Algebraic Varieties: Minimal Models and Finite Generation

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August 28, 2020

Preface

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Introduction

Chapter 1

Algebraic varieties with boundaries

{Chapter 1}

In this chapter, we introduce basic concepts of algebraic varieties with boundaries, where a boundary of an algebraic variety in this book is a divisor with real coefficients. Using the language of numerical geometry, we define cones of curves and cones of divisors. According to the Hironaka desingularization theorem, it is possible to use birational modifications to make algebraic varieties smooth and divisors normal crossing. We focus on adjoint divisors of algebraic varieties with boundaries, and introduce definitions of KLT pairs and DLT pairs. We explain how to use the covering trick to generalize the Kodaira vanishing theorem for smooth projective varieties to KLT or DLT pairs. Also we discuss the classification of algebraic varieties and singularities in lower dimensions.

1.1 Q-divisors and R-divisors

{section 1.1}

The linear equivalence class of a divisor defines a coherent sheaf associated to this divisor which is called its divisorial sheaf. In many situations in algebraic geometry, we deal with coherent sheaves. But in this book, we mainly focus on divisors. It is like dealing with differential forms themselves instead of cohomology classes of differential forms in differential geometry.

Fix a base field k. An *algebraic variety* X is an irreducible reduced separated scheme of finite type over k.

An algebraic variety X is said to be *non-singular* if for every point P on X, the local ring $\mathcal{O}_{X,P}$ of the structure sheaf $\mathcal{O}_X$ at P is a regular local ring. A point P with this property is called a non-singular point of X. In this book we mostly work over a field of characteristic zero, and we will mainly use the word *smooth* instead of non-singular. When $\dim X = n$, X is smooth if and only if for every closed point P on X, the maximal ideal $\mathfrak{m}_P$ of $\mathcal{O}_{X,P}$

is generated by n elements $x_1, \ldots, x_n$. Such $x_1, \ldots, x_n$ are called the *regular system of parameters* or *local coordinates*. If $k = \mathbf{C}$, this is equivalent to that the set of closed points of X forms a complex manifold. For an algebraic variety X, the set of all smooth points $\mathrm{Reg}(X)$ is a non-empty open subset of X, and its complement $\mathrm{Sing}(X) = X \setminus \mathrm{Reg}(X)$, which is a proper closed subset of X, is called the *singular locus*.

An algebraic variety X is said to be *normal* if the local ring at every point is an integrally closed domain. Since normal local rings of dimension 1 are regular, the singular locus of a normal algebraic variety is a closed subset of codimension at least 2. Every algebraic variety X can be easily modified into a normal one: there is a unique finite morphism $f : X^\nu \to X$ from a normal algebraic variety which is isomorphic over $\mathrm{Reg}(X)$, which is called the *normalization* of X. Normality can be determined by Serre's criterion ([102]):

Theorem 1.1.1. *An algebraic variety X is normal if and only if the following* 2 *conditions are satisfied:*

(1) (R_1) Its singular locus is a closed subset of codimension at least 2.

(2) (S_2) For any open subset U and any closed subset Z of codimension at least 2, the restriction map $\Gamma(U, \mathcal{O}_X) \to \Gamma(U \setminus Z, \mathcal{O}_X)$ is bijective.

From now on we always assume that X is a normal algebraic variety.

A *prime divisor* on X is a closed subvariety of codimension 1. A *divisor* is a formal finite sum of prime divisors $D = \sum d_i D_i$. Unless otherwise stated, the coefficients d_i are integers, and D_i are distinct prime divisors. In other words, divisors are elements in the free abelian group $Z^1(X)$ generated by all prime divisors on X. D is said to be *effective* if all coefficients d_i are non-negative. D is said to be *reduced* if all coefficients $d_i = 1$. For two divisors D, D', we write the inequality $D \geq D'$ if $D - D'$ is an effective divisor.

Let D be a prime divisor on X and P be the generic point of D, then the local ring $\mathcal{O}_{X,P}$ is a *discrete valuation ring* with quotient field $k(X)$. For a rational function $h \in k(X)^*$[1.1.0.1], the divisor $\mathrm{div}(h)$ is defined as

$$\mathrm{div}(h) = \sum v_P(h) D,$$

which is known to be a finite sum. Here the sum runs over all prime divisors D, P is the generic point of the prime divisor D, and v_P is the valuation of the discrete valuation ring $\mathcal{O}_{X,P}$. Any divisor of the form $\mathrm{div}(h)$ for some $h \in k(X)$ is called a *principal divisor*.

For a divisor D, the corresponding *divisorial sheaf* $\mathcal{O}_X(D)$ is defined as the following: for any open subset U of X,

$$\Gamma(U, \mathcal{O}_X(D)) = \{h \in k(X)^* \mid \mathrm{div}(h)|_U + D|_U \geq 0\} \cup \{0\}.$$ [1.1.0.2]

[1.1.0.1] corrction
[1.1.0.2] correction

Also we define

$$H^0(X, D) = H^0(X, \mathcal{O}_X(D)).$$

If a non-zero global section s of $\mathcal{O}_X(D)$ corresponds to a rational function h, we define the divisor of s by

$$\mathrm{div}(s) = \mathrm{div}(h) + D,$$

which is an effective divisor. Generally we can also define the divisor $\mathrm{div}(s)$ of a rational section s of $\mathcal{O}_X(D)$ similarly, but in this case $\mathrm{div}(s)$ is not necessarily effective. For example, if we take s_1 to be the rational section corresponding to the rational function $h = 1$, then $\mathrm{div}(s_1) = D$. Let η be the generic point of X, then there is an isomorphism $(\mathcal{O}_X(D))_\eta \cong \mathcal{O}_{X,\eta}$. Also since by taking dual, we have

$$\mathcal{O}_X(D)^* := \mathrm{Hom}(\mathcal{O}_X(D), \mathcal{O}_X) \cong \mathcal{O}_X(-D),$$

the divisorial sheaf $\mathcal{O}_X(D)$ is a *reflexive sheaf of rank one.* A reflexive sheaf is a coherent sheaf which is isomorphic to its double dual.

A divisor is called a *Cartier divisor* if its divisorial sheaf is invertible. In other words, this is to say that this divisor is a principal divisor in a neighborhood of each point P. To distinguish from Cartier divisors, we call a divisor by a *Weil divisor* or an *integral divisor.* Denote by $\mathrm{Div}(X)$ the set of all Cartier divisors. There is an inclusion $\mathrm{Div}(X) \subset Z^1(X)$, and they coincide if X is smooth.

Two divisors D, D' on X are said to be *linearly equivalent*, denoted by $D \sim D'$, if $D - D'$ is a principal divisor. Note that $D \sim D'$ if and only if there is an isomorphism $\mathcal{O}_X(D) \cong \mathcal{O}_X(D')$. In other words, divisorial sheaves can be viewed as linear equivalence classes of divisors. Here D, D' are not necessarily Cartier divisors.

We also have the relative version as follows. Given a morphism $f : X \to S$ between algebraic varieties, two divisors D, D' on X are said to be *relatively linearly equivalent* over S, and denoted by $D \sim_S D'$, if there exists an open covering $\{S_i\}$ of S such that $D|_{S_i} \sim D'|_{S_i}$ after restriction over each S_i. Here we remark that in some other references, D, D' are defined to be relatively linearly equivalent over S if there exists a Cartier divisor B on S such that $D \sim D' + f^*B$. In general these two definitions are not the same and our definition is weaker. But under certain condition, for example when f is proper surjective with connected geometric fibers, it is easy to see that these two definitions coincide. This condition on f is very natural in applications.

A closed subset B on a smooth algebraic variety X is called a *normal crossing divisor* if at each closed point P there are local coordinates $z_1, \dots, z_n$ of the local ring $\mathcal{O}_{X,P}$ and an integer $1 \leq r \leq n$ such that B is defined by the equation $z_1 \dots z_r = 0$ locally around P. In this case, every irreducible component of B is smooth, and the union of several irreducible components of B is again a normal crossing divisor. Given an algebraic variety X and a closed subset B, the set of points in a neighborhood of which X is smooth and B is a normal crossing divisor is an open subset of X, which is denoted by $\mathrm{Reg}(X, B)$. The complement set $\mathrm{Sing}(X, B) = X \setminus \mathrm{Reg}(X, B)$ is called the *singular locus* of (X, B).

{SNC divisor}

Remark 1.1.2. A normal crossing divisor defined above is also called a *simple normal crossing divisor* in many references.

If X is a complex algebraic manifold and $z_1, \dots, z_n$ are local coordinates on X^{an} as the complex manifold associated to X, then a normal crossing divisor B defined as above is not necessarily a simple normal crossing divisor in the algebraic setting. In fact, irreducible components of B may have self-intersection. So we use the term "simple" in the algebraic setting in order to distinguish with the analytic setting.

For example, consider the closed subset defined by the equation $x^2 + y^2 + y^3 = 0$ in the affine plane $\mathbf{C}^2$ with coordinates x, y. It is irreducible and has self-intersection at the point $(0, 0)$. It is a normal crossing divisor on the complex manifold $\mathbf{C}^2$, but not a simple normal crossing divisor.

One feature of this book is to consider divisors with not necessarily integer coefficients. Let X be a normal algebraic variety. If the coefficients d_i in $D = \sum d_i D_i$ are rational numbers (respectively, real numbers), then D is called a $\mathbf{Q}$-*divisor* (respectively, an $\mathbf{R}$-*divisor*). Note that a $\mathbf{Q}$-divisor is also an $\mathbf{R}$-divisor. Those are elements in $Z^1(X) \otimes \mathbf{Q}$ or $Z^1(X) \otimes \mathbf{R}$ respectively, and these vector spaces are usually denoted by $Z^1(X)_{\mathbf{Q}}$ are $Z^1(X)_{\mathbf{R}}$. We will see soon that the range of discussion of birational geometry is expanded widely by considering $\mathbf{Q}$-divisors and $\mathbf{R}$-divisors.

Let $D = \sum d_i D_i$ be an $\mathbf{R}$-divisor on X, where D_i are distinct prime divisors. D is said to be *effective* if all coefficients d_i are non-negative. D is said to be *reduced* if all coefficients $d_i = 1$. For two $\mathbf{R}$-divisors D, D', we write the inequality $D \geq D'$ if $D - D'$ is an effective divisor. The *support* of D is the union of all D_i with $d_i \neq 0$, and is denoted by $\mathrm{Supp}(D)$. In this situation, set $D^+ = \sum_{d_i > 0} d_i D_i$ and $D^- = \sum_{d_i < 0} (-d_i) D_i$, then D^+ and D^- are effective $\mathbf{R}$-divisors with no common components and $D = D^+ - D^-$. D^+, D^- are called the *positive part* and *negative part* of D respectively.

Given two $\mathbf{R}$-divisors $D = \sum_i d_i D_i$ and $D' = \sum_i d'_i D_i$, define their maximum to be $\max\{D, D'\} = \sum_i \max\{d_i, d'_i\} D_i$. For example, $D^+ = \max\{D, 0\}$, $D^- = \max\{-D, 0\}$. Similarly we can define $\min\{D, D'\} = \sum_i \min\{d_i, d'_i\} D_i$. The *round up* (respectively, *round down*) of an $\mathbf{R}$-divisor is defined via the round up (respectively, round down) of coefficients:

$$\ulcorner D \urcorner = \sum \ulcorner d_i \urcorner D_i, \quad \llcorner D \lrcorner = \sum \llcorner d_i \lrcorner D_i.$$

A $\mathbf{Q}$-divisor (respectively, an $\mathbf{R}$-divisor) is said to be $\mathbf{Q}$-*Cartier* (respectively, $\mathbf{R}$-*Cartier*) if it is an element of $\mathrm{Div}(X) \otimes \mathbf{Q}$ (respectively, $\mathrm{Div}(X) \otimes$

R). Note that if a **Q**-divisor is **R**-Cartier, then it is automatically **Q**-Cartier. For a **Q**-Cartier divisor D, there exists a positive integer m such that mD is a Cartier divisor. However, this is not true for **R**-Cartier divisors in general. X is said to be *factorial* (respectively, **Q**-*factorial*), if all Weil divisors on X are Cartier divisors (respectively, **Q**-Cartier divisors), in other words, if $\mathrm{Div}(X) = Z^1(X)$ (respectively, $\mathrm{Div}(X)_{\mathbf{Q}} = Z^1(X)_{\mathbf{Q}}$).

Two **R**-divisors D, D' are said to be **R**-*linearly equivalent*, denoted by $D \sim_{\mathbf{R}} D'$, if $D - D'$ can be written as an **R**-linear combination of principal divisors. The relative version and **Q**-linear equivalence can be defined similarly.

Remark 1.1.3. Recently considering **R**-divisors becomes essential to the development of the minimal model theory. This book can be viewed as a revised version of [82], in which only **Q**-divisors are treated. Already in [64], **R**-divisors played a central role. The *divisorial Zariski decomposition* (which is called the sectional decomposition in [64]) is defined via limits of **Q**-divisors, where **R**-divisors appear naturally. Moreover, it is proved in [63] that the existence of Zariski decomposition (in a good sense that the positive part is nef) as **R**-Cartier divisors implies the finite generation of canonical rings. Unfortunately, this paper was classified in "Affine algebraic geometry" session instead of "Threefolds" session since the word "log" did not have a proper citizenship at that time.

{ODP}

Example 1.1.4. We give examples for a **Q**-Cartier Weil divisor which is not Cartier and a Weil divisor which is not **Q**-Cartier.

(1) Let X be the hypersurface defined by the equation $xy = z^2$ in 3-dimensional affine space $\mathbf{A}^3$ with coordinates x, y, z. It is a surface with an ordinary double point at the origin $(0, 0, 0)$. The line D defined by $x = z = 0$ is a prime divisor on X. At lease 2 equations are needed to define D in X, so D is not a Cartier divisor. On the other hand, we have $\mathrm{div}(x) = 2D$ on X, so D is **Q**-Cartier.

(2) Let X be the hypersurface defined by $xy = zw$ in $\mathbf{A}^4$ with coordinates x, y, z, w. It is a 3-fold with an ordinary double point at the origin $(0, 0, 0, 0)$. The 2-dimensional linear subspace D_1 defined by $x = z = 0$ is a prime divisor on X, which is not a **Q**-Cartier divisor (see Example 1.2.4). It is the same for D_2 defined by $x = w = 0$. However, the sum $D_1 + D_2 = \mathrm{div}(x)$ is a Cartier divisor. See Example 2.5.4(2) for related discussions.

1.2 Rational maps and birational maps

{subsection 1.2}

Let X, Y be two algebraic varieties. A *rational map* $f : X \dashrightarrow Y$ is a morphism $f : U \to Y$ from a non-empty open subset U of X. Since f might not be defined on the whole X, we use dashed arrow to denote this map. If there is another non-empty open subset U' and a morphism $f' : U' \to Y$ such that f and f' coincide on $U \cap U'$, then we consider $f = f'$ as the same rational map. The *domain of definition* of a rational map f is defined to be the largest non-empty open subset U such that there is a morphism $f : U \to Y$ representing f. The *graph* of a rational map $f : X \dashrightarrow Y$ is defined to be the closure of the graph $\Gamma \subset U \times Y$ of the morphism $f : U \to Y$ in $X \times Y$.

A rational map $f : X \dashrightarrow Y$ is said to be a *birational map* if there exist non-empty open subsets U, V on X, Y such that f induces an isomorphism $U \cong V$. In this situation, the inverse map $f^{-1} : Y \dashrightarrow X$ is also a (bi-)rational map. X and Y are said to be *birationally equivalent* if there exists a birational map $f : X \dashrightarrow Y$. In this case, we also say that one is the *birational model* to the other.

A morphism $f : X \to Y$ is said to be a *birational morphism* if it is a birational map. If U is the largest open subset of X on which f induces an isomorphism $U \cong V$, then $\mathrm{Exc}(f) = X \setminus U$ is called the *exceptional set* of f. In this situation, V is the domain of definition of f^{-1}. A prime divisor contained in the exceptional set is called an *exceptional divisor* over Y or an *f-exceptional divisor*. Generally, a divisor with all components contained in the exceptional set is also called an exceptional divisor over Y or an f-exceptional divisor.

For a morphism $f : Y \to X$ and a closed subset D of X, the inverse image $f^{-1}(D)$ is a closed subset of Y. In this book, $f^{-1}(D)$ only means the set-theoretic inverse image, and we forget about its scheme structure. However, for a divisor we can define its direct image and inverse image as the following.

Firstly we define the *inverse image* or *pullback* of a Cartier divisor.

Given a morphism $f : Y \to X$ and an invertible sheaf L on X, we can always define the pullback f^*L which is an invertible sheaf on Y. On the other hand, for a Cartier divisor D on X, we can define its pullback only if the image $f(Y)$ is not contained in the support of D. In this situation, the pullback f^*D is defined by pulling back the local functions defining D. If D is given by a rational section s of the invertible sheaf $\mathcal{O}_X(D)$, then the pullback f^*D is give by the rational section f^*s of the invertible sheaf $f^*\mathcal{O}_X(D)$.

For an $\mathbf{R}$-Cartier divisor D, if we write it as an $\mathbf{R}$-linear combination of Cartier divisors $D = \sum d_i D_i$, then we can define the pullback by $f^*D = \sum d_i f^* D_i$. Here D_i are Cartier divisors, not prime divisors. In other words, the pullback of $\mathbf{R}$-Cartier divisors can be defined by extending the coefficients of the pullback morphism $f^* : \mathrm{Div}(X) \to \mathrm{Div}(Y)$ of Cartier divisors. Note that this definition does not depend on the expression of D. The pullback f^*D is also called the *total transform* of D.

On the other hand, we can not define the pullback for non-$\mathbf{R}$-Cartier divisors in general. However, if the morphism $f : Y \to X$ is a birational map, we can define another form of "pullback" (the strict transform by inverse map f^{-1}) as the following.

Let $f : X \dashrightarrow Y$ be a birational map and D a prime divisor on X. For the domain of definition of U, if $D \cap U \neq \emptyset$, then the image $(f|_U)(D \cap U)$ is a locally closed subvariety of Y. If its closure is a prime divisor on Y, then we denote the closure by f_*D; If $D \cap U = \emptyset$ or the image $(f|_U)(D \cap U)$ has codimension at least 2, then we set $f_*D = 0$. Here f_*D is called the *strict transform* or *birational transform* of D. Generally for $\mathbf{R}$-divisors, we consider the linear map $f_* : Z^1(X)_\mathbf{R} \to Z^1(Y)_\mathbf{R}$ by extending the coefficients, the definition is extended by linearity $f_*(\sum d_i D_i) = \sum d_i f_*(D_i)$.

Example 1.2.1. Given a projective birational morphism $f : Y \to X$, for any prime divisor D on X, the strict transform $f_*^{-1}D$ on Y is again a prime divisor, which is not 0. In fact, the inverse map f^{-1} is well-defined at the generic point of D, and there is no prime divisor contracted by f^{-1}, hence the strict transform is a prime divisor.

Remark 1.2.2. A birational map $f : X \dashrightarrow Y$ between normal algebraic varieties induces an isomorphism between function fields $k(X) \cong k(Y)$. For a prime divisor D on X whose strict transform f_*D is non-zero, this isomorphism identifies the local rings at generic points of D and f_*D. When regarding birationally equivalent algebraic varieties as the same, we identify the divisors defining the same discrete valuation ring, which is equivalent to identifying divisors connected by strict transforms.

A birational map $f : X \dashrightarrow Y$ is said to be *surjective in codimension* 1 if the map $f_* : Z^1(X) \to Z^1(Y)$ is surjective, that is, for any prime divisor $E \subset Y$ there is a prime divisor D on X such that $E = f_*D$. Moreover, it is said to be *isomorphic in codimension* 1 if the map $f_* : Z^1(X) \to Z^1(Y)$ is bijective. The minimal model theory mainly deals with the phenomenon in codimension one, so these maps play important roles.

{blowup}

Example 1.2.3. A classical example of biratonal maps is a *blowing up*. A blowing up is obtained by glueing the following local construction.

(1) Define the rational map $f : X = \mathbf{A}^n \dashrightarrow Y = \mathbf{P}^{r-1}$ by $f(x_1, \ldots, x_n) = [x_1 : \cdots : x_r]$. Let Z be the linear subspace of X defined by $x_1 = \cdots = x_r = 0$, then the domain of definition of f is $U = X \setminus Z$. The graph $X' \subset X \times Y$ of f is defined by $x_i y_j = x_j y_i$ $(1 \leq i, j \leq r)$ where $y_1, \ldots, y_r$ are the homogenous coordinates of Y. The first projection $p : X' \to X$ is the blowing up along center Z. $E = p^{-1}(Z)$ is the exceptional set of the birational morphism p, which is a prime divisor. Moreover, $E \cong Z \times \mathbf{P}^{r-1}$, and p induces an isomorphism $X' \setminus E \to X \setminus Z$. In this case p is surjective in codimension 1, but p^{-1} is not.

(2) Let X_1 be a subvariety of X which is not contained in Z. The strict transform $X_1' = p_*^{-1}(X_1)$ of X_1 is the closure of $p^{-1}(X_1 \setminus Z)$. In this case, $p_1 = p|_{X_1'} : X_1' \to X_1$ is the blowing up of X_1 along center $Z \cap X_1$. In particular, the case $Z \subset X_1$ is important. Since $X_1 \not\subset Z$, p_1 is a birational morphism. However, the exceptional set $\mathrm{Exc}(p_1)$ does not necessarily coincide with $E \cap X_1'$. For example, consider $n = 4$, $r = 2$, $X_1 \subset \mathbf{A}^4$ is the subvariety defined by $x_1x_3 + x_2x_4 = 0$. This is the situation in Example 1.1.4(2). In this case, $Z \subset X_1$, the exceptional set C of $p_1 : X_1' \to X_1$ is isomorphic to $\mathbf{P}^1$, and $p_1(C)$ is the origin. Hence p_1 is isomorphic in codimension 1, and so is p_1^{-1}.

{ODP2}

Example 1.2.4. Consider the situations in Example 1.1.4.

(1) For a $\mathbf{Q}$-Cartier Weil divisor which is not Cartier, the pullback might not be a Weil divisor but only a $\mathbf{Q}$-divisor. The blowing up $f : X' \to X$ of X along the origin $Z = (0,0,0)$ gives a resolution of singularities. The exceptional set $C \subset X'$ is isomorphic to $\mathbf{P}^1$. We have $f^*D = f_*^{-1}D + \frac{1}{2}C$. The projection formula $(f^*D \cdot C) = (D \cdot f_*C)$ stated later (before Proposition 1.4.3) can be confirmed by the following facts: $(f_*^{-1}D \cdot C) = 1$, $(C^2) = -2$, $f_*C = 0$.

(2) Non-$\mathbf{Q}$-Cartier divisor can not be pulled back according to the projection formula. Consider the blowing up $p_1 : X_1' \to X_1$ in the end of Example 1.2.3(2). We change the notation by $f : X' \to X$. Then X' is smooth, the exceptional set $C \subset X'$ is isomorphic to $\mathbf{P}^1$, and p_1 is isomorphic in codimension 1. If the pullbacks f^*D_1, f^*D_2 of D_1, D_2 would exist, they would have to coincide with the strict transforms $f_*^{-1}D_1, f_*^{-1}D_2$ since there is no exceptional divisor. However, intersecting with C, $(f_*^{-1}D_1 \cdot C) = -1$, $(f_*^{-1}D_2 \cdot C) = 1$. This violates the projection formula $(f^*D \cdot C) = (D \cdot f_*C)$ since $f_*C = 0$.

A coherent sheaf F on an algebraic variety X is said to be *generated by global sections* if the natural homomorphism $H^0(X, F) \otimes \mathcal{O}_X \to F$ is surjective.

For a Cartier divisor D, its *complete linear system* is defined by $|D| = \{D' \mid D \sim D' \geq 0\}$, and its *base locus* is defined by $\mathrm{Bs}\,|D| = \bigcap_{D' \in |D|} \mathrm{Supp}(D')$. When $\mathrm{Bs}\,|D| = \emptyset$, $|D|$ is said to be *free*, which is equivalent to that the corresponding coherent sheaf $\mathcal{O}_X(D)$ is generated by global sections. Here D is also said to be free if $|D|$ is free, and D is said to be *semi-ample* if there exists a positive integer m such that mD is free.

More generally, a finite dimensional linear subspace $V \subset H^0(X, D)$ corresponds to a (not necessarily complete) *linear system* $\Lambda = \{\mathrm{div}(s) \mid s \in V \setminus \{0\}\}$. The base locus of Λ is defined similarly by $\mathrm{Bs}\,\Lambda = \bigcap_{D' \in \Lambda} \mathrm{Supp}(D')$, and Λ is said to be free if $\mathrm{Bs}\,\Lambda$ is empty, which is equivalent to that the natural homomorphism $V \otimes \mathcal{O}_X \to \mathcal{O}_X(D)$ is surjective.

The *fixed part* of a linear system Λ is the effective divisor $F = \min_{D' \in \Lambda} D'$. In other words, F is the maximal divisor such that $F \leq D'$ for all $D' \in \Lambda$. In this case, the image of the natural injection $H^0(X, D - F) \to H^0(X, D)$ contains V. Being viewed as a subspace of $H^0(X, D - F)$, V corresponds to the linear system $\Lambda' = \{D' - F \mid D' \in \Lambda\}$, which is called the *movable part* of Λ. We write $\Lambda = \Lambda' + F$. Usually Λ' and F are denoted by $\operatorname{Mov} \Lambda$ and $\operatorname{Fix} \Lambda$ respectively. By definition, the support of F coincides with the codimension one components of $\operatorname{Bs} \Lambda$.

If we assume moreover that X is proper, then Λ is isomorphic to the projective space $\mathbf{P}(V^*) := (V \setminus \{0\})/k^*$ as an algebraic variety. A non-empty linear system Λ induces a rational map $\Phi_\Lambda : X \dashrightarrow \mathbf{P}(V) := (V^* \setminus \{0\})/k^*$ to its dual projective space. The domain of definition of Φ_Λ contains $U = X \setminus \operatorname{Bs} \Lambda$; for $P \in U$, $\Phi_\Lambda(P)$ is the point in $\mathbf{P}(V)$ corresponding to the hyperplane $\{s \in V \mid s(P) = 0\}$ of V. In other words, if we take a basis $s_1, s_2, \ldots, s_m \in V$, then we can define $\Phi_\Lambda(P) = [s_1(P) : s_2(P) : \cdots : s_m(P)] \in \mathbf{P}(V)$. Note that here $s_i(P)$ is not a well-defined value, but $[s_1(P) : s_2(P) : \cdots : s_m(P)]$ is a well-defined point as long as $P \in U$. In particular, when Λ is free, Φ_Λ is a morphism. The rational map given by the movable part of a linear system coincides with the rational map given by the original linear system.

For a morphism $f : Y \to X$ and a linear system Λ on X, the *pullback* is defined by $f^*\Lambda = \{f^*D' \mid D' \in \Lambda\}$. If there is a morphism to a projective space, a free linear system can be obtained by pulling back the linear system consisting of all hyperplanes.

The base locus of a linear system can be removed in the following sense:

Proposition 1.2.5. *Let Λ be a linear system of Cartier divisors on a normal algebraic variety X. Then there exists a projective birational morphism $f : Y \to X$ from a normal algebraic variety Y such that the pullback has the form $f^*\Lambda = \Lambda_1 + F$ where F is the fixed part of $f^*\Lambda$ and the movable part Λ_1 is free.*

Proof. Let $V \subset H^0(X, D)$ be the linear subspace corresponding to Λ. The image of the natural map $V \otimes \mathcal{O}_X \to \mathcal{O}_X(D)$ can be written as $I\mathcal{O}_X(D)$ where I is an ideal sheaf on X. Take f to be the normalization of the blowing up of I, then the inverse image ideal sheaf $I\mathcal{O}_Y$ is an invertible sheaf on Y, and the image of $f^*V \otimes \mathcal{O}_Y \to \mathcal{O}_Y(f^*D)$ can be written as $\mathcal{O}_Y(f^*D - F)$ for some effective divisor F. Since the natural map $f^*V \otimes \mathcal{O}_Y \to \mathcal{O}_Y(f^*D - F)$ is surjective, the linear system $\Lambda_1 = f^*\Lambda - F$ is free and F is the fixed part of $f^*\Lambda$. □

For an $\mathbf{R}$-divisor D on a normal proper algebraic variety X, the set of global sections $H^0(X, \llcorner D \lrcorner)$ is a finite dimensional k-linear space. Considering all natural number multiples mD and taking direct sum, we define the *section ring* of D by

$$R(X, D) = \bigoplus_{m=0}^{\infty} H^0(X, \llcorner mD \lrcorner).$$

Here m runs over all non-negative integers. It admits a graded k-algebra structure defined by

$$H^0(X, \llcorner mD \lrcorner) \otimes H^0(X, \llcorner m'D \lrcorner) \to H^0(X, \llcorner (m+m')D \lrcorner)$$

since

$$\llcorner mD \lrcorner + \llcorner m'D \lrcorner \leq \llcorner (m+m')D \lrcorner.$$

The *Iitaka–Kodaira dimension* of an $\mathbf{R}$-divisor can be defined by the transcendental degree of the section ring:

$$\kappa(X, D) = \begin{cases} \text{tr.deg}_k\, R(X, D) - 1 & \text{if } R(X, D) \neq k, \\ -\infty & \text{otherwise.} \end{cases}$$

The Iitaka–Kodaira dimension takes value among $-\infty, 0, 1, \ldots, n = \dim X$. When it takes the maximal value, that is, when $\kappa(X, D) = \dim X$, D is said to be *big*. For example, ample divisors are big.

If $R(X, D) = k$, that is, $H^0(X, \llcorner mD \lrcorner) = 0$ for any $m > 0$, then $\kappa(X, D)$ is defined to be $-\infty$ instead of -1. This definition is reasonable by the following lemma:

Lemma 1.2.6 ([49, Theorem 10.2], [123, Theorem II.3.7])**.** *There exist positive real numbers c_1, c_2 such that for any sufficiently large and sufficiently divisible integer m,*

$$c_1 m^{\kappa(X,D)} \leq \dim H^0(X, \llcorner mD \lrcorner) \leq c_2 m^{\kappa(X,D)}.$$

Remark 1.2.7. Canonical ring is the section ring of the canonical divisor, which is proved to be finitely generated for smooth projective varieties ([15]), and one of the main goals of this book is to explain the proof. However, in general the section ring $R(X, D)$ of a divisor D is not necessarily finitely generated. There exist examples such that the *anti-canonical ring* (i.e. the section ring of the *anti-canonical divisor* $-K_X$) of a smooth projective surface is not finitely generated ([132], see also Example 2.4.8). Also, the anti-canonical ring $R(X, -K_X)$ is not a birational invariant.

The relative version is as follows. Let $f : X \to S$ be a proper morphism from a normal algebraic variety. The relative global sections of a coherent sheaf F on X are given by the direct image sheaf f_*F. F is said to be *generated by relative global sections* if the natural homomorphism $f^*f_*F \to F$ is surjective. A Cartier divisor D on X is said to be *relatively free* if the corresponding coherent sheaf $\mathcal{O}_X(D)$ is generated by relative global sections. D is said to be *relatively semi-ample* if there exists a positive integer m such that mD is relatively free.

For an $\mathbf{R}$-divisor D on X, the direct image sheaf $f_*(\mathcal{O}_X(\llcorner D \lrcorner))$ is a coherent $\mathcal{O}_S$-module. The *relative section ring* of D is defined by the direct sum

$$R(X/S, D) = \bigoplus_{m=0}^{\infty} f_*(\mathcal{O}_X(\llcorner mD \lrcorner)),$$

which is a graded $\mathcal{O}_S$-algebra.

The *relative Iitaka–Kodaira dimension* is defined by the Iitaka–Kodaira dimension of the generic fiber. Here we always assume that f is surjective with irreducible geometric generic fiber, and define

$$\kappa(X/S, D) = \kappa(X_{\bar{\eta}}, D|_{X_{\bar{\eta}}}).$$

Here X_η is the *generic fiber* which is the fiber of f over the generic point η of S, and $X_{\bar{\eta}}$ is the *geometric generic fiber* which is the base change of X_η to the algebraic closure of $k(S)$. D is said to be *relatively big* or *f-big* if $\kappa(X/S, D) = \dim X_{\bar{\eta}}$. In Section 1.5.1 we will give an equivalent definition for (relative) bigness using Kodaira's lemma (Corollary 1.5.10).

1.3 Canonical divisors

A normal algebraic variety X is automatically associated with a Weil divisor K_X which is called the canonical divisor. K_X is the key player of this book. The canonical ring is the section ring of the canonical divisor. The minimal model program is a sequence of operations that "minimizes" the canonical divisor.

As X is normal, the *singular locus* $\mathrm{Sing}(X)$ is a closed subset of X of codimension at least 2. Since the complement set $U = X \setminus \mathrm{Sing}(X)$ is smooth, the sheaf of differentials $\Omega^1_{X/k}$ is a locally free sheaf of rank $n = \dim X$ over U. The determinant $\omega_U = \det(\Omega^1_{X/k}|_U)$ is an invertible sheaf on U. Taking a non-zero rational section θ_U of ω_U, we get a canonical divisor $K_U = \mathrm{div}(\theta_U)$ of U. Since $X \setminus U$ contains no prime divisors of X, the restriction map of divisors $Z^1(X) \to Z^1(U)$ is bijective. Denote by $K_X \in Z^1(X)$ the corresponding divisor of $K_U \in Z^1(U)$, which is called the *canonical divisor* of X.

Remark 1.3.1. (1) By construction, K_X depends on the choice of θ_U. However, traditionally arguments proceed as if the canonical divisor is a fixed one. Anyway, in this book, all discussions are independent of the choice of θ_U. On the other hand, the corresponding divisorial sheaf $\omega_X = \mathcal{O}_X(K_X)$ is uniquely determined. It is called the *canonical sheaf.* The canonical sheaf ω_X is a natural subject.

(2) In this book, the following situation appears frequently: let $f : Y \to X$ be a birational morphism between normal algebraic varieties and B an $\mathbf{R}$-divisor on X such that $K_X + B$ is $\mathbf{R}$-Cartier. Consider the pullback $f^*(K_X + B)$. By using the isomorphism between function fields $f^* : k(X) \to k(Y)$, we can take the same rational differential form θ which defines K_X and K_Y (in particular, $K_X = f_*K_Y$), then the $\mathbf{R}$-divisor C can be defined by $f^*(K_X+B) = K_Y+C$. Here C is uniquely determined as the sum of the strict transform $f_*^{-1}B$ and an $\mathbf{R}$-divisor supported on the exceptional set of f.

We will discuss general boundary divisors later. Here we consider such a pair when X is a smooth algebraic variety and $B = \sum B_i$ is a normal crossing divisor. Let $n = \dim X$. The sheaf of differentials $\Omega^1_X(\log B)$ with at most *logarithmic poles* along B is naturally defined as a locally free sheaf of rank n with the following property. For any closed point $P \in X$, choose a regular system of parameters $x_1, \dots, x_n$ of the local ring $\mathcal{O}_{X,P}$ such that the local equation of B is $x_1 \cdots x_r = 0$ for some integer r. In this case, the stalk $\Omega^1_X(\log B)_P$ is a free $\mathcal{O}_{X,P}$-module with basis $dx_1/x_1, \dots, dx_r/x_r, dx_{r+1}, \dots, dx_n$. The determinant $\Omega^n_X(\log B)$ of $\Omega^1_X(\log B)$ is isomorphic to $\mathcal{O}_X(K_X + B)$. Therefore $K_X + B$ is called the *logarithmic canonical divisor* or just *log canonical divisor*. This is the origin of the terminology "log".

In general, a log canonical divisor K_X+B is a sum of the canonical divisor and an effective $\mathbf{R}$-divisor. Usually certain conditions on singularities will be imposed on the pair (X, B), which will be discussed later. The *log canonical ring* is defined to be $R(X, K_X+B)$, and the *log Kodaira dimension* is defined to be $\kappa(X, K_X + B)$.

Let X be a smooth projective variety. $R(X) = R(X, K_X)$ is the *canonical ring* of X. $P_m(X) = \dim H^0(X, mK_X)$ is called the *m-genus*, which is an important birational invariant having been studied for a long time. Its growth order $\kappa(X, K_X)$ is called the *Kodaira dimension*, sometimes is simply denoted by $\kappa(X)$. X is said to be *of general type* if K_X is big.

When doing induction on dimensions, one key is the adjunction formula.

Let D be a smooth prime divisor on a smooth algebraic variety X. Then the log canonical divisor on X and the canonical divisor of D are connected by the following *adjunction formula*:

$$(K_X + D)|_D = K_D.$$

In this formula, $K_X|_D$ and $D|_D$ have no natural meaning, but their sum does. The adjunction is given by the map

$$\mathrm{Res}_D : \Omega^n_X(\log D) \to \Omega^{n-1}_D$$

which is induced by the residue map

$$\mathrm{Res}_D : \Omega^1_X(\log D) \to \mathcal{O}_D$$

and the restriction map $\Omega^1_X \to \Omega^1_D$. The residue map is a natural map which is independent of the choice of coordinates. Therefore, the adjunction formula is also a natural formula. Note that this adjunction formula still holds if D is normal and $D \cap \mathrm{Sing}(X)$ has codimension at least 2 in D, as we can first apply the above adjunction formula to $D \setminus \mathrm{Sing}(X) \subset X \setminus \mathrm{Sing}(X)$, then extend it to D by normality. [1.3.0.1]

When D is not a prime divisor but a normal crossing divisor, if taking an irreducible component D_1 of D and write $E = (D - D_1)|_{D_1}$, then we have the adjunction formula

$$(K_X + D)|_{D_1} = K_{D_1} + E.$$

Here the restriction E is well-defined since the intersection of $D - D_1$ and D_1 is of codimension one on D_1.

More generally, we can consider the adjunction formula as a relation between canonical divisors of relevant varieties. For example, consider a surjective finite morphism $f : Y \to X$ between smooth algebraic varieties whose ramification locus is a smooth prime divisor D on X with ramification index m. The set-theoretic inverse image $E = f^{-1}(D)$ is a prime divisor on Y and $f^*D = mE$. In this case, the *ramification formula* or the adjunction formula related to the ramification is the following:

$$K_Y = f^*K_X + (m-1)E.$$

If written as

$$K_Y = f^*(K_X + \frac{m-1}{m}D),$$

then it looks like the adjunction formula for subvarieties. The latter formula is the origin of considering boundary divisors with rational coefficients. Also, if you write

$$K_Y + E = f^*(K_X + D),$$

you will find that "ramification is killed by log setting".

As another example of the adjunction formula, consider the blowing up of an n-dimensional smooth algebraic variety X along an r-codimensional smooth subvariety Z. The blowing up $f : Y \to X$ is a birational morphism with exceptional set E a prime divisor isomorphic to a $\mathbf{P}^{r-1}$-bundle over Z. The relation of canonical divisors is given by

$$K_Y = f^*K_X + (r-1)E.$$

[1.3.0.1] I added this sentence.

As can be seen in the following example, if X is a singular normal algebraic variety and a prime divisor D on X intersects $\mathrm{Sing}(X)$ such that $D \cap \mathrm{Sing}(X)$ contains an irreducible component of codimension 1 on D, then the singularities contribute to the adjunction formula. This phenomenon is called the *subadjunction formula*, which is very important.

{SAF example}

Example 1.3.2. Let X be the quadric surface defined by $xy + z^2 = 0$ in projective space $\mathbf{P}^3$ with homogenous coordinates x, y, z, w. X has a singularity at $[0 : 0 : 0 : 1]$. Let H be a hyperplane section, then $K_X \sim -2H$. The projective line L defined by $x = z = 0$ is a prime divisor on X. We have $\mathrm{div}(x) = 2L$ on X, hence $L \sim_{\mathbf{Q}} \frac{1}{2}H$. Therefore $(K_X + L)|_L \sim_{\mathbf{Q}} -\frac{3}{2}H|_L$ since $L|_L \sim_{\mathbf{Q}} \frac{1}{2}H|_L$. On the other hand, $K_L \sim -2H|_L$. Therefore we have the subadjunction formula $(K_X + L)|_L = K_L + \frac{1}{2}H|_L$ (see Remark 1.11.14).

1.4 Intersection numbers and numerical geometry

Problems in algebraic geometry are equivalent to solving simultaneous polynomial equations, which are highly nonlinear. *Numerical geometry* attempts to linearize those using intersection numbers. In the following two sections, we explain basic definitions in numerical geometry. In Chapter 2, we explain the base point free theorem and the cone theorem which are important in numerical geometry. The explanation here is according to Kleiman [86].

All definitions here will be for a proper morphism $f : X \to S$ between algebraic varieties over a field k. In the case $S = \mathrm{Spec}\, k$, the definitions are for a proper algebraic variety X. We use words "relative" or "over S" to keep in mind this setting. In the case $S = \mathrm{Spec}\, k$, those words will be removed. For simplicity, one can just consider $S = \mathrm{Spec}\, k$ and ignore the word "relative", the context will be almost the same. However, it is indispensable to consider the relative version in applications.

In the following definition, k is an arbitrary field, and X is of finite type over k, not necessarily irreducible or reduced. However, when considering Cartier divisors, X is always assumed to be a normal algebraic variety.

A closed subvariety Z on X is called a *relative subvariety* over S if $f(Z)$ is a closed point of S. In particular, if $\dim Z = 1$, it is called a *relative curve* over S. Denote $\dim Z = t$ and take t invertible sheaves $L_1, \dots, L_t$ on X. Then the *intersection number* $(L_1 \cdots L_t \cdot Z)$ is defined as the coefficient of the following polynomial ([86, p.296])

$$\chi(Z, L_1^{\otimes m_1} \otimes \cdots \otimes L_t^{\otimes m_t} \otimes \mathcal{O}_Z) = (L_1 \cdots L_t \cdot Z) m_1 \cdots m_t + (\text{other terms}).$$

Here $m_1, \dots, m_t$ are variables with integer values, and

$$\chi(Z, \bullet) = \sum (-1)^p \dim_k H^p(Z, \bullet)$$

is the *Euler–Poincaré characteristic.* Here note that X itself is not necessarily proper, but Z is proper as $f(Z)$ is a point, hence the cohomology groups of coherent sheaves on Z are finite dimensional.

