \leq r\}$ whose coefficients belong to F. Then we have the following theorem.

Theorem 1.23. *Assume $r > r_K$.*

(1) *The map $f \mapsto f(\pi_K)$ induces an isomorphism of topological rings from $(\mathcal{A}_F^{\frac{1}{re_K}}, s(r)\omega_{\frac{1}{re_K}})$ to $(\mathbf{A}_K^{\dagger,r}, \nu_r)$ such that $s(r)\omega_{\frac{1}{re_K}}(f) = \nu_r(f(\pi_K))$.*

(2) *The map $f \mapsto f(\pi_K)$ induces an isomorphism from $\mathcal{B}_F^{\frac{1}{re_K}}$ to $\mathbf{B}_K^{\dagger,r}$.*

Proof. The (2) is a consequence of (1). So we only need to prove (1).

Assume $f = \sum_{n\in\mathbb{Z}} a_nT^n \in \mathcal{A}_F^{\frac{1}{re_K}}$. By Lemma1.16, $a_n\pi_K^n = p^{\nu_p(a_n)}[\bar{\pi}_K]^n u_n$ for some unit $u_n \in \mathbf{A}_K^{\dagger,r}$. Therefore, $\nu_r(a_n\pi_K^n) = \nu_p(a_n)s(r) + n\nu_{\mathbf{E}}(\bar{\pi}_K)$. Recall $\nu_{\mathbf{E}}(\bar{\pi}_K) = \frac{1}{e_K}\cdot\frac{p}{p-1} = s(\frac{1}{e_K})$. It follows that

$$\nu_r(a_n\pi_K^n) = s(r)\nu_p(a_n) + ns(\frac{1}{e_K}) = s(r)(\nu_p(a_n) + \frac{n}{re_K}) = s(r)\omega_{\frac{1}{re_K}}(a_nT^n).$$

Therefore, for such an $f = \sum_{n\in\mathbb{Z}} a_nT^n \in \mathcal{A}_F^{\frac{1}{re_K}}$, we see that $f(\pi_K) \in \mathbf{A}_K^{\dagger,r}$ and that $\nu_r(f(\pi_K)) \geq \inf_n \nu_r(a_n\pi_K^n) = s(r)\omega_{\frac{1}{re_K}}(f)$.

Conversely, assume $x \in \mathbf{A}_K^{\dagger,r}$. By the proof of Lemma1.19, $x = \sum_{n\geq 0} p^n s(\bar{x}_n)$. Put $d_n = \frac{\nu_{\mathbf{E}}(\bar{x}_n)}{\nu_{\mathbf{E}}(\bar{\pi}_K)}$. By definition of s, there exists a unique $f_n \in T^{d_n}\mathcal{O}_F[[T]]$ such that $s(\bar{x}_n) = f_n(\pi_K)$. Therefore $x = \sum_{n\geq 0} p^n f_n(\pi_K)$.

We need to show that $p^nf_n \in \mathcal{A}_F^{\frac{1}{re_K}}$. Assume $f_n = T^{d_n}\sum_{j\geq 0} b_jT^j$ with $b_i \in \mathcal{O}_F$. Then $\omega_{\frac{1}{re_K}}(b_jp^nT^{d_n+j}) \geq n + \frac{d_n+j}{re_K}$. Recall $d_n = \frac{\nu_{\mathbf{E}}(\bar{x}_n)}{\nu_{\mathbf{E}}(\bar{\pi}_K)}$. So $\frac{d_n}{re_K} = \frac{\nu_{\mathbf{E}}(\bar{x}_n)}{s(r)}$. By Corollary1.22,

$$\frac{d_n}{re_K} \geq \frac{1}{s(r)}\inf_{0\leq i\leq n}(w_i(x) - (n-i)s(r_K)).$$

Then we deduce that

$$\omega_{\frac{1}{re_K}}(b_ip^nT^{d_n+j}) \geq \frac{j}{re_K} + \frac{1}{s(r)}\inf_{0\leq i\leq n}(w_i(x) + is(r) + (n-i)s(r-r_K))$$

and thus $p^nf_n \in \mathcal{A}_F^{\frac{1}{re_K}}$. Moreover, from above formula, we also see that

$$s(r)\omega_{\frac{1}{re_K}}(p^nf_n) \geq \inf(w_n(x) + ns(r), \nu_r(x) + s(r-r_K)) \geq \nu_r(x).$$

Therefore $f := \sum_{n\geq 0} p^nf_n \in \mathcal{A}_F^{\frac{1}{re_K}}$ satisfying $f(\pi_K) = x$ and $s(r)\omega_{\frac{1}{re_K}}(f) \geq \nu_r(x)$.

These complete the proof. □

2. Colmez-Tate-Sen conditions and proof of Theorem 0.1

2.1. **Colmez-Tate-Sen Condition for $\tilde{\mathbf{B}}^{\dagger,r}$.** Recall ([Ber1, Section 19]): let $\widetilde{\Omega}$ be a $\mathbb{Q}_p$-algebra endowed with a map

$$\nu_\omega : \Omega \to \mathbb{R}\cup\{+\infty\}$$

such that

(1) $\nu_\Omega(x) = 0$ if and only if $x = 0$;

(2) $\nu_\Omega(x+y) \geq \inf(\nu_\Omega(x), \nu_\Omega(y))$;

(3) $\nu_\Omega(xy) \geq \nu_\Omega(x) + \nu_\Omega(y)$;

(4) $\nu_\Omega(p) > 0$ and $\nu_\Omega(px) = \nu_\Omega(p) + \nu_\Omega(x)$.

We assume that $(\widetilde{\Omega}, \nu_\Omega)$ is a completed Banach space over $\mathbb{Q}_p$ and that Gal_K acts on Ω as isometries. Then we say $\widetilde{\Omega}$ satisfies Colmez-Sen-Tate conditions if there exists c_1, c_2 and c_3 in $\mathbb{R}_{\geq 0}$ such that followong conditions are fulfilled.

$(CST\ 1)$ For every finite extensions M/L of K, there exists $\alpha \in \widetilde{\Omega}^{H_M}$ such that $\nu_\Omega(\alpha) > -c_1$ and that $\mathrm{Tr}_{M_\infty/L_\infty}(\alpha) = 1$;

$(CST\ 2)$ For every finite extension finite L/K, there is an increasing sequence $\{\Omega_{L,n}\}_{n\geq 0}$ of closed sub-$\mathbb{Q}_p$-algebra of $\widetilde{\Omega}$ together with maps $R_{L,n} : \widetilde{\Omega}^{H_L} \to \Omega_{L,n}$ satisfying following properties:
(i) if $x \in \widetilde{\Omega}^{H_L}$, then $\nu_\Omega(R_{L,n}(x)) \geq \nu_\Omega(x) - c_2$ and $\lim_n R_{L,n}(x) = x$;
(ii) if $L_1 \subset L_2$, then $\Omega_{L_1,n} \subset \Omega_{L_2,n}$ and the restriction of $R_{L_2,n}$ to $\Omega_{L_1,n}$ is $R_{L_1,n}$;
(iii) $R_{L,n}$ is $\Omega_{L,n}$-linear and is the identify on $\Omega_{L,n}$;
(iv) if $\sigma \in \mathrm{Gal}_K$, then $\sigma(\Omega_{L,n}) = \Omega_{\sigma(L),n}$ and $R_{\sigma(L),n} \circ \sigma = g \circ R_{L,n}$.

$(CST\ 3)$ For every finite extension L/K, there exists an $m(L) \geq n(L)$ such that if $\gamma \in \Gamma_L$ and $n \geq \sup(n(\gamma), m(L))$, then $(1-\gamma)$ is invertible on $X_{L,n} = (1 - R_{L,n})(\widetilde{\Omega}^{H_L})$ and $\nu_\Omega((\gamma - 1)^{-1}(x)) \geq \nu_\Omega(x) - c_3$ for $x \in X_{L,n}$.

Example 2.1. $(\mathbb{C}_p, \nu_p)$ satisfies CST-conditions.

In this section, we shall give another example.

