Sketch of the proof to ThA

Jiahong Yu

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Notation: K is a local field of character 0. $\mathcal{B}_K^\dagger$ is the ring of overconvergent elements and $\mathcal{B}_{rig,K}^\dagger$ is the Robba ring of K.

1 Finiteness of $\mathcal{B}_{rig,L}^\dagger/\mathcal{B}_{rig,K}^\dagger$

In this section, we will prove that

Thm 1.1. *For finite extension L/K, $\mathcal{B}_{rig,L}^\dagger$ is finite over $\mathcal{B}_{rig,K}^\dagger$. More preciously, $\mathcal{B}_{rig,L}^\dagger = \mathcal{B}_L^\dagger \otimes_{\mathcal{B}_K^\dagger} \mathcal{B}_{rig,K}^\dagger$.*

We only need to prove the case L/K is Galois. So we make this assumption from now on.

lemma 1.1. *If L/K is finite Galois, then $(\mathcal{B}_{rig,L}^\dagger)^{H_{L/K}} = \mathcal{B}_{rig,K}^\dagger$*

Proof. Let $x \in (\mathcal{B}_{rig,L}^\dagger)^{H_{L/K}}$, we may choose $x_i \in \mathcal{B}_L^\dagger$ tend to x under Frechet topology. Then $\frac{Tr(x_i)}{|H_{L/K}|}$ tend to x and $\in \mathcal{B}_K^\dagger$ since $\mathcal{B}_L^{\dagger\, H_{L/K}} = \mathcal{B}_K^\dagger$. Thus $x \in \mathcal{B}_{rig,K}^\dagger$ □

As a corollary, $\mathcal{B}_{rig,L}^\dagger$ is integral over $\mathcal{B}_{rig,K}^\dagger$

proof to the theorem. Step 1: We prove that $\mathcal{B}_L^\dagger \otimes_{\mathcal{B}_K^\dagger} \mathcal{B}_{rig,K}^\dagger$ is a domain.

In fact, it is sufficient to prove $\mathcal{B}_{rig,K}^\dagger$ is transcendental over $\mathcal{B}_K^\dagger$. We use the power series definition.

Recall that $\mathcal{B}_K^\dagger$ is the ring of bounded analytic functions on $\{x \in \mathbb{C}_p : r < |x| < 1\}$ ($\Gamma_{con,K}^r$) for some $r < 1$ with coefficients in K_0' and $\mathcal{B}_{rig,K}^\dagger$ is the ring of analytic functions on $\{x \in \mathbb{C}_p : r < |x| < 1\}$ for some $r < 1$ with coefficients in K_0' ($\Gamma_{con,K}^{an,r}$). (Following Kedlaya's notation in [1])

If we have $X^n + a_{n-1}X^{n-1} + ... + a_0 = 0$ for an $X \in \mathcal{B}_{rig,K}^{\dagger,r}$ and $a_i \in \mathcal{B}_K^{\dagger,r}$ $\forall i$, then one can prove that X is bounded by $\sum sup|a_i|$.

Step 2: $\mathcal{B}_L^\dagger \otimes_{\mathcal{B}_K^\dagger} \mathcal{B}_{rig,K}^\dagger$ is a normal domain.

In fact, we can prove the following statment.

lemma 1.2. *Suppose k is a field and A is an $k-$algebra which is also a normal domain. Let l is a separable finite extension of k, and $l \otimes_k A$ is also a domain. Then $l \otimes_k A$ is normal.*

proof to the lemma. Since everything remains the same after taking direct limit, we may assume A is finitely generated.

Thus we only need to check Serre's (R1) and (S2) conditions.

(R1) holds since $l \otimes_k A/A$ is unramified.

Now we check (S2). Let $\mathcal{P}$ is an ideal of height 2 (height $0, 1$ is trivial). Then $\mathcal{P} \cap A$ is also of height 2 since $l \otimes_k A/k$ is finite. The (S2) condition as well as $l \otimes_k A \cap K = A$, while K is the quotient field of A, imply the statement. □

The theorem is now easy to prove. The lemma1 tells us $\mathcal{B}^{\dagger}_{rig,L}$ is integral over $\mathcal{B}^{\dagger}_{rig,K}$, and so is integral over $\mathcal{B}^{\dagger}_{L} \otimes_{\mathcal{B}^{\dagger}_{K}} \mathcal{B}^{\dagger}_{rig,K}$. Comparing the degree of extension one may prove that they have the same fractional field. Then the lemma1 implies they are same. □

Cor 1.1. *$H^n(H_{L/K}, \mathcal{B}^{\dagger}_{rig,L}) = 0$ for all $n > 0$.*

2 Galois descent

Let L/K be finite Galois.

Thm 2.1. *Let M be a finite free $\mathcal{B}^{\dagger}_{rig,L}$ module with a semi-linear $H_{L/K}$ action. Then $M = M^{H_{L/K}} \otimes_{\mathcal{B}^{\dagger}_{rig,K}} \mathcal{B}^{\dagger}_{rig,L}$ as twisted $H_{L/K}$ module.*

lemma 2.1. $(\mathcal{B}^{\dagger}_{rig,K})^{\times} = (\mathcal{B}^{\dagger}_{K})^{\times}$

proof to the Galois descent. We induct on the rank of M.