The intersection number $(L_1 \cdots L_t \cdot Z)$ takes integer value, and it is a symmetric t-linear form with respect to $L_1, \ldots, L_t$ ([86, p.296]). That is, it is independent of order of L_i, and

$$((L_1^{\otimes n_1} \otimes L_1'^{\otimes n_1'}) \cdots L_t \cdot Z) = n_1(L_1 \cdots L_t \cdot Z) + n_1'(L_1' \cdots L_t \cdot Z).$$

For Cartier divisors $D_1, \ldots, D_t$, define

$$(D_1 \cdots D_t \cdot Z) = (\mathcal{O}_X(D_1) \cdots \mathcal{O}_X(D_t) \cdot Z).$$

In particular, when $\dim Z = 1$, taking $\nu : Z^\nu \to Z$[1.4.0.1] to be the normalization where Z^ν is a smooth projective curve, then by the Riemann–Roch theorem,

$$(D_1 \cdot Z) = \deg_{Z^\nu}(\nu^*(\mathcal{O}_X(D_1)|_Z)).$$

When $Z = X$, we simply write $(D_1 \cdots D_t) = (D_1 \cdots D_t \cdot X)$. If moreover all D_i are the same D, then write $(D_1 \cdots D_t) = (D^t)$.

By multi-linearity, the definition of $(D_1 \cdots D_t \cdot Z)$ can be extended to the case when D_i are $\mathbf{R}$-Cartier divisors, which takes value in real numbers.

Remark 1.4.1. (1) Here we use Euler–Poincaré characteristic to give a simple definition for intersection numbers, but the correct geometric definition is by adding up local intersection numbers to get the global intersection number. This is how intersection number (the number of "intersection points") is defined originally. Using the geometric definition, for effective $\mathbf{R}$-Cartier divisors D_i and a t-dimensional relative subvariety Z, if the intersection $\bigcap_{i=1}^t \operatorname{Supp}(D_i) \cap Z$ is non-empty and of dimension 0, then the intersection number is positive, and if the intersection is empty, then the intersection number is zero.

(2) By using intersection numbers of divisorial sheaves, we can define the *self-intersection number* of a divisor, which seems to be a weird name. For example, for an effective Cartier divisor D on an n-dimensional algebraic variety, the self-intersection number (D^n) can be either positive or non-positive.

(3) In this book, a *curve* is an irreducible reduced projective variety of dimension 1. The only intersection number considered in this book is the intersection number of a Cartier divisor with a curve. Among all curves, *rational curve* plays a very important role in the minimal model theory (see Sections 2.7 and 2.8). A rational curve is a curve whose normalization is isomorphic to $\mathbf{P}^1$. In general a rational curve might have singularities and not necessarily isomorphic to $\mathbf{P}^1$.

[1.4.0.1] notation changed

{ODP3}

Example 1.4.2. The intersection number of a divisor and a curve can be defined if this divisor is a **Q**-Cartier divisor. However, the intersection number is not necessarily an integer if the divisor is not Cartier. In general it can not be defined if the divisor is not **Q**-Cartier.

Consider X as in Examples 1.1.4 or 1.2.4, and let $\bar{X}$ be its compactification in projective space $\mathbf{P}^3$ or $\mathbf{P}^4$.

(1) $\bar{X}$ is defined by $xy = z^2$ in $\mathbf{P}^3$ with homogenous coordinates u, x, y, z. The compactification $\bar{D}$ of D is a prime divisor defined by $x = z = 0$. In this case, $(\bar{D}^2) = \frac{1}{2}$. In fact, take a plane $\bar{H}$, then $\bar{H}|_{\bar{X}} \sim \operatorname{div}(x) = 2\bar{D}$, $(\bar{H} \cdot \bar{D}) = 1$.

(2) $\bar{X}$ is defined by $xy = zw$ in $\mathbf{P}^4$ with homogenous coordinates u, x, y, z, w. The compactifications $\bar{D}_1, \bar{D}_2$ of D_1, D_2 are prime divisors defined by $x = z = 0$, $x = w = 0$. Take curve C defined by $y = z = w = 0$. $D_1 + D_2$ is a Cartier divisor and $((D_1 + D_2) \cdot C) = 1$. The blowing up $f_1 : Y_1 \to \bar{X}$ is isomorphic in codimension one. If intersection numbers $(D_i \cdot C)$ $(i = 1, 2)$ could be defined, by the projection formula stated later (before Proposition 1.4.3), $(D_i \cdot C) = (f_{1*}^{-1} D_i \cdot f_{1*}^{-1} C)$ as there is no exceptional divisor. The right hand side can be calculated to be $1, 0$ for $i = 1, 2$. This is absurd since the relations between D_1, D_2 and C are symmetric.

Two invertible sheaves L, L' are called *relatively numerically equivalent*, denoted by $L \equiv_S L'$, if $(L \cdot C) = (L' \cdot C)$ for any relative curve C. When the base is clear, we just write $L \equiv L'$. The abelian group consisting of isomorphism classes of all invertible sheaves is denoted by $\operatorname{Pic}(X)$, and the subgroup consisting of all invertible sheaves relatively numerically equivalent to $\mathcal{O}_X$ is denoted by $\operatorname{Pic}^\tau(X/S)$. The quotient group $\operatorname{Pic}(X)/\operatorname{Pic}^\tau(X/S)$ is a finitely generated abelian group ([86, p.323]), which is called the *relative Neron–Severi group*, and is denoted by $\operatorname{NS}(X/S)$. $\rho(X/S) = \operatorname{rank} \operatorname{NS}(X/S)$ is called the *relative Picard number*. When $S = \operatorname{Spec} k$, it is just called the *Picard number* and is denoted by $\rho(X)$.

If $L_1 \equiv_S \mathcal{O}_X$, $(L_1 \cdot L_2 \cdots L_t \cdot Z) = 0$ holds for arbitrary $L_2, \dots, L_t, Z$ ([86, p.304]). Also, for any invertible sheaf F on a relative subvariety Z, $\chi(Z, F) = \chi(Z, F \otimes L_1)$ holds ([86, p.311]).

Two **R**-Cartier divisors D, D' are called *relatively numerically equivalent*, denoted by $D \equiv_S D'$ or $D \equiv D'$, if $(D \cdot C) = (D' \cdot C)$ for any relative curve C. The numerical equivalence class of D is denoted by $[D]$. The set of all numerical equivalence classes of **R**-Cartier divisors coincides with $\operatorname{NS}(X/S) \otimes \mathbf{R}$, which is a $\rho(X/S)$-dimensional real vector space, and is denoted by $N^1(X/S)$.

If X is a smooth complete complex manifold, $D \equiv D'$ is equivalent to having the same cohomology class $[D] = [D'] \in H^2(X, \mathbf{R})$.

Fix an integer t, a finite formal linear sum of t-dimensional relative subvarieties $Z = \sum a_j Z_j$ is called a *relative t-cycle*. The coefficients a_i can be integers, rational numbers, or real numbers depending on the situation. By linearity, intersection numbers can be defined for relative t-cycles. In this book we only consider the case $t = 1$ or $\dim X - 1$.

Two relative 1-cycles C, C' are called *numerically equivalent*, denoted by $C \equiv_S C'$, if $(D \cdot C) = (D \cdot C')$ for any Cartier divisor D. The set $N_1(X/S)$ of all numerical equivalence classes of relative 1-cycles with real coefficients is a finite dimensional real vector space. $N_1(X/S)$ and $N^1(X/S)$ are dual to each other.

Let $g : Y \to X$ be a proper morphism from another algebraic variety. For a relative subvariety Z on Y over S, the *direct image* g_*Z as an algebraic cycle is defined as the following: if $\dim g(Z) = \dim Z$, then $g_*Z = [k(Z) : k(g(Z))]g(Z)$; if $\dim g(Z) < \dim Z$, then $g_*Z = 0$. Here $g(Z)$ is the set-theoretic image of Z, and $[k(Z) : k(g(Z))]$ is the extension degree of function fields. If g is a birational morphism, then g_*Z coincides with the strict transform defined before in Section 1.2.

For a relative t-cycle Z and invertible sheaves $L_1, \dots, L_t$ on X, the *projection formula*

$$(g^*L_1 \cdots g^*L_t \cdot Z) = (L_1 \cdots L_t \cdot g_*Z)$$

holds ([86, p.299]). In this book we often use this formula for $t = 1$ in which case

$$(g^*L \cdot C) = (L \cdot g_*C).$$

{pullback1}

Proposition 1.4.3 ([86, p.304])**.** *Let $f : X \to S$ and $g : Y \to X$ be two proper morphisms and L an invertible sheaf on X.*

(1) If $L \equiv_S 0$, then $g^*L \equiv_S 0$. Therefore, g induces a natural linear map $g^* : N^1(X/S) \to N^1(Y/S)$.

(2) Conversely, if g is surjective and $g^*L \equiv_S 0$, then $L \equiv_S 0$, that is, the pullback map g^* is injective.

Proof. (1) For any relative curve C' on Y,

$$\begin{aligned} &(g^*L \cdot C') = (L \cdot g_*C') \\ = &\begin{cases} [k(C') : k(g(C'))](L \cdot g(C')) & \text{if } \dim g(C') = 1; \\ 0 & \text{if } \dim g(C') = 0. \end{cases} \end{aligned}$$

which implies the statement.

(2) If g is surjective, for any relative curve C on X, there exists a relative curve C' on Y such that $C = g(C')$, which proves the statement. □

Let $h : S \to T$ be a proper morphism, the identity map on $\text{Div}(X)$ induces a surjective linear map $(1/h)^* : N^1(X/T) \to N^1(X/S)$. By taking dual, $(1/h)_* : N_1(X/S) \to N_1(X/T)$ is injective. For proper morphisms $f : X \to S$ and $g : Y \to X$, the composition of $g^* : N^1(X/S) \to N^1(Y/S)$ and $(1/f)^* : N^1(Y/S) \to N^1(Y/X)$ is zero map.

1.5 Cones of curves and cones of divisors

Cones and polytopes contained in finite dimensional vector spaces play important roles in this book. In Chapter 2, morphisms from algebraic varieties can be constructed by using faces of convex cones (the cone theorem). Also in Chapter 3, sequences of rational maps can be analyzed by looking at a cluster of polytopes.

{subsec cones}

1.5.1 Pseudo-effective cones and nef cones

We will define the closed cone generated by numerical equivalence classes of curves in the real vector space $N_1(X/S)$, and the closed cones generated by numerical equivalence classes of effective divisors and nef divisors in the dual space $N^1(X/S)$.

A subset $\mathcal{C}$ in a finite dimensional vector space V is called a *convex cone* if for any $a, a' \in \mathcal{C}$ and $r > 0$, $a + a' \in \mathcal{C}$ and $ra \in \mathcal{C}$ hold. It is called a *closed convex cone* if moreover it is a closed subset.

For an element $u \in V^*$ in the dual space, define $\mathcal{C}_{u \ge 0} = \{v \in \mathcal{C} \mid (u \cdot v) \ge 0\}$. $\mathcal{C}_{u=0}$ and $\mathcal{C}_{u<0}$ can be defined similarly. The *dual closed convex cone* of a closed convex cone $\mathcal{C}$ is defined by

$$\mathcal{C}^* = \bigcap_{v \in \mathcal{C}} V^*_{v \ge 0} = \{u \in V^* \mid \text{for any } v \in \mathcal{C}, (u \cdot v) \ge 0\}.$$

As $\mathcal{C}$ is a closed convex cone, $v \in \mathcal{C}$ is equivalent to $(u \cdot v) \ge 0$ for all $u \in \mathcal{C}^*$. That is, $\mathcal{C} = \mathcal{C}^{**}$.

Given a morphism $f : X \to S$, an invertible sheaf L on X is called *relatively ample*, or *ample over* S, or f*-ample*, if there exists an open covering $\{S_i\}$ of S, positive integers m, N, and *locally closed* immersion $g_i : X_i = f^{-1}(S_i) \to \mathbf{P}^N \times S_i$ such that $L^{\otimes m}|_{X_i} \cong g_i^* p_1^* \mathcal{O}_{\mathbf{P}^N}(1)$ where $p_1 : \mathbf{P}^N \times S_i \to \mathbf{P}^N$ is the first projection. Here the left hand side is the m-th tensor power of L, and the right hand side is the pullback of the invertible sheaf corresponding to a hyperplane section by the first projection and g_i. A Cartier divisor D is called *relatively ample* if its divisorial sheaf is so. A morphism admitting a relatively ample invertible sheaf is called *quasi-projective*. In particular, if all immersions g_i are closed immersion, the morphism is called *projective*.

Here we recall the following useful fact. Let $f : X \to S$ and $g : Y \to X$ be two projective morphisms, L an invertible sheaf on X, and M an invertible sheaf on Y. Suppose that L is f-ample and M is g-ample, then $ng^*L + M$ is ample over S for sufficiently large n ([46, II.7.10]).

In the following, X is assumed to be normal, and the morphism $f : X \to S$ is assumed to be projective.

In general, the convex cone consisting of numerical equivalence classes of all effective $\mathbf{R}$-Cartier divisors is neither closed nor open. This is because there might be infinitely many prime divisors showing up when considering a limit of effective divisors in $N^1(X/S)$. The closure of this cone is denoted by $\overline{\mathrm{Eff}}(X/S)$, which is called the *relative pseudo-effective cone*, in some literature it is denoted by $\mathrm{Psef}(X/S)$. An $\mathbf{R}$-Cartier divisor D is called *relatively pseudo-effective* if its numerical equivalence class $[D]$ is contained in $\overline{\mathrm{Eff}}(X/S)$.

The set of interior points of the closed convex cone $\overline{\mathrm{Eff}}(X/S)$ is called the *relative big cone* and is denoted by $\mathrm{Big}(X/S)$. Recall that in Section 1.2, we introduced the definition of an $\mathbf{R}$-Cartier divisor D being *relatively big* or *f-big*. By Kodaira's lemma later (Corollary 1.5.8), it can be shown that an $\mathbf{R}$-Cartier divisor D is relatively big if and only if its numerical equivalence class $[D]$ is contained in $\mathrm{Big}(X/S)$.

An $\mathbf{R}$-Cartier divisor D is called *relatively nef* or *f-nef* if $(D \cdot C) \geq 0$ for any relative curve C. This is also called *relatively numerically effective*. "Nef" is an abbreviation, but commonly used now. The set of numerical equivalence classes of all nef $\mathbf{R}$-Cartier divisors is a closed convex cone of $N^1(X/S)$, which is denoted by $\overline{\mathrm{Amp}}(X/S)$, and called the *relative nef cone*, sometimes it is denoted by $\mathrm{Nef}(X/S)$.

The set of interior points of the relative nef cone is called the *relative ample cone* and is denoted by $\mathrm{Amp}(X/S)$. An $\mathbf{R}$-Cartier divisor D is called *relatively ample* or *f-ample* if its numerical equivalence class $[D]$ is contained in $\mathrm{Amp}(X/S)$. This notation will be justified by Kleiman's criterion described later (Theorem 1.5.4): for a Cartier divisor D, being f-ample in this sense is equivalent to being f-ample in the original sense. By definition, the sum of a relatively ample $\mathbf{R}$-Cartier divisor and a relatively nef $\mathbf{R}$-Cartier divisor is again a relatively ample $\mathbf{R}$-Cartier divisor.

In the dual space $N_1(X/S)$, the cone of relative curves is the convex cone generated by numerical equivalence classes of all relative curves, which is in general neither open nor closed. Its closure is called the *closed cone of relative curves*, which is denoted by $\overline{\mathrm{NE}}(X/S)$. By definition, the latter one is the dual closed convex cone of the relative nef cone and the relative ample cone:

$$\overline{\mathrm{Amp}}(X/S) = \{u \in N^1(X/S) \mid (u \cdot v) \geq 0 \text{ for all } v \in \overline{\mathrm{NE}}(X/S)\},$$
$$\mathrm{Amp}(X/S) = \{u \in N^1(X/S) \mid (u \cdot v) > 0 \text{ for all } v \in \overline{\mathrm{NE}}(X/S)\}.$$

Remark 1.5.1. The cones $\overline{\text{Amp}}(X/S)$ and $\overline{\text{NE}}(X/S)$ considered here contain interior points, but contain no linear subspaces. This is a consequence of $f : X \to S$ being projective and Kleiman's criterion. For example, $\overline{\text{NE}}(X/S)$ contains no lines since the intersection number of a relatively ample divisor with a curve class in $\overline{\text{NE}}(X/S)$ is alway positive by Theorem 1.5.4. A relatively ample divisor is also called a *polarization* as it gives the positive direction.

The structures of the relative nef cone and the closed cone of relative curves are important themes of this book.

{pullback2}

Proposition 1.5.2 ([86, p.337])**.** *Let $f : X \to S$ and $g : Y \to X$ be two projective morphisms, and L an invertible sheaf on X.*

(1) If L is f-nef, then the pullback g^*L is $f \circ g$-nef.

(2) If g is surjective and g^*L is $f \circ g$-nef, then L is f-nef.

(3) If g is surjective, then

$$g^*\overline{\text{Amp}}(X/S) = \overline{\text{Amp}}(Y/S) \cap g^*N^1(X/S).$$

(4) Assume that g is surjective. If moreover g is a finite morphism*, then*

$$g^*\text{Amp}(X/S) = \text{Amp}(Y/S) \cap g^*N^1(X/S),$$

otherwise

$$g^*\overline{\text{Amp}}(X/S) = (\partial\overline{\text{Amp}}(Y/S)) \cap g^*N^1(X/S).$$

Here ∂ is the boundary of the closed convex cone.

Proof. The proof of (1) and (2) is similar to that of Proposition 1.4.3. (3) follows from (2).

(4) When g is a finite morphism, the pullback of a relatively ample invertible sheaf is again a relatively ample invertible sheaf, hence the former statement follows. On the other hand, when g is not a finite morphism, the pullback of a relatively ample invertible sheaf is never a relatively ample invertible sheaf, hence the latter statement follows from (3). □

It was shown that a non-finite morphism gives a face of the relative nef cone. Conversely, there are cases where it is possible to construct a non-finite morphism from a face of the relative nef cone; this is the contraction theorem in the minimal model theory.

Example 1.5.3. (1) Let X be a smooth projective *complex algebraic surface* and C a curve on X with negative self-intersection $(C^2) < 0$. For any curve C' different from C, the intersection number is always non-negative: $(C \cdot C') \geq 0$. Denote by $\mathcal{C}' \subset N_1(X)$ the closed convex cone generated by the numerical equivalence classes of all curves C' different from C, then the closed cone of curves $\overline{\mathrm{NE}}(X)$ is generated by $\mathcal{C}'$ and $[C]$. $[C] \notin \mathcal{C}'$ since $(C \cdot C') \geq 0$ for all $C' \in \mathcal{C}'$. Therefore, one can see that $[C]$ generates an *extremal ray* of $\overline{\mathrm{NE}}(X)$. Here an extremal ray ℓ in a convex cone $\mathcal{C}$ is a 1-dimensional subcone such that if $\alpha + \alpha \in \ell$ and $\alpha, \alpha \in \mathcal{C}$, then $\alpha, \alpha \in \ell$. Taking a dual, we get a face $F = \overline{\mathrm{Amp}}(X)_{C=0}$ of $\overline{\mathrm{Amp}}(X)$. According to a result of Grauert ([35]), there exists a compact complex analytic surface Y with only normal singularities and a birational morphism $f : X \to Y$ between complex analytic surfaces such that C is contracted to a point. That is, $f(C)$ is a point and there is an isomorphism $f : X \setminus C \to Y \setminus f(C)$. However, Y is in general not an algebraic variety. But according to a result of Artin ([7]), if $C \cong \mathbf{P}^1$, then Y is a projective algebraic surface and f becomes a birational morphism between algebraic varieties. In this sense, it may or may not be possible to construct a morphism from a face of the nef cone.

(2) Let X be an *Abelian variety*, that is, a smooth projective algebraic variety with an algebraic group structure. In this case, any prime divisor D on X is nef, and

$$\mathrm{Amp}(X) = \{v \in N^1(X) \mid (v^n) > 0\}^0.$$

Here $n = \dim X$ and ^{0} on the right hand side means a connected component.

1.5.2 Kleiman's criterion and Kodaira's lemma

In this subsection, we introduce Kleiman's ampleness criterion. Also we prove Kodaira's lemma, which characterizes big divisors.

{Kleiman}

Theorem 1.5.4 (*Kleiman's criterion* ([86]))**.** *For a projective morphism $f : X \to S$ between algebraic varieties, a Cartier divisor D on X is relatively ample if and only if its numerical equivalence class is contained in the relative ample cone* $\mathrm{Amp}(X/S)$.

Remark 1.5.5. Kleiman's criterion is a paraphrase of Nakai's criterion for projectivity and ampleness using the language of cones of divisors instead of intersection numbers with subvarieties. In Kleiman's criterion as well as Nakai's criterion, X is not necessarily assumed to be irreducible or reduced. It is not necessarily assumed to be projective, and whether a proper scheme is projective can be determined by whether $\mathrm{Amp}(X)$ is not empty.

As ampleness is an algebro-geometric property which is non-linear, we can say that it is linearized by Kleiman's criterion using conditions in numerical geometry. This is a typical example of numerical geometry.

An invertible sheaf L on a projective algebraic variety X induces a functional h_L on the dual space $N_1(X)$. By Kleiman's criterion, L is ample if and only if h_L is positive on the closed cone of curves $\overline{\mathrm{NE}}(X)$.

This condition is strictly stronger than the condition that $h_L(C) = (L \cdot C) > 0$ for any curve C. We explain this by the following example:

Example 1.5.6 (Mumford's example)**.** Let Γ be a smooth complex algebraic curve of genus at least 2 and F a locally free sheaf on Γ of rank 2 and of degree 0. The last condition means that $\bigwedge^2 F \equiv \mathcal{O}_\Gamma$. Assume that F is *stable*, that is, $\deg(M) < 0$ for any invertible subsheaf M of F. Such F can be constructed by using unitary representations of the fundamental group $\pi_1(\Gamma)$. In this case, for any surjective morphism $f : C \to \Gamma$ from a smooth projective curve, f^*F is also stable. Let $X = \mathbf{P}(F)$ be the corresponding $\mathbf{P}^1$-bundle over Γ and $L = \mathcal{O}_{\mathbf{P}(F)}(1)$. Let C_0 be a curve on X. If it is not a fiber of f, take $f : C \to \Gamma$ to be the composition of normalization $g : C \to C_0$ and the projection $C_0 \to \Gamma$. In this case, g^*L is an invertible sheaf which is a quotient of f^*F, hence its degree is positive. If C_0 is a fiber of f, then $(L \cdot C_0) = 1$. That is, $(L \cdot C_0) > 0$ holds for any curve C_0 on X. On the other hand, $(L^2) = 0$ since $\deg(F) = 0$, which means that L is not ample.

The following Kodaira's lemma gives a characterization of big divisors.

Theorem 1.5.7 (*Kodaira's lemma*)**.** *(1) A Cartier divisor D on a normal projective algebraic variety X is big if and only if there exists a positive integer m, an ample Cartier divisor A, and an effective Cartier divisor E such that $mD = A + E$.*

(2) For a surjective projective morphism $f : X \to S$ from a normal algebraic variety to a quasi-projective algebraic variety, a Cartier divisor D on X is relatively big if and only if there exists a positive integer m, a relatively ample Cartier divisor A, and an effective Cartier divisor E such that $mD = A + E$.

In other words, big divisors are divisors bigger than ample divisors.

Proof. (1) As ample divisors are big, the condition is sufficient.

Conversely, assume that D is big. Denote $n = \dim X$. Take a very ample divisor A and a general element in its complete linear system $Y \in |A|$. Consider the exact sequence

$$0 \to \mathcal{O}_X(mD - Y) \to \mathcal{O}_X(mD) \to \mathcal{O}_Y(mD|_Y) \to 0.$$

Look at the corresponding exact sequence

$$0 \to H^0(X, mD - Y) \to H^0(X, mD) \to H^0(Y, mD|_Y),$$

as $\dim Y = n - 1$, the dimension of the last term is bounded by cm^{n-1} for some constant c. Since the central term increases by order m^n by bigness, the first term is not 0 for sufficiently large m. Hence there exists an effective divisor E with linear equivalence $mD-Y \sim E$. In this case, $A' = mD-E \sim Y$ is ample and the proof is completed.

(2) As the restriction of relatively ample divisors (resp. effective) divisors on the generic fiber are ample (resp. effective), the condition is sufficient.

Conversely, assume that D is relatively big. By the argument of (1), for a relatively ample Cartier divisor A, there exists a sufficiently large m such that the direct image sheaf $f_*(\mathcal{O}_X(mD-A)) \neq 0$. Take a sufficiently ample Cartier divisor B on S such that

$$H^0(X, mD - A + f^*B) = H^0(S, f_*(\mathcal{O}_X(mD-A)) \otimes \mathcal{O}_S(B)) \neq 0.$$

Then there exists an effective Cartier divisor E with linear equivalence $mD - A + f^*B \sim E$. In this case, $mD - E \sim A - f^*B$ is relatively ample and the proof is completed. □

As a corollary, together with Kleiman's criterion, the definition of relative big cone is justified.

{equiv definition big}

Corollary 1.5.8. *For a surjective projective morphism $f : X \to S$ from a normal algebraic variety to a quasi-projective algebraic variety, a Cartier divisor D on X is relatively big if and only if the numerical equivalence class $[D]$ is contained in the relative big cone* $\mathrm{Big}(X/S)$.

Proof. By Kleiman's criterion and Kodaira's lemma, D is relatively big if and only if $[D]$ is an interior point of closed convex cone generated by relatively effective divisors. □

Corollary 1.5.9. $\overline{\mathrm{Amp}}(X/S) \subset \overline{\mathrm{Eff}}(X/S)$.

Proof. As ample divisors are big, we have an inclusion of cones $\mathrm{Amp}(X/S) \subset \mathrm{Big}(X/S)$. The conclusion follows by taking closures. □

Kodaira's lemma can be generalized as the following:

{kod lemma R-div}

Corollary 1.5.10. *An $\mathbf{R}$-Cartier divisor D on a normal projective algebraic variety X is big if and only if there exists a positive integer m, an ample $\mathbf{R}$-Cartier divisor A, and an effective $\mathbf{R}$-Cartier divisor E such that $D = A+E$.*

Proof. Assume that $D = A + E$. Then there exists an ample $\mathbf{Q}$-Cartier divisor A' and an effective $\mathbf{R}$-Cartier divisor E', such that we can write $A = A' + E'$, hence D is big.

Conversely, assume that D is big. By the proof of Kodaira's lemma, for sufficiently large m, there exists an ample Cartier divisor A and an effective divisor E, such that $\llcorner mD \lrcorner = A + E$. Since $mD - \llcorner mD \lrcorner$ is effective, the statement is proved. □

Proposition 1.5.11. *Let $f : Y \to X$ be a birational morphism between normal projective algebraic varieties and D an* **R***-Cartier divisor on X. Then D is big if and only if the pullback f^*D is big.*

Proof. For a rational function $h \in k(X) \cong k(Y)$, $\mathrm{div}_X(h) + \llcorner mD \lrcorner \geq 0$ is equivalent to $\mathrm{div}_X(h)+mD \geq 0$. Here the subscript X means taking divisor on X. The latter one is equivalent to $\mathrm{div}_Y(h) + mf^*D \geq 0$, which is then equivalent to $\mathrm{div}_Y(h) + \llcorner mf^*D \lrcorner \geq 0$. Therefore, the natural homomorphism $H^0(X, \llcorner mD \lrcorner) \to H^0(Y, \llcorner mf^*D \lrcorner)$ is bijective, and the statement is concluded. ☐

{nef big +}

Theorem 1.5.12 ([100, Theorem 2.2.16])**.** *Let X be an n-dimensional projective algebraic variety and D a nef* **R***-Cartier divisor. Then D is big if and only if $(D^n) > 0$.*

Proof. If D is big, we can write $D = A + E$ for some ample **Q**-divisor A and effective **R**-divisor E. In this case, since D and A are nef,

$$\begin{aligned}(D^n) &= (D^{n-1} \cdot A) + (D^{n-1} \cdot E) \geq (D^{n-1} \cdot A) \\ &= (D^{n-2} \cdot A^2) + (D^{n-2} \cdot A \cdot E) \geq \cdots \geq (A^n) > 0.\end{aligned}$$

Conversely, to show that B is big provided that D is nef and $(D^n) > 0$, we will show the following slightly generalized statement: if for two nef **R**-Cartier divisors L, M we have $(L^n) > n(L^{n-1} \cdot M)$, then $L - M$ is big. The theorem follows by taking $M = 0$.

Firstly we assume that L, M are ample **Q**-Cartier divisors. We may assume that they are very ample by taking multiples. Taking m general elements $M_i \in |M|$ $(1 \leq i \leq m)$, by the exact sequence

$$0 \to \mathcal{O}_X(m(L-M)) \to \mathcal{O}_X(mL) \to \bigoplus_i \mathcal{O}_{M_i}(mL),$$

and the Riemann–Roch theorem[1.5.2.1], when $m \to \infty$, we have

$$\begin{aligned}&\dim H^0(X, m(L-M)) \\ \geq\ &\dim H^0(X, mL) - \sum_{i=1}^{m} \dim H^0(M_i, mL|_{M_i}) \\ =\ &\frac{(L^n)}{n!}m^n - \sum_{i=1}^{m} \frac{(L^{n-1} \cdot M_i)}{(n-1)!}m^{n-1} + O(m^{n-1}) \\ =\ &\frac{(L^n) - n(L^{n-1} \cdot M)}{n!}m^n + O(m^{n-1}).\end{aligned}$$

[1.5.2.1] in original text, it says "by the Serre vanishing", but I think this is not needed

Here note that we have the estimate $O(m^{n-1})$ since the dimension of $H^0(M_i, mL|_{M_i})$ is of order $O(m^{n-2})$ and independent of the choice of M_i. Therefore, $L - M$ is big.

Then we consider the general case. We may take two sufficiently small ample **R**-Cartier divisors H, H' such that $H' - H$ is big and $L + H$ and $M + H'$ are ample **Q**-Cartier divisors. Here H, H' can be taken sufficiently small in the sense that $((L+H)^n) > n((L+H)^{n-1} \cdot (M+H'))$ holds. Then we already showed that $L + H - M - H'$ is big, which implies that $L - M$ is big. □

We can investigate how cones of divisors are changed under birational maps:

Lemma 1.5.13. *Let $\alpha : X \dashrightarrow X'$ be a birational map between* **Q***-factorial normal varieties which is isomorphic in codimension* 1 *and $f : X \to S$ and $f' : X' \to S$ projective morphisms with $f = f' \circ \alpha$.*

(1) α induces an isomorphism $\alpha_* : N^1(X/S) \to N^1(X'/S)$ between real linear spaces.

(2) $\alpha_*(\overline{\mathrm{Eff}}(X/S)) = \overline{\mathrm{Eff}}(X'/S)$.

(3) If α is not an isomorphism, then $\alpha_*(\mathrm{Amp}(X/S)) \cap \mathrm{Amp}(X'/S) = \emptyset$.

Proof. (1) Since α is isomorphic in codimension 1, there is a 1-1 correspondence between prime divisors on X, X'. Hence $Z^1(X) \cong Z^1(X')$.

Take a divisor D on X and take its strict transform $D' = \alpha_* D$. Applying the desingularization theorem described in the next subsection, there exists a smooth algebraic variety W and projective birational morphisms $g : W \to X$, $g' : W \to X'$, such that we can write $g^*D = (g')^*D' + E$ where $g_*E = 0$, $g'_*E = 0$. Assume that $D \equiv_S 0$, then $g^*D \equiv_S 0$.

In the following we will show that $D' \equiv_S 0$. We may assume that $E \neq 0$ otherwise it is obvious. Write $E = E^+ - E^-$ into the positive part and negative part. If $E^+ \neq 0$, by the negativity lemma (Lemma 1.6.3), there exists a curve C contracting by g' such that $(E^+ \cdot C) < 0$ and $(E^- \cdot C) \geq 0$. On the other hand, $((g')^*D' \cdot C) = (D' \cdot g'_*C) = 0$ and $(g^*D \cdot C) = 0$, a contradiction. We can get a contraction similarly if $E^- \neq 0$.

(2) follows from (1) as the strict transform of an effective divisor is again effective.

(3) As the intersection is an open cone, if the intersection is non-empty, there exists a relatively ample divisor D on X such that $\alpha_* D$ is a relatively ample divisor on X'. Since α is isomorphic in codimension 1, for any integer m, $\alpha_* : f_*\mathcal{O}_X(mD) \to f'_*\mathcal{O}_{X'}(mD')$ is an isomorphism. Therefore,

$$X = \mathrm{Proj}_S \left(\bigoplus_{m=0}^{\infty} f_*\mathcal{O}_X(mD) \right) \cong \mathrm{Proj}_S \left(\bigoplus_{m=0}^{\infty} f'_*\mathcal{O}_{X'}(mD') \right) = X',$$

and α is an isomorphism. □

1.6 The Hironaka desingularization theorem

The desingularization theorem is established by Hironaka for algebraic varieties in characteristic 0. Although it is expected that the same theorem holds for positive characteristics and mixed characteristics, it is only shown in dimension 2 and for positive characteristics in dimension 3, while it remains open in general case. The Hironaka desingularization theorem, as well as the Kodaira vanishing theorem explained in the next section, is a very important theorem in characteristic 0. Here we introduce the desingularization theorem ([47]) without proof.

{HDT}

Theorem 1.6.1 (*Hironaka desingularization theorem*)**.** *(1) For any algebraic variety X defined over a field of characteristic 0, there exists a smooth algebraic variety Y and a projective birational morphism $f : Y \to X$.*

(2) For any algebraic variety X defined over a field of characteristic 0 and a proper [1.6.0.1] *closed subset B of X, there exists a smooth algebraic variety Y, a normal crossing divisor C on Y, and a projective birational morphism $f : Y \to X$ with the following properties:*

(a) If B is non-empty[1.6.0.2]*, the set-theoretic inverse image $f^{-1}(B)$ is a union of several irreducible components of C.*

(b) The exceptional set $\mathrm{Exc}(f)$ is a union of several irreducible components of C.

For each statement, we can assume further the following properties hold:

(1') f is isomorphic over the smooth locus $\mathrm{Reg}(X) = X \setminus \mathrm{Sing}(X)$ and the exceptional set $\mathrm{Exc}(f)$ coincides with the set-theoretic inverse image $f^{-1}(\mathrm{Sing}(X))$.

(2') f is isomorphic over $\mathrm{Reg}(X, B)$ and the exceptional set $\mathrm{Exc}(f)$ coincides with the set-theoretic inverse image $f^{-1}(\mathrm{Sing}(X, B))$.

A birational morphism with property in (1) is called a *resolution of singularities* of X. A birational morphism with property in (2) is called a *log resolution* of (X, B). For the definition of normal crossing divisors please refer to Section 1.1.

{resolution remark}

Remark 1.6.2. (1) If replacing two conditions for log resolution by the condition that $f^{-1}(B) \cup \mathrm{Exc}(f)$ is a normal crossing divisor, we call it a *log resolution in weak sense.* This is called a log resolution in some literatures. On the other hand, if we assume furthermore that $\mathrm{Exc}(f)$ is the support of an f-ample divisor in condition (b), we call it a *log resolution in strong sense.* In this case, the f-ample divisor supported on $\mathrm{Exc}(f)$ has negative coefficients according to Lemma 1.6.3 below.

[1.6.0.1] I added "proper"
[1.6.0.2] added

(2) The resolution of singularities in the Hironaka desingularization theorem can be obtained by blowing up along smooth centers finitely many times. Since there exists a relatively ample divisor supported on exceptional divisors with negative coefficients for a blowing up along a smooth center, the resolution of singularities obtained in this way is a log resolution in strong sense. By using Theorem 1.6.4, starting from any log resolution, one can construct a log resolution in strong sense by further taking blowing ups along the exceptional set.

(3) In the latter part of the above theorem, a normal crossing divisor is in the sense of Zariski topology, that is, it is a "simple normal crossing divisor". It does not hold for normal crossing divisors in complex analytic sense. For example, take divisor B defined by $x^2 + y^2 z = 0$ in $X = \mathbf{C}^3$. The singular locus of B is the line defined by $x = y = 0$ and B is a normal crossing divisor in complex analytic sense if $z \neq 0$. However, the origin $P = (0,0,0)$ has the so-called *pinch point* singularity, no blowing up which is isomorphic outside P can make B a normal crossing divisor.

(4) The above theorem is proved in Hironaka's original paper ([47]), but it has been shown that there exists a more precise "canonical resolution" in subsequent development. The canonical resolution admits strong functionality such that any local isomorphism of the pair (X, B) lifts to a local isomorphism of (Y, C). However, the canonical resolution is not unique, it is only shown that there exists a universal choice ([48], [12], [149], [150]).

{neg coeff}

Lemma 1.6.3 (Negativity lemma)**.** *Let $f : X \to Y$ be a projective birational morphism between normal algebraic varieties and D $\mathbf{R}$-Cartier divisors on X supported in the exceptional set* $\mathrm{Exc}(f)$*.*

(1) If D is non-zero and effective, then there exists a family of curves C which are contracted by f and cover an irreducible component of D, such that $(D \cdot C) < 0$.

(2) If D is f-nef and non-zero, then the coefficients of D are all negative.

Proof. We may assume that Y is affine. Take i to be the maximal dimension of irreducible components of $f(\mathrm{Supp}(D))$ and $j = \dim X - 2 - i$. Take Y_i by cutting Y by general hyperplane sections i times, and take X_{ij} by cutting $f^{-1}(Y_i)$ by general hyperplane sections j times. Since $i + j = \dim X - 2$, X_{ij} is a normal algebraic surface. Let Y_{ij} be the normalization of $f(X_{ij})$, then f induces a projective birational morphism $f_{ij} : X_{ij} \to Y_{ij}$. Note that $D_{ij} = D|_{X_{ij}}$ is an non-zero effective $\mathbf{R}$-Cartier divisor supported in the exceptional set $\mathrm{Exc}(f_{ij})$.

(1) By the Hodge index theorem, applying Corollary 1.13.2 to $\pi : \tilde{X}_{ij} \to Y_{ij}$ and $\pi^* D_{ij}$, where $\tilde{X}_{ij}$ is a resolution of X_{ij}, we get $(D_{ij})^2 < 0$. In

particular, for every irreducible component C of D_{ij}, $(D_{ij} \cdot C) < 0$. View C as a curve in X, we have $(D \cdot C) < 0$. Note that by construction, C comes from cutting an irreducible component of D by hyperplane sections, so such C is in a family covering an irreducible component of D.

(2) Note that $E|_{X_{ij}}$ appears as an irreducible component of D_{ij}. We may write $D_{ij} = D_{ij}^+ - D_{ij}^-$ in terms of its positive and negative parts. Since D_{ij} is f_{ij}-nef, $(D_{ij}^+)^2 \geq (D_{ij}^+ \cdot D_{ij}) \geq 0$. By the Hodge index theorem (Corollary 1.13.2), $D_{ij}^+ = 0$. Hence coefficients of D_{ij} are negative, and then the coefficient of E in D is negative. As E is taken arbitrarily, coefficients of D are all negative. □

Let X be a smooth algebraic variety and B a normal crossing divisor on X. A smooth subvariety Z is called a *permissible center* with respect to (X, B) if the following is satisfied: for the local ring $\mathcal{O}_{X,P}$ at every point $P \in X$, there exists a regular system of parameters $z_1, \dots, z_n$ and integers r, s, t, such that the equations of B, Z are $z_1 \cdots z_r = 0$, $z_s = \cdots = z_t = 0$ respectively. Here, $0 \leq r \leq n$ and $0 \leq s \leq t \leq n$, but there is no specific relation between r and s, t.

The blowing up $f : Y \to X$ along a permissible center Z with respect to (X, B) is called a *permissible blowing up*. In this case, the exceptional set E is a smooth prime divisor on Y and coincides with the set-theoretic inverse image $f^{-1}(Z)$. The sum $C = f_*^{-1}B + E$ is a normal crossing divisor on Y. We have $K_Y = f^*K_X + (t - s)E$ and $f^*B = f_*^{-1}B + \max\{r - s + 1, 0\}E$.

The desingularization theorem also contains the following statement:

ational morphism control}

Theorem 1.6.4 ([47]). *Let X be a smooth algebraic variety defined over a field of characteristic 0, B a normal crossing divisor on X, and $f : Y \to X$ a proper birational morphism from another smooth algebraic variety Y. Then there exists a sequence of blowing ups $f_i : X_i \to X_{i-1}$ ($i = 1, \dots, n$) and a birational morphism $g : X_n \to Y$ with the following properties:*

(1) $X = X_0$ and $f \circ g = f_n \circ \cdots \circ f_1$.

(2) f_i is a permissible blowing up with respect to (X_{i-1}, B_{i-1}). Here $B = B_0$ and the normal crossing divisor B_i on X_i is defined inductively by $B_i = f_{i*}^{-1}B_{i-1} + \mathrm{Exc}(f_i)$.

1.7 The Kodaira vanishing theorem

The Kodaira vanishing theorem holds only in characteristic 0. There are counterexamples in positive characteristics [125]. Vanishing theorems and extension theorems are indispensable tools for the minimal model theory over fields of characteristic 0. Here we introduce the Kodaira vanishing theorem ([88]) without proof.

Theorem 1.7.1 (*Kodaira vanishing theorem*)**.** *Let X be a smooth complex algebraic variety and D an ample divisor on X. Then for any positive integer p, $H^p(X, K_X + D) = 0$. Here K_X is the canonical divisor of X.*

The Kodaira vanishing theorem is a theorem in complex differential geometry established for compact complex manifold X. Let L be a holomorphic line bundle. L is always endowed with a C^∞ Hermitian metric h. The curvature of the corresponding connection of h determines a C^∞ $(1,1)$-form on X. In this case, the following holds by the *Kodaira embedding theorem*:

Theorem 1.7.2 ([89])**.** *Let X be a compact complex manifold and L a line bundle with a Hermitian metric h. If the curvature $\sqrt{-1}\Theta$ is positive definite, then X has a projective complex algebraic variety structure and L is a line bundle corresponding to an ample divisor.*

We have the following comparison:

Algebraic geometry $\Rightarrow$ Complex differential geometry $\Rightarrow$ Numerical geometry
Ample divisor $\Rightarrow$ Positive curvature line bundle $\Rightarrow$ Numerically positive divisor

The feature of the Kodaira vanishing theorem is that canonical divisor appears in the argument and it provides a more accurate vanishing comparing to the Serre vanishing theorem below. Hence it has applications to geometry. To be applied in higher dimensional algebraic geometry, the Kodaira vanishing theorem is greatly generalized and used in many directions, as will be discussed later.