Proposition 2.2 $(CST\ 1)$**.** *Let L/K be finite extensions of $\mathbb{Q}_p$. Fix an $r > 0$, for any $\delta > 0$, there exists an $\alpha \in \tilde{\mathbf{B}}_L^{\dagger,r}$ with $\nu_r(\alpha) > -\delta$ such that* $\mathrm{Tr}_{L_\infty/M_\infty}(\alpha) = 1$.

Proof. Since $\tilde{\mathbf{E}}_L/\tilde{\mathbf{E}}_K$ is separated, there exists some $\beta \in \tilde{\mathbf{E}}_L$ satisfying $\mathrm{Tr}(\beta) = 1$. Because $\nu_{\mathbf{E}}(\varphi^{-n}(\beta)) = p^{-n}\nu_{\mathbf{E}}(\beta)$, we may assume that $\nu_{\mathbf{E}}(\beta) > \sup(-s(r), -\delta)$.

Then we see that $\mathrm{Tr}([\beta]) = 1 + \sum_{n\geq 1}[x_n]p^n$ with

$$\nu_{\mathbf{E}}(x_k) \geq \nu_{\mathbf{E}}(\beta) > -ks(r).$$

Therefore, $\mathrm{Tr}([\beta]) \in \tilde{\mathbf{A}}_K^{\dagger,r}$ and $\nu_r(\mathrm{Tr}([\beta]) - 1) > 0$. By Lemma1.10, $\mathrm{Tr}([\beta])$ is a unit in $\tilde{\mathbf{A}}_K^{\dagger,r}$. Put $\alpha = \frac{[\beta]}{\mathrm{Tr}([\beta])}$, then $\nu_r(\alpha) = \nu_{\mathbf{E}}(\beta) > -\delta$. □

Define $I = \mathbb{Z}[\frac{1}{p}] \cap [0, 1)$ and $I_m = \{x \in I \mid \nu_p(x) \geq -m\}$ for $m \geq 0$.

Define $\mathbf{E}_{K,m} = \varphi^{-m}(\mathbf{E}_K)$ for $m \geq 0$. Then $\mathbf{E}_{K,m}/\mathbf{E}_K$ is purely inseparable of degree p^m and $\tilde{\mathbf{E}}_K$ is the completion of $\mathbf{E}_{K,\infty} = \cup_{m\geq 0}\mathbf{E}_{K,m}$ with respect to $\nu_{\mathbf{E}}$.

The following Lemma is obvious and we omit the proof.

Lemma 2.3. *If $m \geq 0$, then $\{\epsilon^i\}_{i\in I_m}$ is a basis of $\mathbf{E}^+_{\mathbb{Q}_p,m}$ over $\mathbf{E}^+_{\mathbb{Q}_p}$.*

Proposition 2.4. *Assume* $c_K = \nu_{\mathbf{E}}(\mathcal{D}_K) + \nu_{\mathbf{E}}(\bar{\pi})$.

(1) *For every element* $x \in \mathbf{E}_{K,m}$*, it can be uniquely written as*

$$x = \sum_{i \in I_m} a_i(x)\epsilon^i, \quad a_i(x) \in \mathbf{E}_K$$

such that $\nu_{\mathbf{E}}(x) - c_K \leq \inf_{i \in I_m} \nu_{\mathbf{E}}(a_i(x)) \leq \nu_{\mathbf{E}}(x)$.

(2) *For every element* $x \in \tilde{\mathbf{E}}_K$*, it can be uniquely written as*

$$x = \sum_{i \in I} a_i(x)\epsilon^i, \quad a_i(x \in \mathbf{E}_K)$$

such that $\lim_i a_i(x) = 0$ *and that* $\nu_{\mathbf{E}}(x) - c_K \leq \inf_{i \in I} \nu_{\mathbf{E}}(a_i(x)) \leq \nu_{\mathbf{E}}(x)$.

Proof. (1) Since $\mathbf{E}_{\mathbb{Q}_p,m}/\mathbf{E}_{\mathbb{Q}_p}$ is purely inseparable and $\mathbf{E}_K/\mathbf{E}_{\mathbb{Q}_p}$ is separable, a basis of $\mathbf{E}_{\mathbb{Q}_p,m}$ over $\mathbf{E}_{\mathbb{Q}_p}$ is also a basis of $\mathbf{E}_{K,m}/\mathbf{E}_K$. So the existence and the uniqueness is clear. It is trivial that $\inf_{i \in I_m} \nu_{\mathbf{E}}(a_i(x)) \leq \nu_{\mathbf{E}}(x)$. By uniqueness, the function a_i is $\mathbf{E}_K$-linear.

In the case where $K = F$ (thus $c_F = \nu_{\mathbf{E}}(\bar{\pi})$), up to a multiplication by some power of $\bar{\pi}$, we may assume $0 \leq \nu_{\mathbf{E}}(x) < \nu_{\mathbf{E}}(\bar{\pi})$. Since $\{\epsilon\}_{i \in I_m}$ is a basis of $\mathbf{E}^+_{F,m}$ over $\mathbf{E}^+_F$, we see that $\nu_{\mathbf{E}}(a_i(x)) \geq 0 \geq \nu_{\mathbf{E}}(x) - c_F$.

In the general case, we choose $\{e_1, \ldots, e_d\}$ to be a basis of $\mathbf{E}^+_K/\mathbf{E}^+_F$ with $d = [\mathbf{E}_K : \mathbf{E}_F] = e_K$. Let $\{e_i^*\}_{1 \leq i \leq d}$ be the dual basis of $\mathbf{E}_K/\mathbf{E}_F$ under the perfect pairing $(x, y) \mapsto \mathrm{Tr}_{\mathbf{E}_K/\mathbf{E}_F}(xy)$ on $\mathbf{E}_K$. Then $\{e_i^*\}_{1 \leq i \leq d}$ is the basis of $\mathcal{D}_K^{-1}$ over $\mathbf{E}^+_F$ (recall $\mathcal{D}_K$ is the idea of relative differentials). In particular, $\nu_{\mathbf{E}}(e_i^*) \geq -\nu_{\mathbf{E}}(\mathcal{D}_K)$. Clearly, for every $m \geq 0$, $\{e_1, \ldots, e_d\}$ is a basis of $\mathbf{E}_{K,m}/\mathbf{E}_{F,m}$ and $\{e_i^*\}_{1 \leq i \leq d}$ is the corresponding dual basis under

$$(x, y) \mapsto \mathrm{Tr}_{\mathbf{E}_{K,M}/\mathbf{E}_{F,m}}(xy) = \mathrm{Tr}_{\mathbf{E}_K/\mathbf{E}_F}(xy).$$

Therefore, if $x \in \mathbf{E}_{K,m}$, $x = \sum_{j=1}^d \mathrm{Tr}(xe_j)e_j^*$. Since $\mathrm{Tr}(xe_j) \in \mathbf{E}_F$ and $\nu_{\mathbf{E}}(\mathrm{Tr}(xe_j)) \geq \nu_{\mathbf{E}}(x)$, if we define $a_i(x) = \sum_{j=1}^d a_i(\mathrm{Tr}(xe_j))e_j^*$, then

$$\nu_{\mathbf{E}}(a_i(x)) \geq \inf_j \nu_{\mathbf{E}}(a_i(\mathrm{Tr}(xe_j))) - \nu_{\mathbf{E}}(\mathcal{D}_K) \geq \nu_{\mathbf{E}}(x) - c_K.$$

(2) By the proof of (1), we see that a_i is continuous and $\mathbf{E}_K$-linear. Then (2) follows from (1) and the fact that $\tilde{\mathbf{E}}_K$ is the completion of $\mathbf{E}_{K,\infty}$. □

Remark 2.1. From the proof of Proposition2.4, when K is unramified over $\mathbb{Q}_p$, $x \in \tilde{\mathbf{E}}^+_K$ if and only if $a_i(x) \in \mathbf{E}^+_K$.