If the rank is 1, we choose a basis e of M. Define $\gamma(e) = \varphi(\gamma)e$. Then φ is a cross homomorphism from $H_{L/K}$ to $(\mathcal{B}^{\dagger}_{rig,K})^{\times}$. By the lemma, φ is a cross homomorphism from $H_{L/K}$ to $(\mathcal{B}^{\dagger}_{K})^{\times}$. By Galois descent of field (Recall $Gal(\mathcal{B}^{\dagger}_{L}/\mathcal{B}^{\dagger}_{K}) = H_{L/K}$), we have done in this case.

If we have done for $rk(M) = n - 1$, assume now $rk(M) = n$.

By the Galois descent of field (use it to the quotient fields of Robba rings), we find that there exists an $H_{L/K}$ invariant element $e \neq 0 \in M$. Let N be the saturated span of e in M (See Kedlaya). Then N is a rank 1 submodule of M, which is closed under the action of $H_{L/K}$.

Thus we have the following commutative diagram.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & N & \longrightarrow & M & \longrightarrow & M/N & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle \alpha} & & \uparrow{\scriptstyle \beta} & & \uparrow{\scriptstyle \gamma} & & \\
0 & \longrightarrow & \mathcal{B}^{\dagger}_{rig,N} \otimes_{\mathcal{B}^{\dagger}_{rig,K}} N^{H_{L/K}} & \longrightarrow & \mathcal{B}^{\dagger}_{rig,L} \otimes_{\mathcal{B}^{\dagger}_{rig,K}} M^{H_{L/K}} & \xrightarrow{f} & \mathcal{B}^{\dagger}_{rig,L} \otimes_{\mathcal{B}^{\dagger}_{rig,K}} (M/N)^{H_{L/K}} & \longrightarrow & 0
\end{array}$$

By the cor1.1, f is surjective, then use 5-lemma and the induction hypothesis, β is isomorphic. □

3 More on $\mathcal{B}^{\dagger}_{K}$ and $\mathcal{B}^{\dagger}_{rig,K}$

We use the method in §1 to prove more properties of $\mathcal{B}^{\dagger}_{K}$ and $\mathcal{B}^{\dagger}_{rig,K}$.

For a finite extension L/K, we fix an element $\bar{\pi_L} \in \mathbb{E}_L{}^+$ such that $\mathbb{E}_L = \mathbb{E}_K(\pi_L)$. Let P be the monic minimal polynomial of π_L, $\tilde{P}$ be a lifting of P in $\mathcal{A}_{inf,K}$. By Hensel's lemma, $\tilde{P}$ has a solution in $\mathcal{B}^{\dagger}_{K}$ and moreover $\mathcal{B}^{\dagger}_{L} = \mathcal{B}^{\dagger}_{K}(\pi_L)$.

Thm 3.1. *We choose a sufficiently large r such that $\pi_L \in \mathcal{B}^{\dagger,r}_{L}$ and $\tilde{P}'(\pi_L)$ is invertible in $\mathcal{B}^{\dagger,r}_{L}$. Then we have,*
(1). $\mathcal{B}^{\dagger,r}_{L} = \mathcal{B}^{\dagger,r}_{K}[\pi_L]$
(2). $\mathcal{B}^{\dagger,r}_{rig,L} = \mathcal{B}^{\dagger,r}_{rig,K}[\pi_L]$

Proof. Just use the same argument in th1.1. □

As an application we use the result to consider the image of ι_n. Recall we have $r_n = p^n(p-1)$ and we have define $\iota_0 : \tilde{\mathcal{B}}^{\dagger,r_0} \to \mathcal{B}^{+}_{dR}$. Let $X = \pi_K$, $t = log(1+X)$. Then for sufficiently large r, there exists an isomorphism between $\mathcal{B}^{\dagger,r}_{rig,K}$ and $\Gamma^{r}_{con,K}$.

Def 3.1. We define $\iota_n = \iota_0 \circ \varphi^{-n}$.

Prop 3.1. *For sufficiently large n, $\iota_n(\mathcal{B}^{\dagger,r_n}_{rig,K}) \subset K_n[[t]]$, while $K_n = K(\mu_{p^n})$.*

Proof. Let F be the maximal unramified extension of $\mathbb{Q}_p$ in K. Then for sufficiently large r, $\mathcal{B}^{\dagger,r}_{rig,F} = \Gamma^{r}_{an,F}$, and π_F can be chosen to be X. Since $\iota_n(X) = \epsilon^{(n)} exp(\frac{t}{p^n}) - 1$, we prove the proposition.

For K, notice that the map $pr \circ \iota_n = \theta \circ \varphi^n$ while pr is the natural projection from $\mathcal{B}^{+}_{dR}$ to $\mathbb{C}_p$. Thus $pr \circ \iota_n(\mathcal{B}^{\dagger}_{rig,K}) \subset \widehat{K_\infty}$.