Remark 1.7.3. The Kodaira vanishing theorem is originally proved for algebraic varieties defined over complex numbers, but it holds also for algebraic varieties defined over any field in characteristic 0, as a field in characteristic 0 finitely generated over the prime field $\mathbf{Q}$ can be always embedded into $\mathbf{C}$.

Theorem 1.7.4 (*Serre vanishing theorem* [133], [46, III.5.2])**.** *Let X be a projective scheme, L an ample sheaf on X, and F a coherent sheaf on X. Then there exists a positive integer m_0 such that for any integer $m \geq m_0$, the following hold:*

(1) $F \otimes L^{\otimes m}$ is generated by global sections.

(2) For any positive integer p, $H^p(X, F \otimes L^{\otimes m}) = 0$.

The Serre vanishing theorem holds without conditions on characteristics of the base field and singularities of X. It has much more applicability than the Kodaira vanishing theorem, but it is weaker.

The log version of the Kodaira vanishing theorem can be proved by the adjunction formula ([124]):

Corollary 1.7.5. *Let X be a smooth projective algebraic variety defined over a field of characteristic 0, B a normal crossing divisor on X, and an ample divisor D on X. Then for any positive integer p, $H^p(X, K_X + B + D) = 0$.*

Proof. We do induction on the dimension n of X and the number r of prime divisors of B. If $r = 0$, this is just the Kodaira vanishing theorem. If $r > 0$, take a prime divisor B_1 of B, denote $B' = B - B_1$ and $C = B'|_{B_1}$. By the adjunction formula, we get the following exact sequence

$$0 \to \mathcal{O}_X(K_X + B' + D) \to \mathcal{O}_X(K_X + B + D) \to \mathcal{O}_{B_1}(K_{B_1} + C + D|_{B_1}) \to 0.$$

By induction hypothesis, for any positive integer p, $H^p(X, K_X + B' + D) = H^p(B_1, K_{B_1} + C + D|_{B_1}) = 0$. This concludes the statement. □

1.8 The covering trick

The *covering trick* is a classical method to construct new algebraic varieties from a given one by using cyclic coverings. However in this method, the new constructed algebraic variety may have singularities even if the given one is smooth. Therefore, we describe how to construct a covering without creating new singularities.

Firstly, we describe the construction of cyclic covering. Let X be an algebraic variety over an algebraically closed field k, h a rational function on X, and m a positive integer coprime to the characteristic of k. When $k = \mathbf{C}$, m can be taken arbitrarily. Consider the extension of function fields $K = k(X)[h^{1/m}]$, take Y to be the normalization of X in K with the natural map $f : Y \to X$. The extension Y/X is a Galois extension as a cyclic group is Galois, the extension degree $m' = [k(Y) : k(X)]$ is a divisor of m. Y can be constructed as the following. Assume that X is covered by affine open subsets $U_i = \mathrm{Spec}(A_i)$. The fractional field of A_i is the function field $k(X)$. Take B_i to be the normalization of A_i in K, then Y is obtained by gluing affine varieties $\mathrm{Spec}(B_i)$.

Example 1.8.1. Let X be a smooth complex algebraic variety, D a divisor on X, and s a global section of $\mathcal{O}_X(mD)$. The divisor of s and the divisor of the rational function h corresponding to s is related by

$$\mathrm{div}(s) = \mathrm{div}(h) + mD.$$

Here $\mathrm{div}(s)$ is an effective divisor and in general $\mathrm{div}(h)$ is not effective and has poles along D.

Assume that $B = \mathrm{div}(s)$ is reduced and it is a smooth subvariety of X. Consider Y to be the cyclic covering of X induced by h. In this case, Y is smooth and $f : Y \to X$ is a finite morphism branched along B. Here D is not contained in the branch locus. In fact, for any point P in B, take a regular system of parameters $z_1, \ldots, z_n$ such that $B = \operatorname{div}(z_1)$, then the regular system of parameters of Q over P can be taken as $z_1^{1/m}, z_2, \ldots, z_n$.

One should be careful that if $B = \operatorname{div}(s)$ has singularities, then Y has singularities correspondingly. When the support of B is a normal crossing divisor, Y has at worst *toric singularities*, which is easier to handle. This will be discussed later.

We can produce a more useful covering by considering the *Kummer covering*, a generalization of cyclic covering.

{covering}

Theorem 1.8.2 ([55]). *Let X be a smooth projective algebraic variety defined over an algebraically closed field*[1.8.0.1] *of characteristic* 0 *and B a normal crossing divisor on X. Fix a positive integer m_i for each irreducible component B_i of B. Then there exists a smooth projective algebraic variety Y and a finite morphism $f : Y \to X$ with the following properties:*

(1) The set-theoretic inverse image $C = f^{-1}(B)$ is a normal crossing divisor.

(2) For each i, there exists a reduced divisor C_i such that the pullback of B_i as a divisor can be written as $f^*B_i = m_iC_i$. Here a reduced divisor is a divisor with all coefficients equal to 1*.*

(3) f is a Galois covering and the Galois group G is an abelian group.

One feature of this covering is that it is a finite morphism branched along a normal crossing divisor such that the covering space is again smooth. Note that the branch locus of f is a normal crossing divisor containing B, but they do not coincide in general. Since X is smooth, f is a *flat morphism.*

Proof. Denote $n = \dim X$. Take a very ample divisor A such that $m_iA - B_i$ is very ample for all i. For each i, take n general global sections $s_{ij} \in H^0(X, m_iA - B_i)$ $(j = 1, \ldots, n)$. We may assume that for each i, j, $M_{ij} = \operatorname{div}(s_{ij})$ is smooth and $\sum_{i,j} M_{ij} + \sum_i B_i$ is a normal crossing divisor.

Take the rational function h_{ij} corresponding to s_{ij} and take $f_{ij} : Y_{ij} \to X$ to be the normalization of X in $k(X)[h_{ij}^{1/m_i}]$. It is easy to see that the branch locus is $M_{ij} + B_i$ and the ramification index is m_i.

Take $f : Y \to X$ to be the normalization of the fiber product of all $f_{ij} : Y_{ij} \to X$. In other words, Y is just the normalization of X in the field $k(X)[h_{ij}^{1/m_i}]_{ij}$. We will check that this Y satisfied the required properties.

For any point P in X, denote by B_{i_l} $(l = 1, \ldots, r)$ and $M_{j_m k_m}$ $(m = 1, \ldots, s)$ the irreducible components of $\sum_{i,j} M_{ij} + \sum_i B_i$ containing P. Note that $r + s \le \dim X = n$.

[1.8.0.1] we need algebraically closed, which is not assumed in original text.

If $r = 0$, that is, P is not contained in the support of B, then by construction, Y_{ij} is smooth over a neighborhood of P, and there is nothing to prove. So we may assume that $r \geq 1$.

By the numbers of M_{ij}, for each i_l, there exists at least one p_l such that $M_{i_l p_l}$ does not contain P. Denote $\bar{h}_{j_m k_m} = h_{j_m k_m}/h_{i_l p_l}$ if $j_m = i_l$; otherwise $\bar{h}_{j_m k_m} = h_{j_m k_m}$. In this case,

$$h_{i_1 p_1} h_A^{m_{i_1}}, \dots, h_{i_r p_r} h_A^{m_{i_r}}, \bar{h}_{j_1 k_1}, \dots, \bar{h}_{j_s k_s}$$

is a part of a regular system of parameters of $\mathcal{O}_{X,P}$, where h_A is a local equation of the divisor A. The localization $Y \times_X \operatorname{Spec} \mathcal{O}_{X,P}$ is étale over the normalization of $\operatorname{Spec} \mathcal{O}_{X,P}$ in

$$\begin{aligned} &k(X)[h_{i_1 p_1}^{1/m_{i_1}}, \dots, h_{i_r p_r}^{1/m_{i_r}}, h_{j_1 k_1}^{1/m_{j_1}}, \dots, h_{j_s k_s}^{1/m_{j_s}}] \\ = \ &k(X)[h_{i_1 p_1}^{1/m_{i_1}} h_A, \dots, h_{i_r p_r}^{1/m_{i_r}} h_A, \bar{h}_{j_1 k_1}^{1/m_{j_1}}, \dots, \bar{h}_{j_s k_s}^{1/m_{j_s}}]. \end{aligned}$$

Therefore Y is smooth. □

The covering in the above theorem preserves smoothness by adding branch locus artificially. The covering below is a natural construction for **Q**-Cartier Weil divisor which is not Cartier.

Proposition 1.8.3. *Let X be a normal algebraic variety defined over an algebraically closed field of characteristic* 0 *and D a divisor on X. Assume that for some positive integer r, rD is Cartier and moreover $\mathcal{O}_X(rD) \cong \mathcal{O}_X$. Take r to be such a minimal one, then there exists a Galois finite morphism $f : Y \to X$ from a normal algebraic variety whose Galois group is the cyclic group of degree r, such that f is étale in codimension one and f^*D is a Cartier divisor on Y.*

Proof. Fix an everywhere non-zero global section s of $\mathcal{O}_X(rD)$. The corresponding rational function h satisfies $\operatorname{div}_X(h) = -rD$. Take Y to be the normalization of X in the function field extension $L = k(X)[h^{1/r}]$. L is a field as r is minimal. Then $-f^*(D) = \operatorname{div}_Y(h^{1/r})$ is Cartier. It is easy to see that f is étale over the locally free locus of $\mathcal{O}_X(D)$, and in particular, f is étale over $X \setminus \operatorname{Sing}(X)$. □

Such $f : Y \to X$ is called the *index* 1 *cover* of the divisor D. In particular, if $D = K_X$, it is called the *canonical cover.*

Remark 1.8.4. (1) This covering is not unique, it depends on the choice of s. Take another global section s', there is a nowhere zero function u such that $s' = us$. The normalization of X in $k(X)[u^{1/r}]$ gives an étale covering $X' \to X$, and the base change to X' gives an isomorphism $Y \times_X X' \cong Y' \times_X X'$. Here Y' is the cyclic covering obtained by s'. Therefore, this covering is unique up to étale base changes.

(2) Fix a point $P \in X$, take r_P to be the minimal positive number such that $r_P D$ is Cartier in a neighborhood of P, then $f^{-1}(P)$ consists of r/r_P points by construction. In particular, f is étale over the points where D is Cartier.

1.9 Generalizations of the Kodaira vanishing theorem

According to [82], we generalize the Kodaira vanishing theorem to different directions in order to apply to higher dimensional algebraic geometry. The generalized vanishing theorems will be used as one key point of the proof in each part of this book.

In this section, we always assume that the base field is of characteristic 0.

Firstly, we extend the Kodaira vanishing theorem to **R**-divisors:

{R-div vanishing 1}

Theorem 1.9.1. *Let X be a smooth projective algebraic variety and D an ample **R**-divisor on X such that the support of $\ulcorner D \urcorner - D$ is a normal crossing divisor. Then for any positive integer p, $H^p(X, K_X + \ulcorner D \urcorner) = 0$.*

Here we prove the following equivalent theorem:

{R-div vanishing 2}

Theorem 1.9.2. *Let X be a smooth projective algebraic variety, B an **R**-divisor on X with coefficients in $(0, 1)$ and supported on a normal crossing divisor, and D an integral divisor on X. Assume that $D - (K_X + B)$ is an ample **R**-divisor. Then for any positive integer p, $H^p(X, D) = 0$.*

Proof. Write $B = \sum b_i B_i$. Here B_i are prime divisors and $\sum B_i$ is a normal crossing divisor. As ampleness is an open condition, for each i take fraction n_i/m_i $(0 < n_i < m_i)$ sufficiently close to b_i, such that $D - (K_X + \sum(n_i/m_i)B_i)$ is an ample **Q**-divisor. In the following we may assume that $B = \sum(n_i/m_i)B_i$.

Taking the covering $f : Y \to X$ as in Theorem 1.8.2 for irreducible components B_i of B with positive integers m_i. By construction, f^*B is a divisor with integral coefficients. As the Galois group G of f acts as automorphism of Y, $-f^*(K_X + B)$ is G-invariant and the invertible sheaf $\mathcal{O}_Y(K_Y - f^*(K_X + B))$ admits a G-action. Since f is flat, the direct image sheaf $f_*\mathcal{O}_Y(K_Y - f^*(K_X + B)))$ is a locally free sheaf with a G-action and the G-invariant part $L = (f_*\mathcal{O}_Y(K_Y - f^*(K_X + B)))^G$ is an invertible sheaf. Hence L can be written as the form of divisorial sheaf $\mathcal{O}_X(E)$. In order to determine E, we only need to look at the generic point of the branched divisor.

Firstly, any prime divisor not contained in B is not an irreducible component of E. In fact, for any finite Galois covering $g : W \to Z$ between smooth varieties with Galois group G, we have a natural isomorphism

$(g_*\omega_W)^G \cong \omega_Z$, which means that over $U = X \setminus B$, $L|_U = (f_*\mathcal{O}_Y(K_Y - f^*(K_X + B)))^G|_U = (f_*\mathcal{O}_Y(K_Y - f^*(K_X)))^G|_U = \mathcal{O}_U$.

For the generic point P of B_i, set x_1 to be the regular parameter of the discrete valuation ring $\mathcal{O}_{X,P}$. Then for a point Q on Y over P, $y_1 = f^*x_1^{1/m_i}$ is a regular parameter and the invertible sheaf $\mathcal{O}_Y(K_Y - f^*(K_X + B))$ is generated by the section $y_1^{-(m_i-1)+n_i}$. Since $0 < n_i < m_i$, G-invariant sections are generated by 1. Therefore, it turns out that $E = 0$. In summary, $L = (f_*\mathcal{O}_Y(K_Y - f^*(K_X + B)))^G = \mathcal{O}_X$.

As the pullback of an ample divisor by a finite map is ample, the pullback $f^*(D-(K_X+B))$ is again ample. By the Kodaira vanishing theorem, for any positive integer p, $H^p(Y, K_Y + f^*(D - (K_X + B))) = 0$. As f is finite, there is no higher direct image, hence $H^p(X, f_*\mathcal{O}_Y(K_Y + f^*(D-(K_X+B)))) = 0$. As the G-invariant part is a direct summand, $H^p(X, D) = 0$. □

Next, we prove the relative version:

{rel R-div vanishing 2}

Theorem 1.9.3. *Let X be a smooth algebraic variety, B an $\mathbf{R}$-divisor on X with coefficients in $(0, 1)$ and supported on a normal crossing divisor, D an integral divisor on X, and $f : X \to S$ a projective morphism to another algebraic variety. Assume that $D-(K_X+B)$ is a relatively ample $\mathbf{R}$-divisor. Then for any positive integer p,*

$$R^p f_*(\mathcal{O}_X(D)) = 0.$$

We will prove the following equivalent theorem:

{rel R-div vanishing 1}

Theorem 1.9.4. *Let X be a smooth algebraic variety, $f : X \to S$ a projective morphism to another algebraic variety, and D a relatively ample $\mathbf{R}$-divisor on X such that the support of $\ulcorner D \urcorner - D$ is a normal crossing divisor. Then for any positive integer p,*

$$R^p f_*(\mathcal{O}_X(K_X + \ulcorner D \urcorner)) = 0.$$

Proof. As the statement is local on S, we may assume that S is affine. Replacing the integral part of D by a linear equivalent one while keeping $\ulcorner D \urcorner - D$ unchanged, we may assume that the support of D is a normal crossing divisor. However, D is not necessarily effective. We may assume that D is a $\mathbf{Q}$-divisor as ampleness is an open condition.

Shrinking S if necessary, we can find a sufficiently large m such that mD is an integral divisor and there exists a closed immersion $g : X \to \mathbf{P}^N \times S$ such that $\mathcal{O}_X(mD) \cong g^*p_1^*\mathcal{O}_{\mathbf{P}^N}(1)$, where p_1 is the first projection.

Next, take projective algebraic variety $\bar{S}$ to be the compactification of S, and take $\bar{X}$ to be the normalization of closure of X in $\mathbf{P}^N \times \bar{S}$. Note that the projective morphism $\bar{f} : \bar{X} \to \bar{S}$ and the finite morphism $\bar{g} : \bar{X} \to \mathbf{P}^N \times \bar{S}$ are naturally induced.

Here $\bar{X}$ is possibly singular, the extension of D is a $\mathbf{Q}$-Cartier divisor $\bar{D}$ defined by $\mathcal{O}_{\bar{X}}(m\bar{D}) \cong \bar{g}^* p_1^* \mathcal{O}_{\mathbf{P}^N}(1)$. Since $\bar{D}$ is relatively ample over $\bar{S}$, we can choose an ample $\mathbf{Q}$-Cartier divisor A_1 on $\bar{S}$ such that $\bar{D} + \bar{f}^* A_1$ is ample. As S is affine, we may assume that the support of A_1 is contained in $\bar{S} \setminus S$.

Take $h : Y \to \bar{X}$ to be a log resolution of the pair $(\bar{X}, \bar{D} + \bar{f}^* A_1)$ in strong sense. As X is smooth and the support of D is a normal crossing divisor, h can be assumed to be identity over X. We may choose a $\mathbf{Q}$-Cartier divisor A_2 supported in the exceptional set of h such that $\bar{D}' = h^* \bar{D} + h^* \bar{f}^* A_1 + A_2$ is ample. By construction, the support of $\bar{D}'$ is a normal crossing divisor, and by Theorem 1.9.1, for any positive integer p, $H^p(Y, K_Y + \ulcorner \bar{D}' \urcorner) = 0$. Note that the support of $h^* \bar{f}^* A_1 + A_2$ is contained in $Y \setminus X$.

Consider the following spectral sequence:

$$E_2^{p,q} = H^p(\bar{S}, R^q(\bar{f} \circ h)_*(\mathcal{O}_Y(K_Y + \ulcorner \bar{D}' \urcorner))) \Rightarrow H^{p+q}(Y, K_Y + \ulcorner \bar{D}' \urcorner).$$

For any positive integer m_1, replacing A_1 by $m_1 A_1$, the above argument still works. When m_1 is sufficiently large, by the Serre vanishing theorem, for any positive integer p and any integer q,

$$H^p(\bar{S}, R^q(\bar{f} \circ h)_*(\mathcal{O}_Y(K_Y + \ulcorner \bar{D}' \urcorner))) = 0.$$

Also the coherent sheaf $R^q(\bar{f} \circ h)_*(\mathcal{O}_Y(K_Y + \ulcorner \bar{D}' \urcorner))$ is generated by global sections.

By the spectral sequence, when $q > 0$, $H^0(\bar{S}, R^q(h \circ \bar{f})_*(\mathcal{O}_Y(K_Y + \ulcorner \bar{D}' \urcorner))) = 0$. Therefore, $R^q(\bar{f} \circ h)_*(\mathcal{O}_Y(K_Y + \ulcorner \bar{D}' \urcorner)) = 0$. We conclude the theorem by restricting on S. □

The next lemma shows that the conditions as KLT and LC defined later are birational properties:

{log dis}

Lemma 1.9.5. *Let $f : Y \to X$ be a proper birational morphism between smooth algebraic varieties and B, C $\mathbf{R}$-divisors on X, Y supported on normal crossing divisors such that $f^*(K_X + B) = K_Y + C$. Then coefficients of B are all contained in $(-\infty, 1)$ if and only if so are coefficients of C.*

Also the same holds for the condition that all coefficients are contained in $(-\infty, 1]$. Moreover, in this case, assume that the irreducible components of B with coefficient exactly 1 are disjoint, then coefficients of $C - f_*^{-1} B$ are all contained in $(-\infty, 1)$.

Proof. As $B = f_* C$, if coefficients of C are all contained in $(-\infty, 1)$, then coefficients of B are all contained in $(-\infty, 1)$.

Conversely, assume that coefficients of B are all contained in $(-\infty, 1)$. Firstly we consider the case that f is a permissible blowing up with respect to (X, B). Set $B = \sum b_i B_i$. Suppose that the center Z of the blowing up is of codimension r and contained in $B_1, \ldots, B_s$. Note that $r \geq s$. The coefficient e of the exceptional divisor E of f in C is given by

$$e = \sum_{j=1}^{s} b_j + 1 - r.$$

As $b_j < 1$, we have $e < 1$. Since coefficients of other prime divisors of C coincide with those of B, coefficients of C are all contained in $(-\infty, 1)$.

The general case can be proved by Theorem 1.6.4 and induction on numbers of permissible blowing ups. The later part can be proved similarly. □

We can also prove the following lemma which will be used later:

{lemma P not LC}

Lemma 1.9.6. *Fix an n-dimensional pair (X, B) and a point P. Take effective Cartier divisors $D_1, \ldots, D_n$ passing through P such that P is an irreducible component of $\bigcap D_i$. Then there exists a log resolution $f : Y \to (X, B + \sum D_i)$ such that if we write $K_Y + C = f^*(K_X + B + \sum D_i)$, then there exists an irreducible component C_1 of C with coefficient at least 1 and $f(C_1) = \{P\}$.*

Proof. We may assume that X is affine. Write $D_i = \operatorname{div}(h_i)$ where h_i are regular functions on X. Define the morphism $h : X \to Z = \mathbf{A}^n$ by $h = (h_1, \ldots h_n)$. By the assumption, h is quasi-finite in a neighborhood of P. Take $E_1, \ldots, E_n$ to be coordinate hyperplanes of Z, and $h^*E_i = D_i$ by construction. Take $g : Z' \to Z$ to be the blowing up at the origin and F the exceptional divisor, we get $g^*(K_Z + \sum E_i) = K_{Z'} + F + \sum g_*^{-1} E_i$. As differential forms on Z with poles along $\sum E_i$ can be pullback by h, $h^*(K_Z + \sum E_i) \leq K_X + B + \sum D_i$. By taking a log resolution $f : Y \to (X, B + \sum D_i)$ properly, we may assume that the exceptional set contains an irreducible divisor C_1 over F, and this satisfies the requirements. □

Using the relative version of the vanishing theorem, it is easy to show the following generalization:

l R-div nef big vanshing}

Theorem 1.9.7 ([82, Theorem 1.2.3])**.** *Let X be a smooth algebraic variety, $f : X \to S$ a projective morphism to another algebraic variety, and D a relatively nef and relatively big $\mathbf{R}$-divisor on X such that the support of $\ulcorner D \urcorner - D$ is a normal crossing divisor. Then for any positive integer p,*

$$R^p f_*(\mathcal{O}_X(K_X + \ulcorner D \urcorner)) = 0.$$

Proof. We can write $D = A + E$ for some relatively ample $\mathbf{R}$-Cartier divisor A and effective $\mathbf{R}$-Cartier divisor E. For any positive number ϵ, $D - \epsilon E = (1-\epsilon)D + \epsilon A$ is relatively ample.

Take $g : Y \to X$ to be a log resolution of $(X, D + E)$ in strong sense, and $h : Y \to S$ is the composition with f. We can choose a sufficiently small effective **R**-divisor A' supported on the exceptional set of g such that $-A'$ is g-ample and $D' = g^*(D - \epsilon E) - A'$ is h-ample. By Theorem 1.9.4, for any positive integer p,

$$R^p h_*(\mathcal{O}_Y(K_Y + \ulcorner D' \urcorner)) = R^p g_*(\mathcal{O}_Y(K_Y + \ulcorner D' \urcorner)) = 0.$$

By the spectral sequence

$$E_2^{p,q} = R^p f_*(R^q g_*(\mathcal{O}_Y(K_Y + \ulcorner D' \urcorner))) \Rightarrow R^{p+q} h_*(\mathcal{O}_Y(K_Y + \ulcorner D' \urcorner)),$$

$R^p f_*(g_*(\mathcal{O}_Y(K_Y + \ulcorner D' \urcorner))) = 0$ holds for $p > 0$.

Take ϵ and A' to be sufficiently small, then $\ulcorner D' \urcorner = \ulcorner g^* D \urcorner$. Take $B = \ulcorner D \urcorner - D$ and $g^*(K_X + B) = K_Y + C$, by Lemma 1.9.5, coefficients of C are less than 1. Therefore, by

$$g^*(K_X + \ulcorner D \urcorner) = g^*(K_X + B + D) = K_Y + C + g^* D \leq K_Y + \ulcorner g^* D \urcorner$$

(here note that $C + g^* D$ is integral by construction) and $g_*(K_Y + \ulcorner g^* D \urcorner) = K_X + \ulcorner D \urcorner$, we have

$$g_*(\mathcal{O}_Y(K_Y + \ulcorner D' \urcorner)) = \mathcal{O}_X(K_X + \ulcorner D \urcorner),$$

which proves the theorem. Here the last inequality of the first equation is because $C + g^* D$ is an integral divisor and $\mathrm{Supp}(C) \subset \mathrm{Supp}(g^* D)$. □

Higher dimensional algebraic variety got great developed since the following result was proved:

Corollary 1.9.8 (*Kawamata–Viehweg vanishing theorem*, [57], [148]). *Let X be a smooth projective algebraic variety and D a nef and big* **R**-*divisor on X such that the support of $\ulcorner D \urcorner - D$ is a normal crossing divisor. Then for any positive integer p,*

$$H^p(X, K_X + \ulcorner D \urcorner) = 0.$$

1.10 KLT singularities

We can define various singularities for a pair (X, B) where X is a normal algebraic variety and B is an **R**-divisor on X. B is called the *boundary* of the pair for historical reasons. These singularities appear naturally in the minimal model theory. Vanishing theorems can be also generalized to these singularities. The characteristic of the base field is always assumed to be 0 if not specified.

Firstly, we define KLT condition. This is a very natural condition corresponding to L^2 condition in complex analysis. It does not depend on the choice of log resolution. Furthermore, it is easy to handle since it satisfies so-called "open condition" in the sense that it is stable under *perturbation* of divisors. KLT condition defines a category in which the minimal model theory works most naturally and easily.

For simplicity, sometimes we denote a pair (X, B) and a morphism $f : X \to S$ together by a morphism $f : (X, B) \to S$.

{KLT}

Definition 1.10.1. A pair (X, B) is *KLT* (short for *kawamata log terminal*) if it satisfies the following conditions:

(1) $K_X + B$ is $\mathbf{R}$-Cartier.

(2) Coefficients of B are contained in $(0, 1)$.

(3) There exists a log resolution $f : Y \to (X, B)$ such that if we write $f^*(K_X + B) = K_Y + C$, then the coefficients c_j of $C = \sum c_j C_j$ are contained in $(-\infty, 1)$. Here, C_j are distinct prime divisors.

Condition (1) is necessary in order to define the $\mathbf{R}$-divisor C in condition (3). The support of C is contained in the union of set-theoretic inverse image of the support of B and the exceptional set of f, which is a normal crossing divisor. Coefficients c_j of C play an important role in higher dimensional algebraic geometry. $-c_j$ is called the *discrepancy coefficient*, and $1 - c_j$ is called the *log discrepancy coefficient*.

Historically, KLT singularity is just called *log terminal singularity* in [60].

Condition (3) in the definition of KLT does not depend on the choice of log resolution:

{KLT indep log res}

Proposition 1.10.2. *Assume (X, B) satisfies conditions (1), (2) in Definition 1.10.1 and there exists a log resolution $f : Y \to (X, B)$ in weak sense satisfies condition (3). Then (X, B) is KLT. Moreover, for any log resolution $f' : Y' \to (X, B)$ in weak sense, condition (3) in Definition 1.10.1 holds.*

Proof. For two log resolutions $f_1 : Y_1 \to X$, $f_2 : Y_2 \to X$, there exists a third log resolution $f_3 : Y_3 \to X$ dominating them. That is, there exist morphisms $g_i : Y_3 \to Y_i$ $(i = 1, 2)$ such that $f_3 = f_i \circ g_i$. Therefore the statement follows from Lemma 1.9.5. □

The following proposition is obvious:

{KLT obvious}

Proposition 1.10.3. *(1) A pair (X, B) is KLT if and only if there exists an open covering $\{X_i\}$ of X such that pairs $(X_i, B|_{X_i})$ are all KLT.*

(2) Let (X, B) be a KLT pair and B' another effective $\mathbf{R}$-divisor such that $B \geq B'$ and $B - B'$ is $\mathbf{R}$-Cartier, then (X, B') is again KLT.

(3) *When X is a normal complex analytic variety, we can define complex analytic KLT condition similarly by using complex analytic resolution of singularities. When X is a complex algebraic variety, for a pair (X, B), the algebraic KLT condition and analytic KLT condition coincide.*

Remark 1.10.4. Take regular functions $h_1, \dots, h_r$ on polydisk $X = \Delta^n = \{(z_1, \dots, z_n) \in \mathbf{C}^n \mid |z_i| < 1\}$, and write the corresponding divisors by $B_i = \operatorname{div}(h_i)$. Take real numbers $b_i \in (0,1)$. Then $(X, B = \sum b_i B_i)$ is KLT if and only if the function $h = \prod |h_i|^{-b_i}$ is L^2 everywhere.

In fact, the integrability condition L^2 can be studied via resolution of singularities. When the support of B is a normal crossing divisor, the absolute value of a regular function with poles along B satisfies L^2 condition if and only if coefficients of B are in $(-\infty, 1)$, which is exactly the KLT condition.

We introduce quotient singularities as an important example of KLT pairs.

An algebraic variety X is said to have only *quotient singularities* if it is a quotient of a smooth algebraic variety in an étale neighborhood of each point P. That is, there exists a neighborhood U of P, an étale morphism $g : V \to U$ such that $P \in g(V)$, and a smooth algebraic variety $\tilde{V}$ with a finite group action G, such that $V \cong \tilde{V}/G$.

{type quot sing}

Example 1.10.5. Fix a positive integer r and integers $a_1, \dots, a_n$. Define the action of cyclic group $G = \mathbf{Z}/(r)$ on affine space $\tilde{X} = \mathbf{A}^n$ by $z_i \to \zeta^{a_i} z_i$. Here $(z_1, \dots, z_n)$ are coordinates of $\tilde{X}$ and ζ is a primitive r-th root of 1. Then the quotient space $X = \tilde{X}/G$ has only quotient singularities. The image P_0 of the origin might or might not be an isolated singularity, depending on the choice of a_i. X is said to have a quotient singularity of *type* $\frac{1}{r}(a_1, \dots, a_n)$ at P_0.

Proposition 1.10.6. *For an algebraic variety X with only quotient singularities, the pair $(X, 0)$ is KLT.*

Proof. As discrepancy coefficients remain unchanged under étale morphisms, we may assume that X is a global quotient variety. That is, there is a smooth algebraic variety $\tilde{X}$ and a finite group G such that $X = \tilde{X}/G$. It is not hard to see that K_X **Q**-Cartier, in fact, X is **Q**-factorial. Take a log resolution $f : Y \to X$ and write $f^* K_X = K_Y + C$. Take $\tilde{Y}$ to be the normalization of Y in the function field $k(\tilde{X})$ and $\tilde{f} : \tilde{Y} \to \tilde{X}$, $\pi_Y : \tilde{Y} \to Y$ the induced maps, write $\tilde{f}^* K_{\tilde{X}} = K_{\tilde{Y}} + \tilde{C}$. Take a prime divisor E on Y contained in the exceptional set of f, take a prime divisor $\tilde{E}$ on $\tilde{Y}$ such that $\pi_Y(\tilde{E}) = E$. Denote coefficients of $E, \tilde{E}$ is $C, \tilde{C}$ by $c, \tilde{c}$ respectively, denote the ramification index of $\tilde{E}$ with respect to π_Y by e, then we have

$$ce = \tilde{c} + e - 1.$$

Here $\tilde{c} \leq 0$ as $\tilde{X}$ is smooth, hence $c < 1$. □

KLT pairs admit the following special log resolutions. We call it the *very log resolution* in this book.

{very log}

Proposition 1.10.7. *Let (X, B) be a KLT pair consisting of a normal algebraic variety and an* $\mathbf{R}$*-divisor. Then there exists a log resolution $f : Y \to (X, B)$ such that if we write $f^*(K_X + B) = K_Y + C$, then the support of the* $\mathbf{R}$*-divisor $C' = \max\{C, 0\}$ is a disjoint union of smooth prime divisors.*

Proof. Fix a log resolution $f_0 : Y_0 \to (X, B)$ and write $f_0^*(K_X + B) = K_{Y_0} + C_0$. Choose two prime divisor in C_0 and blowing up along their intersection, we get $g_1 : Y_1 \to Y_0$. The composition with f_0 gives a new log resolution $f_1 : Y_1 \to Y_0$. We will show that a very log resolution can be constructed by repeating this operation.

Write $C_0 = \sum c_{0j} C_{0j}$. Fix a positive number n such that $c_{0j} \le 1 - \frac{1}{n}$ for all j.

For any log resolution $f : Y \to (X, B)$, write $f^*(K_X + B) = K_Y + C$ and $C = \sum c_j C_j$. Note that it is easy to see that $c_j \le 1 - \frac{1}{n}$ for all j by induction and Theorem 1.6.4. We define the sequence of integers $r(f) = (r_3(f), \ldots, r_{2n}(f))$ by the formula

$$r_i(f) = \#\{(j_1, j_2) \mid j_1 < j_2, C_{j_1} \cap C_{j_2} \neq \emptyset, 2 - \frac{i}{n} < c_{j_1} + c_{j_2} \le 2 - \frac{i-1}{n}\}.$$

We consider the lexicographical order for sequences of integers. As $r_i \ge 0$, the set of sequences of non-negative integers $(r_3, \ldots, r_{2n})$ satisfies the *DCC* (short for *descending chain condition*). That is, there is no infinite strictly decreasing chain.

For a given f, take the minimal i such that $r_i(f) \neq 0$ and take a pair (j_1, j_2) realizing it. That is, $j_1 < j_2$, $C_{j_1} \cap C_{j_2} \neq \emptyset$, and $2 - \frac{i}{n} < c_{j_1} + c_{j_2} \le 2 - \frac{i-1}{n}$. Take $g : Y' \to Y$ to be the blowing up along $Z = C_{j_1} \cap C_{j_2}$, denote $f' = f \circ g$ and write $(f')^*(K_X + B) = K_{Y'} + C'$. The coefficient e of the exceptional divisor $E = \mathrm{Exc}(g)$ in C' satisfies $1 - \frac{i}{n} < e \le 1 - \frac{i-1}{n}$. Note that for $l = 1, 2$, $e + c_{j_l} \le 1 - \frac{i}{n}$ as $c_{j_l} \le 1 - \frac{1}{n}$. The construction of Y' kills the intersection of C_{j_1} and C_{j_2}, and produces the intersections of E with the strict transforms of C_{j_1} and C_{j_2}. But these two intersections do not contribute to $r_k(f')$ for $k \le i$ by the coefficient computation. Therefore $r_k(f') = r_k(f) = 0$ for $k < i$ and $r_i(f') = r_i(f) - 1$, which means that $r(f') < r(f)$. Since there is no infinite strictly decreasing chain for the sequence $r(f)$, eventually we can get a log resolution f such that $r_i(f) = 0$ for all i. This conclude the proof. □

Note that the log resolution in the above proposition is obtained by blowing up repeatedly, it does not satisfies condition (2') in Theorem 1.6.1. Also the proposition can not be extended to DLT pairs.

We can generalize the vanishing theorem to KLT pairs:

{KLT vanishing}

Theorem 1.10.8 ([82, 1.2.5])**.** *Let X be a normal algebraic variety, $f : X \to S$ a projective morphism, B an $\mathbf{R}$-divisor on X, and D a $\mathbf{Q}$-Cartier integral divisor on X. Assume that (X, B) is KLT and $D - (K_X + B)$ is relatively nef and relatively big. Then for any positive integer p, $R^p f_*(\mathcal{O}_X(D)) = 0$.*

Proof. Take a log resolution $g : Y \to (X, B)$, denote $h = f \circ g$ and write $g^*(K_X + B) = K_Y + C$. Note that $g^*D - (K_Y + C)$ is h-nef and h-big. Here note that coefficients of g^*D are not necessarily integers. By Theorem 1.9.7, for any positive integer p, $R^p g_*(\mathcal{O}_Y(\ulcorner g^*D - C \urcorner)) = R^p h_*(\mathcal{O}_Y(\ulcorner g^*D - C \urcorner)) = 0$. Hence $R^p f_*(g_*(\mathcal{O}_Y(\ulcorner g^*D - C \urcorner))) = 0$.

For a rational function $r \in k(X) \cong k(Y)$, if $\operatorname{div}_X(r) + D \geq 0$, then $\operatorname{div}_Y(r) + g^*D \geq 0$. In this case, $\operatorname{div}_Y(r) + \llcorner g^*D \lrcorner \geq 0$, and then $\operatorname{div}_Y(r) + \ulcorner g^*D - C \urcorner \geq 0$ since coefficients of C are contained in $(-\infty, 1)$. This shows that the natural inclusion

$$g_*(\mathcal{O}_Y(\ulcorner g^*D - C \urcorner)) \subset g_*(\mathcal{O}_Y(\ulcorner g^*D \urcorner)) = \mathcal{O}_X(D)$$

is in fact an identity $g_*(\mathcal{O}_Y(\ulcorner g^*D - C \urcorner)) = \mathcal{O}_X(D)$ and the proof is finished. □

Remark 1.10.9. In a KLT pair (X, B), X has only *rational singularity*, and hence is *Cohen–Macaulay* ([82, 1.3.6]). This asserts that KLT is a “good” singularity. On the other hand, LC to be introduced in the next section is not “good” in this sense. This fact will not be used in this book.

Consider a pair (X, B) consisting of a normal algebraic variety and an effective $\mathbf{R}$-divisor such that $K_X + B$ is $\mathbf{R}$-Cartier. In Chapter 2, we introduce the multiplier ideal sheaf in order to measure how far this pair is from being KLT. The set of points $P \in X$ in whose neighborhood the pair (X, B) is not KLT is a closed subset of X. It is called the *non-KLT locus* of the pair (X, B). The support of the multiplier ideal sheaf coincides with the non-KLT locus. Also, the vanishing theorem can be generalized using multiplier ideal sheaves (see Section 2.11).

1.11 LC, DLT, PLT singularities

KLT condition is easy to handle since it is an open condition with respect to change of coefficients of divisors. However, in the minimal model theory, as it is necessary to consider the limits of divisors, it is necessary to consider the closed condition so-called LC condition. Among LC pairs, we call by $\overline{\text{KLT}}$ pairs the pairs obtained by increasing boundaries of KLT pairs. The property of general LC pairs is not so good, but for $\overline{\text{KLT}}$ pairs it is possible to have similar discussions as for KLT pairs. Besides, there are conditions called DLT and PLT between KLT and LC, which are a little complicated but very useful. In this book, we develop the minimal model theory mainly for DLT pairs. The characteristic of the base field is always assumed to be 0 if not specified.

1.11.1 Various singularities

{LC}

Definition 1.11.1. A pair (X, B) is *LC* (short for *log canonical*) if it satisfies the following conditions:

(1) $K_X + B$ is $\mathbf{R}$-Cartier.

(2) Coefficients of B are contained in $(0, 1]$.

(3) There exists a log resolution $f : Y \to (X, B)$ such that if we write $f^*(K_X + B) = K_Y + C$, then the coefficients c_j of $C = \sum c_j C_j$ are contained in $(-\infty, 1]$. Here, C_j are distinct prime divisors.

When (X, B) is an LC pair, (X, B) is said to have *log canonical singularities*. Same as Proposition 1.10.2, condition (3) above does not depend on the choice of log resolution. Also same statement as in Proposition 1.10.3 holds for LC pairs.

Example 1.11.2. The property of singularities of LC pairs is not always good. Let Z be a smooth projective n-dimensional algebraic variety such that $K_Z \sim 0$, i.e., $\omega_Z \cong \mathcal{O}_Z$. Take an ample invertible sheaf L and take the total space $Y = \mathrm{Spec}_Z(\bigoplus_{m=0}^{\infty} L^{\otimes m})$ of the dual sheaf L^*. Y admits an $\mathbf{A}^1$-bundle structure over Z. Denote $X = \mathrm{Spec}(\bigoplus_{m=0}^{\infty} H^0(Z, L^{\otimes m}))$, there is a natural birational morphism $f : Y \to X$ which maps the 0-section E of $Y \to Z$ to a point $P = f(E)$. By the adjunction formula $(K_Y + E)|_E \sim K_E \sim 0$, we have $K_Y + E \sim 0$ and $K_X \sim 0$, which implies that $f^*K_X \sim K_Y + E$. Hence $(X, 0)$ is LC. The higher direct images of $\mathcal{O}_Y$ are supported on the singular point P of X:

$$R^p f_* \mathcal{O}_Y \cong \bigoplus_{m=0}^{\infty} H^p(Z, L^{\otimes m}) \supset H^p(Z, \mathcal{O}_Z).$$

For $p = n$, $H^n(Z, \mathcal{O}_Z) \neq 0$, hence X is not a *rational singularity*. Moreover, if Z is an Abelian variety, then for $0 < p \leq n$, the right hand side is not 0, and X is not *Cohen-Macaulay*.

As the property of singularities of LC pairs is not always good, we consider intermediate conditions:

{DLT,PLT}

Definition 1.11.3. A pair (X, B) is *DLT* (short for *divisorially log terminal*) if it satisfies the following conditions:

(1) $K_X + B$ is $\mathbf{R}$-Cartier.

(2) Coefficients of B are contained in $(0,1]$.

(3) There exists a log resolution $f : Y \to (X, B)$ such that if we write $f^*(K_X + B) = K_Y + C$, then the coefficients c_j of $C = \sum c_j C_j$ are contained in $(-\infty, 1)$ for those C_j contained in the exceptional set of f.

A pair (X, B) is *PLT* (short for *purely log terminal*) if it satisfies the above conditions (1), (2) and the following condition (3'):

(3') For any log resolution $f : Y \to (X, B)$, if we write $f^*(K_X + B) = K_Y + C$, then the coefficients c_j of $C = \sum c_j C_j$ are contained in $(-\infty, 1)$ for those C_j contained in the exceptional set of f.

{WLT=DLT}

Remark 1.11.4. (1) In [82], a condition called *WLT* (short for *weak log terminal*) is considered. The definition of WLT is by assuming further that the log resolution in condition (3) of definition of DLT is in strong sense. By using similar argument as in Proposition 1.10.2, it can be shown that DLT and WLT are in fact equivalent ([144]). In this book, we will just use DLT rather than WLT.