For $m \geq 0$, we define $\mathbf{A}_{K,m} = \varphi^{-m}(\mathbf{A}_K)$, which is a Cohen ring of $\mathbf{E}_{K,m}$. Define $\mathbf{A}_{K,\infty} = \cup_{m\geq 0}\mathbf{A}_{K,m}$. Then $\tilde{\mathbf{A}}_K$ is the completion of $\mathbf{A}_{K,\infty}$ with respect to the canonical topology on $\tilde{\mathbf{A}}$.

Then it is conceivable that the following proposition is true.

Proposition 2.5. (1) *For every $x \in \mathbf{A}_{K,m}$, x can be uniquely written by formula*

$$x = \sum_{i\in I_m} a_i(x)[\epsilon]^i, \qquad a_i(x) \in \mathbf{A}_K.$$

(2) *For every $x \in \tilde{\mathbf{A}}_K$, x can be uniquely written by formula*

$$x = \sum_{i\in I} a_i(x)[\epsilon]^i, \qquad a_i(x) \in \mathbf{A}_K$$

such that $a_i(x) \to 0$ for the canonical topology on $\tilde{\mathbf{A}}$.

(3) *When $K/\mathbb{Q}_p$ is unramified, $x \in \tilde{\mathbf{A}}_K^+$ if and only if $a_i(x) \in \mathbf{A}_K^+$ for all i.*

Proof. It suffices to prove (1).

We define $s : \mathbf{E}_{K,m} \to \mathbf{A}_{K,m}$ by $s(\sum_{i\in I_m} a_i(\bar{x})\epsilon^i) = \sum_{i\in I_m}[a_i(\bar{x})][\epsilon]^i$, which is a section of the natural projection $\mathbf{A}_{K,m} \to \mathbf{E}_{K,m}$. Put $x_0 = x$. For $n \geq 0$, we put $x_{n+1} = \frac{(x_n - s(\bar{x}_n))}{p}$ inductively. If we define $a_i(x) = \sum_{n\geq 0} p^n[a_i(\bar{x}_n)]$, then we deduce that

$$x = \sum_{i\in I_m} a_i(x)[\epsilon]^i.$$

The uniqueness is clear by the construction and the uniqueness criterion of Proposition2.4.

Clearly, a_i is $\mathbf{A}_K$-linear and continuous under the canonical topology. □

Corollary 2.6. *For $n \geq 0$, put $R_{K,m} : \tilde{\mathbf{A}}_K \to \mathbf{A}_{K,m}$ by $R_{K,m}(x) = \sum_{i\in I_m} a_i(x)[\epsilon]^i$. Then we have*

(1) $\lim_{m\to+\infty} R_{K,m}(x) = x$;

(2) $R_{K,m} = \varphi^{-m} \circ R_{K,0} \circ \varphi^m$;

(3) *$R_{K,m}$ is an $\mathbf{A}_{K,m}$-linear, continuous section of the inclusion $\mathbf{A}_{K,m} \hookrightarrow \tilde{\mathbf{A}}_K$;*

(4) *if $\sigma \in \mathrm{Gal}_{\mathbb{Q}_p}$, then $\sigma \circ R_{K,m} = R_{\sigma(K),m} \circ \sigma$.*

Proof. The (1) is trivial. For (2), since $a_0 = R_{K,0}$ is $\mathbf{A}_K$-linear, we deduce that

$$R_{K,0}(\varphi^m(x)) = \sum_{i\in I_m} \varphi^m(a_i(x))[\epsilon]^{p^m i}.$$

This shows (2) and thus (3) (by applying (2)).

For (4), one can prove Proposition2.5 by replacing ϵ by $\sigma(\epsilon)$. Then (4) follows from the uniqueness criterion. □

Clearly, we can extend $R_{K,n}$ to $\tilde{\mathbf{B}}_K$.

For $r > 0$ and $m \geq 0$, we define $\mathbf{A}^{\dagger,r}_{K,m} = \tilde{\mathbf{A}}^{\dagger,r}_K \cap \mathbf{A}_{K,m}$. Then we also have $\mathbf{A}^{\dagger,r}_{K,m} = \varphi^{-m}(\mathbf{A}^{\dagger,p^m r}_K)$. We want to show that for suitable r, if we restrict $R_{K,n}$ to $\tilde{\mathbf{B}}^{\dagger,r}_K$, then the image of $R_{K,n}$ is contained in $\mathbf{B}^{\dagger,r}_{K,n}$. Thus, $(CSD\ 2)$ holds for $\tilde{\mathbf{B}}^{\dagger,r}$ for suitable r and hence for $\tilde{\mathbf{B}}^{\dagger}$.

Lemma 2.7. *If $\alpha \in \tilde{\mathbf{E}}$ and $l \in \mathbb{Z}$ satisfying $\nu_{\mathbf{E}}(\alpha) \geq -l\nu_{\mathbf{E}}(\bar{\pi})$, then $[\alpha]$ can be uniquely written as*

$$[\alpha] = \sum_{n\geq 0} \frac{p^n}{\pi^{l+a(n)}}[\beta_n]$$

with $\beta_n \in \tilde{\mathbf{E}}^+$, where $a(n) = \lfloor \frac{p-1}{p} n \rfloor$ is the smallest integer $\geq \frac{p-1}{p} n$.

Proof. Put $r = \frac{p-1}{p}$. We note that if $x = \sum_{n\geq 0}[\alpha_n]p^n \in \tilde{\mathbf{A}}$ and $b \in \mathbb{Z}$, then

$$\nu_r(\frac{[\bar{\pi}]^b}{p}(x - [\alpha_0])) = \inf_{k\geq 0}(s(b) + \nu_{\mathbf{E}}(\alpha_{k+1}) + ks(r)) \geq s(b) - 1 + \nu_r(x).$$

Now, we construct β_n inductively. Put $x_0 = \pi^l[\alpha], \beta_n = \bar{x}_n$ and

$$x_{n+1} = \frac{\pi^{a(n+1)-a(n)}}{p}(x_n - [\beta_n]) = (\frac{\pi}{[\bar{\pi}]})^{a(n+1)-a(n)}\frac{[\bar{\pi}]^{a(n+1)-a(n)}}{p}(x_n - [\beta_n]).$$

By example1.13, $\nu_r(\frac{\pi}{[\bar{\pi}]}) = 0$. Therefore, we deduce that

$$\nu_r(x_{n+1}) \geq s(a(n+1) - a(n)) - 1 + \nu_r(x_n).$$

By hypothesis, $\nu_r(x_0) = \nu_r([\alpha]\pi^l) \geq 0$. By induction on n, we see that

$$\nu_r(x_n) \geq s(a(n)) - n \geq 0, \quad \forall n \geq 0,$$

because $a(n) = \lfloor \frac{p-1}{p} n \rfloor$. Therefore $\nu_{\mathbf{E}}(\beta_n) \geq \nu_r(x_n) \geq 0$. The uniqueness comes from the construction. □

Proposition 2.8. *If $r > r_K$ and if $x \in \tilde{\mathbf{A}}^{\dagger,r}_K$, then $a_i(x) \in \mathbf{A}^{\dagger,r}_K[\frac{1}{[\pi]}]$ and for all $i \in I$,*

$$\nu_r(a_i(x)) \geq \nu_r(x) - c_K \quad \text{and} \quad \lim_i \nu_r(a_i(x)) = +\infty.$$

Proof. We assume $x \neq 0$.

Case 1: $K = F$.