Now use the theorem3.1, $\iota_n(\mathcal{B}^{\dagger}_{rig,K})$ contains in $F_n[[t]][\iota_n(\pi_K)]$, which is finite etale over $F_n[[t]]$ for n large enough. By commutative algebra, $F_n[[t]][\iota_n(\pi_K)]$ equals to $K'[[t]]$ for some finite extension $K' \subset \widehat{K_\infty}$. Thus $K' \subset K_n$ for large enough n. □

4 Recover D_{dif}

Let D be a φ module over $\mathcal{B}^{\dagger}_{rig,K}$.

lemma 4.1. *For $r >> 0$, there exists a unique $\mathcal{B}^{\dagger,r}_{rig,K}$ submodule D_r. Such that $\mathcal{B}^{\dagger}_{rig,K} \otimes_{\mathcal{B}^{\dagger,r}_{rig,K}} D_r = D$ and $\varphi(D_r) \subset \mathcal{B}^{\dagger,r}_{rig,K} \otimes_{\mathcal{B}^{\dagger,r}_{rig,K}} D_r$*

Proof. Not hard, see [3]Th1.3.3. □

We define $D_n = D_{r_n}$ for $n >> 0$. Consider $\mathbf{D}_n = K_n[[t]] \otimes^{\iota_n}_{\mathcal{B}^{\dagger,r_n}_{rig,K}} D_n$. We have a natural map

$$\mathbf{D}_n \xrightarrow{id \otimes \varphi} K_n[[t]] \otimes^{\iota_n}_{\mathcal{B}^{\dagger,r_n}_{rig,K}} \left(\mathcal{B}^{\dagger,r_{n+1}}_{rig,K} \otimes^{\varphi}_{\mathcal{B}^{\dagger,r_n}_{rig,K}} D_n\right) = K_n[[t]] \otimes^{\iota_n}_{\mathcal{B}^{\dagger,r_n}_{rig,K},\varphi} D_{n+1}$$

Notice that if we consider $\mathbf{D}_{n+1}$ as a $\mathcal{B}^{\dagger,r_n}_{rig,K}$ module via $a * x := \varphi(a)x$, then the map

$$K_n[[t]] \times D_{n+1} \to \mathbf{D}_{n+1}$$
$$(a,x) \mapsto a \otimes x$$

is bilinear. Thus we have $K_n[[t]] \otimes^{\iota_n}_{\mathcal{B}^{\dagger,r_n}_{rig,K},\varphi} D_{n+1} \to \mathbf{D}_{n+1}$.

Now we have constructed a natural map $\mathbf{D}_n \to \mathbf{D}_{n+1}$ which is $K_n[[t]]$ linear. Taking direct limit, we get a $K_\infty[[t]]$ module.

There is another way to understand this construction better.

Let $\mathfrak{R} = \varinjlim_{\varphi} \mathcal{B}^{\dagger,r_n}_{rig,K}$, then $\varinjlim_{\iota_n} : \mathfrak{R} \to K_\infty[[t]]$ is a ring homomorphism. The module can also be defined as $K_\infty[[t]] \otimes_{\mathfrak{R}} \varinjlim_{\varphi} D_n$. One can see that it is of the same rank as D by this definition. The Γ_K action provides a connection on it (Luo).

5 Compare with D_{dif}

Recall, given a representation $\rho : G_K \to GL_{\mathbb{Q}_p}(V)$, while V is a $\mathbb{Q}_p$ space of dimension n, we can construct a p-adic differential equation $D_{dif}(V)$

Luo Jinyue has prove that

Thm 5.1. *The p-adic differential equation associated to $D^{\dagger}_{rig}(V)$ is naturally isomorphic to $D_{dif}(V)$.*

6 φ-compatible lattice and (φ, Γ)-modules

In this section, we will have a glimpse of the reason why we need 'filtered'.

Recall for any $D \in \Phi_{\mathcal{B}^{\dagger}_{rig,K}}$ of rank d, we constructed a sequence of free modules $\mathbf{D}_n$ over $K_n[[t]]$ for $n >> 0$. Moreover, by the construction, we have $K_{n+1}[[t]] \otimes_{K_n[[t]]} \mathbf{D}_n = \mathbf{D}_{n+1}$.

Def 6.1. A φ-compatible lattice of $\mathbf{D}_* \left[\frac{1}{t}\right]$ is a sequence of $K_n[[t]]$- lattice $\mathbf{M}_n$ of $\mathbf{D}_n \left[\frac{1}{t}\right]$ for $n >> 0$ such that $\mathbf{M}_{n+1} = \mathbf{M}_n \otimes_{K_n[[t]]} K_{n+1}[[t]]$. We say two such lattices are equal if they are equal for sufficiently large n.

One can see that $\mathbf{D}_n$ itself is such a lattice. In general, if D' is a sub-φ-module of $D\left[\frac{1}{t}\right]$ (finite rank of course), then $\mathbf{D}'_n$ is a φ-compatible lattice.

In fact, all φ-compatible lattice comes from a unique D'.

We give a construction of D', for details, see [3]2.1.