(2) For a log resolution $f : Y \to X$ of (X, B), when considering the relation $f^*(K_X + B) = K_Y + C$, sometimes we just write "a morphism $f : (Y, C) \to (X, B)$".

{LC example}

Example 1.11.5. (1) Take affine space $X = \mathbf{A}^n$ and coordinates hyperplanes $B_1, \ldots, B_n$, denote $B = \sum b_i B_i$. Then (X, B) is KLT (resp. PLT, DLT) if and only if $0 \le b_i < 1$ for all i (resp. $0 \le b_i \le 1$ for all i and $b_i < 1$ except for at most one i, $0 \le b_i \le 1$ for all i). Furthermore, DLT and LC coincide.

(2) Let $X = \mathbf{A}^2/\mathbf{Z}_2$ be the quotient of the 2-dimensional affine space $\mathbf{C}^2$ with coordinates x, y by the degree 2 cyclic group action $(x, y) \mapsto (-x, -y)$. That is, it is a quotient singularity of type $\frac{1}{2}(1,1)$. This singularity is the same as the ordinary double point in Example 1.1.4(1). Denote the image of coordinate axes in X to be B_1, B_2 and take $B = \sum b_i B_i$. Then (X, B) is KLT (resp. PLT, LC) if and only if $0 \le b_i < 1$ for all i (resp. $0 \le b_{i_1} \le 1$ for one i_1 and $0 \le b_{i_2} < 1$ for the other i_2, $0 \le b_i \le 1$ for all i). Furthermore, PLT and DLT coincide. In fact, the blowing up $f : Y \to X$ along the image of the origin $(0,0)$ is a log resolution. The exceptional set E is isomorphic to $\mathbf{P}^1$, $f^*B_i = f_*^{-1}B_i + \frac{1}{2}E$, and $f^*K_X = K_Y$. So the claim can be checked easily.

(3) Take $X = \mathbf{A}^2$ to be the 2-dimensional affine space with coordinates x, y and a prime divisor $D = \operatorname{div}(x^2 + y^3)$. We determine the necessary and sufficient condition for the pair (X, dD) to be KLT, LC for a real number d (see Figure **??**).

We can construct a log resolution of (X, dD) in the following way. Firstly, take the blowing up $f_1 : Y_1 \to X$ along the origin $P_0 = (0, 0)$, the exceptional set E_1 is a prime divisor isomorphic to $\mathbf{P}^1$. The strict transform $D_1 = f_{1*}^{-1} D$ is smooth, E_1 and D_1 intersect at one point P_1. Take the blowing up $f_2 : Y_2 \to Y_1$ along P_1, the exceptional set E_2 is a prime divisor isomorphic to $\mathbf{P}^1$. 3 smooth prime divisors E_2, $D_2 = f_{2*}^{-1} D_1$, $E_1' = f_{2*}^{-1} E_1$ intersect at one point P_2. Take the blowing up $f_3 : Y = Y_3 \to Y_2$ along P_2, the exceptional set E_3 is a prime divisor isomorphic to $\mathbf{P}^1$. The union of 4 smooth prime divisors E_3, $D_3 = f_{3*}^{-1} D_2$, $E_1'' = f_{3*}^{-1} E_1'$, $E_2' = f_{3*}^{-1} E_2$ is a normal crossing divisor. The composition $f : Y \to X$ is a log resolution of (X, dD). We have $K_Y = f^* K_X + E_1'' + 2E_2' + 4E_3$ and $f^* D = D_3 + 2E_1'' + 3E_2' + 6E_3$. Therefore the pair (X, dD) is KLT (resp. LC) if and only if $0 \le d < 5/6$ (resp. $0 \le d \le 5/6$).

(4) Consider the example in Examples 1.1.4(2) or 1.2.4(2). In addition to prime divisors D_1, D_2, consider prime divisors D_3, D_4 defined by $y = z = 0$ or $y = w = 0$. Note that $D_3 + D_4$ and K_X are Cartier divisors. Take $B = \sum_{i=1}^4 D_i$ and consider the pair (X, B). Take the resolution of singularities $f : X' \to X$ as in Example 1.2.4(2), then $B' = \sum_{i=1}^4 f_*^{-1} D_i$ is a normal crossing divisor. As f is isomorphic in codimension 1, $f^*(K_X + B) = K_{X'} + B'$. The pair (X, B) is LC but not DLT. Here, as the exceptional set of f is not a normal crossing divisor, f is a log resolution in weak sense, but not a log resolution in the sense of Theorem 1.6.1(2). In order to obtain a log resolution, we need to do further blowing up on X' along the exceptional locus of f and that will induce an exceptional divisor with log discrepancy coefficient 1. The blowing up $g : Y \to X$ along the origin $(0, 0, 0, 0)$ is a log resolution. The exceptional set E is a prime divisor isomorphic to $\mathbf{P}^1 \times \mathbf{P}^1$, and $C = \sum_{i=1}^4 g_*^{-1} D_i + E$ is a normal crossing divisor. Since $g^*(K_X + B) = K_Y + C$, (X, B) is LC.

(5) Take a smooth projective algebraic curve C of genus 1 and two line bundles L_1, L_2 of negative degrees. Take Y to be the total space of the vector bundle $L = L_1 \oplus L_2$, denote by C_1, C_2, E the subvarieties of Y corresponding to subbundles $L_1 \oplus 0$, $0 \oplus L_2$, $0 \oplus 0$ respectively. Note that $E \cong C$. Denote $X = \mathrm{Spec}(\bigoplus_{m=0}^{\infty} H^0(C, L^{\otimes -m}))$, there is a natural birational morphism $f : Y \to X$ which maps E to a point $P = f(E)$. Write $B_i = f(C_i)$. Then $f^*(K_X + B_1 + B_2) = K_Y + C_1 + C_2$ and the pair $(X, B_1 + B_2)$ is not DLT but LC. In fact X is not a rational singularity. The pairs $(B_i, 0)$ are also LC.

{barKLT} We introduce one more definition:

Definition 1.11.6. A pair (X, B) is $\overline{KLT}$ if it satisfies the following conditions:

(1) (X, B) is LC.

(2) There is another effective $\mathbf{R}$-divisor B' such that $B' \leq B$ and (X, B') is KLT.

In this situation, for any positive real number $\epsilon < 1$, $(X, (1-\epsilon)B + \epsilon B')$ is KLT. That is, $\overline{\text{KLT}}$ is the limit of KLT. For this reason, different from general LC pairs, it shares similar properties as a KLT pair.

Toric varieties provide good examples (see [85], [33] for details).

Proposition 1.11.7. *Take an algebraic torus T, and $T \subset X$ a* toric variety*, that is, a T-equivariant open immersion into a normal algebraic variety with a T-action. Consider the complement set $B = X \setminus T$ as a reduced divisor. Then the following statements hold:*

(1) The pair (X, B) is LC. Moreover, it is $\overline{KLT}$.

(2) X is $\mathbf{Q}$-factorial if and only if the corresponding fan *consists of simplicial cones.*

Proof. (1) Take a T-equivariant resolution of singularities $f : Y \to X$ such that $f^{-1}(T) \cong T$ and $C = Y \setminus f^{-1}(T)$ is a normal crossing divisor.

Denote $\dim T = n$, take coordinates $x_1, \dots, x_n$ by pulling back by the standard embedding $T \subset \mathbf{A}^n$. The regular differential form $\theta = dx_1/x_1 \wedge \dots \wedge dx_n/x_n$ on T can be extended to a logarithmic differential form on X and gives a non-zero global section of $K_X + B$. Therefore $K_X + B \sim 0$. Similarly θ extends to a non-zero global section of $K_Y + C$. Therefore, $f^*(K_X + B) = K_Y + C$, and hence (X, B) is LC. As T is affine, there exists an effective Cartier divisor B' with the same support as B. For a sufficiently small real number $\epsilon > 0$, $(X, B - \epsilon B')$ is KLT, and hence (X, B) is $\overline{\text{KLT}}$.

(2) We may assume that X is affine and its fan consists of a single cone σ. Irreducible components B_i of B correspond to points P_i on one dimensional rays of σ. The condition for B_i becoming a $\mathbf{Q}$-Cartier divisor is that there exists a regular function on X such that the corresponding principal divisor is a non-zero multiple of B_i. This is equivalent to that there exists a linear function on σ which takes value 1 at P_i and 0 at all points on other rays, which is equivalent to σ being simplicial. □

The following is a corollary of Lemma 1.9.6.

{P is not LC}

Corollary 1.11.8. *Fix an n-dimensional KLT pair (X, B) and a point P. Take sufficiently general effective Cartier divisors $D_1, \dots, D_n, E$ passing through P and a positive number $1 > \epsilon > 0$. Then there exists a sufficiently small number $\delta > 0$ such that $(X, B + \sum (1-\delta) D_i + \epsilon E)$ is KLT in a punctured neighborhood of P, but not LC at P.*

Proof. As $D_1, \ldots, D_n, E$ are general, take a log resolution $\bar{f} : Y \to (X, B)$ and write $\bar{f}^*(K_X + B) = K_Y + \bar{C}$, we may assume that $\bar{C} + \bar{f}^*(\sum D_i + E)$ is normal crossing outside $\bar{f}^{-1}(P)$. Coefficients of $D_1, \ldots, D_n, E$ in $(X, B + \sum(1-\delta)D_i + \epsilon E)$ is strictly smaller than 1 for $\delta > 0$, hence the pair is KLT in a punctured neighborhood of P.

On the other hand, take the log resolution f and prime divisor C_1 as in Lemma 1.9.6, then the coefficient of C_1 in f^*E is at least 1, and the coefficient of C_1 in $f^*(K_X + B + \sum D_i + \epsilon E)$ is strictly larger than 1. Hence $(X, B + \sum(1-\delta)D_i + \epsilon E)$ is not LC at P for sufficiently small $\delta > 0$. □

1.11.2 The sub-adjunction formula

We will look at the behavior of singularities when restricting a given pair to lower dimensions.

Firstly we show Shokurov's connectedness lemma ([135], [99, Theorem 17.4]), which is a consequence of the vanishing theorem:

{connectedness}

Lemma 1.11.9 (*Connectedness lemma).* *Let (X, B) be a pair of a normal variety and an $\mathbb{R}$-divisor such that $K_X + B$ is* **R**-*Cartier, and let $f : (Y, C) \to (X, B)$ be a log resolution in weak sense. Write $C = C^+ - C^-$ where C^+, C^- are effective* **R**-*divisors with no common components. Then the induced morphism* $\mathrm{Supp}(\llcorner C^+)\lrcorner \to f(\mathrm{Supp}(\llcorner C^+ \lrcorner))$ *has connected fibers.*

Proof. Note that

$$-\llcorner C \lrcorner - (K_Y + C - \llcorner C \lrcorner) \equiv -f^*(K_X + B)$$

is f-nef and f-big. As coefficients of $C - \llcorner C \lrcorner$ are contained in $(0, 1)$, by the vanishing theorem,

$$R^1 f_*(\mathcal{O}_Y(-\llcorner C \lrcorner)) = 0.$$

Since $\llcorner C \lrcorner = \llcorner C^+ \lrcorner - \ulcorner C^- \urcorner$, the natural homomorphism

$$f_*(\mathcal{O}_Y(\ulcorner C^- \urcorner)) \to f_*(\mathcal{O}_{\llcorner C^+ \lrcorner}(\ulcorner C^- \urcorner))$$

is surjective. Since the support of the effective divisor $C^-$is contained in the exceptional set, the natural homomorphism $f_*\mathcal{O}_Y \to f_*(\mathcal{O}_Y(\ulcorner C^- \urcorner))$ is bijective. In the commutative diagram

$$\begin{array}{ccc} \mathcal{O}_X \cong f_*\mathcal{O}_Y & \longrightarrow & f_*\mathcal{O}_{\llcorner C^+ \lrcorner} \\ \downarrow & & \downarrow \\ f_*(\mathcal{O}_Y(\ulcorner C^- \urcorner)) & \longrightarrow & f_*(\mathcal{O}_{\llcorner C^+ \lrcorner}(\ulcorner C^- \urcorner)). \end{array}$$

The left vertical arrow is bijective, the bottom horizontal arrow is surjective, and the right vertical arrow is injective, hence the top horizontal arrow is surjective. We conclude the proof. □

Corollary 1.11.10. *A DLT pair (X, B) is PLT if and only if $\llcorner B \lrcorner$ is a disjoint union of its irreducible components.*

Proof. The sufficiency is easy. Conversely, suppose that two irreducible components B_1, B_2 of $\llcorner B \lrcorner$ intersect. Take a log resolution $f : (Y, C) \to (X, B)$ as in Lemma 1.11.9, then the strict transforms $f_*^{-1}B_1, f_*^{-1}B_2$ are contained in the same connected components of the support of $\llcorner C^+ \lrcorner$. Then there exists an irreducible component of $\llcorner C^+ \lrcorner - f_*^{-1}B_1$ intersecting $f_*^{-1}B_1$. Blowing up along the intersection, the coefficient of the exceptional divisor is 1, which means that (X, B) is not PLT. □

Corollary 1.11.11. *For a DLT pair (X, B), every irreducible component of $\llcorner B \lrcorner$ is normal.*

Proof. We may assume that X is affine. Take H to be an ample divisor on X. Take D to be an irreducible component of $\llcorner B \lrcorner$. Take a log resolution $f : (Y, C) \to (X, B)$ in strong sense. By the definition of DLT, we may assume that the coefficients of exceptional divisors in C are strictly less than 1, note that here we use the fact that DLT is equivalent to WLT (see Remark 1.11.4). Take a sufficiently small effective $\mathbf{Q}$-divisor E supported on the exceptional set of g such that $-E$ is g-ample and $f^*H - E$ is ample on Y.

Write $B = D + \sum b_i B_i$ where B_i are distinct prime divisors, and write $f_*^{-1}B = D' + \sum b_i B_i'$ the strict transform on Y. We can choose a positive integer m such that for every i, the divisorial sheaf $\mathcal{O}_Y(B_i' + m(f^*H - E))$ is generated by global sections. By taking a general global section, we can find a prime divisor $D_i' \sim B_i' + m(f^*H - E)$. Take a sufficiently small positive real number δ and take $C' = C - \delta \sum b_i B_i' + \delta \sum b_i D_i' + m\delta \sum b_i E \sim_{\mathbf{R}} C + m\delta \sum b_i (f^*H)$. Note that the support of C' is a normal crossing divisor as D_i' are general, and the coefficients of $C' - D'$ are less than 1 as δ is sufficiently small. Then we can take $B' = f_*C' = D' + (1 - \delta) \sum b_i B_i + \delta \sum b_i f_* D_i' \sim_{\mathbf{R}} B + m \sum b_i H$. Note that $K_X + B'$ is $\mathbf{R}$-Cartier and $f^*(K_X + B') = K_Y + C'$, which implies that (X, B') is DLT. Also by construction, we have $\llcorner C' \lrcorner = D'$ and $\llcorner B' \lrcorner = D$. Therefore, by Lemma 1.11.9, $D' \to D$ has connected fibers, which means that D is normal. □

Remark 1.11.12. According to this corollary, the irreducible components of $\llcorner B \lrcorner$ have no "self-intersection". For example, if X is a smooth complex algebraic variety and B is a reduced divisor normal crossing in analytic sense but not simple normal crossing, then (X, B) is not DLT. This follows from the definition of normal crossing divisors and log resolutions.

Induction arguments on dimensions using the adjunction formula is compatible with the property of DLT. The reason is the following result:

{SAF}

Theorem 1.11.13 (*Subadjunction formula*)**.** *Let (X, B) be a DLT pair and Z an irreducible component of $\llcorner B \lrcorner$. Then we can naturally define an effective $\mathbf{R}$-divisor B_Z on Z satisfying*

$$(K_X + B)|_Z = K_Z + B_Z,$$

and the pair (Z, B_Z) is again DLT. Moreover, if (X, B) is PLT in a neighborhood of Z, then (Z, B_Z) is KLT.

Proof. Take a log resolution $f : (Y, C) \to (X, B)$ such that coefficients of exceptional prime divisors in C are less than 1. Write $W = f_*^{-1} Z$, $C_W = (C - W)|_W$, and $B_Z = (f|_W)_* C_W$. Here coefficients of C_W are at most 1, so are those of B_Z.

Here we claim that the coefficients of B_Z are contained in $(0, 1]$. To see this, after cutting X by general hyperplanes, we may assume that $\dim X = 2$. In this case, $f : (Y, C) \to (X, B)$ factors through the minimal resolution of X (see Proposition 1.13.8). Hence there exists a pair (Y_1, C_1) and birational morphisms $f_1 : Y \to Y_1$, $f_2 : Y_1 \to X$ such that $f = f_2 \circ f_1$ and $K_{Y_1} + C_1 = f_2^*(K_X + B)$, and moreover $C_1 \geq 0$. Then it is easy to see that $B_Z \geq 0$.

As $(K_Y + C)|_W = K_W + C_W$, we get $(K_X + B)|_Z = K_Z + B_Z$. Hence $K_Z + B_Z$ is $\mathbf{R}$-Cartier. Note that $f|_W$ is a log resolution of (Z, B_Z) and $(f|_W)^*(K_Z + B_Z) = K_W + C_W$.

Recall that every irreducible component of C with coefficient 1 is a strict transform of an irreducible component of $\llcorner B \lrcorner$. Take D to be an irreducible component of C_W with coefficient 1, then D is contained in the intersection of $f_*^{-1} \llcorner B \lrcorner - W$ and W and hence it is not contained in $\mathrm{Exc}(f)$. In fact, if D is contained in $\mathrm{Exc}(f)$, then it is an irreducible component of $\mathrm{Exc}(f)$, which contradicts the fact that $\mathrm{Exc}(f)$ is a normal crossing divisor. Therefore D is not contained in the exceptional set of $f|_W$ and hence (Z, B_Z) is DLT.

The latter part is obvious. □

{SAF remark}

Remark 1.11.14. We may have $B_Z \neq 0$ even if $B = Z$, that is, K_Z might be smaller than expected, and this is why we use the word "sub". For example, consider the quadric surface X defined by $xy = z^2$ in affine space $\mathbf{C}^3$ with coordinates x, y, z and the divisor Z on X defined by $x = z = 0$. Then the pair (X, Z) is DLT and the subadjunction formula is $(K_X + Z)|_Z = K_Z + \frac{1}{2}P$ (see Example 1.3.2).

For a pair (X, B), a subvariety Z of X is called an *LC center* if there exists a log resolution $f : (Y, C) \to (X, B)$ such that there is an irreducible component C_i of $\llcorner C^+ \lrcorner$ with $Z = f(C_i)$.

Lemma 1.11.15. *Fix a log resolution $f : (Y, C) \to (X, B)$ of an LC pair (X, B). The its LC centers consist of images of irreducible components of intersections of several irreducible components of $\llcorner C^+ \lrcorner$.*

Proof. Take the blowing up Y along an irreducible component of the intersection of several irreducible components of $\llcorner C^+ \lrcorner$, we get a new log resolution and the exceptional divisor has coefficient 1 in the new boundary. Hence the image is an LC center. On the other hand, by easy computation, the blowing up along other centers gives an exceptional divisor with coefficient strictly smaller than 1. By Theorem 1.6.4, any log resolution is dominated by a log resolution obtained in this way, which concludes the proof. □

In particular, when (X, B) is DLT, there exists a log resolution $f : (Y, C) \to (X, B)$ with $\llcorner C^+ \lrcorner = f_*^{-1} \llcorner B \lrcorner$, hence an LC center is nothing but an irreducible component of the intersection of several irreducible components of $\llcorner B \lrcorner$. In other words, the reduced part of the boundary obtained by applying the subadjunction formula several times to (X, B) are LC centers.

We extend the vanishing theorem to DLT pairs. Note that the condition "relatively ample" can not be replaced by "relatively nef and big" as DLT is not an open condition.

{DLT vanishing}

Theorem 1.11.16. *Let X be a normal algebraic variety, $f : X \to S$ a projective morphism, B an $\mathbf{R}$-divisor on X, and D a $\mathbf{Q}$-Cartier integral divisor on X. Assume that (X, B) is DLT and $D - (K_X + B)$ is relatively ample. Then for any positive integer p, $R^p f_*(\mathcal{O}_X(D)) = 0$.*

Proof. Take a log resolution $g : (Y, C) \to (X, B)$ in strong sense, denote $h = f \circ g$. By the definition of DLT, we may assume that the coefficients of exceptional divisors in C are strictly less than 1, note that here we use the fact that DLT is equivalent to WLT (see Remark 1.11.4). Take a sufficiently small effective $\mathbf{R}$-divisor A supported on the exceptional set of g such that $-A$ is g-ample, $\llcorner C + A \lrcorner = \llcorner C \lrcorner$, and $g^*D - (K_Y + C + A)$ is h-ample. Take a sufficiently small number $\epsilon > 0$ such that $g^*D - (K_Y + (1 - \epsilon)C + A)$ is again h-ample.

Write $D' - C' = g^*D - ((1 - \epsilon)C + A)$ where D' is a divisor with integral coefficients and C' is an $\mathbf{R}$-divisor with coefficients in $(0, 1)$, in other words, take $D' = \ulcorner g^*D - ((1 - \epsilon)C + A) \urcorner$. Since the support of C' is a normal crossing divisor, by Theorem 1.9.3, for $p > 0$, $R^p g_*(\mathcal{O}_Y(D')) = R^p h_*(\mathcal{O}_Y(D')) = 0$. Therefore, for $p > 0$, $R^p f_*(g_*(\mathcal{O}_Y(D'))) = 0$. Since $g_*D' = D$ by definition and $D' \geq \llcorner g^*D \lrcorner$ as coefficients of $(1 - \epsilon)C + A$ are smaller than 1, we have $g_*(\mathcal{O}_Y(D')) = \mathcal{O}_X(D)$ and the theorem is proved. □

Here we remark that we can avoid using WLT in this proof by applying Lemma 2.1.7 to replace (X, B) by a KLT pair.

1.11.3 Terminal singularities and canonical singularities

We conclude this section by introducing terminal singularities and canonical singularities. These singularities are not considered in the main part of this book. However, they are important in applications and have longer history than KLT, DLT, LC, etc in dimension 3 or higher. Originally 3-dimensional algebraic geometry was successful because these singularities can be classified. But classification of singularities is impossible in higher dimensions, and it is replaced by using pairs and induction on dimensions.

Definition 1.11.17. A normal algebraic variety X is said to have *canonical singularities* if the following conditions are satisfied:

(1) K_X is $\mathbf{Q}$-Cartier.

(2) For a resolution of singularities $f : Y \to X$, if write $f^*K_X = K_Y + C$, then $-C$ is effective.

Furthermore, X is said to have *terminal singularities* if the following is satisfied:

(3) the support of $-C$ coincides with the divisorial part of $\mathrm{Exc}(f)$.

In terms of discrepancy coefficients, terminal singularities (canonical singularities) have all discrepancy coefficients positive (non-negative). It is easy to see that conditions (2) and (3) do not depend on the choice of resolution of singularities. The concept of terminal singularities and canonical singularities can also be extended to pairs.

Definition 1.11.18. A pair (X, B) consisting of a normal algebraic variety X and an effective $\mathbf{R}$-divisor B is said to have *canonical singularities* if the following conditions are satisfied:

(1) $K_X + B$ is $\mathbf{R}$-Cartier.

(2) For any resolution of singularities $f : Y \to X$, if write $f^*(K_Y + B) = K_Y + C$, then $-C + f_*^{-1}B$ is effective.

Furthermore, (X, B) is said to have *terminal singularities* if the following is satisfied:

(3) the support of $-C+f_*^{-1}B$ coincides with the divisorial part of $\mathrm{Exc}(f)$.

It is easy to see that in conditions (2) and (3) it is not sufficient to check for one log resolution.

As will be explained later, discrepancy coefficients are not decreasing under the minimal model program (MMP), hence the MMP preserves singularities. That is, when applying a birational map in MMP to an algebraic variety with certain singularities, we get an algebraic variety with the same type of singularities. In other words, MMP can be considered within the category of varieties having certain singularities. In particular, when considering MMP starting from a smooth algebraic variety, everything is within the category of terminal singularities. Note that 2-dimensional terminal singularities without boundaries are just smooth, that is the reason why singularities are not considered in classical 2-dimensional MMP.

1.12 Minimality and log minimality

{section 1.12}

The minimality in the minimal model theory is defined by the minimality of canonical divisors. Log minimal model is the log version of minimal model, where the log canonical divisor is minimized. The minimal model program (MMP) is a process to find a "minimal model" which is a birational model with good properties for a given algebraic variety.

Firstly, we define "minimality" by the property of singularities and numerical property of canonical divisors:

{minimality}

Definition 1.12.1. (1) A projective morphism $f : X \to S$ from a normal algebraic variety to another algebraic variety is said to be relatively *minimal* over S if it satisfies the following conditions (a), (b). It is said to be relatively *minimal in weak sense* over S if it satisfies the following conditions (a'), (b).

(a) X has $\mathbf{Q}$-factorial terminal singularities.

(a') X has canonical singularities.

(b) K_X is relatively nef over S.

(2) A projective morphism $f : (X, B) \to S$ for a pair consisting of a normal algebraic variety X and an $\mathbf{R}$-divisor B to another algebraic variety is said to be relatively *log minimal* over S if it satisfies the following conditions (a), (b). It is said to be relatively *log minimal in weak sense* over S if it satisfies the following conditions (a'), (b).

(a) X is $\mathbf{Q}$-factorial and (X, B) is DLT.

(a') (X, B) is LC.

(b) $K_X + B$ is relatively nef over S.

The minimality in weak sense defined above leads to the minimality of canonical divisor K_X and log canonical divisor $K_X + B$:

{canonical divisor minimal

Proposition 1.12.2. *(1) Let $f : X \to S$ be a morphism minimal in weak sense. Consider a projective morphism $g : Y \to S$ from another normal algebraic variety and birational projective morphisms $f' : Z \to X$, $g' : Z \to Y$ from a third normal algebraic variety with $f \circ f' = g \circ g'$. Then if K_Y is $\mathbf{Q}$-Cartier, the inequality $(f')^* K_X \le (g')^* K_Y$ holds. That is, K_X is minimal in birational equivalence classes.*

(2) Let $f : (X, B) \to S$ be a morphism log minimal in weak sense. Consider a projective morphism $g : (Y, C) \to S$ from anther pair of a normal algebraic variety and an $\mathbf{R}$-divisor and birational projective morphisms $f' : Z \to X$, $g' : Z \to Y$ from a third normal algebraic variety with $f \circ f' = g \circ g'$. Furthermore, assume the following conditions:

(a) *For each irreducible component B_i of B, the strict transform $C_i = g'_*(f')^{-1}_* B_i$ is an irreducible component of C. If denote coefficients of B_i, C_i to be b_i, c_i, then $b_i \leq c_i$.*

(b) *For each irreducible component C_j of C satisfying $f'_*(g')^{-1}_* C_j = 0$, its coefficent c_j is 1.*

Then if $K_Y + C$ is **R***-Cartier, the inequality $(f')^*(K_X + B) \leq (g')^*(K_Y + C)$ holds. That is, $K_X + B$ is minimal in birational equivalence classes.*

Proof. (1) By the desingularization theorem we may assume that Z is smooth. Write $(f')^* K_X = K_Z + E$, $(g')^* K_Y = K_Z + F$.

Since X has canonical singularities, $-E$ is effective. That is, K_X is smaller than K_Z. So the condition on singularities guarantees the minimality locally. In order to see the global properties, we apply the negativity lemma (Lemma 1.6.3). Write $F - E = G^+ - G^-$ where G^+, G^- are effective **Q**-divisors with no common components. Our goal is to show $G^- = 0$. Suppose that $G^- \neq 0$. As $-E$ is effective, the support of G^- is contained in the support of F, which is contracted by g'. By Lemma 1.6.3, there exists a family of curves C contracted by g' and covering a component of G^-, such that $(G^- \cdot C) < 0$. Note that $((K_Z + F) \cdot C) = 0$ and $(G^- \cdot C) \geq 0$. On the other hand, since K_X is nef,

$$0 \leq ((K_Z + E) \cdot C) = ((E - F) \cdot C) = -(G^+ \cdot C) + (G^- \cdot C) < 0$$

a contradiction. Therefore $G^- = 0$ and $F - E$ is effective.

(2) We may assume that f', g' are log resolutions. Write $(f')^*(K_X + B) = K_Z + E$, $(g')^*(K_Y + C) = K_Z + F$.

Since (X, B) is LC, coefficients of E are at most 1. That is, if denote by $\bar{E}$ the sum of the strict transform $(f')^{-1}_* B$ and all exceptional divisors of f' with coefficient 1, then $K_X + B$ is smaller than $K_Z + \bar{E}$. So LC condition guarantees the minimality locally.

Let us look at the global property. Write $F - E = G^+ - G^-$ where G^+, G^- are effective **R**-divisors with no common components. Our goal is to show $G^- = 0$. Once it is shown that the support of G^- is contracted by g', the conclusion follows exactly as the proof of (1). In order to show that the support of G^- is contracted by g', for any prime divisor R on Z, we are going to show that R is not an irreducible component of G^- if $g'_* R = Q$ is a prime divisor on Y.

If $f'_* R = P$ is a prime divisor on X, by assumption (a), the coefficient of P in B is not greater than that of Q in C. This holds even if P is not a component of B in which case we just formally set the coefficient to be 0. Therefore the coefficient of R in $F - G$ is non-negative and it is not a component of G^-.

If $f'_*R = 0$, by assumption (b), the coefficient of of Q in C is 1 while that of R in E is at most 1. Therefore the coefficient of R in $F - G$ is non-negative and it is not a component of G^-.

□

Remark 1.12.3. (1) In the minimal model theory in classical algebraic surface theory, a minimal model is defined to be the minimal one under the following relation using birational maps: for two smooth projective algebraic surfaces X, Y, we define $X \leq Y$ if there exists a birational morphism $Y \to X$.

However, in dimension 3 or higher, there are examples showing that such definition does not work [27], [26]. Therefore, in the minimal model theory discussed in this book, we consider projective algebraic varieties with singularities, and define the minimal model by the size of canonical divisors; the relation $X \leq Y$ between two birational equivalent algebraic varieties is defined by the inequality $K_X \leq K_Y$. Here the inequality of divisors is by comparing the pullbacks on a birational model: we write $K_X \leq K_Y$ if $f^*K_X \leq g^*K_Y$ for projective birational morphisms $f : Z \to X$, $g : Z \to Y$. The relation $(X, B) \leq (Y, C)$ for log pairs is defined by $f^*(K_X+B) \geq g^*(K_Y+C)$ for projective birational morphisms $f : Z \to X$, $g : Z \to Y$ together with 2 conditions of (2) of the above proposition.

Such kind of change of viewpoint has already been observed in algebraic surfaces in logarithmic situation ([54]). The importance of considering logarithmic situation showed up at that time. Furthermore, extending to the logarithmic situation is indispensable for inductive proof of the existence of minimal models in this book.

(2) Form the above proposition, the minimality in weak sense is equivalent to the minimality of canonical divisors. Furthermore, according to Corollary 3.6.10, minimal models are maximal among minimal models in weak sense under the relation defined by birational morphisms.

Looking at this locally, we can say that: canonical singularities are characterized by the property that the canonical divisors are minimal locally. Furthermore, **Q**-factorial terminal singularities are maximal, among those with canonical divisors minimal locally, under the relation defined by birational morphisms.

For pairs, the log minimality in weak sense is equivalent to the minimality of log canonical divisors. But as a DLT blowing up can be blown up any times, it is impossible to construct a "maximal minimal model". However, if the minimal model is KLT, then we can construct a maximal minimal model by Corollary 3.6.10. This is a pair with **Q**-factorial terminal singularities.

Looking at this locally, we can say that: LC pairs are characterized by the property that the log canonical divisors are minimal locally. Furthermore, by only looking at KLT pairs, $\mathbf{Q}$-factorial terminal pairs are maximal, among pairs with canonical divisors minimal locally, under the relation defined by birational morphisms.

Therefore, the situation requiring $\mathbf{Q}$-factorial terminal singularities can be called "maximality theory" and the situation requiring canonical singularities or LC singularities can be called "minimality theory". A model expected to be obtained using a minimal model program gets into the "maximality theory".

(3) Let $\alpha : X \dashrightarrow Y$ be a birational morphism between projective normal algebraic varieties over S. X, Y are said to be *crepant* or *K-equivalent* to each other if there are birational projective morphisms $f : Z \to X$, $g : Z \to Y$ from a third normal algebraic variety with $g = \alpha \circ f$ such that $f^*K_X = g^*K_Y$. Here the comparison of canonical divisors is by using rational differential forms identified by the birational map. By the above proposition, birational equivalent minimal models are crepant to each other.

Furthermore, given effective $\mathbf{R}$-divisors B, C on X, Y, assume that $K_X + B$, $K_Y + C$ are $\mathbf{R}$-Cartier. Pairs (X, B), (Y, C) are said to be *log crepant* or *K-equivalent* to each other if $f^*(K_X + B) = g^*(K_Y + C)$, or just *crepant* for simplicity. When considering minimal models with boundaries, only being birational is not enough, we should also pay attention to how to define the boundaries. This is settled in Section 2.5.5.

1.13 The curve case and the surface case

In this section, we describe known results such as the finite generation of canonical rings in dimension up to 2. Many of them are special phenomena which only happen in dimension up to 2. In particular, we describe the classification of DLT pairs in dimension 2. We obtain a subadjunction formula which will be useful later. For a DLT pair in general dimensions, its structure in codimension two can be considered by cutting down the dimension by general hyperplanes and reducing to the classification of DLT pairs in dimension 2.

1.13.1 The curve case

Firstly we discuss dimension 1 case briefly. Take an algebraic curve X, that is, a smooth projective 1-dimensional algebraic variety. Denote its genus by g. If $g = 0$, then $X \cong \mathbf{P}^1$ and $R(X, K_X) \cong k$. If $g = 1$, then $K_X \sim 0$ and $R(X, K_X) \cong k[t]$. These cases are simple.

In the following we consider $g \geq 2$. This is equivalent to X being of general type. It is also equivalent to $\deg(K_X) > 0$ since the degree of the canonical divisor K_X is $2g-2$. The plurigenera are given by $\dim H^0(X, mK_X) = (2m-1)(g-1)$ for $m \geq 2$. As K_X is ample, the canonical ring $R(X, K_X)$ is finitely generated and

$$X = \operatorname{Proj} R(X, K_X).$$

X is called a *hyperelliptic curve* if there exists a finite morphism $\pi : X \to \mathbf{P}^1$ of degree 2. The canonical linear system $|K_X|$ is always free, but it is very ample if and only if X is not a hyperelliptic curve. When X is a hyperelliptic curve,

$$|K_X| = \pi^*|\mathcal{O}_{\mathbf{P}^1}(g-1)|$$

and π is the morphism corresponding to $|K_X|$. In this case, $|3K_X|$ is very ample ([46, IV.5]).

To be more specific, if X is not a hyperelliptic curve, then the canonical ring is generated by the degree 1 part $H^0(X, K_X)$ (Max Noether [6, p.117]). On the other hand, if X is a hyperelliptic curve, then degree up to 3 parts are required to generate the canonical ring.

1.13.2 Minimal model of algebraic surfaces

In the following we consider 2-dimensional case. For details please refer to [11]. An *algebraic surface* is a 2-dimensional algebraic variety.

Numerical geometry is particularly effective on algebraic surfaces. This is because the intersection number becomes a symmetric bilinear form as prime divisors are the same as curves. The following powerful theorem is often used in algebraic surface theory.

{HIT}

Theorem 1.13.1 (*Hodge index theorem*, [46, Theorem V.1.9])**.** *Let A, B be Cartier divisors on a proper 2-dimensional algebraic variety X. If $(A^2) > 0$, $(A \cdot B) = 0$, and $B \not\equiv 0$, then $(B^2) < 0$.*

{negative definite}

Corollary 1.13.2. *Let $f : Y \to X$ be a resolution of singularities of an algebraic surface and D a non-zero divisor on Y supported in the exceptional set* $\operatorname{Exc}(f)$. *Then $(D^2) < 0$. Therefore, if exceptional divisors of f are $E_1, \ldots, E_r$, then the matrix of intersection numbers $[(E_i \cdot E_j)]$ is negative definite.*

Proof. We may assume that X is projective. Take an ample divisor H on X, then $(f^*H \cdot f^*H) > 0$ and $(f^*H \cdot D) = 0$. If $D \geq 0$, as Y is projective, $D \neq 0$ implies $D \not\equiv 0$. Therefore $(D^2) < 0$. In general, we can write $D = D^+ - D^-$ in terms of the positive part and the negative part, then $(D^2) \leq (D^+)^2 + (D^-)^2 < 0$. □

The Hodge index theorem can be used even for problems in higher dimensional algebraic geometry, because we can cut by hyperplane sections and reduce to algebraic surfaces (see Lemma 1.6.3).

In general, given a resolution of singularities $f : Y \to X$, the *dual graph* Γ can be constructed from the exceptional set as the following:

(1) Take vertices $v_1, \dots, v_r$ of Γ corresponding to prime divisors $E_1, \dots, E_r$ in $\mathrm{Exc}(f)$.

(2) Join v_i, v_j with an edge if two distinct prime divisors E_i, E_j intersect, and associate the edge with weight $(E_i \cdot E_j)$.

(3) Associate each vertex v_i with weight (E_i^2).

First of all, we recall the minimality of algebraic surfaces. The definition of minimal models in algbraic surface theory is different from that in higher dimensional algebraic geometry. Hence here we use "minimal in the classical sense". Given two smooth algebraic surfaces X, Y, the relation $X \geq Y$ is defined by that there is a projective birational morphism $f : X \to Y$. An algebraic surface minimal under this relation is defined to be *minimal in the classical sense*.

A curve C on a smooth projective algebraic surface X is called a (-1)-*curve* if $C \cong \mathbf{P}^1$ and $(C^2) = -1$. If take a blowing up of a smooth algebraic surface Y at a point P, then the exceptional set is a (-1)-curve. Conversely, a (-1)-curve can be contracted to a smooth curve:

{Castelnuovo}

Theorem 1.13.3 (*Castelnuovo's contraction theorem*, [46, Theorem V.5.7])**.** *For a smooth algebraic surface X and a (-1)-curve C on X, there exists a projective birational morphism $f : X \to Y$ to another smooth algebraic surface, such that $f(C)$ is a point and f induces an isomorphism $X \setminus C \cong Y \setminus f(C)$.*

Minimality is characterized by the absence of (-1)-curve:

Theorem 1.13.4 ([46, Proposition V.5.3])**.** *A smooth algebraic surface X is minimal in the classical sense if and only if there is no (-1)-curve on X.*

Corollary 1.13.5. *For a smooth projective algebraic surface X, its minimal model in the classical sense always exists.*

Proof. In the case that $f : X \to Y$ is a contraction of a (-1)-curve, the Picard number decreases exactly by one: $\rho(X) = \rho(Y) + 1$. As Picard number is always positive, a minimal model in the classical sense can be obtained by taking contractions finitely many times until there is no more (-1)-curve. □

Minimal projective algebraic surfaces in the classical sense are classified into the following 3 types:

(1) A surface with K_X nef.

(2) A $\mathbf{P}^1$-bundle over a curve.

(3) $\mathbf{P}^2$.

In this book, (1) is called a *minimal model*, and (2) or (3) is called a *Mori fiber space*. In case (1), the minimal model is unique. However in cases (2) and (3), the minimal model (in the classical sense) is not unique, so such a model is sometimes said to be *relatively minimal*, but to avoid confusion we will not use this terminology.

Combining the existence of resolution of singularities and Castelnuovo's contraction theorem, we get the *minimal resolution of singularities* of a normal algebraic surface. It is a minimal model in relative setting, which is obtained by considering $\rho(Y/X)$ instead of $\rho(X)$:

Corollary 1.13.6 ([46, Theorem V.5.8])**.** *Let X be a normal algebraic surface. Then among all projective birational map $g : Y \to X$ from a smooth algebraic surface, there exists a unique minimal one in the classical sense.*

We also have the following *minimal log resolution of singularities* which is the log version of minimal resolution of singularities:

Proposition 1.13.7. *Let (X, B) be a pair consisting of a normal algebraic surface and a reduced divisor on it. Then among all projective birational map $g : Y \to X$ from a smooth algebraic surface such that the sum of $f_*^{-1}B$ and all exceptional divisors is a normal crossing divisor, there exists a unique minimal one in the classical sense.*

For a projective algebraic curve C on a smooth algebraic surface X, the following *genus formula* holds ([46, Example V.3.9.2]):

$$(K_X \cdot C) + (C^2) = 2\bar{g} - 2 \geq -2.$$

Here $\bar{g}$ is called the *virtual genus* of C, which is a non-negative integer. Take g to be the genus of the smooth projective curve C^ν obtained from normalization of C, then $\bar{g} \geq g$. The difference $\bar{g} - g$ comes from singularities of C. In particular, the equality holds if and only if C is smooth.

Minimal resolution of singularities is characterized by relative nefness. This coincide with the definition of minimality in this book:

{min resol property}

Proposition 1.13.8. *(1) A projective birational morphism $f : Y \to X$ from a smooth algebraic surface to a normal algebraic surface is the minimal resolution of singularities if and only if K_Y is relatively nef.*

(2) Let $f : Y \to X$ be the minimal resolution of singularities of a normal algebraic surface. If write $f^*K_X = K_Y + C$, then C is effective.

Proof. (1) If there is a (-1)-curve C such that $f(C)$ is a point, then $(K_Y \cdot C) = -1$ and K_Y is not relatively nef.

Conversely, if K_Y is not relatively nef, then there is a curve C such that $(K_Y \cdot C) < 0$ and $f(C)$ is a point. By the Hodge index theorem (Corollary 1.13.2), $(C^2) < 0$. On the other hand, by the genus formula, $(K_Y \cdot C) + (C^2) \geq -2$. Hence we have $((K_Y + C) \cdot C) = -2$ and $(K_Y \cdot C) = (C^2) = -1$. This means that $C \cong \mathbf{P}^1$ and C is a (-1)-curve.