We assume $x=[\alpha]$ at first. Let l be the smallest integer such that $\nu_{\mathbf{E}}(\alpha)\geq l\nu_{\mathbf{E}}(\bar{\pi})$. Then $l\geq 0$. Applying above Lemma2.7, we can write $x=\sum_{n\geq 0}\frac{p^n}{\pi^{l+a(n)}}[\beta_n]$ for $\beta_n\in\tilde{\mathbf{E}}_K^+$ (by uniqueness, β_n is H_K-invariant). For $i\in I$, we put

$$a_i([\alpha])=\sum_{n\geq 0}\frac{p^n}{\pi^{l+a(n)}}a_i([\beta_n]).$$

It remains to check that $a_i([\alpha])\in\mathbf{A}_K^{\dagger,r}$. By Proposition2.5 (3), $a_i([\beta_n])\in\mathbf{A}_K^+$. Put $n=q(n)p+r(n)$ for $0\leq r(n)\leq p-1$ and then $a(n)=q(n)(p-1)+r(n)$. Therefore,

$$\nu_r(\frac{p^n}{\pi^{l+a(n)}})=\nu_r(\frac{p^n}{[\bar{\pi}]^{l+a(n)}})=ns(r)-s(l+a(n))=s(nr-l-a(n)).$$

In this case, $r_K=\frac{p-1}{p}$, we see that

$$nr-l-a(n)=q(n)(rp-(p-1))+r(n)(r-1)-l=n(r-r_K)-\frac{r(n)}{p-1}-l\geq n(r-r_K)-1-l.$$

Since $a_i([\beta_n])\in\mathbf{A}_K^+$, we deduce that $a_i([\alpha])[\bar{\pi}]^{l+1}\in\mathbf{A}_K^{\dagger,r}$ and that

$$\nu_r(a_i([\alpha]))\geq -s(l+1)\geq\nu_r([\alpha])-\nu_{\mathbf{E}}(\bar{\pi}).$$

Because $a_i([\beta_n])\in\mathbf{A}_K^+$ tends to 0 under the weak topology and $\mathbf{A}_K^+\subset\tilde{\mathbf{A}}^+$, $\lim_i\nu_r(a_i([\beta_n]))=+\infty$ and thus $\lim_i\nu_r(a_i([\alpha]))=+\infty$.

In general, if $x=\sum_{n\geq 0}p^n[\alpha_n]$, then we define $a_i(x)=\sum_{n\geq 0}p^n a_i([\alpha_n])$. Thus, we are reduced to the above special case.

Case 2: the general case.

Let $\{f_j^*\}_{1\leq j\leq e_K}$ be the basis of $\mathbf{B}_K/\mathbf{B}_F$ described in Lemma1.17. Then $P_K'(\pi_K)f_j^*\in\mathbf{A}_F^+[\pi_K]$. Therefore, by Lemma1.16, for $r>r_K$, $[\bar{P}_K'(\bar{\pi}_K)]f_j^*\in\mathbf{A}_K^{\dagger,r}$. For every $x\in\tilde{\mathbf{A}}_K^{\dagger,r}$, $x=\sum_{1\leq j\leq e_K}\mathrm{Tr}(x\pi_K^j)f_j^*$ for $\mathrm{Tr}=\mathrm{Tr}_{\mathbf{E}_K/\mathbf{E}_F}$ and $[\bar{P}_K'(\bar{\pi}_K)]x\in\mathbf{A}_K^{\dagger,r}$. Furthermore, $\nu_r(\mathrm{Tr}(x\pi_K^j))\geq\nu_r(x)$. Put $a_i(x)=\sum_{1\leq j\leq e_K}a_i(\mathrm{Tr}(x\pi_K^j))f_j^*$. Then

$$\nu_r(a_i(x))\geq\inf_j\nu_r(a_i(\mathrm{Tr}(x\pi_K^j)))-\nu_r([\bar{P}_K'(\bar{\pi}_K)])\geq\nu_r(x)-\nu_{\mathbf{E}}(\bar{\pi})-\nu_{\mathbf{E}}(\mathcal{D}_K)=\nu_r(x)-c_K.$$

□

Now, the following corollary is straightforward.

Corollary 2.9 (CST 2)**.** *If $r>0$ and $p^n r>r_K$, then $R_{K,n}(x)\in\mathbf{A}_{K,n}^{\dagger,r}[\frac{1}{[\pi]}]$. Moreover, we have that $\lim_n R_{K,n}(x)\to x$ in $\mathbf{A}_K^{\dagger,r}[\frac{1}{[\pi]}]$ and that*

$$\nu_r(R_{K,n}(x))\geq\nu_r(x)-p^{-n}c_K.$$

As a consequence, the condition (CST 2) holds for $(\tilde{\mathbf{B}}^{\dagger,r},\nu_r)$ for maps $\{R_{K,m}:\tilde{\mathbf{B}}_K^{\dagger,r}\to\mathbf{B}_{K,m}^{\dagger,r}\}_{m\geq 0}$ when $r>r_K$.

Proof. If $n \geq 0$, we have seen that $R_{K,n} = \varphi^{-n} \circ R_{K,0} \circ \varphi^n$. If $x \in \tilde{\mathbf{A}}_K^{\dagger,r}[\frac{1}{[\pi]}]$, $\varphi^n(x) \in \tilde{\mathbf{A}}_K^{\dagger,p^n r}[\frac{1}{[\pi]}]$. Thus, by above Proposition2.8, $R_{K,0}(\varphi^n(x)) \in \mathbf{A}_K^{\dagger,p^n r}[\frac{1}{[\pi]}]$ and furthermore $R_{K,n}(x) \in \mathbf{A}_{K,n}^{\dagger,r}[\frac{1}{[\pi]}]$. Now, by Lemma1.5 (4), we have

$$\nu_r(R_{K,n}(x)) = p^{-n}\nu_{p^n r}(R_{K,0}(\varphi^n(x))) \geq p^{-n}\nu_{p^n r}(\varphi^n(x)) - p^{-n}c_K = \nu_r(x) - p^{-n}c_K$$

as expected. □

We define $\mathbf{X}_{K,m}^{\dagger,r} = (1 - R_{K,m})(\tilde{\mathbf{B}}_K^{\dagger,r})$, then $\tilde{\mathbf{B}}_K^{\dagger,r} = \mathbf{B}_{K,m}^{\dagger,r} \oplus \mathbf{X}_{K,m}^{\dagger,r}$ for all $m \geq 0$.

Now, we study the action of Γ_K on $\tilde{\mathbf{A}}_K^{\dagger,r}$.

Recall we have proved in [Ber1, Section 9] (or [Col, Section 4]) that there exists an $n_0(K) \geq 0$ such that for all $n \geq n_0(K)$,
(1) K_{n+1}/K_n is totally ramified of degree p and $1 + p^n\mathbb{Z}_p \subset \Gamma_K$;
(2) $F \subset K_n$;
(3) $e(K_n/\mathbb{Q}_p(\zeta_{p^n})) = e_K$ and $f(K_n/\mathbb{Q}_p(\zeta_{p^n})) = f_K$.
(4) $\nu_{\mathbf{E}}(\mathcal{D}_K) = p^n\nu_p(\mathcal{D}_{K_n/\mathbb{Q}_p(\zeta_{p^n})}) = p^{n+1}\nu_p(\mathcal{D}_{K_{n+1}/\mathbb{Q}_p(\zeta_{p^{n+1}})}) \leq \frac{p^{n_0(K)}}{p-1}$.

Lemma 2.10. *If $\gamma \in \Gamma_K$ has infinite order, then*
(1) $\mathbf{E}_K^{\gamma=1} = \tilde{\mathbf{E}}_K^{\gamma=1} = k_F^{\gamma=1}$ *and*
(2) $\mathbf{A}_K^{\gamma=1} = \tilde{\mathbf{A}}_K^{\gamma=1} = \mathcal{O}_F^{\gamma=1}$.

Proof. The (2) follows from (1) by p-adic completeness.

If $k_F^{\gamma=1} \neq \mathbf{E}_K^{\gamma=1}$, then there exists $x \in \mathbf{E}_K^{\gamma=1}$ such that $\nu_{\mathbf{E}}(x) > 0$. Therefore, $k_F^{\gamma=1}((x))$ is a subfield of $\mathbf{E}_K^{\gamma=1}$. Since both of $\mathbf{E}_K$ and $k_F^{\gamma=1}((x))$ have transcendent degree 1 (over $\mathbb{F}_p$), $\mathbf{E}_K/k_F^{\gamma=1}((x))$ is an algebraic extension. In particular, $\mathbf{E}_K/\mathbf{E}_K^{\gamma=1}$ is algebraic. Thus, the Galois closure of $\mathbf{E}_K^{\gamma=1}(\epsilon)$ in $\mathbf{E}_K$ is a finite extension of $\mathbf{E}_K$. It follows that there is a $k \in \mathbb{N}$ such that $\gamma^k(\epsilon) \in \mathbf{E}_K^{\gamma=1}$. This is impossible!