Let $\mathbf{M}_n \subset \mathbf{D}\left[\frac{1}{t}\right]$ be a φ-compatible lattice.

lemma 6.1. *There exists an $h \geq 0$ such that $t^h \mathbf{D}_n \subset \mathbf{M}_n \subset t^{-h}\mathbf{D}_n$ for all $n >> 0$.*

Now for $n >> 0$, let $M_n = \{x \in t-h\mathbf{D}_n : 1 \otimes^{\iota_m}_{\mathcal{B}^{\dagger,r_m}_{rig,K}} x \in \mathbf{M}_m \ \forall m \geq n\}$. (Cautions: $1 \otimes^{\iota_m}_{\mathcal{B}^{\dagger,r_m}_{rig,K}} x \in \mathbf{M}_m$ dose not imply $1 \otimes^{\iota_{m+1}}_{\mathcal{B}^{\dagger,r_{m+1}}_{rig,K}} x \in \mathbf{M}_{m+1}$ but $1 \otimes^{\iota_m}_{\mathcal{B}^{\dagger,r_m}_{rig,K}} \varphi(x) \in \mathbf{M}_m$)

lemma 6.2. *M_n is a free module over $\mathcal{B}^{\dagger,r_n}_{rig,K}$ of rank d*

lemma 6.3. $K_n[[t]] \otimes^{\iota_n}_{\mathcal{B}^{\dagger,r_n}_{rig,K}} M_n = \mathbf{M}_n$

We omit the proofs, since we to analyze the Frechet topology carefully. See [3]2.1 and [2]4.2. Let $D' = \varinjlim M_n$, the previous lemma implies that $\mathbf{D}'_n = M_n$.

7 Filtered (φ, N, G_K)-modules

In this section, we introduce the language of filtered $(\phi, N, G_K)-$modules. For a local field K over $\mathbb{Q}_p$, define K_0 be the maximal unramified extension of $K/\mathbb{Q}_p$ and σ be the frobenius $W(x \mapsto x^p)$.

7.1 Why isocrystals?

This part comes from [4]2.7 (Page 83-101).

For simply, we only consider elliptic curves, all things also hold for general abelian varieties.

For an elliptic curve E/K and any prime $l \neq p$, we have:

Thm 7.1. *E has a good reduction if and only if $T_l(E)$ is an unramified Galois representation. In fact, we have*

$$\begin{array}{ccc} G_K & \rightsquigarrow & T_l(E) \\ \downarrow & & \downarrow \\ G_k & \rightsquigarrow & T_l(\bar{E}) \end{array}$$

But when we consider the case when $l = p$, the previous theorem files since the reduction of E does not have so mach $p-$torsion points. Grothendieck gave a good analogue of the criterion.

Thm 7.2. *E has a good reduction if and only if $E[p^n]$ admits an integral model $\mathscr{G}_n$ (i.e.there exists a finite flat group scheme $\mathscr{G}_n/\mathcal{O}_K$ such that $E[p^n] = K \otimes_{\wr_K} \mathscr{G}_n$) for any n.*

The previous $\mathscr{G}_n$ satisfies:

(1). $\mathscr{G}_n$ is of order p^{2n}.

(2). There exists $i_n : \mathscr{G}_n \to \mathscr{G}_{n+1}$ comes from the inclusion $E[p^n] \to E[p^{n+1}]$.

(3). i_n is an isomorphism from $\mathscr{G}_n$ to $\mathscr{G}_{n+1}[p^n]$.

These properties make us to consider a new object, so called 'p-divisible group', and the number 2 is called the height. A theorem by Dieudonné tells us:

Thm 7.3. *If k is a perfect field of character $p > 0$. There exists an anti-equivalence between the category of p-divisible groups over k and the category of free $W(k)$-modules D equipped with a Frobenius semi-linear action $\mathcal{F}$ such that $pD \subset \mathcal{F}(D)$.*

These facts provide us a covariant functor from elliptic curves with good reduction to $\varphi-$modules over $W(k)$, denoted by $\mathbf{D}$

Recall we have

Thm 7.4. *For two elliptic curves E_1, E_2, $l \neq p$, the natural map*

$$\mathbb{Z}_p \otimes_{\mathbb{Z}} Hom(E_1, E_2) \to Hom_{G_K}(T_l(E_1), T_l(E_2))$$

is injective.

Likewise, we have:

Thm 7.5. **D** *is faithful.*

7.2 Definitions

For details, see [5]6.4, or [4]2.8 (page 101-127).

We have already seen that φ can be used to classify abelian varieties which have good reductions. For those with bad reductions, we need another operator N.

Def 7.1. Let $L/K/\mathbb{Q}_p$ be two local fields such that L/K is Galois. A $(\varphi, N, G_{L/K})$-module is a finite dimensional vector space V over L_0 with a σ-semilinear action ϕ, a $G_{L,K}-$semilinear action and a linear endomorphism N, such that:

(1).φ is invertible.

(2).$p\varphi \circ N = N \circ \varphi$

(3).The action of $G_{L/K}$ is commute with N and φ.

Def 7.2. A filtered $(\varphi, N, G_{L/K})$-module is a $(\varphi, N, G_{L/K})$-module D as well as a separable and exhaustive descending filtration on D_L, which is compatible with $G_{L/K}$. We do not assume anything between the filtration and (φ, N) action.

A $(\varphi, N, G_{L/K})$-module is considered to be the same as its base changes.