(2) Write $C = C^+ - C^-$ where C^+, C^- are effective divisors with no common components. If $C^- \neq 0$, then $(K_Y \cdot C^-) = -(C^+ \cdot C^-) + (C^- \cdot C^-) < 0$, which contradicts the fact that K_Y is relatively nef. □

For Euler characteristic $\chi(\mathcal{O}_X) = \sum (-1)^i \dim H^i(X, \mathcal{O}_X)$ of a smooth projective algebraic surface X, we have the following *Noether's formula*

$$\chi(\mathcal{O}_X) = \frac{1}{12}((K_X^2) + c_2(X)).$$

Here $c_2(X)$ is the *second Chern class* of the tangent bundle of X, and $-K_X = c_1(X)$ is the *first Chern class*.

1.13.3 Finite generation and the classification of algebraic surfaces

Let us consider the finite generation of canonical rings of smooth projective algebraic surfaces. The important thing here is that canonical rings are invariants under contractions of (-1)-curves: $f^* : R(X', K_{X'}) \cong R(X, K_X)$. Therefore, in the following we consider X to be minimal.

In the classification of minimal models in the classical sense, for a Mori fiber space in case (2) or (3), its canonical ring is just k, and the finite generation is trivial. In the following we just consider case (1). The following is a deep result called the *Kodaira–Enriques classification theory* for algebraic surfaces. In addition, Kodaira also classified (not necessarily algebraic) compact complex surface, but we will not discuss them here ([10]).

The Kodaira dimension $\kappa(X)$ takes value among $0, 1, 2$. When $\kappa(X) = 0$, there exists a positive integer r such that $rK_X \sim 0$. If we take r to be the smallest one with such property, then $r = 1, 2, 3, 4, 6$. In particular, $R(X, K_X) \cong k[t^r]$.

When $\kappa(X) = 1$, there exists a surjective morphism $f : X \to Y$ to a smooth projective algebraic curve such that the generic fiber is an elliptic curve. The following *Kodaira's canonical bundle formula* holds:

$$K_X \sim_{\mathbf{Q}} f^*(K_Y + B).$$

Moreover, $\deg(K_Y + B) > 0$. Here B is a $\mathbf{Q}$-divisor on Y determined by singular fibers of f. Singular fibers are completely classified and the corresponding coefficients of B are determined. Here coefficients of B are not necessarily contained in $(0,1)$. This is because it includes also a part induced from the J-function $J : Y \to \mathbf{P}^1$ coming from the fibers of f. Anyway, there exists a positive integer r such that $rK_X \sim f^*(r(K_Y + B))$ and $R(X, rK_X) \cong R(Y, r(K_Y + B))$. The latter one is finitely generated as $r(K_Y + B)$ is an ample divisor, which implies that $R(X, K_X)$ is finitely generated. Here note that $R(X, K_X)$ is finitely generated if and only if $R(X, rK_X)$ is so.

Consider the case $\kappa(X) = 2$. A minimal model X is of general type if and only if $(K_X^2) > 0$. For $m \geq 2$, by a vanishing theorem of Kodaira type, we have the following plurigenus formula

$$\dim H^0(X, mK_X) = \frac{1}{2}m(m-1)(K_X^2) + \chi(\mathcal{O}_X).$$

We discuss the canonical models. A curve C on X is called a (-2)-*curve* if $C \cong \mathbf{P}^1$ and $(C^2) = -2$. On a minimal surface of general type, a (-2)-curve is characterized by the condition $(K_X \cdot C) = 0$. This is because we get $(C^2) < 0$ from the Hodge index theorem (Corollary 1.13.2) and $(K_X \cdot C) + (C^2) \geq -2$ from the genus formula. According to *Artin's contraction theorem* ([7] or Theorem 1.13.10), we can contract all (-2)-curves by a birational morphism; there exists a birational morphism $g : X \to Y$ to a normal algebraic surface such that the exceptional set of g coincides with the union of all (-2)-curves. Y is called the *canonical model*. The canonical divisor K_Y of Y is a Cartier divisor and $K_X = g^*K_Y$. Therefore there is an isomorphism $g^* : R(Y, K_Y) \cong R(X, K_X)$. Since all the curves intersecting K_X are contracted by g, we can see that K_Y is ample, and the canonical ring $R(X, K_X)$ is finitely generated and $Y = \operatorname{Proj} R(X, K_X)$. This is the proof of the finite generation of canonical rings in dimension 2 by Mumford ([116]). In more details, on the canonical model, $|5K_Y|$ is very ample ([17]).

1.13.4 Rational singularities

For a minimal model X of general type, its canonical model Y has *canonical singularities*, because the birational morphism $g : X \to Y$ is crepant ($K_X = f^*K_Y$). Canonical singularities in dimension 2 is known to be the same as *rational double points*, that is, rational singularities of multiplicity 2. Such singularities are investigated in many different situations from long ago, they are also called *Du Val singularities*, *Klein singularities*, *simple singularities*, *ADE singularities*. Here we summarize the classification of 2-dimensional canonical singularities:

Theorem 1.13.9. *Let $P \in X$ be a canonical singularity in dimension* 2.

(1) Take $f : Y \to X$ to be the minimal resolution of singularities, then the exceptional set $\operatorname{Exc}(f)$ *is a normal crossing divisor whose irreducible*

components are all (-2)*-curves and the dual graph defined by their intersections is among* Dynkin diagrams *of type* A_n, D_n, E_6, E_7, E_8 *(see Figure* **??***). Conversely, on a smooth algebraic surface, a normal crossing divisor whose irreducible components are all* (-2)*-curves with dual graph of type* A_n, D_n, E_6, E_7, E_8 *can be contracted to a canonical singularity by a projective birational morphism.*

(2) *When the base field is* $\mathbf{C}$*, there exists an analytic neighborhood of* P *isomorphic to the neighborhood of the origin of hypersurface in* $\mathbf{C}^3$ *defined by one of the following equations:*

$$\begin{aligned} A_n &: \ x^2 + y^2 + z^{n+1} = 0, \quad n \geq 1; \\ D_n &: \ x^2 + y^2 z + z^{n-1} = 0, \quad n \geq 4; \\ E_6 &: \ x^2 + y^3 + z^4 = 0; \\ E_7 &: \ x^2 + y^3 + yz^3 = 0; \\ E_8 &: \ x^2 + y^3 + z^5 = 0. \end{aligned}$$

Here, (x, y, z) *are coordinates of* $\mathbf{C}^3$*.*

(3) *When the base field is* $\mathbf{C}$*, it is analyticaly isomorphic to the singularity of the origin of the quotient space* $\mathbf{C}^2/G$ *for a finite subgroup* G *of* $\mathrm{SL}(2, \mathbf{C})$.

More generally, rational singularities on algebraic surfaces are defined by Artin [8]. Please refer to the original paper for the proof. The theorem is characteristic free:

{Artin}

Theorem 1.13.10. *Let* X *be a smooth algebraic surface and* E_i $(i = 1, \ldots, r)$ *projective curves on* X *such that the union* $E = \bigcup E_i$ *is connected. Assume that the matrix of intersections* $[(E_i \cdot E_j)]$ *is negative definite. Then the following statements hold:*

(1) *There exists a smallest effective integral divisor* $F = \sum e_i E_i \neq 0$ *satisfying the property that* $(F \cdot E_i) \leq 0$ *for all* i*. It is called the* fundamental cycle.

(2) *Inequality* $(K_X \cdot F) + (F^2) \geq -2$ *holds.*

(3) *If equality* $(K_X \cdot F) + (F^2) = -2$ *holds, there exists a projective birational morphism* $f : X \to Y$ *to a normal algebraic surface and the exceptional set* $\mathrm{Exc}(f)$ *coincides with* E*. In this case, the singularity of* Y *is called a* rational singularity.

(4) *Rational singularities are* $\mathbf{Q}$*-factorial. Moreover,* $R^1 f_* \mathcal{O}_X = 0$*. Conversely, a normal singularity on an algebraic surface* Y *with resolution of singularities* $f : X \to Y$ *satisfying* $R^1 f_* \mathcal{O}_X = 0$ *is a rational singularity.*

The condition $R^1 f_* \mathcal{O}_X = 0$ is independent of the choice of resolution of singularities since for $g : X' \to X$ a blowing up of a smooth algebraic surface at a point, $R^1 g_* \mathcal{O}_{X'} = 0$ and $g_* \mathcal{O}_{X'} \cong \mathcal{O}_X$ hold.

Example 1.13.11. (1) On a smooth algebraic surface, a curve satisfying $C \cong \mathbf{P}^1$ and $(C^2) = -n$ can be contracted to a rational singularity.

(2) Dual graphs obtained by taking resolution of singularities of 2-dimensional DLT pairs (see Figure **??**) can be contracted to rational singularities.

Proposition 1.13.12. *Let X be a normal algebraic surface with at most rational singularities and $f : Y \to X$ a resolution of singularities. Then prime divisors in the exceptional set of f are all isomorphic to $\mathbf{P}^1$ and the dual graph is a* tree. *Here a tree is a graph with all edges having weigh one and with no cycles.*

Proof. Since $R^1 f_* \mathcal{O}_Y = 0$, $\lim_E H^1(E, \mathcal{O}_E) = 0$ by [46, Theorem III.11.1]. Here the limit is the inverse limit for all subschemes E supported on the exceptional set of f. Since the exceptional set of f is 1-dimensional, for any f-exceptional effective divisor E, we have $H^1(E, \mathcal{O}_E) = 0$. This concludes the proof. □

Remark 1.13.13. According to a theorem of Grauert ([35]), for a smooth complex analytic surface X and projective curves E_i $(i = 1, \dots, r)$ on X such that the union $E = \bigcup E_i$ is connected and the matrix of intersections $[(E_i \cdot E_j)]$ is negative definite, there always exists a proper birational morphism $f : X \to Y$ to a normal complex analytic surface such that the exceptional set of f coincides with E. However, Y does not necessarily admit an algebraic structure and f is not necessarily algebraic.

1.13.5 The classification of DLT surface singularities 1

Numerical geometry becomes easy for normal algebraic surfaces. Even for non-$\mathbf{R}$-Cartier divisors, intersection numbers and pullback by a morphism can be well-defined.

Let X be a normal algebraic surface and D an $\mathbf{R}$-divisor on X. Take a resolution of singularities $f : Y \to X$ and denote E_i $(i = 1, \dots, r)$ to be the exceptional divisors. The *Mumford's numerical pullback* $f^*D = f_*^{-1}D + \sum e_i E_i$ is defined as the following ([115]): coefficients e_i are the solution of the equations $(f^*D \cdot E_i) = 0$ for all i, which is unique since $[(E_i \cdot E_j)]$ is negative definite. If D is effective, it is easy to see that f^*D is again effective. For two $\mathbf{R}$-divisors D, D', their intersection number can be defined by $(D \cdot D') = (f^*D \cdot f^*D')$.

From now on, we work on the classification of 2-dimensional DLT pairs. Here all discussions are over a base field of characteristic 0. There is also a classification in positive characteristics ([54]).

As the definition of pullback extends to all **R**-divisors, for a pair (X, B), we can define the concept such as KLT, DLT without assuming that $K_X + B$ is **R**-Cartier. Therefore, in the following, this assumption is removed. However, as will be shown later, it turns out that $K_X + B$ automatically becomes **R**-Cartier.

Firstly, we generalize the vanishing theorem slightly. For algebraic surfaces, the normal crossing condition which is important in Theorem 1.9.7 can be removed:

{surface vanishing}

Proposition 1.13.14. *Let X be a smooth projective algebraic surface defined over an algebraically closed field of characteristic* 0*, $f : X \to S$ a projective morphism to another algebraic variety, and D a relatively nef and relatively big* **R***-divisor on X. Then $R^1 f_*(\mathcal{O}_X(K_X + \ulcorner D \urcorner)) = 0$.*

Proof. Take a log resolution $g : Y \to (X, D)$. By Theorem 1.9.7, $R^1(f \circ g)_*(\mathcal{O}_Y(K_Y + \ulcorner g^*D \urcorner)) = R^1 g_*(\mathcal{O}_Y(K_Y + \ulcorner g^*D \urcorner)) = 0$. Then, arguing by spectral sequence, we get $R^1 f_*(g_*(\mathcal{O}_Y(K_Y + \ulcorner g^*D \urcorner))) = 0$. In the exact sequence

$$0 \to g_*(\mathcal{O}_Y(K_Y + \ulcorner g^*D \urcorner)) \to \mathcal{O}_X(K_X + \ulcorner D \urcorner) \to Q \to 0,$$

the cokernal Q has 0-dimensional support, hence its higher cohomologies vanish. Therefore the proof is completed. □

DLT pairs has rational singularities:

Proposition 1.13.15. *Let (X, B) be a* 2*-dimensional DLT pair defined over an algebraically closed field of characteristic* 0*. Then X has rational singularities. If (X, B) is only LC, then X has rational singularities at points in the support of B.*

Proof. Since (X, B) is DLT, $(X, 0)$ is again DLT. Here note that the condition $K_X + B$ being **R**-Cartier is removed in the definition of DLT. As $(X, 0)$ has no boundary, it is KLT. Take the minimal resolution of singularities $f : Y \to X$ and write $f^*K_X = K_Y + C$. As it is minimal, C is effective. Since $(X, 0)$ is KLT, $\ulcorner -C \urcorner = 0$. Applying Proposition 1.13.14 to $D = -f^*K_X$, we get $R^1 f_* \mathcal{O}_Y = R^1 f_*(\mathcal{O}_Y(\ulcorner -C \urcorner)) = 0$.

For the latter statement, when the pair (X, B) is LC, $(X, 0)$ is KLT at points in the support of B. □

Rationality of singularities implies **Q**-factoriality:

Proposition 1.13.16. *Algebraic surfaces defined over the complex number field with only rational singularities are* **Q***-factorial.*

Proof. Take a resolution of singularities $f : Y \to X$. Consider Y as a complex analytic variety, consider its sheaves in classical topology instead of Zariski topology. Then there exists an *exponential exact sequence*

$$0 \to \mathbf{Z}_Y \to \mathcal{O}_Y \to \mathcal{O}_Y^* \to 0.$$

Here the map $\mathcal{O}_Y \to \mathcal{O}_Y^*$ is defined by the exponential function $z \to e^{2\pi i z}$. Note that such kind of exact sequence does not exist in Zariski topology.

By the assumption, $R^1 f_* \mathcal{O}_Y = 0$, hence the map $R^1 f_* \mathcal{O}_Y^* \to R^2 f_* \mathbf{Z}_Y$ is injective.

For any divisor D on X, its numerical pullback f^*D is a $\mathbf{Q}$-divisor, so we can take a positive integer m such that mf^*D is integral. Note that $\mathcal{O}_Y(mf^*D)$ determines an element in $R^1 f_* \mathcal{O}_Y^*$ whose image in $R^2 f_* \mathbf{Q}_Y$ is zero since $(mf^*D \cdot E) = 0$ for every f-exceptional curve E. Therefore there exists a positive integer m', such that the image of $\mathcal{O}_Y(mm'f^*D)$ in $R^2 f_* \mathbf{Z}_Y$ is 0. This induces an isomorphism

$$\mathcal{O}_Y(mm'f^*D) \cong \mathcal{O}_Y.$$

The global section of the left hand side corresponding to 1 of the right hand side determines a rational function h on Y such that $\operatorname{div}(h)_Y = -mm'f^*D$. Hence $\operatorname{div}(h)_X = -mm'D$ which means that $mm'D$ is Cartier. □

As 2-dimensional DLT pairs are rational singularities, they are $\mathbf{Q}$-factorial, and hence numerical pullback is actually the same as pullback. For LC pairs the same holds true on the support of boundaries.

Next we show that KLT or LC property is preserved under taking covering:

Lemma 1.13.17. *Let $f : Y \to X$ be a finite surjective morphism étale in codimension 1 between normal algebraic varieties defined over an algebraically closed field of characteristic 0. Let B be an effective $\mathbf{R}$-divisor on X such that $K_X + B$ is $\mathbf{R}$-Cartier, write $f^*(K_X + B) = K_Y + C$. Then the pair (X, B) is LC if and only if so is (Y, C). The same holds true for KLT pairs.* {LC covering}

Proof. As f is étale in codimension 1, C is effective. Take a log resolution $g : X' \to X$ of (X, B) and take Y' to be the normalization of X' in the function field $k(Y)$. Denote the induced maps by $h : Y' \to Y$ and $f' : Y' \to X'$. Write $g^*(K_X + B) = K_{X'} + B'$, $h^*(K_Y + C) = K_{Y'} + C'$.

Firstly, we show that (X, B) is LC assuming that (Y, C) is LC. Take an arbitrary prime divisor D contracted by g and denote its coefficient in B' by d. Take a prime divisor E on Y' such that $f'(E) = D$ and denote the ramification index of E with respect to f' by r. Then the coefficient of E in $(f')^*D$ and $K_{Y'} - (f')^*K_{X'}$ are r and $r - 1$ respectively. Therefore, take e to be the coefficient of E in C', we get the relation

$$dr = r - 1 + e.$$

Since $e \le 1$ by the assumption, we get $d \le 1$. Moreover, if $e < 1$, then $d < 1$.

Conversely, we show that (Y, C) is LC assuming that (X, B) is LC. By using the result we just proved in the first part, we may replace Y by taking the Galois closure and assume that $k(Y)/k(X)$ is Galois. As the Galois group G acts on Y, we take $h : Y' \to Y$ to be a *G-equivariant* log resolution. For example, a canonical resolution (Remark 1.6.2(4)) is automatically G-equivariant. The quotient space $X' = Y'/G$ has quotient singularities. Denote by $g : X' \to X$ and $f' : Y' \to X'$ the induced maps. Take a prime divisor E contracted by h, define D, e, d in the same way as the first part. Although X' is not smooth, we still have $dr = r - 1 + e$. Since $d \le 1$ by the assumption, we get $e \le 1$. Moreover, if $d < 1$, then $e < 1$. □

Remark 1.13.18. Here we discuss about topology of algebraic varieties defined over the complex number field. In general the topology of algebraic varieties is Zariski topology, but when the base field is the complex number field, classical Euclid topology is also useful. For example, exponential exact sequence appeared before makes sense only in the latter topology.

As an open subset in Zariski topology is large, it admits non-trivial structure itself, on the other hand, classical topology has polydisks as base, its local structure is trivial. Since there are many open subsets, even the constant sheaf has non-trivial cohomology groups.

For algebraic varieties defined over the complex number field, many definitions and results hold both for Zariski topology and classical topology. Furthermore, in many cases they can be generalized to non-algebraic complex analytic varieties. For example, definitions of DLT pairs and LC pairs can be generalized using resolution of complex analytic singularities. The same is true for DLT pairs being rational singularities. The fact that LC and KLT are preserved by étale in codimension one covering can be also generalized since it is a consequence of the ramification formula.

The construction of *index* 1 *cover* can be also generalized. For example, for an effective divisor D on a complex analytic variety X such that $\mathcal{O}_X(rD) \cong \mathcal{O}_X$, take a regular function h such that $\mathrm{div}(h) = rD$, take the normalization of the subvariety defined by the equation $z^r = h$ in the trivial line bundle $X \times \mathbf{C}$ over X, we get the index 1 cover. Here z is the coordinate in the fiber direction. When D is not effective, we can consider similar construction in $X \times \mathbf{P}^1$.

However, as stated in Remark 1.1.2, we should take care of the concept of normal crossing divisor. We should also take care of $\mathbf{Q}$-factoriality. A complex analytic variety X is *analytically* $\mathbf{Q}$*-factorial* if for any analytic neighborhood U of any point $P \in X$ and any codimension 1 subvariety D on X, there exists a neighborhood U' of P in U, a positive integer r, and a regular function h on U' such that $\mathrm{div}_{U'}(h) = r(D \cap U')$. As the algebraic $\mathbf{Q}$-factoriality is a condition for globally defined prime divisors, analytical $\mathbf{Q}$-factoriality is a stronger condition.

1.13.6 The classification of DLT surface singularities 2

We describe the classification of DLT pairs for algebraic surfaces. The results are established in a sufficiently small analytic neighborhood near the singularity.

Firstly, consider the structure near points in the support of boundaries:

{surface DLT classificati

Theorem 1.13.19 ([64]). *Let X be an algebraic surface defined over the complex number field and B a reduced divisor on X. Assume that (X, B) is DLT. Then for any point $P \in X$ in the support of B, there exists an analytic neighborhood U such that one of the following statements holds:*

(1) U is smooth and $B|_U$ is a normal crossing divisor in complex analytic sense.

(2) U has a quotient singularity of type $\frac{1}{r}(1, s)$ and $B|_U$ is irreducible. Here r, s are coprime positive integers. In more details, there exists a neighborhood U_0 of the origin of affine space $\mathbf{C}^2$ with coordinates x, y, a group action by $G = \mathbf{Z}/(r)$ as $x \mapsto \zeta x$, $y \mapsto \zeta^s y$ such that the pair $(U, B|_U)$ is analytically isomorphic to $(U_0/G, B_0/G)$. Here ζ is a primitive r-th root of 1 and $B_0 = \mathrm{div}(x)$. In this case, $(U, B|_U)$ is PLT.

Conversely, pairs satisfying (1) or (2) are DLT.

Proof. Take a sufficiently small analytic neighborhood U of P, take an analytic irreducible component B_1 of $B \cap U$. We may assume that B_1 remains irreducible when replacing U by smaller neighborhoods. Here note that it is possible that an (algebraic) irreducible component of B containing B_1 is strictly bigger than B_1 when restricting to U.

Since X has rational singularities, it is analytically $\mathbf{Q}$-factorial. Hence B_1 is $\mathbf{Q}$-Cartier. Take r_1 to be the smallest positive integer such that $r_1 B_1$ is Cartier. Then we may assume that $\mathcal{O}_U(r_1 B_1) \cong \mathcal{O}_U$. Take $\pi_1 : Y_1 \to U$ to be the index 1 cover of B_1. As π_1 is étale in codimension 1, by Lemma 1.13.17, $(Y_1, \pi_1^* B)$ is LC. If one of the analytic irreducible component of $\pi_1^* B$ is not Cartier, note that Y_1 has again rational singularities, we can construct an index 1 cover again. Therefore, we can construct a finite cover $\pi : Y \to U$ étale in codimension 1 such that any analytically irreducible component of $C = \pi^* B$ is Cartier. By construction, $Q = \pi^{-1}(P)$ is one point.

We will show that Y is smooth. Suppose not, take the minimal resolution of singularities $g : Z \to Y$. Take C_j to be an analytically irreducible component of C, as C_j is Cartier, $g^* C_j$ is an integral divisor. Note that the support of $g^* C_j$ contains the exceptional set of g.

Take s to be the number of such C_j. If $s \geq 2$, then any exceptional divisor of g has coefficients at least 1 in $g^* C_1, g^* C_2$. Since $K_Z \leq g^* K_Y$, this contradicts to the fact that (Y, C) is LC.

Now $s = 1$. Take $E_1, \dots, E_r$ to be the exceptional divisors of g. Since Y has rational singularities, the dual graph of the exceptional divisors is a tree. Since (Y, C) is LC, we get $g^*C_1 = g_*^{-1}C_1 + \sum E_i$ and $K_Z = g^*K_Y$. Note that since C_1 is analytically irreducible, set-theoretically $g_*^{-1}C_1$ intersects the support of $\sum E_i$ at one point. If the graph of $g_*^{-1}C_1 + \sum E_i$ is not a tree, we need more blowing ups to get a log resolution of (Y, C), but this procedure will produce an exceptional divisor with log discrepancy coefficient at least 2, which is a contradiction.

On the other hand, If the graph of $g_*^{-1}C_1 + \sum E_i$ is a tree, then there exists an irreducible component E_1 intersecting $g_*^{-1}C_1 + \sum_{i \neq 1} E_i$ at just one point. But by $(K_Z \cdot E_1) = 0$ we get $(E_1^2) = -2$, which contradicts to $(g^*C_1 \cdot E_1) = 0$.

In summary, we showed that Y is smooth. Note that $Y \setminus Q$ is connected and simply connected, it coincides with the universal covering of $U \setminus P$. In particular, $\pi : Y \to U$ is a Galois covering. Take G to be the Galois group.

Embed Y into affine space $\mathbf{C}^2$ with coordinates x, y such that Q is the origin. Since (Y, C) is LC and Q is contained in the support of C, we may assume that the equation of C is $xy = 0$ or $x = 0$. By construction, C is invariant under the action of G.

If the equation of C is $x = 0$, $B \cap U$ is analytically irreducible, and hence G is the Galois group of an index 1 cover which is isomorphic to $\mathbf{Z}/(r_1)$. We get into case (2) by diagonalizing the generator of G. Here if r, s are not coprime, there is a non-trivial subgroup of G with fixed locus outside Q, which contradicts to the fact that $\pi : Y \to U$ is étale in codimension 1.

Consider the case that the equation of C is $xy = 0$. Firstly, consider the case that every irreducible component of C is invariant under the action of G. By choosing coordinates properly, the log canonical form $dx/x \wedge dy/y$ is invariant under the action of G, and determines a log canonical form $\theta \in H^0(U, K_U + B)$ on $Y/G \cong U$. Since θ has no zeros, $K_U + B$ is Cartier on U. Suppose that U is not smooth, take $h : V \to U$ to be the minimal resolution of singularities and write $h^*(K_U + B) = K_V + B_V$, then the coefficients of B_V are integers. Since $h^*K_U \geq K_V$, the coefficients of B_V are at least 1. This contradicts to the fact that (X, B) is DLT. Hence U is smooth and we get into case (1).

Next, suppose that there exists an element in G exchanging irreducible components of C. Then $B \cap U$ is again analytically irreducible. Hence the DLT pair (U, B) is PLT. Take G' to be the subgroup of G consisting of all elements preserving irreducible components of C, then $G_1 = G/G' \cong \mathbf{Z}/(2)$ and the log canonical divisor $K_{Y'} + C'$ on $Y' = Y/G'$ is Cartier. Here C' is the image of C, which is a reduced divisor with two irreducible components. If Y' is not smooth, take $g' : Z' \to Y'$ to be the minimal resolution of singularities and write $(g')^*(K_{Y'} + C') = K_{Z'} + C'_Z$, then the coefficients of C'_Z all equal to 1. The action of G_1 on Y' extends to Z' and induces a birational morphism $h : V = Z'/G_1 \to U = Y'/G_1$. This is not necessarily the minimal resolution of singularities, but if write $h^*(K_U + B) = K_V + B_V$, then by the ramification formula, the coefficients of B_V all equal to 1, which contradicts to that (U, B) is PLT. Therefore, Y' is smooth. Then $G' = \{1\}$ and the action of G_1 exchanging irreducible components of C is étale in codimension 1, which is absurd. □

As an application in general dimensions, we can show the subadjunction formula for DLT pairs (see Theorem 1.11.13):

Corollary 1.13.20. *Let (X, B) be a DLT pair and Z an irreducible component of $\llcorner B \lrcorner$. Define the $\mathbf{R}$-divisor B_Z on Z by $(K_X + B)|_Z = K_Z + B_Z$. Take an irreducible component P of B_Z with coefficient p. Denote by b_i the coefficients of irreducible components of B containing P. Then there exist positive integers m_i, r such that*

$$p = (r - 1 + \sum b_i m_i)/r.$$

Proof. As we can check the coefficient of P on its generic point, we may assume that $\dim X = 2$ and P is a point. The coefficient remains the same when X is considered as a complex analytic variety, hence we just need to consider two cases in Theorem 1.13.19. Case (1) is trivial, we only consider case (2).

Let $Y = \mathbf{C}^2$, $W = \mathrm{div}(x)$, $G = \mathbf{Z}/(r)$, $X = Y/G$, $Z = W/G$. Denote the projection by $\pi : Y \to X$. Take the origin $Q \in Y$ and denote $P = \pi(Q)$. In the DLT pair (X, B), $B = Z + \sum b_i B_i$. Take $C_i = \pi^* B_i$ and $m_i = (C_i \cdot W)$. When B_i passes through P, m_i is a positive integer.

Since $\pi : Y \to X$ is étale outside the origin, $\pi^*(K_X + Z) = K_Y + W$. On the other hand, $\pi|_W : W \to Z$ is ramified over Q with index r, hence $\pi^* P = rQ$, $K_W = (\pi|_W)^* K_Z + (r-1)Q$. On Y we have the usual adjunction formula $(K_Y + W)|_W = K_W$. Then the coefficient of P can be easily computed by the above relations. □

Next we consider points outside the boundary:

{surface DLT classificati

Theorem 1.13.21 ([64])**.** *Let X be an algebraic surface defined over the complex number field. Assume that $(X, 0)$ is DLT. Then any point $P \in X$ is a quotient singularity. That is, there exists an analytic neighborhood U of P which is analytically isomorphic to the quotient of a neighborhood of the origin $(0, 0)$ of $\mathbf{C}^2$ by the linear action of a finite subgroup G of the general linear group $\mathrm{GL}(2, \mathbf{C})$.*

Conversely, if X has quotient singularities, then $(X, 0)$ is DLT.

Proof. Since $B = 0$, $(U, 0)$ is KLT. Firstly take the index 1 cover $\pi_1 : Y_1 \to U$ of K_X. Since $(Y_1, 0)$ is also KLT and K_{Y_1} is Cartier, Y_1 has canonical singularities. Therefore, $Y_1 = U_0/G_1$ where U_0 is a neighborhood of the origin of $\mathbf{C}^2$ and G_1 is a finite subgroup of $\mathrm{SL}(2, \mathbf{C})$. Now $U_0 \setminus \{0\}$ is the universal cover of $U \setminus \{P\}$ and we get the conclusion.

The converse statement follows from the ramification formula and holds for any dimension. □

Birational geometry of algebraic surfaces works for arbitrary characteristics. The classification of minimal models works under certain replacement ([117], [18], [19]). The theory of rational singularities remains true, also the contraction theorem remains true ([7], [8]). The graph of the resolution of singularities of a DLT pair is completely classified, which is the same as in characteristic 0 ([54], Figure **??**). However, in characteristic 0 the singularity can be determined by the graph of the resolution of singularities, which turns out to be a quotient singularity, but in positive characteristics it is only known to be a rational singularity and the structure of the singularity is not determined only by the graph of the resolution of singularities, the classification seems to be more complicated. In addition, [54] is the origin where the second author was involved in the minimal model theory.

1.13.7 The Zariski decomposition

Finally, we state the Zariski decomposition for divisors on algebraic surfaces:

Theorem 1.13.22. *Let D be an integral divisor on a smooth projective surface X. Assume that there exists a positive integer m such that $|mD| \neq \emptyset$. Then there exists an effective $\mathbb{Q}$-divisor N satisfying the following conditions:*

(1) $P = D - N$ is nef;

(2) $(P \cdot E_i) = 0$ for every i, where $E_1, \dots, E_m$ are irreducible components of N;

(3) the matrix $[(E_i \cdot E_j)]_{i,j}$ is negative-definite.

Moreover, N is uniquely determined by the above conditions.

Such a decomposition $D = P + N$ is called the *Zariski decomposition* of D ([153]).

Proposition 1.13.23. *Let X a smooth projective surface and $f : X \to Y$ be a morphism to a minimal model in the classical sense. Assume that K_Y is nef. Set $N = K_X - f^*K_Y$, then $K_Y = f^*K_Y + N$ is the Zariski decomposition of K_Y.*

That is, we can say that the Zariski decomposition in fact gives the minimal model without taking a birational model. This is the reason why Zariski decomposition has been drawn a lot of attention.

Example 1.13.24. We give an example of log minimal model in dimension 2. The correspondence between Zariski decompositions and log minimal models holds in general ([54]).

Consider an irreducible curve B of degree 4 with 3 ordinary cusp singularities on the projective plane $X = \mathbf{P}^2$. Here an ordinary cusp singularity is a singularity analytically equivalent to that given by the equation $x^2 - y^3 = 0$ at the origin. By the genus formula, B is a rational curve, that is, its normalization is isomorphic to $\mathbf{P}^1$. Let $f : Y \to X$ be the minimal log resolution of the pair (X, B), and $C_0 = f_*^{-1}B$ be the strict transform. Let P_i $(i = 1, 2, 3)$ be the singularities on B, over each there are 3 exceptional divisors E_{ij} $(i, j = 1, 2, 3)$ on Y. It is easy to calculate the intersection numbers $(C_0^2) = -2$ and $(E_{ij}^2) = -j$. $C = C_0 + \sum_{i,j} E_{ij}$ is a normal crossing divisor with all components isomorphic to $\mathbf{P}^1$. The dual graph is shown in Figure **??**.

The Zariski decomposition $K_Y + C = P + N$ is given by

$$P = K_Y + C_0 + \sum_i \left(E_{i1} + \frac{1}{2}E_{i2} + \frac{2}{3}E_{i3} \right), \ N = \sum_i \left(\frac{1}{2}E_{i2} + \frac{1}{3}E_{i3} \right).$$

Here P is nef and big with $(P^2) = 1/2$.

Denote by $g : Y \to Z$ the contraction of 6 curves E_{i2}, E_{i3} $(i = 1, 2, 3)$ in the support of N and $D = g_*C$. Then $K_Z + D$ is ample and $P = g^*(K_Z + D)$. The pair (Z, D) is a log minimal model of (Y, C) which is also the log canonical model.

In Section 2.9 “Divisorial Zariski decomposition” of Chapter 2, we generalize the definition of Zariski decomposition in a weak sense for pseudo-effective $\mathbb{R}$-divisors in arbitrary dimensions.

1.14 The three-dimensional case

Let us consider the 3-dimensional case. In this situation, results in higher dimensional algebraic geometry discussed in subsequent chapters are necessary. In fact, higher dimensional algebraic geometry starts from dimension 3. However, there are also special phenomena and results only in dimension 3. We will describe them briefly as comparison to results in dimension up to 2. This section will not be used in subsequent sections.

The minimal model program, including existence of flips, termination of flips, and abundance conjecture, is completely understood in dimension 3 even for log version. As a consequence of the minimal model theory, the following theorem holds:

Theorem 1.14.1. *Let X be a smooth projective 3-dimensional algebraic variety over a field of characteristic 0. Then there exists a projective algebraic variety X' with at most terminal singularities and a birational map*

$f : X \dashrightarrow X'$ surjective in codimension 1 such that one of the following statements holds:

(1) X' is a minimal model. That is, the canonical divisor $K_{X'}$ is nef.

(2) X' admits a Mori fiber space structure. That is, there exists a surjective morphism $g : X' \to Y$ to the third normal algebraic variety Y with $\dim Y < \dim X$ with connected geometric fibers such that $-K_X$ is g-ample.

Remark 1.14.2. (1) f is not necessarily a morphism and X' is not necessarily smooth, this is a feature in dimension 3 and higher.

(2) X' has terminal singularities means that the pair $(X', 0)$ has terminal singularities. The concept of terminal singularities was originally defined by Reid in dimension 3 ([127]). However, log terminal singularities for algebraic surfaces already appeared before this ([54]). In dimension 2, terminal singularities are impossible to be aware of since they are automatically smooth.

(3) Any terminal singularity can appear in a minimal model. Terminal singularities in dimension 3 are isolated singularities and are completely classified (Theorem 1.14.5). For example, for two coprime positive integers r, b with $b < r$, a quotient singularity of type $\frac{1}{r}(1, -1, b)$ is a terminal singularity (see Example 1.10.5 for notation). The *Cartier index* of a singularity $P \in X$ is the minimal positive integer m such that mK_X is Cartier in a neighborhood of P. The Cartier index of a quotient singularity of type $\frac{1}{r}(1, -1, b)$ is r. In particular, the Cartier index of a minimal model can be arbitrarily large.

(4) Existence of flips in dimension 3 was proved by Mori via an almost complete classification of small contractions ([109]). As will be discussed later, existence of flips in general dimensions is proved in a completely different way by induction on dimensions. Here the generalization to log version is essential.

(5) Termination of flips in dimension 3 was proved by Shokurov ([134]). Termination of log flips in dimension 3 was proved in [70]. It remains open in general dimensions.

The abundance theorem holds in dimension 3 ([103], [104], [105], [69]):

Theorem 1.14.3. *Let X be a 3-dimensional minimal model. That is, X is a projective algebraic variety with terminal singularities and K_X is nef. Then there exists a positive integer m such that the pluricanonical system $|mK_X|$ is free. As a consequence, there exists a surjective morphism $f : X \to Y$ to a normal projective algebraic variety with connected geometric fibers such that*

$K_X \sim_{\mathbf{Q}} f^*H$ for an ample $\mathbf{Q}$-divisor H on Y. By definition, $\dim Y = \kappa(X)$. In particular, the canonical ring is finitely generated.

Remark 1.14.4. (1) The log version of abundance conjecture is also proved in dimension 3 ([84]).

(2) As can be shown in subsequent chapters, the finite generation of canonical rings is much weaker that the abundance theorem.

Terminal singularities in dimension 3 are completely classified as complex analytic singularities ([127], [108], [130]):

{classification terminal 3

Theorem 1.14.5. *Let X be a 3-dimensional algebraic variety defined over the complex number field with terminal singularities and take $P \in X$ be a singular point. Then (X, P) is an isolated singularity. Take r to be the Cartier index, then there exists an analytic neighborhood of P isomorphic to the neighborhood of origin of one of the following singularities:*

(1) a quotient singularity of type $\frac{1}{r}(a, -a, 1)$. Here r, a are coprime positive integers (see Example 1.10.5 for notation).

(2) general type*: the quotient space of the hypersurface in $\mathbf{C}^4$ defined by $xy + f(z^r, w) = 0$ at the origin by the cyclic group $\mathbf{Z}/(r)$. In other words, the prime divisor in 4-dimensional quotient singularity defined by*

$$\{(x, y, z, w) \in \frac{1}{r}(a, -a, 1, 0) \mid xy + f(z^r, w) = 0\}.$$

Here r, a are coprime positive integers and f has no constant term and w term.

The following (2), (3) are also prime divisors in 4-dimensional quotient singularity.

(3) special type*:*

$$\{(x, y, z, w) \in \frac{1}{2}(1, 0, 1, 1) \,|\, x^2 + y^2 + f(z, w) = 0\}, f \in \mathfrak{m}^4, r = 2;$$

$$\{(x, y, z, w) \in \frac{1}{2}(1, 0, 1, 1) \,|\, x^2 + f(y, z, w) = 0\},$$
$$f \in \mathfrak{m}^3 \setminus \mathfrak{m}^4, f_3 \neq y^3, r = 2;$$

$$\{(x, y, z, w) \in \frac{1}{3}(0, 1, 2, 2) \,|\, x^2 + f(y, z, w) = 0\},$$
$$f \in \mathfrak{m}^3, f_3 = y^3 + z^3 + w^3, y^3 + zw^2 \textit{ or } y^3 + z^3, r = 3;$$

$$\{(x, y, z, w) \in \frac{1}{2}(1, 0, 1, 1) \,|\, x^2 + y^3 + yf(z, w) + g(z, w) = 0\},$$
$$f \in \mathfrak{m}^4, g \in \mathfrak{m}^4 \setminus \mathfrak{m}^5, r = 2.$$

Here $\mathfrak{m}$ is the maximal ideal of the origin.

(4) exceptional type*:*

$$\{(x, y, z, w) \in \frac{1}{4}(1, 3, 1, 2) \mid x^2 + y^2 + f(z^2, w) = 0\}.$$

Here f has no constant term and w term.

The exceptional type is different since f is not invariant under group action.

Example 1.14.6. A terminal singularity appears in a divisorial contraction from a smooth 3-dimensional algebraic variety is either smooth or among one of the following cases:

(1) a quotient singularity of type $\frac{1}{2}(1, 1, 1)$.

(2) the hypersurface defined by $xy + zw = 0$ in $\mathbf{C}^4$.

(3) the hypersurface defined by $xy + z^2 + w^3 = 0$ in $\mathbf{C}^4$.

In cases (2) and (3), K_X is Cartier.

More complicated terminal singularities appear when making divisorial contractions from singular 3-dimensional algebraic varieties. Conversely, for the equation of each singularity above, we can construct a divisorial contraction $f : Y \to X$ explicitly by a *weighted blowing up* of X.

Let X be a minimal projective algebraic variety. When $\kappa(X) = 3$, we want to have a formula for plurigenera. Being of general type for X is equivalent to $(K_X^3) > 0$ (Theorem 1.5.12). Note that as K_X is not necessarily Cartier, (K_X^3) is in general only a rational number.

By the finite generation of canonical rings, we can define the *canonical model* $Y = \operatorname{Proj} R(X, K_X)$. There exists a birational morphism $g : X \to Y$ such that $K_X = g^*K_Y$ which is the same as in dimension 2. Here this equality is in the following sense: for an integer m, mK_X is Cartier if and only if mK_Y is Cartier, moreover, in this case $mK_X = g^*(mK_Y)$. In particular, $|mK_X|$ is free if and only if $|mK_Y|$ is free.

In order to state the plurigenera formula, we introduce the concept of baskets of singularities. Take $\{P_1, \dots, P_t\}$ to be the singular points of X. Each singular point (X, P_i) is associated with a set of couples of integers $\{(b_{ij}, r_{ij})\}$ which is called the *basket*. Here r_{ij}, b_{ij} are coprime positive integers with $b_{ij} < r_{ij}$. For example, when (X, P_i) is a quotient singularity of type $\frac{1}{r}(1, -1, b)$, its basket just consists of one couple $\{(b, r)\}$, which coincides with the type of the singularity. In general, a 3-dimensional terminal singularity can be deformed into several quotient singularities, in this case its basket is the collection of types of those quotient singularities. The Cartier index r_i of (X, P_i) coincides with the least common multiple of r_{ij} in its basket. By considering baskets, terminal singularities can be replaced by a set of virtual quotient singularities.

The plurigenera formula for $m \geq 2$ is the following:

$$\begin{aligned}\dim H^0(X, mK_X) =& \frac{1}{12}m(m-1)(2m-1)(K_X^3) + (1-2m)\chi(\mathcal{O}_X) \\ &+ \sum_{i,j}\sum_{k=0}^{m-1}\frac{\overline{b_{ij}k}(r_{ij} - \overline{b_{ij}k})}{2r_{ij}}.\end{aligned}$$

Here $\overline{b_{ij}k}$ denotes the residue of $b_{ij}k$ modulo r_{ij} ([130]). This formula is a sum of a polynomial of m and a periodic correction term with respect to m (see [153]). The correction term runs over baskets of all singularities. As plurigenera are birational invariants, the left hand side are the same on a smooth model, but the right hand side can be only computed on a minimal model with singularities. In other words, when computing plurigenera on a smooth model, the singularities of its minimal model appear, which is a surprising phenomenon.