If $x \in \tilde{\mathbf{E}}_K$, by Corollary2.6 (4), for all $n \geq 0$, $R_{K,n}(x) \in \mathbf{E}_{K,n}^{\gamma=1}$. Thus,

$$\varphi^n(R_{K,n}(x)) \in \mathbf{E}_K^{\gamma=1} = k_F^{\gamma=1}.$$

It follows that $x = \lim R_{K,n}(x) \in k_F^{\gamma=1}$. □

Lemma 2.11. *Assume $\gamma \in \Gamma_K$ with $n(\gamma) \geq n_0(K)$, then*

$$\nu_{\mathbf{E}}(\gamma(\bar{\pi}_K) - \bar{\pi}_K) = p^{n(\gamma)}\nu_{\mathbf{E}}(\bar{\pi}) - \nu_{\mathbf{E}}(\mathcal{D}_K).$$

Proof. Put $n = n(\gamma)$. Because $n \geq n_0(K)$, K_{n+1}/K_n is totally ramified of degree p and γ is the generator of $\mathrm{Gal}(K_{n+1}/K_n)$. By [Ser, IV.§1.Proposition 4], if ω is a uniformizer of K_{n+1}, then $\nu_p(\mathcal{D}_{K_{n+1}/K_n}) = (p-1)\nu_p(\gamma(\omega)-\omega)$. Recall

$$\mathbf{E}_K^+ = \{(x_m)_{m\geq 0} \in \tilde{\mathbf{E}}^+ \mid x_m \in \mathcal{O}_{K_m} \text{and } N(x_{m+1}) \equiv x_m \mod \mathfrak{a} \text{ for } m \gg 0\},$$

where $\mathfrak{a} = \{x \in \mathcal{O}_{\widehat{K_\infty}} \mid \nu_p(x) \geq \frac{1}{p}\}$. Then $\bar{\pi}_K = (\pi_{K,m})_{m\geq 0}$ such that for $m \geq n_0(K)+1$, $\pi_{K,m}$ is a uniformizer of K_m. So

$$\begin{aligned}\nu_{\mathbf{E}}(\gamma(\bar{\pi}_K)-\pi_K) = p^{n+1}\nu_p(\gamma(\pi_{K,n+1})-\pi_{K,n+1}) &= \frac{p^{n+1}}{p-1}\nu_p(\mathcal{D}_{K_{n+1}/K_n}) \\ &= \frac{p^{n+1}}{p-1}(\nu_p(\mathcal{D}_{K_{n+1}/F_{n+1}}) + \nu_p(\mathcal{D}_{F_{n+1}/F_n}) - \nu_p(\mathcal{D}_{K_n/F_n})) \\ &= \frac{p^{n+1}}{p-1}(1 + p^{-n-1}\nu_{\mathbf{E}}(\mathcal{D}_K) - p^{-n}\nu_{\mathbf{E}}(\mathcal{D}_K)) = p^n\nu_{\mathbf{E}}(\bar{\pi}) - \nu_{\mathbf{E}}(\mathcal{D}_K)\end{aligned}$$

as expected. □

Lemma 2.12. *If $m \geq 0$, $u \in \mathbb{Z}_p^\times$ and $r > \frac{p-1}{p}p^m \geq \frac{p-1}{p}$, then $\frac{[\epsilon]^{p^m u}-1}{[\bar{\pi}]^{p^m}}$ is a unit in $\mathbf{A}_{\mathbb{Q}_p}^{\dagger,r}$.*

Proof. Recall $[\epsilon] = 1+\pi$. When $m = 0$, $\frac{[\epsilon]^{p^m u}-1}{[\bar{\pi}]^{p^m}} = \frac{(1+\pi)^u-1}{\pi}\frac{\pi}{[\bar{\pi}]}$. Because $u \in \mathbb{Z}_p^\times$, the element $\frac{(1+\pi)^u-1}{\pi}$ is a unit in $\mathbb{Z}_p[[\pi]] = \mathbf{A}_{\mathbb{Q}_p}^+$. For $r > \frac{p-1}{p}$, $\frac{\pi}{[\bar{\pi}]}$ is a unit in $\mathbf{A}_{\mathbb{Q}_p}^{\dagger,r}$. Therefore, $\frac{(1+\pi)^u-1}{\pi}\frac{\pi}{[\bar{\pi}]}$ is a unit.

For general m, we see $\frac{[\epsilon]^{p^m u}-1}{[\bar{\pi}]^{p^m}} = \varphi(\frac{(1+\pi)^u-1}{\pi}\frac{\pi}{[\bar{\pi}]})$. Since $\varphi^m : \mathbf{A}_{\mathbb{Q}_p}^{\dagger,r} \to \mathbf{A}_{\mathbb{Q}_p}^{\dagger,rp^m}$ is an isomorphism, the lemma follows. □

Lemma 2.13. *If $\gamma \in \Gamma_K$ satisfying $n(\gamma) \geq n_0(K)$ and if $r > \sup(r_K, \frac{p-1}{p}p^{n(\gamma)})$, then*

$$\nu_r(\gamma(\pi_K)-\pi_K) = p^{n(\gamma)}\nu_{\mathbf{E}}(\bar{\pi}) - \nu_{\mathbf{E}}(\mathcal{D}_K).$$

Proof. Since $\bar{P}_K$ is an Eisenstein polynomial on $\mathbf{E}_F^+[T]$ (because $\mathbf{E}_K/\mathbf{E}_F$ is totally ramified), the constant term of P_K is a multiplication of π by some unit in $\mathbf{A}_F^+$. Therefore, $\gamma(P_K) - P_K = (\gamma(\pi)-\pi)Q$ for some $Q \in \mathbf{A}_F^+[T]$ whose constant term is unit in $\mathbf{A}_F^+$. So $Q(\gamma(\pi_K))$ is also unit in $\mathbf{A}_K^{\dagger,r}$. We note that

$$(\gamma(\pi)-\pi)Q(\gamma(\pi_K)) = -(\gamma(\pi_K)-\pi_K)\frac{P_K(\gamma(\pi_K)) - P_K(\pi_K)}{\gamma(\pi_K)-\pi_K}.$$

Define $\alpha = \frac{P_K(\gamma(\pi_K))-P_K(\pi_K)}{\gamma(\pi_K)-\pi_K}$. Similar to the proof of Corollary1.15, for all $k \geq 1$, $w_k(\alpha) \geq -(2k-1)\nu_{\mathbf{E}}(\bar{P}_K(\bar{\pi}_K))$. Because $\bar{\alpha} = \bar{P}'_K(\bar{\pi}_K)$, similar to Lemma1.16 (2), $\frac{\alpha}{[\bar{P}'_K(\bar{\pi}_K)]}$ is also a unit in $\tilde{\mathbf{A}}_K^{\dagger,r}$. Therefore, there is a unit $u \in \tilde{\mathbf{A}}_K^{\dagger,r}$ such that

$$(\gamma(\pi_K)-\pi_K)[\bar{P}'_K(\bar{\pi}_K)] = (\gamma(\pi)-\pi)u.$$

Now $(\gamma(\pi)-\pi)=[\epsilon]([\epsilon]^{p^{n(\gamma)v}}-1)$ for some $v\in\mathbb{Z}_p^\times$, by above Lemma2.12,

$$(\gamma(\pi_K)-\pi_K)\frac{[\bar{P}'_K(\bar{\pi}_K)]}{[\bar{\pi}]^{p^{n(\gamma)}}}$$

is a unit in $\tilde{\mathbf{A}}_K^{\dagger,r}$. Therefore, $\nu_r(\gamma(\pi_K)-\pi_K)=p^{n(\gamma)}\nu_{\mathbf{E}}(\bar{\pi})-\nu_{\mathbf{E}}(\mathcal{D}_K)$ as desired. □