Def 7.3. Given two filtered $(\varphi, N, G_{L/K})$-modules D_1, D_2, we define their tensor product $D_1 \otimes D_2$ as:

(1). The vector space $D_1 \otimes_{L_0} D_2$;

(2). $\varphi(x \otimes y) = \varphi_1(x) \otimes \varphi_2(y)$ the same as $G_{L/K}$-action;

(3). $N(x \otimes y) = N_1(x) \otimes y + x \otimes N_2(y)$;

(4). $Fil^k(X_L \otimes_L Y_L) = \sum\limits_{i+j=k} Fil^i(X_L) \otimes_L Fil^j(Y_L)$

Moreover, for a $(\varphi, N, G_{L/K})$-module D, define the filtration of $\bigwedge^k D_L$ to be the images of $Fil^l(\bigotimes^k D_L),\ l \in \mathbb{Z}$.

For a $(\varphi, N, G_{L/K})$-module D of dimension 1, choose a basis e and suppose $\varphi(e) = \lambda e$, define $t_N(D) = v_p(\lambda),\ t_H(D) = \max\{k :\ Fil^k(D_L) \neq 0\}$. If $dim_{L_0} D = d$, define $t_N(D) = t_N(\bigwedge^d D)$ and $t_H(D) = t_H(\bigwedge^d D)$.

lemma 7.1. *N is nilpotent.*

Proof. Let $D' = \cap Im(N^n)$, then $\varphi(D') = D'$ and N is invertible on D'. Choose a basis and write φ, N as matrixes F, A.

Then we have $pFA^{\varphi} = AF$, thus $p^{\dim D} \det F \det A = \det F \det A$ which implies $\dim D = 0$ □

Def 7.4. A filtered $(\varphi, N, G_{L/K})$-module D is called weakly admissible if $t_N(D) = t_H(D)$ and for all submodule D' of D, $t_N(D') \leq t_H(D')$.

8 Filtered (φ, N, G_K)-modules and (φ, Γ)-modules

8.1 From filtered (φ, N, G_K)-modules to (φ, Γ)-modules

Let ℓ_X be a variable which is considered as 'log(X)' and we prolong the $\tilde{\mathcal{B}}^{\dagger}_{rig}$ to $\tilde{\mathcal{B}}^{\dagger}_{rig}[\ell_X]$. Let log be the p-adic logarithm such that $\log(p) = 0$. Given an $f \in \mathbb{Q}_p[[X]]^*$, we define $\log(f)$ as $\log(f(0)) + \log\left(\frac{f}{f(0)}\right)$.

Def 8.1. We prolong the (φ, Γ_K)-action as following:

(1). $\varphi(\ell_X) = \ell_X + \log\left(\frac{\varphi(X)}{X}\right)$.

(2). $\gamma(\ell_X) = \ell_X + \log\left(\frac{\gamma(X)}{X}\right)$.

Prolong the ι_n as:

(3). $\iota_n(\ell_X) = \log(\iota_n(X))$

Finally define the monodromy operator N as:

(4). $N(f) = -\frac{p}{p-1}\frac{d}{d\ell_X}$

Let D be a filtered $(\varphi, N) - module$ (over K), consider $V = (\mathcal{B}^{\dagger}_{rig,K}[\ell_X] \otimes_{K_0} D)^{N=0}$. (Recall, $\mathbf{N}$ is defined to be $\mathbf{N}(a \otimes x) = N(a) \otimes x + a \otimes N_D(x)$.)

lemma 8.1. *V is a finite free module over $\mathcal{B}^{\dagger}_{rig,K}$ of rank* $\dim(D)$*.*

Proof. Recall, the operator N on D is nilpotent.

We induct on $\dim(D)$. If $\dim(D) = 1$, then $N_D = 0$, thus the lemma holds since $(\mathcal{B}^{\dagger}_{rig,K}[\ell_X])^{N=0} = \mathcal{B}^{\dagger}_{rig,K}$. If the lemma holds for $\dim \leq n$, let $\dim(D) = n$

We consider an exact sequence

$$0 \to D' \to D \to K_0 \to 0$$

while D' is a subspace of dimension $n-1$ contains $N_D(D)$.

Thus by snake lemma, we have an exact sequence

$$0 \to (\mathcal{B}^{\dagger}_{rig,K}[\ell_X] \otimes_{K_0} D')^{\mathbf{N}=0} \to (\mathcal{B}^{\dagger}_{rig,K}[\ell_X] \otimes_{K_0} D)^{\mathbf{N}=0} \to (\mathcal{B}^{\dagger}_{rig,K}[\ell_X] \otimes_{K_0} K_0)^{\mathbf{N}=0}$$

We only need to prove that the last one is surjective. In fact, let $e \in D$ maps to 1. Then $\sum_{i \geq 0} \left(\frac{p}{p-1}\right)^i \ell_X^i \otimes N_D^i(e)$ is a preimage of 1. □

Thus V is a (φ, Γ_K)-module. We then use the given filtration to 'twist' V, which is what we want.