Also we have the following formula ([65])

$$\chi(\mathcal{O}_X) = -\frac{1}{24}(K_X \cdot c_2(X)) + \sum_{i,j}\frac{r_{ij}^2 - 1}{24r_{ij}}.$$

Here as X has only isolated singularities, the intersection number $(K_X \cdot c_2(X))$ can be defined properly.

Remark 1.14.7. In this book, we will show the finite generation of canonical rings. However, it is impossible to find a bound of the degrees of generators depending only on the dimension. This can already be observed in dimension 3. Let P be a singular point on a minimal model X. If m is not divisible by the Cartier index r of P, then P is a base point of $|mK_X|$. Hence for arbitrary large m, we can construct examples such that $|mK_X|$ is not free.

For example, if $\dim X = 3$ and P is a quotient singularity of type $\frac{1}{r}(a, -a, 1)$, then the canonical ring can not be generated by elements of degree less than r. This is a completely different phenomenon from that in dimension up tp 2, because singularities appear in minimal models in dimension 3 and higher.

Chapter 2

The minimal model program

{Chapter 2}

The purpose of this chapter is to formulate the minimal model program. The base point free theorem and the cone theorem are two main pillars for the minimal model program, which are known results at the time of [82]. We will also discuss subsequent developments as effective version of the base point free theorem, the MMP with scaling, length of extremal rays, divisorial Zariski decomposition, Shokurov polytopes. Extension theorems obtained by multiplier ideal sheaves is an important result which leads to the newest developments of the minimal model program described in the next chapter.

Numerical geometry plays an important role in the minimal model theory. But different from Kleiman's criterion, the base point free theorem and the cone theorem do not hold for arbitrary schemes. A feature of the minimal model theory is that canonical divisor plays an important role.

2.1 The base point free theorem

The base point free theorem is one of the two pillars supporting the minimal model theory. It is an important consequence of the vanishing theorem of cohomologies.

For algebraic surfaces, minimal models are obtained by applying Castelnuovo's contraction theorem repeatedly. Contracting topological spaces is always possible. Also, as in Grauert's theorem, contraction morphisms in complex geometry are proved to exist by only assuming numerical conditions. However, as in Artin's theorem, contraction morphisms in algebraic geometry are more subtle. In order to construct a contraction morphism in algebraic geometric, one needs a free linear system. By using the base point free theorem, one can construct a free linear system in a general setting.

2.1.1 Proof of the base point free theorem

{KLT BPF thm}

Theorem 2.1.1 (*Base point free theorem*)**.** *Let (X, B) be a KLT pair, $f : X \to S$ a projective morphism, and D, E Cartier divisors on X. Assume the following conditions.*

(1) D is relatively nef.

(2) There exists a positive integer m_1 such that $m_1D + E - (K_X + B)$ is relatively nef and relatively big.

(3) E is effective and there exists a positive integer m_2 such that for any integer $m \geq m_2$, the natural homomorphism $f_*(\mathcal{O}_X(mD)) \to f_*(\mathcal{O}_X(mD+E))$ is an isomorphism.

Then there exists a positive integer m_3 such that for any integer $m \geq m_3$, mD is relatively free. That is, the natural homomorphism $f^*f_*(\mathcal{O}_X(mD)) \to \mathcal{O}_X(mD)$ is surjective.

Remark 2.1.2. For a given divisor, assuming its numerical equivalent class is in the closure of ample cone, that is, assuming it is nef, to show that it is semi-ample is beyond the limit of Kleiman's criterion. The base point free theorem can be generalized in many different directions, but it is not true if one completely remove the condition on singularities and the condition about canonical divisor. This reflects the complicated geometry of algebraic varieties.

Proof. **Step 0**. As the statement is relative over S, we may assume that S is affine. Then the statement of the theorem is that the natural homomorphism $H^0(X, mD) \otimes \mathcal{O}_X \to \mathcal{O}_X(mD)$ is surjective, in other words, the linear system corresponding to $H^0(X, mD)$ has no base points. Note that if S is not a point, $H^0(X, mD)$ may be infinite dimensional.

Step 1. We will show that we may assume that $m_1D + E - (K_X + B)$ is relatively ample and B is a $\mathbf{Q}$-divisor.

By assumption (2), we can write $m_1D + E - (K_X + B) = A + B'$ for a relatively ample $\mathbf{R}$-divisor A and an effective $\mathbf{R}$-divisor B'. Then for a real number ϵ with $0 < \epsilon \leq 1$, $m_1D + E - (K_X + B + \epsilon B')$ is relatively ample, and if ϵ is sufficiently small, $(X, B + \epsilon B')$ is KLT. We can just replace B by $B + \epsilon B'$. As ampleness is an open condition, we can adjust the coefficients of B to become rational numbers.

Step 2. The statement "under the assumption of the theorem, there exists a positive integer m_3' such that for any integer $m \geq m_3'$, $H^0(X, mD) \neq 0$" is a part of the base point free theorem, and is called the *non-vanishing theorem* independently. According to the historical order, we will give the proof of the non-vanishing theorem later, and show the base point free theorem assuming the non-vanishing theorem in this step.

Fix an integer $m \geq m_3'$, suppose that the linear system $|mD|$ corresponding to $H^0(X, mD)$ has base point. Take a general divisor $M \in |mD|$. Take a log resolution $g : Y \to X$ of $(X, B+E+M)$ in strong sense, denote $h = f \circ g$ and write $g^*(K_X + B) = K_Y + C$. We may assume that $g^*M = M_1 + M_2$ where $|M_1|$ is free and M_2 is the fixed part of $|g^*M|$.

Take an effective **Q**-divisor C' such that $\mathrm{Exc}(g) \cup \mathrm{Supp}(C + g^*E + M_2) = \mathrm{Supp}(C')$ and $-C'$ is g-ample. The construction of C' is as following: by the definition of log resolution in strong sense, we can take an effective **Q**-divisor C'' such that $\mathrm{Exc}(g) = \mathrm{Supp}(C'')$ and $-C''$ is g-ample, then we can perturb it to extend the support by openness of ampleness.

We can take a sufficiently small positive rational number ϵ such that $g^*(m_1 D + E - (K_X + B)) - \epsilon C'$ is h-ample and $\ulcorner -C - \epsilon C' \urcorner = \ulcorner -C \urcorner \geq 0$.

One key point of the proof is to consider the following *threshold*:

$$c = \sup\{t \in \mathbf{R} \mid \llcorner t M_2 - g^*E + C + \epsilon C' \lrcorner \leq 0\}.$$

This is a kind of *LC threshold*. By definition, the maximal coefficient of $cM_2 - g^*E + C + \epsilon C'$ is exactly 1. By perturbing coefficients of C' while preserving the ampleness of $-C'$, we may assume that there is only one prime divisor attaining the maximal coefficient 1.

This idea of breaking the balance of coefficients by perturbing coefficients of **Q**-divisors is called the *tie breaking*. This is the advantage of considering **Q**-divisors and **R**-divisors instead of only integral divisors.

Denote Z to be the prime divisor with coefficient 1 in $cM_2 - g^*E + C + \epsilon C'$. As coefficients of $C + \epsilon C'$ are less than 1, Z is contained in the support of M_2. Hence $g(Z)$ is contained in $\mathrm{Bs}\,|mD|$. Write

$$cM_2 - g^*E + C + \epsilon C' = F + Z.$$

By construction, F does not contain Z and $\ulcorner -F \urcorner \geq 0$.

Let m' be an integer, as $mg^*D \equiv_S M_1 + M_2$, we get the following equation:

$$m' g^*D - F - Z - K_Y \equiv_S (m' - cm) g^*D + cM_1 + g^*E - (K_Y + C + \epsilon C').$$

If $m' \geq m_1 + cm$, as M_1 is free, the right hand side is h-ample. Applying Theorem 1.9.3, we get

$$H^1(Y, m' g^*D + \ulcorner -F \urcorner - Z) = 0.$$

Therefore, the natural homomorphism

$$H^0(Y, m' g^*D + \ulcorner -F \urcorner) \to H^0(Z, m' g^*D|_Z + \ulcorner -F \urcorner|_Z)$$

is surjective. Here the restriction of F can be defined as F does not contain Z. Also D can be replaced by a (not necessarily effective) linearly equivalent divisor which does not contain Z.

On the other hand, as the negative coefficient part of $F + g^*E$ comes from the negative coefficient part of C, its support is contained in the exceptional set of g. Therefore, $g_*F + E \geq 0$ and there are natural injective homomorphisms

$$H^0(X, m'D) \to H^0(X, m'D + \ulcorner -g_*F \urcorner) \to H^0(X, m'D + E).$$

If $m' \geq m_2$, then they become bijective by condition (3). Hence $H^0(Y, m'g^*D) \to H^0(Y, m'g^*D + \ulcorner -F \urcorner)$ is bijective.

Define the boundary $B_Z = (F + \ulcorner -F \urcorner)|_Z$ on Z, then the pair (Z, B_Z) is KLT. Let us check that the projective morphism $h|_Z : Z \to S$, the Cartier divisors $g^*D|_Z, \ulcorner -F \urcorner|_Z$ on Z satisfy conditions of the theorem. Obviously, (1) holds, and (2) holds since $m'g^*D|_Z - F|_Z - K_Z$ is relatively ample. Consider the following commutative diagram:

$$\begin{array}{ccc} H^0(Y, m'g^*D) & \longrightarrow & H^0(Y, m'g^*D + \ulcorner -F \urcorner) \\ \downarrow & & \downarrow \\ H^0(Z, m'g^*D|_Z) & \longrightarrow & H^0(Z, m'g^*D|_Z + \ulcorner -F \urcorner|_Z). \end{array}$$

If $m' \geq m_2$, the top horizontal arrow is bijective. Moreover if $m' \geq m_1 + cm$, the right vertical arrow is surjective. Hence the bottom horizontal arrow is surjective and (3) holds.

By applying the non-vanishing theorem to Z, there exists a positive integer m_3'' such that if $m' \geq m_3''$, then $H^0(Z, m'g^*D|_Z) \neq 0$. By the above commutative diagram, this implies that $g(Z)$ is not contained in the base locus of $|m'D|$. If m' is a multiple of m, the we have a strict inclusion of base loci $\operatorname{Bs}|m'D| \subsetneq \operatorname{Bs}|mD|$.

Fix a prime number p and take m, m' to be powers of p. As there is no strictly decreasing sequence of closed subsets in X, by repeating the above argument, for any sufficiently large power p^t, $\operatorname{Bs}|p^tD| = \emptyset$. This argument is called the *Noetherian induction*. For another prime number q, by the same argument, there exists a sufficiently large positive integer s such that $\operatorname{Bs}|q^sD| = \emptyset$. As p^t and q^s are coprime, there exists a positive integer m_3, such that for any integer $m \geq m_3$ there exist positive integers a, b such that $m = ap^t + bq^s$. In this case, $\operatorname{Bs}|mD| = \emptyset$. Therefore, assuming the non-vanishing theorem, we proved the base point free theorem.

Step 3. We will show the non-vanishing theorem by induction on $\dim X$. The method is similar to the proof of the base point free theorem, but we create base point artificially.

It suffices to show the non-vanishing of a general fiber of f, hence we may assume that $S = \operatorname{Spec} k$.

The statement of the non-vanishing theorem is that for all sufficiently large m, $H^0(X, mD + E) \neq 0$. By Theorem 1.10.8, for any integers $p > 0$ and $m \geq m_1$, $H^p(X, mD + E) = 0$, and hence $\dim H^0(X, mD + E) = \chi(X, mD + E)$. The latter one is a polynomial, so it suffices to show that it is not identically 0.

In general proving the existence of global sections is a difficult problem. In our situation we reduce the problem to a problem for the Euler–Poincaré characteristic, and prove it as the following.

Firstly, consider the case $D \equiv 0$. In this case, as $E - (K_X + B)$ is nef and big, by Theorem 1.10.8, for any integers $p > 0$, $H^p(X, E) = 0$. Since E is effective, $\chi(X, E) = \dim H^0(X, E) \neq 0$, the the proof is finished.

Step 4. Finally we show the non-vanishing theorem in the case $D \not\equiv 0$.

As in Step 1, we may assume that $m_1 D + E - (K_X + B) = A$ is an ample **Q**-divisor. Take a positive integer a such that aA is a Cartier divisor.

Since $D \not\equiv 0$, there exists a curve Γ such that $(D \cdot \Gamma) > 0$. Denote by I_Γ the ideal sheaf of Γ, replacing a by a sufficiently large multiple, we may assume that $\mathcal{O}_X(aA) \otimes I_\Gamma$ is generated by global sections. Denote $d = \dim X$, by taking intersections of zeros of $d-1$ general sections of this sheaf, we get $A^{d-1} \equiv c\Gamma + \Gamma'$. Here $c > 0$ and Γ' is a linear sum of curves distinct from Γ. Hence $(D \cdot A^{d-1}) > 0$. We can take a sufficiently large integer m such that $(mD + aA)^d > a^d(d+1)^d$.

By the Serre vanishing theorem, there exists an integer k_1 such that for all integers $k \geq k_1$ and $p > 0$, $H^p(X, k(mD + aA)) = 0$. Therefore, $\dim H^0(X, k(mD + aA)) = \chi(X, k(mD + aA))$ is a polynomial in k of degree d, and the coefficient of the highest degree term is larger than $a^d(d+1)^d/d!$.

Fix a smooth point P in X not contained in the support of $E + B$. Take $\mathfrak{m}_P$ the maximal ideal of $\mathcal{O}_{X,P}$, the length $\mathrm{length}(\mathcal{O}_{X,P}/\mathfrak{m}_P^{a(d+1)k})$ is a polynomial in k of degree d, and the coefficient of the highest degree term is $a^d(d+1)^d/d!$. Hence for any sufficiently large k,

$$H^0(X, \mathcal{O}_X(k(mD + aA)) \otimes \mathfrak{m}_P^{a(d+1)k}) \neq 0.$$

Therefore we showed that there exists an element $M' \in |k(mD + aA)|$ such that $\mathrm{mult}_P M' \geq a(d+1)k$. This is called the *concentration method*. Denote $M = \frac{M'}{k}$, then $M' \sim_{\mathbf{Q}} mD + aA$ and $\mathrm{mult}_P M' \geq a(d+1)$.

From now on, the proof are the same as that of the base point free theorem. Take a log resolution $g : Y \to X$ of $(X, B+E+M)$ in strong sense and write $g^*(K_X + B) = K_Y + C$. Note that here we first take the blowing up at P, then construct g by further blowing ups. We can take an effective **Q**-divisor C' such that $\mathrm{Exc}(g) \cup \mathrm{Supp}(g^*(B+E+M)) = \mathrm{Supp}(C')$ and $-C'$ is g-ample. We can take a sufficiently small positive real number ϵ such that $g^*(m_1 D + E - (K_X + B)) - (d+1)\epsilon C'$ is ample and $\ulcorner -C - \epsilon C' \urcorner = \ulcorner -C \urcorner \geq 0$.

Consider the threshold

$$c = \sup\{t \in \mathbf{R} \mid \llcorner tg^*M - g^*E + C + \epsilon C' \lrcorner \leq 0\}.$$

By perturbing coefficients of C', we may assume that there is only one prime divisor Z attaining the maximal coefficient 1.

Take C_0 to be the strict transform of the exceptional divisor of the first blowing up. C_0 and Z may or may not coincide. The coefficient of C_0 in $tg^*M - g^*E + C + \epsilon C'$ is larger than $a(d+1)t - (d-1)$, hence $ac < d/(d+1)$ by definition. Write

$$cg^*M - g^*E + C + \epsilon C' = F + Z.$$

By construction, F does not contain Z and $\ulcorner -F \urcorner \geq 0$.

Take integer m' such that $m' \geq m_1 + cm$, as $1 - ac > 1/(d+1) > 0$,

$$\begin{aligned} m'g^*D - F - Z - K_Y \equiv &(m' - m_1 - cm)g^*D \\ &+ (1 - ac)(m_1 g^*D + g^*E - (K_Y + C + \epsilon C'/(1 - ac))) \end{aligned}$$

is ample. By Theorem 1.10.8,

$$H^1(Y, m'g^*D + \ulcorner -F \urcorner - Z) = 0,$$

hence the natural homomorphism

$$H^0(Y, m'g^*D + \ulcorner -F \urcorner) \to H^0(Z, m'g^*D|_Z + \ulcorner -F \urcorner|_Z)$$

is surjective. Also

$$H^0(Y, m'g^*D) \to H^0(Y, m'g^*D + \ulcorner -F \urcorner) \to H^0(Y, m'g^*D + g^*E)$$

are bijective.

Denote $B_Z = (F + \ulcorner -F \urcorner)|_Z$, the pair (Z, B_Z) is KLT, Cartier divisors $g^*D|_Z, \ulcorner -F \urcorner|_Z$ on Z satisfy the conditions of the non-vanishing theorem. Here recall that S is assumed to be a point. By applying the non-vanishing theorem to Z, there exists a positive integer m_3'' such that if $m' \geq m_3''$, then $H^0(Z, m'g^*D|_Z) \neq 0$. Hence $H^0(X, m'D) \neq 0$. The proof of the the non-vanishing theorem is finished. □

Remark 2.1.3. In the original base point free theorem, $E = 0$, but the proof are exactly the same ([82]). As in the proof we take log resolution, even if $E = 0$ is assumed in the beginning, $F \neq 0$ appears naturally after taking log resolution. Therefore, it is natural to assume that $E \neq 0$ in the beginning. Also, in the statement of the non-vanishing theorem, E appears in the beginning ([134]). In order to apply to the abundance theorem, according to [29], we showed the general form with E.

In the former half of the above proof, we apply the vanishing theorem to linear systems appearing naturally, while in the latter half, we apply the vanishing theorem to linear systems artificially constructed. The latter method is called the *concentration method* of singularities.

The argument of proving the base point free theorem by using the vanishing theorem was originally developed in [58]. In dimension 3, the non-vanishing theorem follows easily from the *Riemann-Roch theorem*. This argument was applied by Shokurov [134] to the proof of the non-vanishing theorem, and hence the base point free theorem was proved in all dimensions. Furthermore, it was shown in [60] that this argument can be applied to the proof of the cone theorem using the rationality theorem described in a subsequent section. It was also used in the establishment of the abundance conjecture [61]. So this argument has been found to have a wide range of applications, and is known as the *X-method*.

2.1.2 Paraphrasing and generalization

The following corollary is equivalent statement of the base point free theorem:

{cor BPF equivalent}

Corollary 2.1.4. *Let $f : (X, B) \to S$, D, E satisfy the assumptions of Theorem 2.1.1. Then there exists a projective morphism $g : Z \to S$ from a normal algebraic variety, a surjective projective morphism $h : X \to Z$ with connected geometric fibers such that $f = g \circ h$, and a g-ample Cartier divisor H such that $h^*H \sim D$.*

Proof. By the base point free theorem, there exists a positive integer m_3 such that for $m \geq m_3$, mD is f-free. Denote by $\phi_m = h'_m \circ h_m : X \to Z_m \to Z'_m$ the *Stein factorization* of the morphism defined by mD over S, and denote by $g_m : Z_m \to S$ the induced morphism. By construction, there exists a g_m-ample Cartier divisor H_m on Z_m such that $mD \sim h_m^* H_m$.

It can be seen that, for a curve C on X such that $f(C)$ is a point in S, $h_m(C)$ is a point in Z_m if and only if $(D \cdot C) = 0$. By Zariski's main theorem, there exists an isomorphism $k_m : Z_m \to Z_{m+1}$ such that $k_m \circ h_m = h_{m+1}$. Take $H = k_m^* H_{m+1} - H_m$, then $h_m^* H \sim D$. □

{ample thm Q}

Corollary 2.1.5. *Let (X, B) be a KLT pair and $f : X \to S$ a projective morphism. Assume that $K_X + B$ is f-nef and B is an f-big $\mathbf{Q}$-Cartier divisor. Then there exists a projective morphism $g : Z \to S$ from a normal algebraic variety, a surjective projective morphism $h : X \to Z$ with connected geometric fibers such that $f = g \circ h$, and a g-ample $\mathbf{Q}$-Cartier divisor H such that $h^*H \sim_{\mathbf{Q}} K_X + B$.*

Proof. By Corollary 2.1.4, it is sufficient to show that there exists a positive integer m_3 such that for $m \geq m_3$, $m(K_X + B)$ is f-free. We may assume that S is affine. There exists a positive integer m_1 such that $D = m_1(K_X + B)$ is an f-nef Cartier divisor. As B is f-big, there exists an f-ample $\mathbf{Q}$-Cartier divisor A and an effective $\mathbf{Q}$-Cartier divisor E such that we can write $B = A+E$. For a sufficiently small positive rational number ϵ, $(X,(1-\epsilon)B+\epsilon E)$ is still KLT. Also $D-(K_X+(1-\epsilon)B+\epsilon E) = (m_1-1)(K_X+B)+\epsilon A$ is f-ample. We get the conclusion by Theorem 2.1.1. □

Remark 2.1.6. The condition the B is a $\mathbf{Q}$-divisor can be removed by using the cone theorem (Corollary 2.4.13).

The following lemma is useful when generalizing statements for KLT pairs to DLT pairs.

{KLT to DLT}

Lemma 2.1.7 ([98]). *Let (X,B) be a DLT pair, $f: X \to S$ a projective morphism, H a relatively ample divisor on X, and ϵ a positive real number. Assume that S is quasi-projective. Then there exists an ample divisor A on S and an effective $\mathbf{R}$-divisor B' on X such that $B+\epsilon(H+f^*A) \sim_{\mathbf{R}} B'$ and the pair (X,B') is KLT.*

Proof. We can choose an ample divisor A on S such that $H+f^*A$ is ample on X. Take a log resolution $g:(Y,C) \to (X,B)$ in strong sense, denote $h = f\circ g$. By the definition of DLT, we may assume that the coefficients of exceptional divisors in C are strictly less than 1, note that here we use the fact that DLT is equivalent to WLT (see Remark 1.11.4). Take a sufficiently small effective $\mathbf{Q}$-divisor E supported on the exceptional set of g such that $-E$ is g-ample, $\llcorner C+E \lrcorner = \llcorner C \lrcorner$, and $g^*(H+f^*A)-E$ is ample on Y.

Write $B = \sum b_i B_i$ where B_i are distinct prime divisors, and write $g_*^{-1}B = \sum b_i B_i'$ the strict transform on Y. We can choose a positive integer m such that for every i, the divisorial sheaf $\mathcal{O}_Y(B_i'+m(g^*(H+f^*A)-E))$ is generated by global sections. By taking a general global section, we can find a prime divisor $D_i' \sim B_i' + m(g^*(H+f^*A)-E)$. Take a sufficiently small positive real number δ and take $C' = C-\delta\sum b_iB_i'+\delta\sum b_iD_i'+m\delta\sum b_iE \sim_{\mathbf{R}} C+m\delta\sum b_i(g^*(H+f^*A))$. Note that the support of C' is a normal crossing divisor as D_i' are general, and the coefficients of C' are less than 1 as δ is sufficiently small. Then we can take $B' = g_*C' = (1-\delta)B+\delta\sum b_ig_*D_i' \sim_{\mathbf{R}} B+m\sum b_i(H+f^*A)$. Note that K_X+B' is $\mathbf{R}$-Cartier and $f^*(K_X+B') = K_Y+C'$, which implies that (X,B') is KLT. □

Now we can show the base point free theorem for DLT pairs:

{DLT BPF thm}

Corollary 2.1.8 (*Base point free theorem*). *Let (X,B) be a DLT pair, $f: X\to S$ a projective morphism, and D,E Cartier divisors on X. Assume the following conditions.*

(1) D is relatively nef.

(2) There exists a positive integer m_1 such that $m_1D+E-(K_X+B)$ is relatively ample.

(3) E is effective and there exists a positive integer m_2 such that for any integer $m \geq m_2$, the natural homomorphism $f_*(\mathcal{O}_X(mD)) \to f_*(\mathcal{O}_X(mD+E))$ is an isomorphism.

Then there exists a positive integer m_3 such that for any integer $m \geq m_3$, mD is relatively free. That is, the natural homomorphism $f^*f_*(\mathcal{O}_X(mD)) \to \mathcal{O}_X(mD)$ is surjective.

Proof. We may assume that S is affine. Take $B' \sim_{\mathbf{R}} B + \epsilon(H + f^*A)$ as in Lemma 2.1.7 such that (X, B') is KLT. If ϵ is sufficiently small, then $m_1D + E - (K_X + B')$ is still relatively ample. The corollary follows from Theorem 2.1.1. □

2.2 The effective base point free theorem

The base point free theorem states that a multiple of a certain Cartier divisor is free. Its effective version shows how large this multiple can be taken in practice. The proof is not just a refinement of that of the base point free theorem, but by assuming the base point free theorem and using the conclusion of existence of such a morphism.

Theorem 2.2.1 (*Effective base point free theorem* [91])**.** *Let (X, B) be a KLT pair consisting of an n-dimensional algebraic variety and an $\mathbf{R}$-divisor, E an effective Cartier divisor on X, D a Cartier divisor on X, and $f : X \to S$ a projective morphism. Assume the following conditions hold:*

(1) D is f-nef and $D + E - (K_X + B)$ is f-nef and f-big.

(2) The natural homomorphism

$$f_*\mathcal{O}_X(mD) \to f_*\mathcal{O}_X(mD + E)$$

is bijective for any positive integer m.

Then for any $m \geq 2n + 3$, $|m^{n+1}D|$ is f-free, that is, the natural homomorphism

$$f^*f_*\mathcal{O}_X(m^{n+1}D) \to \mathcal{O}_X(m^{n+1}D)$$

is bijective. Therefore, there exists a positive integer m_0 depending only on n, such that $|mD|$ is f-free for any $m \geq m_0$.

Proof. We may assume that S is affine. Note that in this case "ample over S" or "free over S" is simply the same as "ample" or "free". By slightly perturbing the coefficients of B, we may assume that $D + E - (K_X + B)$ is ample.

By Corollary 2.1.4, there exists a normal algebraic variety Y, a surjective projective morphism $g : X \to Y$ over S with connected geometric fibers, and a relatively ample Cartier divisor H on Y such that $D = g^*H$. Denote $h : Y \to S$ to be the morphism such that $f = h \circ g$, and $d = \dim Y$. Take X_s, Y_s to be the fibers over a general point s in $h(Y)$ and denote $d_s = \dim Y_s \leq d \leq n$.

Firstly, we show the effective version of the non-vanishing theorem (see Step 3 of the proof of Theorem 2.1.1). By the vanishing theorem, for $m > 0$, $h^0(X_s, mD) = h^0(X_s, mD + E) = \chi(X_s, mD + E)$. By the non-vanishing theorem, the latter one is a non-zero polynomial of degree d_s, and has at most d_s distinct roots.

We claim that for $m \geq 2d_s + 2$, $H^0(X_s, mD) \neq 0$. For $1 \leq i \leq d_s + 1$, if $H^0(X_s, mD) = 0$, then either $H^0(X_s, iD) = 0$ or $H^0(X_s, (m-i)D) = 0$. This means that $\chi(X_s, mD + E)$ has at least $d_s + 1$ roots, a contradiction. Therefore, $H^0(X_s, mD) \neq 0$ for $m \geq 2d_s + 2$.

The theorem can be reduced to the following lemma:

{lemma ebpf}

Lemma 2.2.2. *Fix any $m \geq 2d + 2$ and take an irreducible component $\bar{Z}$ of the base locus*

$$\operatorname{Bs}|mH| = \operatorname{Supp}(\operatorname{Coker}(h^*h_*\mathcal{O}_Y(mH) \to \mathcal{O}_Y(mH))).$$

Then for any $k \geq 2d + 2$, $\bar{Z} \not\subset \operatorname{Bs}|kmH|$.

Let us continue the proof by assuming Lemma 2.2.2. Note that $g^{-1}(\operatorname{Bs}|mH|) = \operatorname{Bs}|mD|$. Fix any $m \geq 2d + 2$, Lemma 2.2.2 shows that the dimension of $\operatorname{Bs}|m^jD|$ is at most $d - j$ for $1 \leq j \leq d + 1$, and in particular, the base locus $\operatorname{Bs}|m^{d+1}D|$ is empty, which concludes the first statement. The interesting point of this proof is that the above lemma can be applied to every irreducible component of the base locus at the same time to cut down the dimension.

Take two distinct prime numbers $p, q \geq 2d+2$. Then $|p^{d+1}D|$ and $|q^{d+1}D|$ are free. There exists a positive integer m_0 such that any integer $m \geq m+0$ can be expressed as $m = ap^{d+1} + bq^{d+1}$ $(a, b \in \mathbf{Z}_{>0})$ and hence $|mD|$ is free. This proves the second statement. □

Proof of Lemma 2.2.2. Note that $h(\bar{Z})$ is a subset of $h(Y)$. Denote $Z = g^{-1}(\bar{Z})$. The proof is by applying the argument in the base point free theorem to the neighborhood of the generic point of $\bar{Z}$.

Firstly, we construct singularities in a neighborhood of Z. Denote $\bar{d} = \dim \bar{Z}$. Take $d - \bar{d} + 1$ general global sections of $\mathcal{O}_Y(mH)$, say, $\bar{M}_i$ $(1 \leq i \leq d - \bar{d})$ and $\bar{N}$. Note that the supports of $\bar{M}_i$ and $\bar{N}$ contain $\operatorname{Bs}|mH|$, but they are free outside $\operatorname{Bs}|mH|$. Take sufficiently small numbers $\epsilon, \delta > 0$ and take $M = (1-\delta)\sum_i g^*\bar{M}_i + \epsilon g^*\bar{N}$. We may take a neighborhood $\bar{U}$ of the generic point of $\bar{Z}$ such that $\bar{U}$ does not intersect irreducible components of $\operatorname{Bs}|mH|$ other than $\bar{Z}$ and denote $U = g^{-1}(\bar{U})$. Here note that $\bar{Z}$ is not necessarily contained in $\bar{U}$.

Since $\bar{Z}$ has codimension $d-\bar{d}$, by Corollary 1.11.8, we may assume that the pair $(X, B+M)$ is KLT on $U \setminus \mathrm{Bs}\,|mD|$ but not LC on $U \cap Z$ by taking ϵ, δ appropriately and shrinking U.

Take a log resolution $\mu : X' \to (X, B+M)$ in strong sense and write $\mu^*(K_X+B) = K_{X'} + B'$. The coefficients of B' are all less than 1. Take an effective divisor F supported on the exceptional set of μ such that $-F$ is μ-ample. Take a sufficiently small positive number ϵ' such that the coefficients of $B' + \epsilon' F$ are all less than 1 and $\mu^*(D+E-(K_X+B)) - \epsilon' F$ is ample.

Consider the LC threshold c such that on $\mu^{-1}(U)$ the coefficients $B' + c\mu^* M + \epsilon' F$ are all no greater than 1, and some coefficient is exactly 1. Here $c<1$ as $(X, B+M)$ is not LC on $U \cap Z$, and note that outside $\mu^{-1}(U)$ the coefficients are not necessarily no greater than 1. Write

$$\mu^*(K_X + B + cM) + \epsilon' F = K_{X'} + F_{i_0} + B''.$$

Here F_{i_0} is the sum of all irreducible components with coefficient 1 intersecting $\mu^{-1}(U)$. By perturbing the coefficients of F and shrinking $\bar{U}$, we may assume that F_{i_0} is irreducible, $g \circ \mu(F_{i_0}) = \bar{Z}$, and $\llcorner B'' \lrcorner \le 0$ on $\mu^{-1}(U)$.

For a natural number m', consider the following exact sequence

$$\begin{aligned}0 &\to \mathcal{O}_{X'}(\mu^*(m'D+E) - \llcorner B'' \lrcorner - F_{i_0}) \to \mathcal{O}_{X'}(\mu^*(m'D+E) - \llcorner B'' \lrcorner) \\ &\to \mathcal{O}_{F_{i_0}}((\mu^*(m'D+E) - \llcorner B'' \lrcorner)|_{F_{i_0}}) \to 0.\end{aligned}$$

If $m' \ge c((d-\bar{d})(1-\delta)+\epsilon)m + 1$,

$$\begin{aligned}&\mu^*(m'D+E) - B'' - F_{i_0} - K_{X'} \\ &\equiv (m' - c((d-\bar{d})(1-\delta)+\epsilon)m - 1)\mu^* D + \mu^*(D+E-(K_X+B)) - \epsilon' F\end{aligned}$$

is ample. By the vanishing theorem, higher cohomologies of the first term vanish, and the natural homomorphism

$$H^0(X', \mu^*(m'D+E) - \llcorner B'' \lrcorner) \to H^0(F_{i_0}, (\mu^*(m'D+E) - \llcorner B'' \lrcorner)|_{F_{i_0}})$$

is surjective. On the other hand, as the support of $(B'')^-$ is contained in the exceptional set of μ,

$$H^0(X', \mu^*(m'D+E) - \llcorner B'' \lrcorner) \to H^0(X, m'D+E)$$

is injective and

$$H^0(Y, m'H) \to H^0(X, m'D+E)$$

is bijective. Therefore, for $m' \ge (d-\bar{d}+1)m$, the image of the natural homomorphism

$$H^0(Y, m'H) \to H^0(\bar{Z}, m'H|_{\bar{Z}}) \to H^0(F_{i_0}, \mu^*(m'D+E)|_{F_{i_0}})$$

contains $H^0(F_{i_0}, (\mu^*(m'D+E) - \llcorner B'' \lrcorner)|_{F_{i_0}})$.

Take a general point t in $h(\bar{Z})$ and denote $F_{i_0,t}$ to be the fiber of F_{i_0} over t. By the vanishing theorem, for $m' \geq (d - \bar{d} + 1)m$,

$$h^0(F_{i_0,t}, (\mu^*(m'D+E) - \llcorner B'' \lrcorner)|_{F_{i_0,t}}) = \chi(F_{i_0,t}, (\mu^*(m'D+E) - \llcorner B'' \lrcorner)|_{F_{i_0,t}}).$$

As H is ample on Y, this is a non-zero polynomial of degree at most $\bar{d}$, and hence has at most $\bar{d}$ zero points.

Note that the image of $H^0(Y, m'H) \to H^0(\bar{Z}, m'H|_{\bar{Z}})$ is not 0 implies that $\bar{Z}$ is not contained in the base locus of $|m'H|$. Hence

$$\bar{Z} \subset \mathrm{Bs}\,|(d - \bar{d} + j)mH|$$

can be true for at most $\bar{d}$ values of $j \geq 1$. Using similar argument as in the proof of effective base point free theorem, for $k \geq 2d + 2$,

$$\bar{Z} \not\subset \mathrm{Bs}\,|kmH|.$$

To be more precise, if

$$\bar{Z} \subset \mathrm{Bs}\,|kmH|,$$

then $1 \leq i \leq \bar{d} + 1$, either

$$\bar{Z} \subset \mathrm{Bs}\,|(d - \bar{d} + i)mH|$$

or

$$\bar{Z} \subset \mathrm{Bs}\,|(k - d + \bar{d} - i)mH|,$$

a contradiction. □

2.3 The rationality theorem

The rationality theorem to be proved in this section is a key point of the cone theorem in the next section.

The first part of the rationality theorem shows that certain threshold is a rational number. It concludes the existence of extremal rays in the cone theorem. The second part of the rationality theorem gives an estimate of the denominator of the threshold, which concludes the discreteness of extremal rays. As we will explain later, the discreteness of extremal rays can be proved alternatively by the estimate of length of extremal rays. The latter argument uses the theorem on existence of rational curves and algebraic geometry in positive characteristics.

The proof of the cone theorem uses the argument in the base point free theorem. It is developed in [60] which completed the formulation of the minimal model program.

onality theorem}

Theorem 2.3.1 (*Rationality theorem* [82, Theorem 4.1.1])**.** *Let (X, B) be a KLT pair where B is a $\mathbf{Q}$-divisor, $f : X \to S$ a projective morphism, and A a relatively nef and relatively big divisor. Assume that $K_X + B$ is not relatively nef. Then the threshold*

$$r = \max\{t \in \mathbf{R} \mid A + t(K_X + B) \textit{ is relatively nef}\}$$

is a rational number. Moreover, denote a to be the minimal positive integer such that $a(K_X + B)$ is Cartier and b the maximal dimension of fibers of f, if we write $r/a = p/q$ as irreducible fraction, then

$$q \leq a(b + 1).$$

Proof. The proof is by explore that of the base point free theorem in more details. To the contrary, assume that either r is not a rational number, or r is a rational number but $q > a(b + 1)$.

Step 1. Clearly r is a positive real number. We may assume that S is affine. We firstly reduce to the case that A is free.

By Theorem 2.1.1, we can take sufficiently large integers m, n such that $a < mr$, $(mn, q) = 1$ (if r is a rational number), and $A' = n(mA+a(K_X+B))$ is free. Then the threshold

$$r' = \max\{t \in \mathbf{R} \mid A' + t(K_X + B) \text{ is relatively nef}\}$$

satisfies the relation $mnr = an + r'$. So r is rational if and only if r' is so. Moreover, if r' is rational and we write $r'/a = p'/q'$ as irreducible fraction, then $q = q'$. So after replacing A by A', we may assume that A is free.

Step 2. The following lemma plays a similar role as the non-vanishing theorem in the proof of the base point free theorem. Here our assumption is weaker than that in [82], and the proof is irrelevant to the non-vanishing theorem.

{rationality thm lemma}

Lemma 2.3.2 ([82, Lemma 4.1.2])**.** *Let (X, B) be a projective KLT pair, D_1, D_2, E Cartier divisors on X, d a positive integer, and r', s positive real numbers. For integers x, y, denote $D(x, y) = xD_1 + yD_2$. Assume the following conditions:*

(1) E is effective.

(2) There exists a positive integer y_1 such that $D(x, y)+E-(K_X+B)$ is nef and big and the natural homomorphism $H^0(X, D(x, y)) \to H^0(X, D(x, y)+E)$ is bijective if $x > 0$, $y \geq y_1$, and $y - r'x < s$.

(3) The polynomial $\chi(X, D(x, y) + E)$ in two variables x, y is of degree at most d and not identically zero.

(4) r' is an irrational number, or r' is a rational number and $qs > d+1$ where we write $r' = p/q$ as irreducible fraction.

Then there exists a positive integer y_2 such that $H^0(X, D(x,y)+E) \neq 0$ if $y - r'x < s$ and $y \geq y_2$.

Proof. If r' is irrational, then there are infinitely many couples of positive integers (x, y) such that $0 < y - r'x < s/(d+1)$. If r' is rational, then as p, q are coprime, there are infinitely many couples of positive integers (x, y) such that $y - r'x = 1/q < s/(d+1)$. So in either case, there are infinitely many couples of positive integers (x, y) such that $0 < y - r'x < s/(d+1)$. We may assume that $y \geq y_1$ in each couple.

For any such a couple (x_0, y_0), consider the polynomial $\chi(X, D(mx_0, my_0) + E)$ in m. For any integer m such that $1 \leq m \leq d+1$, $my_0 - mr'x_0 < s$ holds, and hence $D(mx_0, my_0) + E - (K_X + B)$ is nef and big. By the vanishing theorem, higher cohomologies vanish and

$$\chi(X, D(mx_0, my_0) + E) = \dim H^0(X, D(mx_0, my_0) + E).$$

On the other hand, if $H^0(X, D(mx_0, my_0)+E) = 0$ for all $1 \leq m \leq d+1$, then the polynomial $\chi(X, D(x, y) + E)$ in x, y is identically 0 on the line $y_0x - x_0y = 0$. By construction, there are infinitely many such lines, and $\chi(X, D(x, y) + E)$ can not be identically 0 on all such lines. Hence there exists a couple (x', y') such that $x' > 0$, $y' \geq y_1$, $0 < y' - r'x' < s$, and $H^0(X, D(x', y') + E) \neq 0$.

If such a positive integer y_2 in the statement does not exist, then there are infinitely many couples of positive integers (x'', y'') such that $y'' - r'x'' < s$, $x'' > dx'$, $y'' \geq y_1 + dy'$, and $H^0(X, D(x'', y'') + E) = 0$. Since $H^0(X, D(x', y')) \cong H^0(X, D(x', y')+E) \neq 0$, $H^0(X, D(x''-mx', y''-my')+E) = 0$ for $0 \leq m \leq d$ and such a couple (x'', y''). So $\chi(X, D(x, y) + E)$ is identically 0 on infinitely many lines $y'(x - x'') - x'(y - y") = 0$, a contradiction. This concludes the lemma. □

Step 3. If r is rational, by the assumption that $q > a(b+1)$, we may take a sufficiently small positive real number δ such that $q(1-\delta) > a(b+1)$. If r is irrational, just take any $0 < \delta < 1$. Take E, d, r', s to be 0, b, r/a, $(1-\delta)/a$ respectively, and take $D(x, y) = xA + ay(K_X + B)$.

Applying Lemma 2.3.2 to a general fiber of f, we know that there exists a couple of positive integers (x, y) such that $0 < ay - rx < 1 - \delta$ and $H^0(X, D(x, y)) \neq 0$. Note that since S is affine, the nonvanishing of H^0 on a general fiber implies the nonvanishing of on X.

Fix such a couple (x, y). As $ay - rx > 0$, $D(x, y)$ is not relatively nef, and therefore $|D(x, y)|$ is not free. Take a general element $M \in |D(x, y)|$, we are going to apply the argument in the base point free theorem to kill the base locus of M and get a contradiction.

Take a log resolution $g : Y \to X$ of $(X, B + M)$ in strong sense, write $h = f \circ g$, $g^*(K_X + B) = K_Y + C$. Write $g^*M = M_1 + M_2$ where $|M_1|$ is free and M_2 is the fixed part of $|g^*M|$. Take an effective divisor C' such that $\mathrm{Exc}(g) \cup \mathrm{Supp}(C + M_2) = \mathrm{Supp}(C')$ and $-C'$ is g-ample. Take a sufficiently small positive real number ϵ, such that $\delta g^*A - r\epsilon C'$ is h-ample and $\ulcorner -C - \epsilon C' \urcorner = \ulcorner -C \urcorner \geq 0$. Consider the following threshold:

$$c = \sup\{t \in \mathbf{R} \mid \llcorner tM_2 + C + \epsilon C' \lrcorner \leq 0\}.$$

We may assume that there exists exactly one prime divisor Z attaining the maximal coefficient 1 in $cM_2 + C + \epsilon C'$ by perturbing the coefficients of C'. Note that $g(Z)$ is contained in the base locus $\mathrm{Bs}\,|D(x, y)|$. Write

$$cM_2 + C + \epsilon C' = F + Z.$$

Here the support of F does not contain Z and $\ulcorner -F \urcorner \geq 0$.