Proposition 2.14. *If $\gamma\in\Gamma_K$ satisfying $n(\gamma)\geq n_0(K)$ and if $r>\sup(r_K,\frac{p-1}{p}p^{n(\gamma)})$, then for any $x\in\mathbf{A}_K^{\dagger,r}$*

$$\nu_r(\gamma(x)-x)\geq\nu_r(x)+p^{n(\gamma)}\nu_{\mathbf{E}}(\bar{\pi})-c_K.$$

Proof. By Theorem1.23, there exists $f(T)=\sum_{k\in\mathbb{Z}}a_kT^k\in\mathcal{A}_F^{\frac{1}{re_K}}$ (i.e. $a_k\in\mathcal{O}_F$, $\nu_p(a_k)+\frac{k}{re_K}\geq 0$ and $\lim_{k\to-\infty}\nu_p(a_k)+\frac{k}{re_K}=+\infty$) such that $f(\pi_K)=x$ and that

$$s(r)\inf_k(\nu_p(a_k)+\frac{k}{re_K})=\nu_r(x).$$

Because $n(\gamma)\geq n_0(K)$, γ acts as identity on F. Thus,

$$\gamma(x)-x=f(\gamma(\pi_K))-f(\pi_K)=\sum_{k\geq 0}\frac{f^{(k)}(\pi_K)}{k!}(\gamma(\pi_K)-\pi_K)^k=\sum_{k\geq 0}\frac{f^{(k)}(\pi_K)\pi_K^k}{k!}(\frac{\gamma(\pi_K)}{\pi_K}-1)^k.$$

Since $\frac{f^{(k)}(\pi_K)\pi_K^k}{k!}=\sum_{n\geq 0}\binom{n}{k}a_n\pi_K^n$, by Theorem1.23 again, $\frac{f^{(k)}(\pi_K)\pi_K^k}{k!}\in\mathbf{A}_K^{\dagger,r}$ with

$$\nu_r(\frac{f^{(k)}(\pi_K)\pi_K^k}{k!})\geq\nu_r(x).$$

Therefore $\nu_r(\gamma(x)-x)\geq\nu_r(x)+\inf_{k\geq 0}\nu_r((\frac{\gamma(\pi_K)}{\pi_K}-1)^k)$. By above Lemma2.13 and Lemma1.16

$$\nu_r(\frac{\gamma(\pi_K)}{\pi_K}-1)=p^{n(\gamma)}\nu_{\mathbf{E}}(\bar{\pi})-\nu_{\mathbf{E}}(\mathcal{D}_K)-\nu_{\mathbf{E}}(\bar{\pi}_K)\geq p^{n(\gamma)}\nu_{\mathbf{E}}(\bar{\pi})-c_K\geq 0.$$

Therefore, we deduce that $\nu_r(\gamma(x)-x)\geq\nu_r(x)+p^{n(\gamma)}\nu_{\mathbf{E}}(\bar{\pi})-c_K$. □

In order to check that $(CST\ 3)$ holds for $\tilde{\mathbf{B}}^{\dagger,r}$, we need to show that $(1-\gamma)$ is invertible on $\mathbf{X}_{K,m}^{\dagger,r}$ for suitable r and $\gamma\in\Gamma_K$. The following proposition plays an important role in the proof.

Proposition 2.15. *If $1\neq\gamma\in\Gamma_K$ with $n(\gamma)\geq\sup(2,n_0(K)+1)$ and if $r>\sup(pr_K,\frac{p-1}{p}p^{n(\gamma)})$, then $(1-\gamma)$ is invertible on $(\mathbf{B}_K^{\dagger,r})^{\psi=0}$ and for every $x\in(\mathbf{B}_K^{\dagger,r})^{\psi=0}$,*

$$\nu_r((1-\gamma)^{-1}x)\geq\nu_r(x)-pc_K-p^{n(\gamma)}\nu_{\mathbf{E}}(\bar{\pi}).$$

Proof. We need to check that for $1 \le i \le p-1$, $(1-\gamma)$ is invertible on $[\epsilon]^i\varphi(\mathbf{B}_K^{\dagger,\frac{r}{p}})$.

Put $m = n(\gamma)$ and then there is a $u \in \mathbb{Z}_p^\times$ such that $\chi(\gamma) = 1 + p^m u$. For any $x \in [\epsilon]^i\varphi(\mathbf{A}_K^{\dagger,\frac{r}{p}}[\frac{1}{[\bar{\pi}]}])$, we may assume $x = [\epsilon]^i\varphi(y)$ for some $y \in \mathbf{A}_K^{\dagger,\frac{r}{p}}[\frac{1}{[\bar{\pi}]}]$. In fact, $y = a_{\frac{i}{p}}(\varphi^{-1}(x))$ by Propostion2.8. Because $r > pr_K$, we have

$$\nu_{\frac{r}{p}}(y) \ge \nu_{\frac{r}{p}}(\varphi^{-1}(x)) - c_K = p^{-1}\nu_r(x) - c_K.$$

Since $\frac{1-[\epsilon]^{p^m u}}{[\bar{\pi}]^{p^m}}$ is invertible in $\mathbf{A}_K^{\dagger,r}$, we can define a bijection

$$f_\gamma : [\epsilon]^i\varphi(\mathbf{A}_K^{\dagger,\frac{r}{p}}[\frac{1}{[\bar{\pi}]}]) \to [\epsilon]^i\varphi(\mathbf{A}_K^{\dagger,\frac{r}{p}}[\frac{1}{[\bar{\pi}]}])$$

by $f_\gamma([\epsilon]^i\varphi(y)) = [\epsilon]^i\frac{\varphi(y)}{1-[\epsilon]^{p^m iu}}$. Then

$$\nu_r(f_\gamma([\epsilon]^i\varphi(y))) = \nu_r([\epsilon]^i\varphi(y)) - p^m\nu_{\mathbf{E}}(\bar{\pi}).$$

Now, noticing that $\gamma([\epsilon]^i) = [\epsilon]^i[\epsilon]^{p^m iu}$, we have

$$[\epsilon]^i\varphi(y) - f((1-\gamma)([\epsilon]^i\varphi(y))) = -[\epsilon]^i\frac{\varphi((1-\gamma)y)}{[\epsilon]^{-p^m iu}-1}.$$

Because $\frac{r}{p} > \sup(r_K, \frac{p-1}{p}p^{n(\gamma)})$, by above Proposition2.14,

$$\nu_r(\varphi((1-\gamma)y)) = p\nu_{\frac{r}{p}}(y-\gamma(y)) \ge p\cdot(\nu_{\frac{r}{p}}(y)+p^m\nu_{\mathbf{E}}(\bar{\pi})-c_K) \ge \nu_r([\epsilon]^i\varphi(y))+p^{m+1}\nu_{\mathbf{E}}(\bar{\pi})-2pc_K.$$

Therefore, we deduce that for $[\epsilon]^i\varphi(\mathbf{A}_K^{\dagger,\frac{r}{p}}[\frac{1}{[\bar{\pi}]}])$

$$\nu_r(x - f_\gamma((1-\gamma)x)) \ge \nu_r(x) + (p^{m+1}-p^m)\nu_{\mathbf{E}}(x) - 2pc_K.$$

By our hypothesis on $m = n(\gamma)$, $(p^{m+1}-p^m)\nu_{\mathbf{E}}(x) - 2pc_K > 0$ and a fortiori $\nu_r(x - f_\gamma((1-\gamma)x)) > \nu_r(x)$.

For every $z \in [\epsilon]^i\varphi(\mathbf{A}_K^{\dagger,\frac{r}{p}}[\frac{1}{[\bar{\pi}]}])$, if we define $g_z : [\epsilon]^i\varphi(\mathbf{A}_K^{\dagger,\frac{r}{p}}[\frac{1}{[\bar{\pi}]}]) \to [\epsilon]^i\varphi(\mathbf{A}_K^{\dagger,\frac{r}{p}}[\frac{1}{[\bar{\pi}]}])$ by

$$g_z(x) = x - f_\gamma((1-\gamma)x - z),$$

then g_z is contractible. Thus, there exists a unique fixed point $z_0 \in [\epsilon]^i\varphi(\mathbf{A}_K^{\dagger,\frac{r}{p}}[\frac{1}{[\bar{\pi}]}])$ of g_z. Since f_γ is bijective, we deduce that $(1-\gamma)(z_0) = z$.