Let $D^{(n)}$ be the (φ, N)-module $K_0 \otimes^{\varphi^{-n}}_{K_0} D$ (i.e. the (φ, N) operators stay the same but the scalar multiplication is given by $a * x = \varphi^{-n}(a)x$). The filtration of $D^n_K = K \otimes_{K_0} D^{(n)}$ is the one passed from $K \otimes_{K_0} D$ by $id \otimes \varphi^n$. We endow $K_n((t))$ with the natural filtration $t^i K_n[[t]]$ and define

$$M_n(D) = Fil^0\left(K_n((t)) \otimes_K D^{(n)}_K\right)$$

lemma 8.2. *$\{M_n(D)\}$ is a $K_n[[t]]$-lattice of $\mathbf{V}_n = K_n((t)) \otimes^{\iota_n}_{\mathcal{B}^{\dagger,r_n}_{rig,K}} V_{r_n}$ for $n >> 0$ and they form a $\varphi-$compatible lattice.*

Proof. Notice that for $n >> 0$, we have

$$K_n((t)) \otimes^{\iota_n}_{\mathcal{B}^{\dagger,r_n}_{rig,K}} V = K_n((t)) \otimes^{\iota_n}_{\mathcal{B}^{\dagger,r_n}_{rig,K}[\ell_X]} \left(\mathcal{B}^{\dagger,r_n}_{rig,K}[\ell_X] \otimes_{\mathcal{B}^{\dagger,r_n}_{rig,K}} \left(\mathcal{B}^{\dagger,r_n}_{rig,K}[\ell_X] \otimes_{K_0} D \right)^{N=0} \right)$$
$$= K_n((t)) \otimes^{\iota_n}_{\mathcal{B}^{\dagger,r_n}_{rig,K}[\ell_X]} \left(\mathcal{B}^{\dagger,r_n}_{rig,K}[\ell_X] \otimes_{K_0} D \right)$$
$$= K_n((t)) \otimes^{\varphi^{-n}}_{K_0} D = K_n((t)) \otimes_K D^{(n)}_K$$

Choose a basis $\{e_i\}$ compatible with filtration and let $h_i = h(e_i)$ (i.e. $Fil^m D^{(n)}_K = \sum_{h_i \geq m} K e_i$). Then $Fil^0 \left(K_n((t)) \otimes_K D^{(n)}_K \right)$ has a $K_n[[t]]-$basis consists of $t^{-h_i} \otimes e_i$.

The $\varphi-$compactiblity can be proved by $\{\varphi(e_i)\}$ forms a basis of $D^{(n+1)}_K$ with $h_{D^{(n+1)}}(\varphi(e_i)) = h_{D^{(n)}}(e_i)$. □

We define $\mathcal{M}(D)$ to be the (φ, Γ_K)-module which is included in $V\left[\frac{1}{t}\right]$ and associated to $M_n(D)$.

Now if D is a $(\varphi, N, G_{L/K})$-module, one can check that $\mathcal{M}_L(D)$ is a $(\varphi, \Gamma_L)-$module with a $G_{L/K}$-action. We define $\mathcal{M}(D) = \mathcal{M}_L(D)^{G_{L/K}}$

8.2 From (φ, Γ)-modules to filtered (φ, N, G_K)-modules

8.2.1 General facts about connections

Let E be a field of character 0. We define $\nabla(f) = t\frac{df}{dt}$ for all $f \in E((t))$. Let M be a finite dimensional $E((t))-$space. A connection on M is an additive map $\nabla_M : M \to M$ such that $\nabla_M(\lambda x) = \nabla(\lambda)x + \lambda\nabla_M(x)$.

lemma 8.3. $\dim_E M^{\nabla_M=0} \leq \dim_{E((t))} M$

Proof. In fact, we may prove that the natural map $M^{\nabla_M=0} \otimes_E E((t)) \to M$ is injective. □

We say the connection is trivial if $\dim_E M^{\nabla_M=0} = \dim_{E((t))} M$ (or $M^{\nabla_M=0} \otimes_E E((t)) \to M$ is bijective).

lemma 8.4. *The connection is trivial if and only if there exists an $E[[t]]-$lattice M_0 such that $\nabla_M(M_0) \subset tM_0$*

Proof. If ∇_M is trivial, then let $M_0 = E[[t]] \otimes_E M^{\nabla_M=0}$.

Now suppose M_0 is such a lattice. For any $x \in M_0$, if $\nabla_M(x) = t^n y$, then

$$\nabla_M(x - \frac{\nabla_M(x)}{n}) = \nabla_M(x) - \nabla_M(\frac{t^n y}{n}) = -\frac{t^n \nabla_M(y)}{n} \in t_0^M$$

.