For a couple of integers (x', y'), consider

$$\begin{aligned} &g^*D(x', y') - F - Z - K_Y \\ &\equiv (x' - cx)g^*A + (ay' - acy)g^*(K_X + B) + cM_1 - (K_Y + C + \epsilon C'). \end{aligned}$$

This $\mathbf{R}$-divisor is h-ample if $x' > cx$, $y' > cy + 1/a$, and $r(x' - cx) \geq a(y' - cy) - 1 + \delta$. In particular, the last one is satisfied if $ay' - rx' < 1 - \delta$.

By Theorem 1.9.1,

$$H^1(Y, g^*D(x', y') + \ulcorner -F \urcorner - Z) = 0$$

and the natural homomorphism

$$H^0(Y, g^*D(x', y') + \ulcorner -F \urcorner) \to H^0(Z, g^*D(x', y')|_Z + \ulcorner -F \urcorner|_Z)$$

is surjective. On the other hand,

$$H^0(Y, g^*D(x', y')) \to H^0(Y, g^*D(x', y') + \ulcorner -F \urcorner)$$

is surjective. By the commutative diagram

$$\begin{array}{ccc} H^0(Y, g^*D(x', y')) & \longrightarrow & H^0(Y, g^*D(x', y') + \ulcorner -F \urcorner) \\ \downarrow & & \downarrow \\ H^0(Z, g^*D(x', y')|_Z) & \longrightarrow & H^0(Z, (g^*D(x', y') + \ulcorner -F \urcorner)|_Z), \end{array}$$

the bottom horizontal arrow is surjective.

Denote $B_Z = (F + \ulcorner -F \urcorner)|_Z$, then (Z, B_Z) is KLT. We may apply Lemma 2.3.2 to the general fiber of $h|_Z : Z \to S$. Here we take D_1, D_2, E to be the restrictions of g^*A, $ag^*(K_X + B)$, $\ulcorner -F \urcorner$, and take d, r', s to be b, r/a, $(1 - \delta)/a$. It is easy to check that the conditions of Lemma 2.3.2 are satisfied, where (3) follows from $\dim g(Z) \le b$. So by Lemma 2.3.2, $g(Z)$ is not contained in $\operatorname{Bs}|D(x', y')|$ if $ay' - rx' < 1 - \delta$ and y' is sufficiently large.

Now consider a couple (x', y') satisfying $0 < ay' - rx' < 1 - \delta$ defined in the following way. If r is irrational, take a sufficiently large integer l such that

$$x' = \llcorner aly/r \lrcorner = lx + \llcorner l(ay - rx)/r \lrcorner, \qquad y' = ly$$

and $ay' - rx' < 1 - \delta$; if r is rational, take a sufficiently large integer l and take

$$x' = x + lq, \qquad y' = y + lp.$$

Note that A is free and in the latter case $l(qA + ap(K_X + B))$ is free by the base point free theorem, hence

$$\operatorname{Bs}|D(x', y')| \subset \operatorname{Bs}|D(x, y)|.$$

To summarize, we constructed a couple (x', y') such that $0 < ay' - rx' < 1 - \delta$ and $\operatorname{Bs}|D(x', y')| \subsetneq \operatorname{Bs}|D(x, y)|$. Applying the Noetherian induction as in the proof of the base point free theorem, there exists a couple of positive integers (x'', y'') such that $0 < ay'' - rx'' < 1 - \delta$ and $D(x'', y'')$ is free. This implies that $x''A + ay''(K_X + B)$ is relatively nef, which contradicts the maximality of r. $\square$

2.4 The cone theorem

The base point free theorem and the cone theorem are two main pillars for the minimal model theory. Higher dimensional minimal model theory started from the introduction of the concept of extremal rays in [107]. Birational geometry becomes visible by looking at cones and polyhedra in finite-dimensional real vector spaces.

The cone theorem states that the cone of curves is locally a rational polyhedral cone in the part with negative value on the canonical divisor. This statement splits into two parts: existence and discreteness of extremal rays. The discreteness of extremal rays can be proved by the rationality theorem proved in the previous section, or the boundedness of length of extremal rays which will be proved later. We will introduce both arguments, the former one stays in characteristic 0, while the latter one uses the positive characteristic method.

2.4.1 The contraction theorem

Generally, a subset $\mathcal{C}$ in a finite dimensional vector space V is called a *cone* if $t\mathcal{C} \subset \mathcal{C}$ for any $t \in \mathbf{R}^*$. It is called *convex* if for any $v, v' \in \mathcal{C}$ and any $t \in [0, 1]$, $tv + (1 - t)v' \in \mathcal{C}$. Consider a convex cone $\mathcal{C}$. A subset F of $\mathcal{C}$ is called a *face* if there exists $u \in V^*$ such that $\mathcal{C} \subset V_{u \ge 0}$ and $F = \mathcal{C}_{u=0}$. Here

$V_{u\geq 0} = \{v \in V \mid (u,v) \geq 0\}$, $\mathcal{C}_{u=0} = \{v \in \mathcal{C} \mid (u,v) = 0\}$. u is called the *supporting function* of F. In particular, a half line which is a face is called an *extremal ray.*

Give $f : X \to S$ and $g : Y \to S$ two projective morphism from normal algebraic varieties, a projective morphism $h : X \to Y$ over S is called a *contraction morphism* if the natural homomorphism $\mathcal{O}_Y \to h_*\mathcal{O}_X$ is bijective. In other words, it is surjective and with connected geometric fibers. Here h is a morphism over S means that $g \circ h = f$. A contraction morphism is also called an *algebraic fiber space.* Usually the former one is used for birational morphisms, and the latter one is mainly used in the case $\dim Y < \dim X$. However these can often be handled uniformly.

Consider a face F of the cone of curves $\overline{\mathrm{NE}}(X/S)$ and a contraction morphism $h : X \to Y$. h is called the *contraction morphism associated to F* if the following conditions are satisfied:

- For a curve C on X, $f(C)$ is a point if and only if $[C] \in F$;
- the smallest closed convex cone containing the equivalence classes of such curves coincides with F.

In particular, h is not an isomorphism if $F \neq 0$.

By the *Zariski main theorem*, the contraction morphism h is determined by the face F and independent of the choice of the supporting function.

The following contraction theorem is a consequence of the base point free theorem (Theorem 2.1.1):

{contraction thm}

Theorem 2.4.1 (*Contraction theorem*). *Let (X, B) be a KLT pair, $f : X \to S$ a projective morphism, and F a face of $\overline{\mathrm{NE}}(X/S)$. Assume that the supporting function u of F is defined over the rational number field and $K_X + B$ takes negative values on $F \setminus \{0\}$. Then the following statements hold:*

(1) The contraction morphism $h : X \to Y$ associated to F exists.

(2) The smallest linear subspace containing F coincides with the image of the injection $N_1(X/Y) \to N_1(X/S)$, and F coincides with the image of $\overline{\mathrm{NE}}(X/Y)$.

(3) $-(K_X + B)$ is h-ample.

(4) If a Cartier divisor D on X is identically 0 on $N_1(X/Y)$, then there exists a Cartier divisor E on Y such that $D \sim h^*E$.

(5) $\rho(X/S) = \rho(Y/S) + \rho(X/Y)$.

Proof. (1) After taking a multiple of u, we may assume that it gives a Cartier divisor L. L is relatively nef by assumption.

As K_X+B is negative on $F\setminus\{0\}$, $\overline{\mathrm{NE}}(X/S)_{K_X+B\geq 0}\cap F=\{0\}$. So L is everywhere non-zero on $\overline{\mathrm{NE}}(X/S)_{K_X+B\geq 0}\setminus\{0\}$ and hence the quotient of functions $(K_X+B)/L$ is well-defined on the compact subset $(\overline{\mathrm{NE}}(X/S)_{K_X+B\geq 0}\setminus\{0\})/\mathbf{R}_{>0}$ in the $(\rho(X/S)-1)$-dimensional sphere $(N_1(X/S)\setminus\{0\})/\mathbf{R}_{>0}$. In particular, $(K_X+B)/L$ is bounded on this subset. Therefore there exists a sufficiently small real number ϵ such that $L-\epsilon(K_X+B)$ is positive on $\overline{\mathrm{NE}}(X/S)\setminus\{0\}$. By Kleiman's criterion, $L-\epsilon(K_X+B)$ is relatively ample.

Applying the base point free theorem, after replacing L by a multiple, we may assume that the natural homomorphism $f^*f_*\mathcal{O}_X(L)\to\mathcal{O}_X(L)$ is surjective. Correspondingly, we get a projective morphism $\bar{h}:X\to\mathbf{P}_S(f_*\mathcal{O}_X(L))$ over S. Here the latter one is the projective scheme over S corresponding to the coherent sheaf $f_*\mathcal{O}_X(L)$. By definition, $\bar{h}^*\mathcal{O}_{\mathbf{P}_S(f_*\mathcal{O}_X(L))}(1)\cong\mathcal{O}_X(L)$.

Take the *Stein factorization* of $\bar{h}$, we get a surjective morphism $h:X\to Y$ to a normal algebraic variety and a finite morphism $Y\to\mathbf{P}_S(f_*\mathcal{O}_X(L))$. Take $g:Y\to S$ to be the induced map.

We claim that h is the contraction morphism associated to F. Firstly, a curve C on X, if $h(C)$ is a point, then $\mathcal{O}_X(L)\otimes\mathcal{O}_C\cong\mathcal{O}_C$ and hence $(L\cdot C)=0$, which implies that $[C]\in F$.

Secondly, take F' to be the closed convex cone spanned by equivalence classes of curves contracted by h, then $F'=\overline{\mathrm{NE}}(X/Y)\subset F$. Assume, to the contrary, that $F'\neq F$, then there exists a Cartier divisor L' on X such that L is positive on $F'\setminus\{0\}$ but negative at some point of F. Note that L' is h-ample and $L=h^*L''$ for some g-ample Cartier divisor L''. Hence for any sufficiently large m, $L'+mh^*L''$ is f-ample and hence positive on $F\setminus\{0\}$. This is a contradiction since h^*L'' is identically 0 on $F\setminus\{0\}$.

(2), (3) are directly from (1).

(4) Since D is h-nef and $D-(K_X+B)$ is h-ample, the base point free theorem (Theorem 2.1.1) can be applied to D and h, which implies that there exists a positive integer m_1, such that mD is h-free for $m\geq m_1$. The corresponding map over Y coincides with h since $mD\equiv 0$ over Y. That is, there exists a Cartier divisor E_m on Y such that $mD\sim h^*E_m$. We can conclude (4) by taking $E=E_{m+1}-E_m$.

From (4), we get the following exact sequence

$$0\to N^1(Y/S)\to N^1(X/S)\to N^1(X/Y)\to 0,$$

which concludes (5). □

Remark 2.4.2. The phenomenon in (4) suggests the fibers of a contraction morphism are special varieties similar to $\mathbf{P}^1$. That is because, for example, on elliptic curves there exist many Cartier divisors which are numerically trivial but not trivial.

Later in Corollary 2.8.4, we will prove that the fibers of a contraction morphism is covered by rational curves. However, rational curves with singularities have similar Cartier divisors as in the case of elliptic curves, so we may expect much stronger statements.

2.4.2 The cone theorem

The shape of the cone of curves $\overline{\mathrm{NE}}(X/S)$ varies, but according to the following cone theorem, if restricted to the part taking negative values on the canonical divisor, then locally it is generated by finitely many extremal rays. By the contraction theorem, those extremal rays associates with contraction morphisms.

{cone thm}

Theorem 2.4.3 (*Cone theorem*)**.** *Let (X, B) be a KLT pair, $f : X \to S$ a projective morphism. Fix a relatively ample divisor A and a positive real number ϵ. Then there exist finitely many extremal rays R_i of $\overline{\mathrm{NE}}(X/S) \subset N_1(X/S)$, such that*

$$\overline{\mathrm{NE}}(X/S) = \overline{\mathrm{NE}}(X/S)_{K_X+B+\epsilon A \geq 0} + \sum R_i.$$

This equation means that the smallest convex cone containing all terms on the right hand side is the left hand side. Moreover, after removing unnecessary terms in the sum, for each i, $K_X + B$ is negative on $R_i \setminus \{0\}$, and there exists a contraction morphism $h_i : X \to Y_i$ associated to the extremal ray R_i.

Proof. We do induction on $\rho(X/S) = \dim N_1(X/S)$. In the proof, the relative setting plays an important role.

Step 1. We may assume that ϵ is a rational number. We will show that we may also assume that B is a $\mathbf{Q}$-divisor.

We may write $K_X + B = \sum r_i D_i$ where $D_1, \ldots, D_t$ are Cartier divisors. We may approximate real numbers r_i by rational numbers r_i', such that $\sum(r_i - r_i')D_i + \epsilon A/3$ is ample.

As $B' = \sum r_i' D_i - K_X$ is not necessarily effective, write $B' = (B')^+ - (B')^-$. Here $(B')^+, (B')^-$ are effective $\mathbf{Q}$-divisors with no common components. If taking $r_i' - r_i$ sufficiently small, then the coefficients of $(B')^-$ are sufficiently small, and there exists an effective $\mathbf{Q}$-divisor B'' with sufficiently small coefficients such that $\epsilon A/3 - (B')^- \sim_{\mathbf{Q}} B''$.

Also by taking $r_i' - r_i$ sufficiently small, we may assume that $(X, (B')^+ + B'')$ is again KLT. Once we proved the statement for $(X, (B')^+ + B'')$, the statement for (X, B) follows from the fact that

$$\overline{\mathrm{NE}}(X/S)_{K_X+(B')^+ + B'' + \epsilon A/3 \geq 0} \subset \overline{\mathrm{NE}}(X/S)_{K_X+B+\epsilon A \geq 0}.$$

Therefore we may assume that B is a $\mathbf{Q}$-divisor.

Step 2. If $\rho(X/S) = 1$, then there is nothing to prove. So we assume that $\rho(X/S) > 1$ in the following. Also we may assume that $K_X + B$ is not relatively nef.

For any relatively ample **Q**-divisor H, by the rationality theorem (Theorem 2.3.1), the threshold

$$r_H = \max\{t \in \mathbf{R} \mid H + t(K_X + B) \text{ is relatively nef}\} \in \mathbf{Q}$$

determines a **Q** divisor $L_H = H + r_H(K_X + B)$. By construction, L_H is relatively nef but not relatively ample. We know that

$$F_H = \overline{\mathrm{NE}}(X/S)_{L_H=0}$$

is a face of the cone or curves and satisfies the contraction theorem (Theorem 2.4.1). Denote $h_H : X \to Y_H$ to be the corresponding contration.

Step 3. Take $\mathcal{C}$ to be the closed convex cone containing $\overline{\mathrm{NE}}(X/S)_{K_X+B\geq 0}$ and all F_H with $L_H \not\equiv 0$. We will show that $\overline{\mathrm{NE}}(X/S) = \mathcal{C}$. Note that in this step there might be infinitely many F_H.

To the contrary, assume that $\mathcal{C} \neq \overline{\mathrm{NE}}(X/S)$. The there exists a **Q**-divisor M such that $(M \cdot v) > 0$ for all $v \in \mathcal{C} \setminus \{0\}$ and $(M \cdot v_0) < 0$ for some $v_0 \in \overline{\mathrm{NE}}(X/S)$. Moreover, M can not be a multiple of $K_X + B$.

The dual closed convex cone $(\overline{\mathrm{NE}}(X/S)_{K_X+B\geq 0})^*$ of $\overline{\mathrm{NE}}(X/S)_{K_X+B\geq 0}$ is just the closed convex cone spanned by $\overline{\mathrm{Amp}}(X/S)$ and $K_X + B$, because the dual of the latter one is $\overline{\mathrm{NE}}(X/S)_{K_X+B\geq 0}$.

Since M is positive on $\overline{\mathrm{NE}}(X/S)_{K_X+B\geq 0} \setminus \{0\}$, it is an interior point of $(\overline{\mathrm{NE}}(X/S)_{K_X+B\geq 0})^*$. Therefore, we can write $M = H + t(K_X + B)$ for some relatively ample **Q**-divisor H and some rational positive number t.

Since M is not relatively nef, $r_H < t$. On the other hand, since $L_H = H + r_H(K_X + B) \not\equiv 0$, we have $F_H \subset \mathcal{C}$ and hence M is positive on $F_H \setminus \{0\}$. This is a contradiction.

Step 4. Take $\mathcal{C}_1$ to be the closed convex cone containing $\overline{\mathrm{NE}}(X/S)_{K_X+B\geq 0}$ and all extremal rays of the form $R_H = F_H$. We will show that $\overline{\mathrm{NE}}(X/S) = \mathcal{C}_1$. Note that in this step there might be infinitely many extremal rays R_H.

For a face F_H with $\dim F_H \geq 2$, we may apply Step 3 to $F_H = \overline{\mathrm{NE}}(X/Y_H) \subset \overline{\mathrm{NE}}(X/S)$. Since $(F_H)_{K_X+B\geq 0} = \{0\}$, F_H is generated by lower dimensional faces.

Step 5. We will show the *discreteness* of extremal rays by applying the estimate of denominators in the rationality theorem, that is, to show that there are only finitely many extremal rays negative on $(K_X + B + \epsilon A)$.

For each extremal ray R_i, take the associated contraction morphism $h_i : X \to Y_i$. Since $-(K_X + B)$ is h_i-ample, there is a unique element $v_i \in R_i$ with $(a(K_X + B) \cdot v_i) = -1$.

Take relatively ample Cartier divisors $H_1, \ldots, H_{\rho(X/S)-1}$ such that together with $a(K_X+B)$ they form a basis of $N_1(X/S)$. Since $\dim N_1(X/Y_i) = 1$, we can define r_{ij} such that $H_j + r_{ij}(K_X + B) \equiv 0$ over Y_i. Applying the rationality theorem (Theorem 2.3.1) to h_i, we can express $r_{ij}/a = p_{ij}/q_{ij}$ as irreducible fraction, and $q_{ij} \leq a(b+1)$. Here a is the minimal positive integer such that $a(K_X + B)$ and b is the maximal dimension of fibers of f. Therefore, $(a(b+1))!(H_j \cdot v_i) \in \mathbf{Z}$.

Take a sufficiently large number N such that $NA - H_j$ is f-ample for all j. If we only look at extremal rays R_i such that $((K_X + B + \epsilon A) \cdot v_i) < 0$, then

$$(H_j \cdot v_i) < (NA \cdot v_i) < N/a\epsilon,$$

and hence there are only finitely many possible values for $(H_j \cdot v_i)$. This mean that there are only finitely many extremal rays generated by v_i.

Step 5'. Let us give another proof of discreteness of extremal rays by applying the estimate of length of extremal rays instead of the rationality theorem.

Keep the notation in last step. By Corollary 2.8.4, there exists an h_i-relative curve C_i such that $(-(K_X + B) \cdot C_i) \leq 2b$. If we only look at extremal rays R_i such that $((K_X + B + \epsilon A) \cdot C_i) < 0$, then $(A \cdot C_i) < 2b/\epsilon$.

As the degree of C_i is bounded, there exists a scheme of finite type W and a closed subscheme $V \subset X \times W$ such that C_i's appear as fibers of the projection $\phi : V \to W$. Therefor there are only finitely many numerical equivalence classes of those C_i.

Also we can use the following argument. Since $(-a(K_X + B) \cdot C_i) \leq 2ab$ and $(H_j \cdot C_i) \in \mathbf{Z}$, we have $(2ab)!(H_j \cdot C_i) \in \mathbf{Z}$. Then we can argue the same as the end of Step 5. □

Remark 2.4.4. (1) The contraction theorem was firstly proved in the case that X is smooth, $B = 0$, and $\dim X \leq 3$ ([107]). The proof is by completely classifying the contraction morphisms. The classification shows for the first time that even if we start from a smooth X, the image of the contraction morphism may have singularities, which is different from the surface case. The general contraction theorem was proved in a completely different way as an application of the base point free theorem ([59],[60]).

(2) The contraction theorem was firstly proved in the case that X is smooth and $B = 0$ (Mori [107]). The proof efficiently uses Frobenius morphisms in positive characteristics (Theorem 2.7.2 is an application of this method). However, this method uses deformation theory, which is difficult to be generalized to algebraic varieties with singularities so it is limited as in the minimal model theory we can not avoid dealing with algebraic varieties with singularities. Therefore, a completely different proof was developed by extending that of the base point free theorem ([60]).

(3) In Step 5' of the proof, it might be possible to get a stronger estimate of length of extremal rays $(-(K_X+B)\cdot C) \le b+1$. This is still an open problem.

(4) When considering an extremal ray R in this book, we always assume that $K_X + B$ is negative on $R \setminus \{0\}$. Such an extremal ray is called a $(K_X + B)$-negative extremal ray.

{big finite}

Corollary 2.4.5. *Keep the assumption of Theorem 2.4.3. Assume that B is $\mathbf{R}$-Cartier and relatively big. Then there are only finitely many $K_X + B$-negative extremal rays in $\overline{\mathrm{NE}}(X/S)$.*

Proof. Write $B = A + E$ for some relatively ample $\mathbf{R}$-divisor A and some effective $\mathbf{R}$-divisor E. We may take a sufficiently small positive real number ϵ such that $(X, (1-\epsilon)B+\epsilon E)$ is KLT. Note that $K_X+(1-\epsilon)B+\epsilon E+\epsilon A = K_X+B$. By Theorem 2.4.3, there are only finitely many $K_X+(1-\epsilon)B+\epsilon E+\epsilon A$-negative extremal rays. □

It is easy to extend the cone theorem to DLT pairs:

Corollary 2.4.6. *Let (X, B) be a DLT pair, $f : X \to S$ a projective morphism. Fix a relatively ample divisor A and a positive real number ϵ. Then there exist finitely many extremal rays R_i of $\overline{\mathrm{NE}}(X/S) \subset N_1(X/S)$, such that*

$$\overline{\mathrm{NE}}(X/S) = \overline{\mathrm{NE}}(X/S)_{K_X+B+\epsilon A\ge 0} + \sum R_i.$$

Moreover, after removing unnecessary terms in the sum, for each i, K_X+B is negative on $R_i \setminus \{0\}$, and there exists a contraction morphism $h_i : X \to Y_i$ associated to the extremal ray R_i.

Proof. By Lemma 2.1.7, there is $B' \equiv_S B + \frac{1}{2}\epsilon A$ such that (X, B') is KLT. The corollary can be reduced to the cone theorem. □

2.4.3 Contraction morphisms in dimensions 2 and 3

In this section, we describe the contraction morphism associated to an extremal ray in dimension at most 3. Firstly let us consider the surface case.

{2d contration}

Example 2.4.7. Consider the case that X is smooth, $S = \mathrm{Spec}(k)$, $B = 0$, and $\dim X = 2$. Here the base field k is algebraically closed of arbitrary characteristic.

The contraction morphism $\phi : X \to Y$ associated to an extremal ray R can be classified as the following ([107]).

(a) There exists a (-1)-curve $C \subset X$ such that $R = \mathbf{R}_+[C]$. Y is smooth, $\phi(C) = P$ is a point, and ϕ is the blowup of Y at P. Conversely, a (-1)-curve always generates an extremal ray.

(b) $\phi : X \to Y$ is a $\mathbf{P}^1$-bundle over a smooth curve Y, and $R = \mathbf{R}_+[C]$ for any fiber C. In this case, X is called a *ruled surface*. Conversely, if X admits a $\mathbf{P}^1$-bundle structure, its fiber generates an extremal ray.

(c) $X \cong \mathbf{P}^2$, $Y = \operatorname{Spec} k$, and R is generated by the equivalence class of a line on $\mathbf{P}^2$.

It is important that, in each case, the extremal ray is generated by a curve isomorphic to $\mathbf{P}^1$.

As we will see later, the theory of extremal rays can be extended to algebraically non-closed base field k. Take the base change $\bar{X} = X \times \operatorname{Spec} \bar{k}$ to algebraic closure, the classification can be generalized as the following.

(a') There exist disjoint (-1)-curves $C_1, \dots, C_t$ on $\bar{X}$ such that a multiple of their sum $C = m \sum C_i$ is defined over k, and $R = \mathbf{R}_+[C]$. Y is smooth, $\phi(C) = P$ is a point, and ϕ is the blowup of Y at P. Here the residue field of P is an extension of k.

(b') $\phi : X \to Y$ is a morphism to a smooth curve Y, and $R = \mathbf{R}_+[C]$ for any fiber C. In this case, every fiber is isomorphic to a curve of degree 2 in $\mathbf{P}^2$, and X is called a *conic surface.*

(c') $-K_X$ is ample and $\rho(X) = 1$. Here $\rho(X) = \dim N^1(X)$ is the *Picard number.* Generally, a smooth projective surface with ample anticanonical divisor is called a *del Pezzo surface.* There is a classical classification of del Pezzo surfaces.

The following example shows that there can be infinitely many extremal rays.

{nagata example}

Example 2.4.8 (Nagata's example)**.** By the cone theorem, there are only finitely many $K_X + B + \epsilon A$-negative extremal rays, but when taking limit $\epsilon \to 0$, it is possible to have infinitely many extremal rays. Here the base field k is algebraically closed of characteristic 0.

Givne two curves C_1, C_2 of degree 3 on $\mathbf{P}^2$ intersecting at 9 distinct points $P_1, \dots, P_9$. The rational function h defined by $\operatorname{div}(h) = C_1 - C_2$ determines a rational map $\bar{h} : \mathbf{P}^2 \dashrightarrow \mathbf{P}^1$. The indeterminacy locus of $\bar{h}$ is $\{P_1, \dots, P_9\}$. The blowup along those points $f : X \to \mathbf{P}^2$ resolves the indeterminacy and gives a morphism $g = \bar{h} \circ f$.

For a smooth curve C of degree 3 passing through these 9 points, its strict transform $F = f_*^{-1}C$ becomes a smooth fiber of g, and $K_X = -F$. In particular, F is an elliptic curve. The exceptional set of f is 9 (-1)-curves E_i $(i = 1, \dots, 9)$, which are sections of g.

The generic fiber F_η of g is an elliptic curve defined over $k(\mathbf{P}^1)$. Take $Q_i = E_i \cap F_\eta$. Consider the additive group structure on F_η with Q_1 as the origin. If C_1, C_2 are chosen generally, Q_2 is not a torsion point, that is, $mQ_2 \neq Q_1$ for all positive integer m. Take G_m to be the closure of mQ_2 in X, which is a section of g. Then $G_m \cong \mathbf{P}^1$ and $(K_X \cdot G_m) = -1$. That is, G_m is a (-1)-curve. In this case, there are infinitely many extremal rays.

Take $S = \{(P_1, \ldots, P_9) \in (\mathbf{P}^2)^9 \mid P_i \neq P_j \ (i \neq j)\}$. The projection $\mathbf{P}^2 \times S \to S$ naturally admits 9 sections. Take $\tilde{f} : \mathcal{X} \to \mathbf{P}^2 \times S$ to be the blowup along those sections, then the above constructed X is a fiber of the smooth morphism $\pi : \mathcal{X} \to S$. That is, π is a *deformation family* of X.

As (-1)-curves are preserved by small deformations, for each m there exists a non-empty open set U_m and a closed subvariety $\tilde{G}_m$ of $\pi^{-1}(U_m)$ such that $\tilde{G}_m \cap X = G_m$ and on each fiber $X_s = \pi^{-1}(s)$ $(s \in U_m)$ $\tilde{G}_m \cap X_s$ is a (-1)-curve. In the case that the base field k is the complex number field, the intersection $U = \bigcap U_m$ is not empty, and for each $s \in U$, X_s has infinitely many extremal rays.

Generally, if there exists a non-empty open set such that a property holds for each point in this set, then we say that this property holds for *general* points; if a property holds for each point in the intersection of countably infinitely many non-empty open sets, like the above U, then we say that this property holds for *very general* points. So a very general fiber of π has infinitely many extremal rays.

The 3-dimensional case is as the following.

{3d contraction}

Example 2.4.9. Consider the case that X is smooth, $S = \operatorname{Spec} k$, $B = 0$, and $\dim X = 3$. The contraction morphism $\phi : X \to Y$ associated to an extremal ray R can be classified as the following ([107]). Here the base field k is algebraically closed of characteristic 0.

(a) The exceptional set of ϕ is a prime divisor E and ϕ is the blowup of Y along $\phi(E)$. However, Y is not necessarily smooth. E and ϕ are classified as the following.

(a-1) $\phi(E) = P$ is a point, $E \cong \mathbf{P}^2$, and $\mathcal{O}_E(E) \cong \mathcal{O}_{\mathbf{P}^2}(-1)$. In this case Y is smooth.

(a-2) $\phi(E) = P$ is a point, $E \cong \mathbf{P}^2$, $\mathcal{O}_E(E) \cong \mathcal{O}_{\mathbf{P}^2}(-2)$. If $k = \mathbf{C}$, then (Y, P) is analytically isomorphic to the quotient singularity of type $\frac{1}{2}(1,1,1)$.

(a-3) $\phi(E) = P$ is a point, E is isomorphic to the quadratic surface in $\mathbf{P}^3$ defined by $xy + zw = 0$, $\mathcal{O}_E(E) \cong \mathcal{O}_E(-1)$. E is isomorphic to $\mathbf{P}^1 \times \mathbf{P}^1$. If $k = \mathbf{C}$, then the singularity (Y, P) is analytically isomorphic to the hypersurface singularity defined by $xy + zw = 0$ in $\mathbf{C}^4$.

(a-4) $\phi(E) = P$ a point, E is isomorphic to the quadratic surface in $\mathbf{P}^3$ defined by $xy + z^2 = 0$, $\mathcal{O}_E(E) \cong \mathcal{O}_E(-1)$. If $k = \mathbf{C}$, then the singularity (Y, P) is analytically isomorphic to the hypersurface singularity defined by $xy + z^2 + w^3 = 0$ in $\mathbf{C}^4$.

(a-5) $\phi(E) = C$ is a smooth projective curve, $\phi|_E : E \to C$ is a $\mathbf{P}^1$-bundle, and $(E \cdot F) = -1$ for each fiber F. In this case Y is smooth.

(b) Y is a smooth projective surface, the geometric generic fiber of ϕ is isomorphic to $\mathbf{P}^1$. Every fiber of ϕ is isomorphic to a conic curve in $\mathbf{P}^2$, hence X is called a *conic bundle*.

(c) Y is a smooth projective curve, the geometric generic fiber of ϕ is a del Pezzo surface.

(d) Y is a point, X is a *Fano manifold* of Picard number $\rho(X) = 1$. Generally, a smooth projective algebraic variety X is called a Fano manifold, if $-K_X$ is ample. 3-dimensional Fano manifolds are classified ([50], [51], [110], [111]).

2.4.4 The cone theorem for cone of divisors

Taking the dual of the contraction theorem and the cone theorem, we can describe them in terms of cones of divisors. The paraphrase is powerful when considering birational models. For example, when the nef cones of two birational models adjoin along a face of both cones, the phenomenon of *wall crossing* is important, and can be described appropriately in space of divisors. Here a wall is a face of codimension 1, which is the dual concept of an extremal ray.

{cone thm for div}

Theorem 2.4.10. *Let (X, B) be a KLT pair and $f : X \to S$ a projective morphism. Fix a relatively ample divisor A and a positive real number ϵ. Assume that $K_X + B + \epsilon A$ is not f-nef. Take R_i $(i = 1, \dots, N)$ to be all $(K_X + B + \epsilon A)$-negative extremal rays, and $h_i : X \to Y_i$ contraction morphism associated to R_i. Fix a non-zero rational point $v_i \in R_i$ for each i. Note that v_i can be viewed as a linear function on $N^1(X/S)$. Then the following statements hold:*

(1) v_i is non-negative on $\overline{\text{Amp}}(X/S)$.

(2) $G_i = \{u \in \overline{\text{Amp}}(X/S) \mid (u \cdot v_i) = 0\}$ is a face of codimension 1 in $\overline{\text{Amp}}(X/S)$ which coincides with $h_i^* \overline{\text{Amp}}(Y_i/S)$.

(3) Take F to be the face of $\overline{\text{NE}}(X/S)$ generated by several extremal rays $R_{i_1}, \dots, R_{i_r}$ and $h : X \to Y$ the associated contraction morphism. Then

$$G = \bigcap_{j=1}^{r} G_{i_j} = \{u \in \overline{\text{Amp}}(X/S) \mid (u \cdot v) = 0 \text{ for all } v \in F\}$$

is $h^*\overline{\mathrm{Amp}}(Y/S)$.

(4) *For any f-ample* $\mathbf{R}$*-divisor H, take*

$$t_0 = \min\{t \mid K_X + B + \epsilon A + tH \text{ is } f\text{-nef}\},$$

then there exists a face G of the form $G = \bigcap_{j=1}^r G_{i_j}$ such that $[K_X + B + \epsilon A + t_0 H]$ is a relative interior point of G. In other words, it is contained in $h^*\mathrm{Amp}(Y/S)$.

Proof. (1) This follows from $R_i \subset \overline{\mathrm{NE}}(X/S)$.

(2) By the contraction theorem, $G_i = h_i^*\overline{\mathrm{Amp}}(Y_i/S)$ is of condimension 1.

(3) This is a consequence of the contraction theorem.

(4) By definition, $u = [K_X + B + \epsilon A + t_0 H]$ is the supporting function of a face F of $\overline{\mathrm{NE}}(X/S)$. By the cone theorem, such a face is generated by extremal rays, say $R_{i_1}, \dots, R_{i_r}$, which implies that u is contained in $G = \bigcap_{j=1}^r G_{i_j}$. As u is the supporting function of F, u is an interior point of G. □

Remark 2.4.11. In other words, the cone theorem can be explained as the following: imagine the nef cone as an opaque planet, and $[K_X + B] \in N^1(X/S)$ as a satellite moving around it. Firstly, if $[K_X + B] \in \overline{\mathrm{Amp}}(X/S)$, then we can observe nothing and hence the statement is empty. If $[K_X + B] \notin \overline{\mathrm{Amp}}(X/S)$, then we can observe the front side V of the surface $\partial\overline{\mathrm{Amp}}(X/S)$ of the nef cone. The back side $\partial\overline{\mathrm{Amp}}(X/S) \setminus V$ can not be observed.

When we look at the planet from a slightly closer observation point $[K_X + B + \epsilon A] \in N^1(X/S)$, the surface V looks like a polyhedron consisting of finitely many faces G_i. If we move the observation point to the limit $[K_X + B]$ as $\epsilon \to 0$, in the case of infinitely many extremal rays, there turns out to be infinitely many faces converging to the horizon.

As a corollary, we get the base point free theorem for $\mathbf{R}$-divisors:

{abundance thm R1}

Corollary 2.4.12. *Let (X, B) be a KLT pair, $f : X \to S$ a projective morphism, and D an* $\mathbf{R}$*-Cartier divisor. Assume that D is f-nef, and $D - (K_X + B)$ is f-nef and f-big. Then there exists a projective morphism $g : Z \to S$ from a normal algebraic variety, a projective surjective morphism $h : X \to Z$ with connected geometric fibers such that $f = g \circ h$, and a g-ample* $\mathbf{R}$*-Cartier divisor H on Z, such that $h^*H \sim_{\mathbf{R}} D$.*

Proof. We may assume that D is not f-ample. Since $D - (K_X + B)$ is f-big, we may write $D - (K_X + B) = A + E$ for some f-ample $\mathbf{R}$-Cartier divisor A and some effective $\mathbf{R}$-Cartier divisor E. Since $D - (K_X + B)$ is also f-nef, for any sufficiently small positive real number ϵ, $L = D - (K_X + B) - \epsilon E$ is f-ample.

By taking ϵ sufficiently small, we may assume that $(X, B + \epsilon E)$ is KLT. Since D is not f-ample, $K_X + B + \epsilon E = D - L$ is not f-nef. Therefore, for a sufficiently small $\delta > 0$, $K_X + B + \epsilon E + \delta L$ is not f-nef. Consider

$$t_0 = \min\{t \mid K_X + B + \epsilon E + \delta L + tL \text{ is } f\text{-nef}\},$$

then $t_0 = 1 - \delta$ and $K_X + B + \epsilon E + \delta L + t_0 L = D$. Then the conclusion follows from Theorem 2.4.10(4). □

As a corollary of the above corollary, we can show the existence of *canonical models* when the boundary is big:

{abundance thm R2}

Corollary 2.4.13. *Let (X, B) be a KLT pair, $f : X \to S$ a projective morphism. Assume that $K_X + B$ is f-nef, and B is an f-big $\mathbf{R}$-Cartier divisor. Then there exists a projective morphism $g : Z \to S$ from a normal algebraic variety, a projective surjective morphism $h : X \to Z$ with connected geometric fibers such that $f = g \circ h$, and a g-ample $\mathbf{R}$-Cartier divisor H on Z, such that $h^*H \sim_{\mathbf{R}} K_X + B$.*

Proof. Take $D = K_X + B$. Then D is f-nef. As B is f-big, we may write $B = A + E$ for some f-ample $\mathbf{R}$-Cartier divisor A and some effective $\mathbf{R}$-Cartier divisor E. Take a sufficiently small $\epsilon > 0$, such that $(X, (1-\epsilon)B+\epsilon E)$ is KLT. Then, $D-(K_X+(1-\epsilon)B+\epsilon E) = \epsilon A$ is f-ample, and we can apply Corollary 2.4.12. □

Also by applying Lemma 2.1.7, we can generalize Corollary 2.4.12 to DLT pairs. The proof is left to the readers.

{DLT BPF R-div}

Corollary 2.4.14. [2.4.4.1] *Let (X, B) be a DLT pair, $f : X \to S$ a projective morphism, and D an $\mathbf{R}$-Cartier divisor. Assume that D is f-nef, and $D - (K_X + B)$ is f-ample. Then there exists a projective morphism $g : Z \to S$ from a normal algebraic variety, a projective surjective morphism $h : X \to Z$ with connected geometric fibers such that $f = g \circ h$, and a g-ample $\mathbf{R}$-Cartier divisor H on Z, such that $h^*H \sim_{\mathbf{R}} D$.*

2.5 Types of contraction morphisms and the minimal model program

The minimal model program is an operation to modify a given pair consisting of a variety and a boundary by applying birational maps repeatedly. The pair we consider is assumed to be KLT or DLT, and the variety is assumed to be $\mathbf{Q}$-factorial and projective over the base variety. This condition is preserved under the operation of the minimal model program.

[2.4.4.1] Added in translation

Such an operation is constructed by the contraction morphism associated to an extremal ray. There are 3 types of contraction morphisms: divisorial contractions, small contractions, and Mori fiber spaces.

The goal of the minimal model program is to obtain either a minimal model (a pair with relatively nef log canonical divisor) or a Mori fiber space.

2.5.1 Classification of contraction morphisms

Firstly, consider the case that the contraction morphism associated to an extremal ray is a birational morphism contracting a divisor:

{divisorial contraction}

Theorem 2.5.1. *Let (X, B) be a DLT pair and $f : X \to S$ a projective morphism. Assume that X is $\mathbf{Q}$-factorial. Let R be a $(K_X + B)$-negative extremal ray of $\overline{\mathrm{NE}}(X/S)$, take $h : X \to Y$ to be the contraction morphism associated to R. Assume that h is birational and its exceptional set contains a prime divisor. Then the following statements hold:*

(1) $-(K_X + B)$ is h-ample.

(2) $\rho(X/Y) = 1$, $\rho(X/S) = \rho(Y/S) + 1$.

(3) The exceptional set of h is a prime divisor, say E.

(4) Y is $\mathbf{Q}$-factorial.

(5) We can write $K_X + B = h^*(K_Y + B_Y) + eE$, $e > 0$. Here $B_Y = h_*B$.

(6) (Y, B_Y) is DLT. Moreover, if (X, B) is KLT, then (Y, B_Y) is KLT.

Proof. (1), (2) follow directly from the contraction theorem.

(3) Let E be a prime divisor contained in the exceptional set of h. Since X is $\mathbf{Q}$-factorial, E is $\mathbf{Q}$-Cartier. Since E is exceptional, by Lemma 1.6.3, there exists a curve C contracted by h such that $(E \cdot C) < 0$. Since C is a relative curve over Y and $\rho(X/Y) = 1$, $-E$ is h-ample. Suppose that E does not coincide with the exceptional set of h, then there exists a relative curve C' not contained in E. This implies that $(E \cdot C') \geq 0$, a contradiction. Therefore, the exceptional set of h is a prime divisor.

(4) Take any prime divisor F of Y. X is $\mathbf{Q}$-factorial, $h_*^{-1}F$ is $\mathbf{Q}$-Cartier. Since $\rho(X/Y) = 1$, there exists a rational number r such that $h_*^{-1}F + rE \equiv 0$ over Y. By the contraction theorem (Theorem 2.4.1), there exists a $\mathbf{Q}$-Cartier divisor F' on Y such that $h_*^{-1}F + rE \sim_{\mathbf{Q}} h^*F'$. Since h is birational, $F \sim_{\mathbf{Q}} F'$, which means that F is $\mathbf{Q}$-Cartier.

(5) Write $h^*(K_Y + h_*B) = K_X + B - eE$, Since $-(K_X + B)$ and $-E$ are h-ample, we know that $e < 0$.

(6) follows from (5). $\square$

Let (X, B) be a $\mathbf{Q}$-factorial DLT pair and $f : X \to S$ a projective morphism. If $K_X + B$ is relatively nef, then $f : (X, B) \to S$ is already minimal. If not, then by the cone theorem, there exists a $(K_X + B)$-negative extremal ray R in $\overline{\mathrm{NE}}(X/S)$. Take $h : X \to Y$ to be the contraction morphism associated to R. By Theorem 2.5.1, we have the following 3 cases:

(1) *Divisorial contraction*: h is birational and the exceptional set is a prime divisor.

(2) *Small contraction*: h birational and the exceptional set is of codimension at least 2.

(3) *Mori fiber space*: $\dim Y < \dim X$.