Finally, since $z_0 = z_0 - f_\gamma((1-\gamma)z_0 - z)$,

$$\nu_r(z_0) = \nu_r(f_\gamma(z)) = \nu_r(z) - p^m\nu_{\mathbf{E}}(\bar{\pi}).$$

In general, if $x \in (\mathbf{B}_K^{\dagger,r})^{\psi=0}$, we may write $x = \sum_{i=1}^{p-1}[\epsilon]^i\varphi(x_i)$ with $x_i = a_{\frac{i}{p}}(\varphi^{-1}(x))$. Put $z_i = (1-\gamma)^{-1}([\epsilon]^i\varphi(x_i))$ and put $x_0 = \sum_{i=1}^{p-1} z_i$. Then $x_0 = (1-\gamma)^{-1}x$ and

$$\nu_r(x_0) \ge \inf_i \nu_r(z_i) = \inf_i \nu_r([\epsilon]^i\varphi(x_i)) - p^m\nu_{\mathbf{E}}(\bar{\pi}) = p\cdot\inf_i \nu_{\frac{r}{p}}(a_{\frac{i}{p}}(\varphi^{-1}(x))) - p^m\nu_{\mathbf{E}}(\bar{\pi})$$

$$\geq p\nu_{\frac{r}{p}}(\varphi^{-1}(x)) - pc_K - p^m\nu_{\mathbf{E}}(\bar{\pi}) = \nu_r(x) - pc_K - p^m\nu_{\mathbf{E}}(\bar{\pi}),$$

as desired. □

Remark 2.2. Since $(1-\gamma)(1+\gamma+\cdots+\gamma^{p^m-1}) = (1-\gamma^{p^m})$ for all m, there exists an $r(K) \geq r_K > 0$ such that for every $r > r(K)$, if $\chi(\gamma) \in 1+2p\mathbb{Z}_p$, then $(1-\gamma)$ is invertible on $(\mathbf{B}_K^{\dagger,r})^{\psi=0}$ and there is a $c(K) > 0$ such that for every $x \in (\mathbf{B}_K^{\dagger,r})^{\psi=0}$

$$\nu_r((1-\gamma)^{-1}x) \geq \nu_r(x) - p^{n(\gamma)}c(K)).$$

Now, for $m \geq 1$, we define $R^*_{K,m} = R_{K,m} - R_{K,m-1}$. Then

$$R^*_{K,m}(x) = \sum_{i \in I_m - I_{m-1}} a_i(x)[\epsilon]^i, \text{ for } \forall x \in \tilde{\mathbf{A}}_K.$$

Lemma 2.16. *If $m \geq 1$ and if $x \in \tilde{\mathbf{A}}_K$, $R^*_{K,m}(x) \in \varphi^{-m}(\mathbf{A}_K^{\psi=0})$.*

Proof. For every $i \in I_m - I_{m_1}$, there exists a unique $1 \leq r(i) \leq p-1$ such that $p^m i \equiv r(i) \mod p$. Put $q(i) = \frac{p^m i - r(i)}{p}$. Thus

$$\varphi^m(R^*_{K,m}(x)) = \sum_{i \in I_m - I_{m-1}} \varphi^m(a_i(x)[\epsilon]^i) = \sum_{i \in I_m - I_{m-1}} \varphi(\varphi^{m-1}(a_i(x))[\epsilon]^{q(i)})[\epsilon]^{r(i)}.$$

Therefore $\varphi^m(R^*_{K,m}(x)) \in \mathbf{A}_K^{\psi=0}$ and we complete the proof. □

Proposition 2.17 (*CST* 3)**.** *If $r > 0$ and $n \in \mathbb{N}$ satisfying $p^n r > \sup(pr(K), \frac{p-1}{p}p^{n(\gamma)})$, for $\gamma \in \Gamma_K$ with $n \geq n(\gamma)$, $(\gamma - 1)$ is invertible on $\mathbf{X}_{K,n}^{\dagger,r}$ and there exists a c'_K such that*

$$\nu_r((\gamma-1)^{-1}x) \geq \nu_r(x) - p^{n(\gamma)-n}c'_K.$$

Proof. By Lemma2.10, $(\gamma - 1)$ is injective on $\mathbf{X}_{K,n}^{\dagger,r}$. If $x \in \mathbf{X}_{K,n}^{\dagger,r}$, we see that $R_{K,n}(x) = 0$. Thus, $x = \sum_{m \geq n+1} R^*_{K,m}(x)$. Because $R^*_{K,m} = R_{K,m} - R_{K,m-1}$, by Corollary2.9,

$$\nu_r(R^*_{K,m}(x)) \geq \nu_r(x) - p^{1-m}c_K.$$

Because $R^*_{K,m}(x) = \sum_{i \in I_m - I_{m-1}} a_i(x)[\epsilon]^i$, we see that $\varphi^m(R^*_{K,m}(x)) \in (\mathbf{B}_K^{\dagger,p^m r})^{\psi=0}$. By remark2.2, there exists a $z_m \in (\mathbf{B}_K^{\dagger,p^m r})^{\psi=0}$ satisfying

$$\varphi^m(R^*_{K,m}(x)) = (\gamma-1)z_m \text{ and } \nu_{p^m r}(z_m) \geq p^m\nu_r(R^*_{K,m}(x)) - p^{n(\gamma)}c(K)).$$

Thus, $\nu_r(\varphi^{-m}(z_m)) \geq \nu_r(R^*_{K,m}(x)) - p^{-m}p^{n(\gamma)}c(K))$. Since $R^*_{K,m}(x) \to 0$ in $\tilde{\mathbf{B}}_K^{\dagger,r}$, we deduce that $z = \sum_{m \geq n+1} \varphi^{-m}(z_m)$ converges in $\tilde{\mathbf{B}}_K^{\dagger,r}$. By construction, $(\gamma - 1)z = x$ and

$$\nu_r(z) \geq \inf_{m \geq n+1} \nu_r(\varphi^{-m}(z_m)) \geq \inf_{m \geq n+1} (\nu_r(R^*_{K,m}(x)) - p^{-m}p^{n(\gamma)}c(K))$$

$$\geq \nu_r(x) - p^{-n} \sup_{m \geq n+1} (p^{1-m+n} c_K + p^{-m+n} p^{n(\gamma)} c(K)).$$

Thus, if we choose $c'_K > 0$ satisfying $p^{n(\gamma)} c'_K \geq \sup_{k \geq 1}(p^{1-k} c_K + p^{-k} p^{n(\gamma)} c(K))$, then the proposition follows. □

Now, the following theorem is obvious.

Theorem 2.18. *There exists an $r'_K > 0$ such that for any $r > r_K$, $(\tilde{\mathbf{B}}^{\dagger,r}, \nu_r)$ satisfies conditions of CST.*

2.2. **Theorem of Cherbonnier-Colmez.** Now, we can prove Theorem0.1 at the beginning of this note.