Now use this fact and $t-$adically approximation, we can choose elements $e_1, ..., e_d \in M_0$ such that their projection to M_0/tM_0 form a basis. □

The proof as well as Nakayama's lemma also imply that the lattice is unique.

lemma 8.5. *If N is a subspace of M which is stable under ∇_M and ∇_M is trivial on M, then ∇_M is trivial on N.*

Proof. Let $M_0 = E[[t]] \otimes_E M^{\nabla_M=0}$, it is a lattice of M since ∇_M is trivial on M. Thus $M_0 \cap N$ is a lattice of N, which implies $\dim_E N^{\nabla_M=0} = \dim_{E((t))} N$ (lemma 8.4) □

Def 8.2. Let $D \in \Phi\Gamma_{\mathcal{B}^{\dagger}_{rig,K}}$, we have constructed a $K_n((t))$-space $\mathbf{D}_n\left[\frac{1}{t}\right]$ for $n >> 0$ with a Γ_K-action, define

$$\nabla_D = \lim_{\gamma \to 1} \frac{\log \gamma}{\log_p \chi(\gamma)}$$

It is a connection on $\mathbf{D}_n\left[\frac{1}{t}\right]$

Def 8.3. We say D is of locally trivial differential if ∇_D is trivial on $\mathbf{D}_n$ for $n >> 0$.

lemma 8.6. *Let D be a (φ, N, G_K)-module, then $\mathcal{M}(D)$ is of locally trivial differential.*

Proof. See the proof of lemma 8.2 □

8.2.2 Construction

We will use the following p-adic local monodromy theorem:

Thm 8.1. *Let $D \in \Phi_{\mathcal{B}^{\dagger}_{rig,K}}$ with a connection ∇_D, then there exists a finite extension L/K such that ∇_D is trivial on $\mathcal{B}^{\dagger,r_n}_{rig,L}[\ell_X] \otimes_{\mathcal{B}^{\dagger,r_n}_{rig,K}} D$.*

Proof. [1] □

For $D \in \Phi\Gamma_{\mathcal{B}^{\dagger}_{rig,K}}$, we define

$$Sol_L(D) = (\mathcal{B}^{\dagger}_{rig,L}[\ell_X] \otimes_{\mathcal{B}^{\dagger}_{rig,K}} D)^{\Gamma_L}; S_L(D) = (\mathcal{B}^{\dagger}_{rig,L}[\ell_X] \otimes_{\mathcal{B}^{\dagger}_{rig,K}} D)^{\nabla=0}$$

It is a fact that there exists an L such that $Sol_L(D) = S_L(D)$ and $\dim_{L_0} S_L(D) = rank(D)$. In this case, $Sol_L(D)$ is a $(\varphi, N, G_{L/K})-$module.

Thm 8.2. *Let $\mathbf{M}$ be a $(\varphi, \Gamma_K)-$module with locally trivial differential, then there exists a $(\varphi, \Gamma_K)-$module $\mathbf{D} \subset \mathbf{M}\left[\frac{1}{t}\right]$ such that $\mathbf{D}\left[\frac{1}{t}\right] = \mathbf{M}\left[\frac{1}{t}\right]$ and $\nabla_{\mathbf{M}}(\mathbf{D}) \subset t\mathbf{D}$.*

Moreover, $\mathbf{D}$ determines a filtration on $L \otimes_{L_0} Sol_L(\mathbf{M})$ whose induced $(\varphi, \Gamma_K)-$module is $\mathbf{M}$

Proof. The first part comes from lemma 8.4 and its remark.

Now we prove the second part.

Notice that

$$L_n[[t]] \otimes^{\varphi^{-n}}_{L_0} Sol_L(\mathbf{M}) = L_n[[t]] \otimes^{\iota_n}_{\mathcal{B}^{\dagger,r_n}_{rig,K}} \mathbf{D}_n$$

For $n >> 0$, $L_n((t)) \otimes^{\iota_n}_{\mathcal{B}^{\dagger,r_n}_{rig,K}} \mathbf{D}_n = L_n((t)) \otimes^{\iota_n}_{\mathcal{B}^{\dagger,r_n}_{rig,K}} \mathbf{M}_n$ has a natural filtration given by $t^k L_n[[t]] \otimes^{\iota_n}_{\mathcal{B}^{\dagger,r_n}_{rig,K}} \mathbf{M}_n$. Restrict the filtration on $L \otimes^{\varphi^{-n}}_{L_0} Sol_L(\mathbf{M})$ then pull back to $L \otimes_{L_0} Sol_L(\mathbf{M})$, we construct the filtration. □

Thm 8.3 (Theorem A in Berger's thesis)**.** *The functor $\mathcal{M}$ is an equivalence between the category of $(\phi, \Gamma_K)-$modules over $\mathcal{B}^{\dagger}_{rig,K}$ with locally trivial differential to the category of filtered $(\phi, N, G_K)-$modules.*

9 Slopes and weakly admissible filtered (ϕ, N, G_K)-modules

In this section, we will prove the following theorem by calculation.

Thm 9.1 (Theorem B in Berger's thesis[3])**.** *The functor $\mathcal{M}$ induces an equivalence between the category of étale $(\phi, \Gamma_K)-$modules over $\mathcal{B}^{\dagger}_{rig,K}$ with locally trivial differential to the category of weakly admissible filtered $(\phi, N, G_K)-$modules.*

In fact, we will prove that:

Thm 9.2. *For a $(\varphi, N, G_{L/K})$-module D, then the slope of $\det D$ is equal to $t_N(D) - t_H(D)$.*

Proof. One can check that $\mathcal{M}$ is an exact tensor functor, so we only need to prove for $\dim D = 1$.

In this case, $N_D = 0$, assume $D = L_0 e$, $\varphi(e) = \lambda e$ where $\lambda \in Ł_0$, $t_H = h$ (so $t_N = v_p(\lambda)$). Then, $\mathcal{M}_L(D)\left[\frac{1}{t}\right] = \mathcal{B}^{\dagger}_{rig,L}\left[\frac{1}{t}\right] e$. A naive calculation provides that $\mathcal{M}_L(D) = t^{-h}\mathcal{B}^{\dagger}_{rig,L} \otimes V$, where $V = \mathcal{B}^{\dagger}_{rig,L} e$.

Thus, $\varphi(t^{-h}e) = p^{-h}\lambda t^{-h} e$, this proves the theorem. □

To prove the theorem, we only need to show that.

lemma 9.1. *Let D is a semi-stable ϕ-module over $\mathcal{B}^{\dagger}_{rig,K}$ of slope 0. Then D is étale.*

Proof. See Ji Yibo's note. □

10 Application

Thm 10.1 (Theorem A by Colmez-Fontaine)**.** *Any weakly admissible $(\phi, N, G_K)-$module comes from a potentially semi-stable representation.*

Proof. Let D be a weakly admissible $(\phi, N, G_K)-$module, then $\mathcal{M}(D)$ is *étale* (ϕ, Γ_K)-module, so comes from a Galois representation V.

Recall [2] $D_{st,L}(V) = \left(\mathcal{B}^{\dagger}_{rig,L}\left[\frac{1}{t}, \ell_X\right] \otimes_{\mathcal{B}^{\dagger}_{rig,K}} D^{\dagger}_{rig}(V)\right)^{\Gamma_L}$. Thus we have

$$\begin{aligned} D_{st,L}(V) &= \left(\mathcal{B}^{\dagger}_{rig,L}\left[\frac{1}{t}, \ell_X\right] \otimes_{\mathcal{B}^{\dagger}_{rig,K}} \mathcal{M}(D)\right)^{\Gamma_L} \\ &= \left(\mathcal{B}^{\dagger}_{rig,L}\left[\frac{1}{t}, \ell_X\right] \otimes_{\mathcal{B}^{\dagger}_{rig,K}} (\mathcal{M}_L(D))^{G_{L/K}}\right)^{\Gamma_L} \\ &= \left(\mathcal{B}^{\dagger}_{rig,L}\left[\frac{1}{t}, \ell_X\right] \otimes_{\mathcal{B}^{\dagger}_{rig,L}} \mathcal{M}_L(D)\right)^{\Gamma_L} \quad \text{(Galois descent)} \\ &= \left(\mathcal{B}^{\dagger}_{rig,L}\left[\frac{1}{t}, \ell_X\right] \otimes_{\mathcal{B}^{\dagger}_{rig,L}\left[\frac{1}{t}\right]} (\mathcal{B}^{\dagger}_{rig,L}\left[\frac{1}{t}, \ell_X\right] \otimes_{L_0} D)^{N=0}\right)^{\Gamma_L} \\ &= \left(\mathcal{B}^{\dagger}_{rig,L}\left[\frac{1}{t}, \ell_X\right] \otimes_{L_0} D\right)^{\Gamma_L} = L_0 \otimes_{L_0} D = D \end{aligned}$$

This proves that V is semi-stable after restricting on G_L

Thus we only need to check the given filtration on D_L is the same as the one comes from $\mathcal{B}_{dR} \otimes_{\mathbb{Q}_p} V$.

In fact, we have

$$\begin{array}{ccc}
L \otimes_{L_0} D_{st,L}(V) & \longrightarrow & L \otimes_{L_0} \left(\mathcal{B}^{\dagger,r_n}_{rig,L} \left[\frac{1}{t}, \ell_X\right] \otimes_{\mathcal{B}^{\dagger,r_n}_{rig,K}} D^{\dagger}_{rig}(V)_{r_n} \right)^{\Gamma_L} \\
\big\downarrow & & \big\downarrow \\
 & & \mathcal{B}^{\dagger,r_n}_{rig,L} \left[\frac{1}{t}, \ell_X\right] \otimes_{\mathcal{B}^{\dagger,r_n}_{rig,K}} D^{\dagger}_{rig}(V)_{r_n} \\
 & & \big\downarrow \\
\mathcal{B}_{dR} \otimes_{\mathbb{Q}_p} V & \xleftarrow{\iota_n} & \mathcal{B}^{\dagger,r_n}_{rig} \left[\frac{1}{t}, \ell_X\right] \otimes_{\mathbb{Q}_p} V
\end{array}$$

for $n >> 0$.

A theorem by Fontaine implies

$$L_n[[t]] \otimes^{\iota_n}_{\mathcal{B}^{\dagger,r_n}_{rig,L}} D^{\dagger,r_n}_{rig}(V) = Fil^0(L_n((t)) \otimes_L D_{dR,L}(V)^{(n)})$$

Hence

$$Fil^0(L_n((t)) \otimes_L D_L^{(n)}) = Fil^0(L_n((t)) \otimes_L D_{dR,L}(V)^{(n)})$$

which proves the two filtrations are the equal. □

References

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[4] Oliver Brinon and Brian Conrad, CMI summer school notes on p-adic hodge theory (preliminary version)

[5] J.M.Fontain and Yi Ouyang, Theory of p-adic Galois Representations