Lemma 2.19. *If V is a p-adic representation of Gal_K of dimension d then there is a finite extension L/K and an $s(V) > 0$ such that if $s \geq s(V)$, then $(\tilde{\mathbf{B}}^{\dagger,s} \otimes V)^{H_L}$ admits a free $\mathbf{B}_L^{\dagger,s}$-submodule $\mathrm{D}_L^{\dagger,s}$ of rank d and stable under the action of Gal_K and such that $\tilde{\mathbf{B}}^{\dagger,s} \otimes V = \tilde{\mathbf{B}}^{\dagger,s} \otimes_{\mathbf{B}_L^{\dagger,s}} \mathrm{D}_L^{\dagger,s}$ and $\mathbf{B}_L^{\dagger} \otimes_{\mathbf{B}_L^{\dagger,s}} \mathrm{D}_L^{\dagger,s} \subset \tilde{\mathbf{B}}^{\dagger} \otimes V$ is stable by φ.*

Proof. We choose an $r > 0$ such that $(\tilde{\mathbf{B}}^{\dagger,r}, \nu_r)$ satisfies CST conditions. By [Ber1, Theorem 19.1], there exists a finite extension L/K and a finite free $\mathbf{B}_{L,n}^{\dagger,r}$-module $\mathrm{D}_{L,n}^{\dagger,r} \subset (\tilde{\mathbf{B}}^{\dagger,r} \otimes V)^{H_L}$ of rank d which is stable under the action of Gal_K such that

$$\mathrm{D}_{L,n}^{\dagger,r} \otimes_{\mathbf{B}_{L,n}^{\dagger,r}} \tilde{\mathbf{B}}^{\dagger,r} = \tilde{\mathbf{B}}^{\dagger,r} \otimes V,$$

for some $n \gg 0$. Therefore, the $\mathbf{B}_L^{\dagger,p^n r}$-module generated by $\varphi^n(\mathrm{D}_{L,n}^{\dagger,r})$, namely $\mathrm{D}_L^{\dagger,p^n r}$, is finite of rank d and is stable under the action of Gal_K. Moreover, we also have $\mathrm{D}_L^{\dagger,p^n r} \otimes_{\mathbf{B}_L^{\dagger,p^n r}} \tilde{\mathbf{B}}^{\dagger,p^n r} = \tilde{\mathbf{B}}^{\dagger,p^n r} \otimes V$ because $\varphi^n(\tilde{\mathbf{B}}^{\dagger,r}) = \tilde{\mathbf{B}}^{\dagger,p^n r}$.

We remain to study the action of φ. By [Ber1, Theorem 19.8], we deduce that

$$\mathrm{D}_{L,\infty}^{\dagger,p^{n+1} r} = \mathrm{D}_L^{\dagger,p^n r} \otimes_{\mathbf{B}_L^{\dagger,p^n r}} \mathbf{B}_{L,\infty}^{\dagger,p^{n+1} r}$$

and that under a basis contained in $\mathrm{D}_L^{\dagger,p^n r}$, the matrix of φ belongs to $\mathrm{M}_d(\mathbf{B}_{L,\infty}^{\dagger,p^{n+1} r})$ and furthermore belongs to $\mathrm{M}_d(\mathbf{B}_{L,m}^{\dagger,p^{n+1} r})$ for some $m \gg 0$.

Now, let $\mathrm{D}_L^{\dagger,p^{n+m+1} r}$ be the $\mathbf{B}^{\dagger,p^{n+m+1} r}$-module generated by $\varphi^m(\mathrm{D}_L^{\dagger,p^n r})$. Then it satisfies all conditions we need. Put $s(V) = rp^{n+m+1}$. We complete the proof. □

Theorem 2.20. (1) *Let V be a p-adic representation of Gal_K. Then V is overconvergent and $\mathrm{D}^{\dagger}(V) \otimes_{\mathbf{B}_K^{\dagger}} \mathbf{B}_K = \mathrm{D}(V)$ is the étale (ϕ, Γ)-module over $\mathbf{B}_K$ associated to V under the equivalence described in* [Ber1, Therorem 18.8].

(2) *The functor* $V \mapsto \mathrm{D}^\dagger(V)$ *induce an equivalence from the category of* p*-adic representations of* Gal_K *to the category of étale* (φ,Γ)*-modules over* $\mathbf{B}_K^\dagger$*. (By an étale* (φ,Γ)*-module over* $\mathbf{B}_K^\dagger$*, we mean a finite free* (φ,Γ)*-module which is étale after base-changing to* $\mathbf{B}_K$*.)*

Proof. (1) By above Lemma2.19, for a given p-adic representation V of Gal_K of dimension d, we can find a finite Galois extension L/K and an $s \geq s(V)$ such that $\mathrm{D}_L^\dagger = \mathrm{D}_L^{\dagger,s} \otimes_{\mathbf{B}_L^{\dagger,s}} \mathbf{B}_L^\dagger$ is a (φ,Γ)-module over $\mathbf{B}_L^\dagger$ together with an action of Gal_K. Define $\mathrm{D}_L = \mathrm{D}_L^\dagger \otimes_{\mathbf{B}_L^\dagger} \mathbf{B}_L$. Then D_L is a (φ,Γ)-module over $\mathbf{B}_L$ satisfying

$$\mathrm{D}_L \otimes_{\mathbf{B}_L} \tilde{\mathbf{B}} \simeq \mathrm{D}_L^\dagger \otimes_{\mathbf{B}_L^\dagger} \tilde{\mathbf{B}}^\dagger \otimes_{\tilde{\mathbf{B}}^\dagger} \tilde{\mathbf{B}} \simeq V \otimes \tilde{\mathbf{B}}^\dagger \otimes_{\tilde{\mathbf{B}}^\dagger} \tilde{\mathbf{B}} \simeq V \otimes \tilde{\mathbf{B}}.$$

Thus, D_L is étale and then there is a p-adic representation W of Gal_L such that

$$W \otimes \tilde{\mathbf{B}} = \mathrm{D}_L \otimes_{\mathbf{B}_L} \tilde{\mathbf{B}} = V \otimes \tilde{\mathbf{B}}.$$

By taking φ-invariant part, we deduce that $W = V$ (as representations of Gal_L). As a consequence, we get $\mathrm{D}_L^\dagger \subset \mathrm{D}_L^\dagger(V) = (\mathbf{B}^\dagger \otimes V)^{H_L}$. Since both of sides are vector spaces over $\mathbf{B}_L^\dagger$, it follows from

$$\dim \mathrm{D}_L^\dagger(V) \leq \dim V = d = \dim \mathrm{D}_L^\dagger$$

that $\mathrm{D}_L^\dagger = \mathrm{D}_L^\dagger(V)$. For the same reason, $\mathrm{D}_L^\dagger \otimes_{\mathbf{B}_L^\dagger} \mathbf{B}_L = \mathrm{D}_L(V)$.

By Corollary1.18, $\mathbf{B}_L^\dagger/\mathbf{B}_K^\dagger$ is a Galois extension with Galois group

$$\mathrm{Gal}(\mathbf{B}_L^\dagger/\mathbf{B}_K^\dagger) = \mathrm{Gal}(H_K/H_L).$$

Therefore, by Hilbert's theorem 90, we see that $\mathrm{D}^\dagger(V) = (\mathrm{D}_L^\dagger)^{H_K}$ is of dimension d and that $\mathrm{D}^\dagger(V) \otimes_{\mathbf{B}_K^\dagger} \mathbf{B}_L^\dagger = \mathrm{D}_L^\dagger$. Hence,

$$\mathrm{D}^\dagger(V) \otimes_{\mathbf{B}_K^\dagger} \mathbf{B} = \mathrm{D}_L^\dagger \otimes_{\mathbf{B}_L^\dagger} \mathbf{B} = \mathrm{D}_L(V) \otimes_{\mathbf{B}_L} \mathbf{B} = V \otimes \mathbf{B}.$$

By taking H_K-invariants, we get $\mathrm{D}^\dagger(V) \otimes_{\mathbf{B}_K^\dagger} \mathbf{B}_K = \mathrm{D}(V)$ as desired.

(2) This follows from (1). □

References

[Ber1] L. Berger: *Galois representations and* (φ,Γ)*-modules*, available at http://perso.ens-lyon.fr/laurent.berger/other.php

[CC] F. Cherbonnier, P. Colmez: *Représentations de p-adiques surconvergentes*, Invent. math. 133, 581-611 (1998).

[Col] P. Colmez: *Espaces vectoriels de dimension finie et représentations de de Rham*,Astérisque No. 319 (2008).

[Ked] Kiran. S. Kedlaya: *New methods for* (φ,Γ)*-modules*

[Ser] J-P. Serre: *Local fields* GTM 67