If h is a divisorial contraction, then the new pair (Y, B_Y) has the same property as (X, B). If $K_Y + B_Y$ is not relatively nef, that is, it is not a minimal model, then we can continue to consider it contraction morphisms. Moreover, since $\rho(Y/S) = \rho(X/S) - 1$, there can not be infinitely many divisorial contractions in this procedure. So we may expect to get a minimal model by induction on the Picard number $\rho(X/S)$.

For example, for a pair where X is a smooth projective surface and $B = 0$, a divisorial contraction is the contraction of a (-1)-curve (see Example 2.4.7). Then after finitely many divisorial contractions, there is no (-1)-curve, and we reach a minimal model in the classical sense. This model is either a minimal model in the sense of this book, or admits a further contraction. By dimension reason, this contraction is not small, hence a Mori fiber space, that is, a ruled surface or $\mathbf{P}^2$.

However, this is not the case in higher dimensions due to the existence of small contractions. In dimension 3, small contractions appear only if X is singular or $B \neq 0$ (see Example 2.4.9). In dimension 4 or higher, small contractions can appear even if X is smooth and $B = 0$ (see [66]).

Although Mori fiber spaces are not birational, but it is interesting to be able to handle them in a same category of contraction morphisms. A Mori fiber space is also called a *Fano fibration*.

In general, an algebraic variety X is called a *uniruled variety* if it is covered by a family of rational curves. In other words, this condition means that there exists an algebraic variety Z with $\dim Z = \dim X - 1$ and a dominant rational map $Z \times \mathbf{P}^1 \dashrightarrow X$. Uniruledness is a property invariant under birational equivalence.

As later described by the length of extremal rays (Section 2.8), each irreducible component of any fiber of a contraction morphism is always uniruled, unless it is a point. One image of the minimal model program is that "if you contract redundant rational curves by contraction morphisms, then you will get a minimal model". In particular, an algebraic variety with a Mori fiber space structure is a uniruled variety. Moreover, Hacon and McKernan showed further that the fibers of contraction morphisms are always *rationally connected* ([38]).

For Mori fiber spaces we have the following result:

Proposition 2.5.2. *Let $h : X \to Y$ be a Mori fiber space. Then Y is $\mathbf{Q}$-factorial.*

Proof. We may assume that $\dim Y > 0$. Take any prime divisor E on Y and take a prime divisor D on X such that $h(D) = E$. Since X is $\mathbf{Q}$-factorial, there exists a positive integer d such that dD is Cartier. Since $\rho(X/Y) = 1$ and there exists a curve C contained in a fiber of h such that $D \cap C = \emptyset$, we get $D \equiv_Y 0$. Apply the contraction theorem to h, there exists a Cartier divisor E' on Y and a rational function on X such that $dD = h^*E' + \mathrm{div}(g)$. Since $\mathrm{div}(g)$ does not intersect general fibers of h, there exists a rational function g' on Y such that $g = h^*(g')$. Since $h(D) = E$, we know that $dE = E' + \mathrm{div}(g')$ and hence E is $\mathbf{Q}$-Cartier. □

2.5.2 Flips

The existence of small contractions is a phenomenon appearing only in dimension 3 and higher, which is completely different from the situation of dimension 2. If $X \to Y$ is a small contraction and we consider the pair (Y, h_*B), then $K_Y + h_*B$ is not $\mathbf{R}$-Cartier. In fact, if $K_Y + h_*B$ is $\mathbf{R}$-Cartier, then we can consider its pullback by h. Since X and Y are isomorphic in codimension one, $h^*(K_Y + h_*B) = K_X + B$. On the other hand, take any curve C contracted by h, then $((K_X + B) \cdot C) < 0$, which contradicts to the projection formula (before Proposition 1.4.3).

By this reason, we need to construct a new pair by an operation called flip. The new pair obtained by flip has the same properties as the original pair. Flips and divisorial contractions are completely different operations in geometry, but they are very similar in the point view of numerical geometry.

Definition 2.5.3. Let (X, B) be a $\mathbf{Q}$-factorial DLT pair and $f : X \to S$ a projective morphism. Assume that $g : X \to Y$ is a small contraction morphism associated to a $(K_X + B)$-negative extremal ray R. Then another projective birational morphism $g^+ : X^+ \to Y$ is called the *flip* of g if the following conditions are satisfied:

(1) g^+ is isomorphic in codimension 1.

(2) $K_{X^+} + B^+$ is g^+-ample, here B^+ is the strict transform of B.

Here note that the positivity of log canonical divisors $K_X + B$ and $K_{X^+} + B^+$ are reversed. The birational transform $(g^+)^{-1} \circ g$ is also called a flip.

When considering the existence of the flip of a small contraction, as ampleness is an open condition, it suffices to consider the case that B is a $\mathbf{Q}$-divisor. In fact, the ampleness of $-(K_X + B)$ and $K_{X^+} + B^+$ is not changed after perturbing B slightly. Similarly, it suffices to consider KLT pairs instead of DLT pairs.

{flip example}

Example 2.5.4. Let us give two examples of flips. Both examples are flips of toric varieties [128].

(1) Let us consider the example by Francia ([26]). Here $\dim X = 3$, $B = 0$, and X is singular. We denote $X = X^-$. Originally, this example intended to claim that "the minimal model theory is impossible in dimension 3 or higher", but later it was included into the development of the minimal model theory, and become the simplest example of flips (see Figure **??**).

Consider the locally free sheaf $F = \mathcal{O}_{\mathbf{P}^1}(-1) \oplus \mathcal{O}_{\mathbf{P}^1}(-2)$ over $C^+ = \mathbf{P}^1$, take X^+ to be the total space of the corresponding vector bundle, that is,

$$X^+ = \operatorname{Spec}_{C^+}(\bigoplus_{m=0}^{\infty} \operatorname{Sym}^m F^*).$$

X^+ is a smooth 3-dimensional algebraic variety which contains C^+ as the 0-section, and the cotangent bundle N_{C^+/X^+} is isomorphic to F. Hence $(K_{X^+} \cdot C^+) = 1$. Set

$$S = \operatorname{Spec} H^0(X^+, \mathcal{O}_{X^+}) = \operatorname{Spec}(\bigoplus_{m=0}^{\infty} H^0(C^+, \operatorname{Sym}^m F^*)),$$

then there is a natural birational morphism $f^+ : X^+ \to S$. The exceptional set of f^+ coincides with the 0-section C^+, and $f^+(C^+) = P$ is a point. Hence K_{X^+} is f^+-ample.

Take $g_1^+ : Y_1^+ \to X^+$ to be the blowing up of X^+ along C^+. The exceptional set E_1^+ of g_1^+ is isomorphic to the ruled surface $\mathbf{P}(F^*)$. Take l_1^+ to be a fiber of $g_1^+|_{E_1^+}$ and C_1^+ the curve with negative intersection on E_1^+. Note that C_1^+ is a section of $g_1^+|_{E_1^+}$. The cotangent bundle $N_{C_1^+/Y_1^+}$ is isomorphic to $\mathcal{O}_{\mathbf{P}^1}(-1)^{\oplus 2}$.

Take $g_2^+ : Y \to Y_1^+$ to be the blowing up of Y_1^+ along C_1^+. The exceptional set E_2 of g_2^+ is isomorphic to $\mathbf{P}^1 \times \mathbf{P}^1$. Take l_2^- to be a fiber of $g_2^+|_{E_2}$ and l_2^- the fiber of the other projection of E_2. On Y_1^+ and Y, g_1^+ and g_2^+ are divisorial contractions. Denote $l_1 = (g_2^+)_*^{-1} l_1^+$.

Since $\dim N^1(Y/S) = 3$, we have

$$\overline{\mathrm{NE}}(Y/S) = \langle l_1, l_2^+, l_2^- \rangle.$$

Here the symbol $\langle\quad\rangle$ means the convex cone generated by the elements in there. We have $(K_Y \cdot l_1) = 0$, $(K_Y \cdot l_2^+) = (K_Y \cdot l_2^-) = -1$. Take R_2^+, R_2^- to be the extremal rays generated by l_2^+, l_2^-. The contraction morphism associated to R_2^+ is just g_2^+. The contraction morphism $g_2^- : Y \to Y_1^-$ associated to R_2^- contracts the exceptional divisor E_2 of g_2^+ in the other direction.

Take $E_1 = (g_2^+)_*^{-1} E_1^+$. Since $((K_Y + E_1) \cdot l_1) = -2$, if we consider the pair (Y, E_1), the l_1 also generates an extremal ray, so the corresponding contraction morphism exists and is a divisorial contraction contracting E_1. But we do not consider this contraction morphism here.

Now let us continue to consider g_2^+. $E_1^- = (g_2^-)_* E_1$ is isomorphic to $\mathbf{P}^2$ and $l_1^- = (g_2^-)_* l_1$ is a line. As $\dim N^1(Y_1^-/S) = 2$, $\overline{\mathrm{NE}}(Y_1^-/S)$ is generated by l_1^- and $C_1^- = (g_2^-)_* l_2^+$. Here $(K_{Y_1^-} \cdot l_1^-) = -1$, $(K_Y \cdot C_1^-) = 0$. Take R_1^- to be the extremal ray generated by l_1^-, the corresponding contraction morphism $g_1^- : Y_1^- \to X^-$ contracts E_1^- to a singular point Q on X^-. As $\mathcal{O}_{E_1^-}(E_1^-) \cong \mathcal{O}_{\mathbf{P}^2}(-2)$, the singular point Q is a quotient singularity of type $\frac{1}{2}(1,1,1)$.

Take $C^- = (g_1^-)_* C_1^-$, then $\overline{\mathrm{NE}}(X^-/S)$ is generated by C^-. It is easy to compute $(K_{X^-} \cdot C^-) = -1/2$. Here it might seem strange that the intersection number is a fractional, but this is because that K_{X^-} is not Cartier. In fact, C^- passes through the singular point Q, and $2K_{X^-}$ becomes Cartier near Q.

In the end, $-K_{X^-}$ is f^--ample and the morphism $f^- : X^- \to S$ is a small contraction. The morphism $f^+ : X^+ \to S$ is just the flip of f^-.

(2) Let us consider an example in dimension at least 4.

Consider the locally free sheaf $F = \mathcal{O}_E(-1)^{\oplus t+1}$ of rank $t+1$ over $E = \mathbf{P}^s$, take its total space $X = \mathrm{Spec}_E(\bigoplus_{m=0}^\infty \mathrm{Sym}^m F^*)$. X is a smooth $(s+t+1)$-dimensional algebraic variety which contains E as the 0-section, and the cotangent bundle $N_{E/X}$ is isomorphic to F. Set

$$S = \mathrm{Spec}\, H^0(X, \mathcal{O}_X) = \mathrm{Spec}(\bigoplus_{m=0}^\infty H^0(E, \mathrm{Sym}^m F^*)),$$

then there is a natural birational morphism $f : X \to S$. The exceptional set of f coincides with the 0-section E. View a line C on E as a curve on X, we have $(K_X \cdot C) = t - s$, and f is a small contraction if $s > t$.

Take homogenous coordinates $x_0, \dots, x_s$ on E and coordinates $y_0, \dots, y_t$ along the direction of fibers of F, then

$$\bigoplus_{m=0}^\infty H^0(E, \mathrm{Sym}^m F^*) \cong k[x_i y_j]_{0 \le i \le s, 0 \le j \le t}$$

is a symmetric form with respect to x_i, y_j, so we can make another construction as the following.

Consider the locally free sheaf $F^+ = \mathcal{O}_{E^+}(-1)^{\oplus s+1}$ of rank $s+1$ over $E^+ = \mathbf{P}^t$, take its total space $X^+ = \operatorname{Spec}_{E^+}(\bigoplus_{m=0}^{\infty} \operatorname{Sym}^m(F^+)^*)$. Then there is an isomorphism

$$S \cong \operatorname{Spec} H^0(X^+, \mathcal{O}_{X^+}) = \operatorname{Spec}(\bigoplus_{m=0}^{\infty} H^0(E^+, \operatorname{Sym}^m(F^+)^*))$$

and a natural birational morphism $f^+ : X^+ \to S$. The exceptional set of f^+ coincides with the 0-section E^+. View a line C^+ on E^+ as a curve on X^+, we have $(K_{X^+} \cdot C^+) = s - t$. If $s > t$, then f^+ is the flip of f.

If $s = t$, then $K_X = f^* K_S$, $K_{X^+} = (f^+)^* K_S$, which is an example of birational transforms so called *flops*. In particular, if $s = t = 1$, then S is the same as in Example 1.1.4(2), and the flop is called *Atiyah's flop*.

The pair obtained by a flip admits the same property as the original one:

Theorem 2.5.5. *Let (X, B) be a $\mathbf{Q}$-factorial DLT pair and $f : X \to S$ a projective morphism. Let R be a $(K_X + B)$-negative extremal ray of $\overline{\mathrm{NE}}(X/S)$, take $g : X \to Y$ to be the contraction morphism associated to R. Assume that $g : X \to Y$ is small and the flip $g^+ : X^+ \to Y$ of g exists. Then X^+ is $\mathbf{Q}$-factorial, (X^+, B^+) is DLT, and $\rho(X/S) = \rho(X^+/S)$.*

Proof. Take any prime divisor E^+ on X^+, denote E to be its strict transform on X. Since X is $\mathbf{Q}$-factorial, E is a $\mathbf{Q}$-Cartier divisor. Since $\rho(X/Y) = 1$, there exists a real number r such that $E + r(K_X + B) \equiv_Y 0$. As g is a birational morphism, by the base point free theorem, $E_0 = g_*(E + r(K_X + B))$ is $\mathbf{R}$-Cartier and $g^* E_0 = E + r(K_X + B)$. Since $K_{X^+} + B^+$ is $\mathbf{R}$-Cartier, $E^+ = (g^+)^* E_0 - r(K_{X^+} + B^+)$ is $\mathbf{R}$-Cartier. Therefore, X^+ is $\mathbf{Q}$-factorial. Then it is easy to see that $\rho(X/S) = \rho(X^+/S)$. The fact that (X^+, B^+) is DLT can be conclude from Theorem 2.5.6 in the next subsection. □

2.5.3 Decrease of canonical divisors

Although flips and divisorial contractions look very different, the following theorem shows that they are similar in the sense that both are operations that make canonical divisors smaller.

{coeff decrease}

Theorem 2.5.6 ([82, Proposition 5.1.11])**.** *Let (X, B) be a $\mathbf{Q}$-factorial DLT pair and $f : X \to S$ a projective morphism. Let R be a $(K_X + B)$-negative extremal ray of $\overline{\mathrm{NE}}(X/S)$, take $g : X \to Y$ to be the contraction morphism associated to R. Consider the following to cases:*

(a) $h : X \to Y$ is a divisorial contraction.

(b) $h : X \to Y$ a small contraction with flip $h^+ : X^+ \to Y$.

In each case, take a normal algebraic variety Z with projective morphisms in the following way: in case (a) take $g : Z \to X$; in case (b) take $g : Z \to X$ and $g^+ : Z \to X^+$ such that $h \circ g = h^+ \circ g^+$. For each case, $\mathbf{R}$-divisors C, C' on Z can be determined as the following:

(a) $g^*(K_X + B) = K_Z + C$, $(h \circ g)^*(K_Y + h_*B) = K_Z + C'$.

(b) $g^*(K_X + B) = K_Z + C$, $(g^+)^*(K_{X^+} + B^+) = K_Z + C'$.

Then we have $C \geq C'$. Moreover, the support of $C - C'$ coincides with $g^{-1}(\mathrm{Exc}(h))$, the inverse image of the exceptional set of h.

Proof. In case (a), take E to be the exceptional divisor of h, then we can write $K_X + B - h^*(K_Y + h_*B) = eE$ with $e > 0$. The statement of the theorem is clear.

Let us consider case (b). Note that

$$C - C' = g^*(K_X + B) - (g^+)^*(K_{X^+} + B^+)$$

is g-exceptional and $C' - C$ is g-nef, hence $C - C' \geq 0$ by the negativity lemma (Lemma 1.6.3). Since $C - C'$ is exceptional over Y, it is easy to see that $\mathrm{Supp}(C - C') \subset g^{-1}(\mathrm{Exc}(h))$. To see that $\mathrm{Supp}(C - C') \supset g^{-1}(\mathrm{Exc}(h))$, it suffices to show that for any curve Γ on X contracted by h, $g^{-1}(\Gamma) \subset \mathrm{Supp}(C - C')$. For any curve Γ' on Z such that $g(\Gamma') = \Gamma$, it is easy to see that $((C - C') \cdot \Gamma') < 0$. This shows that $\mathrm{Supp}(C - C') \cap g^{-1}(\mathrm{Exc}(h))$ is not empty. Assume, to the contrary that, $g^{-1}(\Gamma) \not\subset \mathrm{Supp}(C - C')$, then there exists a curve $\Gamma'' \subset g^{-1}(\Gamma)$ such that Γ'' intersects but is not contained in $\mathrm{Supp}(C - C')$. This implies that Γ is contracted by g and $((C - C') \cdot \Gamma'') \geq 0$, which contradicts the fact that $C' - C$ is g-nef. □

2.5.4 Existence and termination of flips

The existence of flips is equivalent to a special case of the finite generation of canonical rings:

Theorem 2.5.7. *Let (X, B) be a $\mathbf{Q}$-factorial DLT pair where B is a $\mathbf{Q}$-divisor, and $f : X \to Y$ a small contraction. Then the following conditions are equivalent:*

(1) The flip $f^+ : X^+ \to Y$ exists.

(2) The graded $\mathcal{O}_Y$-algebra

$$R(X/Y, K_X + B) = \bigoplus_{m=0}^{\infty} f_*(\mathcal{O}_X(\llcorner m(K_X + B) \lrcorner))$$

is finitely generated.

Moreover,

$$X^+ \cong \mathrm{Proj}_Y R(X/Y, K_X + B).$$

In particular, the flip is unique if exists.

Proof. Assume that the flip $f^+ : X^+ \to Y$ exists. Since X and X^+ are isomorphic in codimension 1, we have

$$R(X/Y, K_X + B) \cong \bigoplus_{m=0}^{\infty} f^+_*(\mathcal{O}_{X^+}(\llcorner m(K_{X^+} + B^+)\lrcorner)).$$

Since $K_{X^+} + B^+$ is a relatively ample $\mathbf{Q}$-divisor, $R(X/Y, K_X + B)$ is finitely generated and

$$X^+ \cong \mathrm{Proj}_Y R(X/Y, K_X + B).$$

Assume that $R(X/Y, K_X+B)$ is finitely generated. Take $X^+ = \mathrm{Proj}_Y R(X/Y, K_X+B)$ and the natural projection $f^+ : X^+ \to Y$. By construction, there exists a positive integer r and a relatively ample divisor H on X^+ such that

$$f^+_*(\mathcal{O}_{X^+}(mH)) \cong f_*(\mathcal{O}_X(mr(K_X + B)))$$

for any positive integer m. Since f is isomorphic in codimension 1, $f^+_*(\mathcal{O}_{X^+}(mH))$ is a reflexive sheaf on Y.

We will show that f^+ is isomorphic in codimension 1. Assume, to the contrary, that f^+ contracts a prime divisor E, consider the coherent sheaf F supported on E satisfying the following exact sequence

$$0 \to \mathcal{O}_{X^+}(mH) \to \mathcal{O}_{X^+}(mH + E) \to F(mH) \to 0.$$

Here E is not assumed to be $\mathbf{Q}$-Cartier. Since H is relatively ample, we can take m sufficiently large such that $R^1 f^+_*(\mathcal{O}_{X^+}(mH)) = 0$ and $f^+_*(F(mH)) \neq 0$. However, as $f^+(E)$ is of codimension at least 2 and $f^+_*(\mathcal{O}_{X^+}(mH))$ is reflexive, $f^+_*\mathcal{O}_{X^+}(mH) \to f^+_*\mathcal{O}_{X^+}(mH + E)$ is an isomorphism. This is a contradiction.

So f^+ is isomorphic in codimension 1. By contraction, $K_{X^+} + B^+$ is f^+-ample and therefore f^+ is the flip. □

The following theorem is called the "existence of flip" conjecture before it was finally proved by Hacon and McKernan ([39]).

Theorem 2.5.8 (*Existence of flip*)**.** *Let (X, B) be a $\mathbf{Q}$-factorial DLT pair and $f : X \to S$ a projective morphism. Assume that $g : X \to Y$ is a small contraction morphism associated to a $(K_X + B)$-negative extremal ray R. Then the flip $g^+ : X^+ \to Y$ always exists.*

The proof is in Chapter 3. This theorem is a special case of the finite generation theorem of canonical rings, but it is also an essential part in the proof of the finite generation theorem.

Divisorial contractions decrease Picard numbers by 1, but flips preserve Picard numbers. Therefore, to make the minimal model program work, we need the following "termination of flip" conjecture.

Conjecture 2.5.9 (*Termination of flips*)**.** *Let (X, B) be a $\mathbf{Q}$-factorial DLT pair and $f : X \to S$ a projective morphism. Then there does not exists any infinite sequence of flips:*

$$\begin{aligned}(X, B) = (X_0, B_0) &\dashrightarrow (X_1, B_1) \dashrightarrow \dots \\ &\dashrightarrow (X_n, B_n) \dashrightarrow (X_{n+1}, B_{n+1}) \dashrightarrow \dots\end{aligned}$$

Here, $\alpha_n : (X_n, B_n) \dashrightarrow (X_{n+1}, B_{n+1})$ is a flip over S and B_n is the strict transform of B on X_n.

{subsection 2.5.5}

2.5.5 Minimal models and canonical models

In Section 1.12, we defined when a morphism $f : X \to S$ or $f : (X, B) \to S$ is called minimal. In this section, for a morphism $f : X \to S$ or $f : (X, B) \to S$, we define its minimal model and canonical model:

Definition 2.5.10. (1) Let X be a normal $\mathbf{Q}$-factorial terminal algebraic variety and $f : X \to S$ a projective morphism. Another normal $\mathbf{Q}$-factorial terminal algebraic variety X' with a projective morphism $f' : X' \to S$ such that there exists a birational map $\alpha : X \dashrightarrow X'$ with $f = f' \circ \alpha$ is called a *minimal model* of $f : X \to S$ if the following conditions are satisfied. Sometimes it is also called a *terminal model*, or more accurately, a $\mathbf{Q}$*-factorial terminal minimal model*.

(a) α is surjective in codimension 1. That is, any prime divisor on X' is the strict transform of a prime divisor on X.

(b) If we take a normal algebraic variety Z with birational projective morphisms $g : Z \to X$ and $g' : Z \to X'$ such that $g' = \alpha \circ g$, then $g^*K_X - (g')^*K_{X'}$ is effective, and its support contains all $g_*^{-1}E$ where E is a prime divisor contracted by α.

(c) $K_{X'}$ is relatively nef.

A normal algebraic variety Y with a projective morphism $f'' : Y \to S$ and a projective morphism $h : X' \to Y$ such that $f' = f'' \circ h$ is called a *canonical model* or an *ample model* of $f : X \to S$ if the following conditions are satisfied.

(d) h is surjective with connected geometric fibers.

(e) There exists an f''-ample $\mathbf{R}$-divisor H such that $h^*H \equiv_S K_{X'}$.

(2) Let (X, B) be a $\mathbf{Q}$-factorial DLT pair and $f : X \to S$ a projective morphism. Another $\mathbf{Q}$-factorial DLT pair (X', B') with a projective morphism $f' : X' \to S$ such that there exists a birational map $\alpha : X \dashrightarrow X'$ with $f = f' \circ \alpha$ is called a *minimal model* of $f : (X, B) \to S$ if the following conditions are satisfied. Sometimes it is also called a *log minimal model*, or more accurately, a $\mathbf{Q}$*-factorial DLT minimal model*.

(a) α is surjective in codimension 1, $B' = \alpha_* B$.

(b) If we take a normal algebraic variety Z with birational projective morphisms $g : Z \to X$ and $g' : Z \to X'$ such that $g' = \alpha \circ g$, then $g^*(K_X + B) - (g')^*(K_{X'} + B')$ is effective, and its support contains all $g_*^{-1}E$ where E is a prime divisor contracted by α.

(c) $K_{X'} + B'$ is relatively nef.

A normal algebraic variety Y with a projective morphism $f'' : Y \to S$ and a projective morphism $h : X' \to Y$ such that $f' = f'' \circ h$ is called a *canonical model*, a *log canonical model* or an *ample model* of $f : (X, B) \to S$ if the following conditions are satisfied.

(d) h is surjective with connected geometric fibers.

(e) There exists an f''-ample $\mathbf{R}$-divisor H such that $h^*H \equiv_S K_{X'} + B'$.

Remark 2.5.11. (1) By condition (a), prime divisors contracted by g are contracted by g'. Hence the support of $g^*(K_X + B) - (g')^*(K_{X'} + B')$ is contracted by g'.

(2) A minimal model defined as above is (log) minimal in the sense of Definition 1.12.1, hence by Proposition 1.12.2 it is easy to see that the effectivity part in condition (b) above automatically holds.

(3) The latter part of condition (b) tells that we can keep track of the prime divisors contracted by α by looking at the difference of canonical divisors.

(4) We say a birational morphism α satisfying (a) and (b) a $(K_X + B)$-*negative contraction*, or $(K_X + B)$ is negative with respect to α. Note that a contraction associated to a $(K_X + B)$-negative extremal ray is always a $(K_X + B)$-negative contraction by Theorem 2.5.6.

(5) The minimal model and canonical model defined in the former part of the definition are special cases of the log version defined in the latter part. In fact, if $B = 0$ and X is terminal in the given pair (X, B), then Y is also terminal by condition (b). Therefore, when considering the

existence of minimal models, it suffices to consider the log version. In this book we will consider the log version in general, and usually the word "log" will be omitted.

For a given morphism $f : (X, B) \to S$, its minimal model is not necessarily unique. But its canonical model is unique if exists:

{exceptional}

Theorem 2.5.12. *Let (X, B) be a $\mathbf{Q}$-factorial DLT pair and $f : X \to S$ a projective morphism. For $i = 1, 2$, assume that there exist minimal models $f'_i : (X'_i, B'_i) \to S$ with birational maps $\alpha_i : X \dashrightarrow X'_i$, and canonical models $f''_i : Y_i \to S$ with projective morphisms $h_i : X'_i \to Y_i$. Then the following statements hold:*

(1) The induced birational map $\beta : X'_1 \dashrightarrow X'_2$ is isomorphic in codimension 1 and (X'_i, B'_i) $(i = 1, 2)$ are K-equivalent to each other.

(2) There exists an isomorphism $e : Y_1 \to Y_2$ such that $f''_1 = f''_2 \circ e$.

Proof. (1) We can take a smooth algebraic variety Z with a birational projective morphism $g : Z \to X$ such that $g_i = \alpha_i \circ g$ is a birational morphism for $i = 1, 2$. Denote $g_1^*(K_{X'_1} + B'_1) - g_2^*(K_{X'_2} + B'_2) = E$. Assume, to the contrary, that $E \neq 0$. Write $E = E^+ - E^-$ into parts with positive and negative coefficients. By symmetry, we may assume that $E^+ \neq 0$. Since $g^*(K_X + B) \geq g_1^*(K_{X'_1} + B'_1) = g_2^*(K_{X'_2} + B'_2) + E$, every component of E^+ is contracted by g_2. By the negativity lemma (Lemma 1.6.3), there exists a family of curves C contracted by g_2 and covering a component of E^+ such that $(E^+ \cdot C) < 0$. As C is in a covering family, $(E^- \cdot C) \geq 0$. Hence $(g_1^*(K_{X'_1} + B'_1) \cdot C) = (g_2^*(K_{X'_2} + B'_2) \cdot C) + (E \cdot C) < 0$. This contradicts to the fact that $K_{X'_1} + B'_1$ is relatively nef. This shows the K-equivalence. Moreover, we know that the set of divisors contracted by α_i is independent of i, which implies that β is isomorphic in codimension 1.

(2) By definition, for each $i = 1, 2$, there exists f''_i-ample $\mathbf{R}$-divisor H_i, such that $h_i^* H_i \equiv_S K_{X'_i} + B'_i$. Hence a curve C on Z is contracted by $h_i \circ g_i : Z \to Y_i$ if and only if $(g_i^* h_i^* H_i \cdot C) = 0$, which is a condition independent of i. Hence we get the conclusion by Zariski's main theorem. □

Example 2.5.13. Consider X_0 to be the hypersurface defined by $x_1x_2 + x_3x_4 = 0$ in $\mathbf{P}^4$ with homogenous coordinates $x_0, \dots, x_4$. X_0 is the projective cone over $\mathbf{P}^1 \times \mathbf{P}^1 \subset \mathbf{P}^3$ with vertex $P = [1 : 0 : 0 : 0 : 0]$, and $P \in X_0$ is a terminal singularity. Take $\bar{B}$ to be a general hypersurface not passing P and $B_0 = \bar{B} \cap X_0$. Assume that the degree $d = \deg(\bar{B})$ is at least 3, then $K_{X_0} + B_0 = \mathcal{O}_{X_0}(d - 3)$ is nef.

Blowing up the ideal (x_1, x_3) or (x_1, x_4) on X_0, we get two small resolution $g_i : X_i \to X_0$ $(i = 1, 2)$. g_i is isomorphic outside P and $g_i^{-1}(P)$ is isomorphic to $\mathbf{P}^1$. Take B_i to be the strict transform of B_0 on X_i. Then (X_i, B_i)

is a minimal model of (X_0, B_0). The induced birational map $\alpha : X_1 \dashrightarrow X_2$ is the Atiyah flop (see Example 2.5.4(2)).

If one would like to have an example without boundaries B_i, one can consider the cyclic covering $\pi_0 : X_0' \to X_0$ of degree $d \geq 4$ ramified along B_0, and do the similar construction. Here if B_0 is defined by the equation $f(x) = 0$, the the covering map π_0 is given by $t^d = f(x)$. In this case, $K_{X_0'} = \pi_0^*(K_{X_0} + (d-1)B_0/d)$ and $K_{X_0'}$ is nef.

2.5.6 The minimal model program

We will introduce the formal definition of minimal model program. Starting from an arbitrary $\mathbf{Q}$-factorial DLT pair (X, B) and a projective morphism $f : X \to S$, in order to get a minimal model or a Mori fiber space, we have the following *minimal model program* (*MMP* for short) which is a process consists of a sequence of birational operations.

(1) Given a $\mathbf{Q}$-factorial DLT pair (X, B) and a projective morphism $f : X \to S$.

(2) If $K_X + B$ is relatively nef, then (X, B) is minimal, and the MMP ends here.

(3) If $K_X + B$ is relatively nef, then there exists a contraction morphism $h : X \to Y$ associated to an extremal ray.

(a) If h is a divisorial contraction, then $(Y, B_Y = h_*B)$ is again a $\mathbf{Q}$-factorial DLT pair and $\rho(Y/S) = \rho(X/S) - 1$. Replace (X, B) by the new pair (Y, B_Y) and go back to (1).

(b) If h is a small contraction, then take the flip $h^+ : X^+ \to Y$ and (X^+, B^+) is again a $\mathbf{Q}$-factorial DLT pair. Here B^+ is the strict transform of B and $\rho(Y/S) = \rho(X/S)$. Replace (X, B) by the new pair (X^+, B^+) and go back to (1).

(c) If h is a Mori fiber space, the MMP ends.

If the termination of flips is true, then the operations in (3-b) stops after finitely many times, and eventually we get into case (2) or (3-c).

Example 2.5.14. Let (X, B) be a $\mathbf{Q}$-factorial DLT pair and $f : X \to S$ a projective morphism. Let us consider the case $\rho(X/S) = 2$. The corresponding MMP is called a 2*-ray game*.

The cone of curves $\overline{\text{NE}}(X/S)$ is a fan generated by two extremal rays R_1, R_2 in $N_1(X/S)$. If $K_X + B$ is not nef over S, then for at least one extremal ray, say R_1, $((K_X + B) \cdot R_1) < 0$.

Assume that the corresponding contraction morphism $\phi : (X, B) \to Y$ is small, and $\phi' : (X', B') \to Y$ is the flip. Then again we have $\rho(X'/S) = 2$ and $\overline{\text{NE}}(X'/S)$ is a fan generated by two extremal rays R_1', R_2'. Suppose

R_2' is the extremal ray generated by curves contracted by ϕ', then by the property of flips, $((K_{X'} + B') \cdot R_2') > 0$. If $K_{X'} + B'$ is not nef over S, then $((K_{X'} + B') \cdot R_1') < 0$. Therefore, the choice of the extremal ray is unique, and we can repeat the same operation.

The 2-ray game can be easily understand using cones of divisors. The nef cone $\overline{\mathrm{Amp}}(X/S)$ is a fan generated by two extremal rays L_1, L_2 in $N^1(X/S)$. Take L_1 to be the extremal ray corresponding to ϕ, that is, $(L_1 \cdot R_1) = 0$.

As the induced map $X \dashrightarrow X'$ is isomorphic in codimension 1, we can identify $N^1(X/S) \cong N^1(X'/S)$. Then after flip the nef cone $\overline{\mathrm{Amp}}(X'/S)$ is a fan generated by two extremal rays L_1', L_2' in $N^1(X/S)$, one of them, say L_2', is just L_1. This is because they all coincides with the pullback of $\overline{\mathrm{Amp}}(Y/S)$.

Therefore, we can view this flip as moving from one room $\overline{\mathrm{Amp}}(X/S)$ to another room $\overline{\mathrm{Amp}}(X'/S)$ by crossing the wall $L_2' = L_1$. The next contraction corresponds to the wall L_1' on the other side. This is similar to the the MMP with scaling introduced in the next section.

Remark 2.5.15. In the formulation of MMP, we can make similar arguments by just assuming the pairs are $\overline{\text{KLT}}$ instead of DLT. In fact, as X is assumed to be $\mathbf{Q}$-factorial, if (X, B) is $\overline{\text{KLT}}$, then for $\epsilon \in (0, 1)$, $(X, (1-\epsilon)B)$ is KLT. If $K_X + B$ is not nef, then $K_X + (1-\epsilon)B$ is not nef for a sufficiently small ϵ.

2.6 Minimal model program with scaling

In each step of the minimal model program, when there exists more than one extremal rays, we just choose one of them arbitrarily. The so-called *MMP with scaling* or *directed MMP* proceeds by choosing the extremal ray in an efficient way. The MMP with scaling goes to the final minimal model straightly in one direction, and its termination is easier to control. Except for lower dimensional cases, to proof the termination of flips is an extremely hard problem, but it is sightly hopeful if we only consider the termination for MMP with scaling.

Originally the MMP uses convex geometry, but the MMP with scaling is particularly compatible with convex geometry. The idea of such MMP was first seen in [129], and it was developed greatly and becomes a basic tool in [15]. As for the termination of flips, it might not be true for the general MMP, but it is expected to be true for the MMP with scaling.

Given a $\mathbf{Q}$-factorial KLT pair (X, B) and a projective morphism $f : X \to S$, a *scale* is an effective $\mathbf{R}$-divisor H satisfying the following properties:

(1) H is effective and relatively big.

(2) $(X, B + H)$ is LC.

(3) $K_X + B + H$ is relatively nef.

The idea is to use H to control the progress of MMP. Starting from $(X, B) = (X_0, B_0)$, we construct the MMP for (X, B) with scaling of H such that in the n-th step we have a $\mathbf{Q}$-factorial KLT pair (X_n, B_n) such that

(1) H_n is relatively big.

(2) $(X_n, B_n + t_{n-1}H_n)$ is LC.

(3) $K_{X_n} + B_n + t_{n-1}H_n$ is relatively nef.

Here H_n the strict transform of H, and t_n is defined as the following threshold:

$$t_n = \min\{t \geq 0 \mid K_{X_n} + B_n + tH_n \text{ is relatively nef}\}.$$

We denote $t_{-1} = 1$. When $n = 0$, by assumption, $t_0 \leq 1$. Assume that $K_X + B$ is not relatively nef, then $t_0 > 0$. When $n > 0$, by construction, $K_{X_n} + B_n + t_{n-1}H_n$ is relatively nef, and hence $t_n \leq t_{n-1}$.

The inductive construction of the MMP is as the following. Take $n \geq 0$. Assume that we already have (X_n, B_n). If $t_n = 0$, then $K_{X_n} + B_n$ is relatively nef and the MMP ends. If $t_n > 0$, then we proceed to the next step by the following lemma:

{measure}

Lemma 2.6.1. *If $t_n > 0$, then there exists a $(K_{X_n} + B_n)$-negative extremal ray R_n such that*

$$((K_{X_n} + B_n + t_nH_n) \cdot R_n) = 0.$$

Proof. Since $B_n + t_nH_n$ is relatively big, for a sufficiently small positive real number ϵ, there are only finitely many $(K_{X_n} + B_n + (t_n - \epsilon)H_n)$-negative extremal rays (Corollary 2.4.5). Since $K_{X_n} + B_n + t_nH_n$ is relatively nef, for $0 < \epsilon' < \epsilon$, a $(K_{X_n} + B_n + (t_n - \epsilon')H_n)$-negative extremal ray is also a $(K_{X_n} + B_n + (t_n - \epsilon)H_n)$-negative extremal ray. So by finiteness, the threshold t_n is determined by one of the extremal rays, that is, there exists such a ray R_n such that $((K_{X_n} + B_n + t_nH_n) \cdot R_n) = 0$. Note that R_n is also a $(K_{X_n} + B_n)$-negative extremal ray. □

Using the extremal ray R_n in the above Lemma to proceed the MMP, we get a new $\mathbf{Q}$-factorial KLT pair (X_{n+1}, B_{n+1}). Since $K_{X_n} + B_n + t_nH_n$ is nef and numerically trivial along R_n, the strict transform $K_{X_{n+1}} + B_{n+1} + t_nH_{n+1}$ is relatively nef. Also note that $(X_n, B_n + t_nH_n)$ is LC, which implies that $(X_{n+1}, B_{n+1} + t_nH_{n+1})$ is LC. In this way, we inductively constructed the MMP with scaling of H. Note that we get a non-increasing sequence $1 \geq t_0 \geq t_1 \geq \dots$.

The MMP with scaling can be virtualized as in the following Figure **??**. For simplicity, let us assume that the MMP consists of flips. In this case, $N^1(X_n/S)$ can be identified with $N^1(X/S)$, and the point corresponding to

$K_{X_n} + B_n$ depends on n. Let us track the changing of nef cones $\overline{\mathrm{Amp}}(X_n/S)$ in $N^1(X/S)$. By the cone theorem, observing from $K_X + B$, the surface of the nef cone $\overline{\mathrm{Amp}}(X_n/S)$ is locally a polyhedron. Choosing an extremal ray corresponds to choosing a face, and taking the flip means that passing through this face and moving from a room $\overline{\mathrm{Amp}}(X_n/S)$ to the other room $\overline{\mathrm{Amp}}(X_{n+1}/S)$. Such an operation is usually called *wall crossing*.

According to the original condition, $K_X + B + H \in \overline{\mathrm{Amp}}(X/S)$. Consider the line in $N^1(X/S)$ connecting $K_X + B + H$ and $K_X + B$. In each step of the MMP with scaling, we choose the face intersecting L. Note that

$$K_X + B + t_n H \in \overline{\mathrm{Amp}}(X_n/S) \cap \overline{\mathrm{Amp}}(X_{n+1}/S) \cap L$$

and all rooms line up along the line L. So such an MMP moves from $K_X + B + H$ to $K_X + B$ on this line straightly, and the termination is easier.

Remark 2.6.2. Here we assume that (X, B) is KLT and H is relatively big in order to apply the finiteness of extremal rays (Corollary 2.4.5) in Lemma 2.6.1. Later we will see that we can replace Lemma 2.6.1 by Corollary 2.10.12, and the MMP with scaling can be generalized to the case that (X, B) is DLT and H is not relatively big.

Birkar, Cascini, Hacon, McKernan showed the termination of flips in the following special but very important case. The proof will be in Chapter 3.

Theorem 2.6.3 ([15]). *Let (X, B) be a $\mathbf{Q}$-factorial KLT pair and $f : X \to S$ a projective morphism. Assume that B is relatively big. Suppose that H is an effective $\mathbf{R}$-divisor such that $(X, B + H)$ is KLT and $K_X + B + H$ is relatively nef. Then the MMP with scaling of H terminates.*

As a interesting corollary, we can show the existence of minimal models for varieties of general type, or oppositely the existence of Mori fiber spaces for varieties with non-pseudo-effective canonical divisors:

Corollary 2.6.4. *Let (X, B) be a $\mathbf{Q}$-factorial KLT pair and $f : X \to S$ a projective morphism.*

(1) Assume that $K_X + B$ is not relatively pseudo-effective over S. Then there exists a Mori fiber space birational to (X, B).

(2) Assume that $K_X + B$ is relatively big over S. Then (X, B) has a minimal model. Moreover, by the base point free theorem, (X, B) has a canonical model.

{section 2.7}

2.7 Existence of rational curves

Given an algebraic variety, whether there exists a rational curve, and how many rational curves there are if exist, are very important questions. We will give a proof of Theorem 2.7.2 which states that there are many rational curves on algebraic varieties with canonical divisors satisfying certain negativity. For example, $\mathbf{P}^1$ is the only smooth projective curve with negative canonical divisor ($-K$ is ample).

In order to prove this theorem, we first take the *reduction* of the given algebraic variety to positive characteristics, and then proceed the discussion by methods specific in positive characteristics. Applying the *Frobenius morphism*, there is a method to get a morphism from $\mathbf{P}^1$ by deforming a given morphism and degenerate it by taking a limit. This method was originally discovered by Mori, and so far is the only method to prove the existence of rational curves in general situation. Existence of rational curves is also a very important problem in complex geometry, but this theorem has no analytic proof. It can be said that this is a theorem of algebraic geometry only.

2.7.1 Deformation of morphisms

Firstly, in order to construct the space of all deformations of morphism, or the *moduli space* of morphisms, we introduce the definition of *Hilbert scheme* by Grothendieck ([36]). For details we refers to [92].

Definition 2.7.1. Fix a projective morphism $f : X \to S$ between Noetherian schemes and a relatively ample sheaf H. For a closed subscheme Z of a fiber $X_s = f^{-1}(s)$ of f, the polynomial

$$P_Z(m) = \chi(Z, mH) = \sum_{p \geq 0} \dim_{k(s)} H^p(Z, mH)$$

in integer m is called the *Hilbert polynomial* of Z. Fixing a polynomial P, there exists a moduli space for all closed subscheme of fibers of f whose Hilbert polynomial coincides with $P(m)$. This moduli space is a projective scheme $g : \mathrm{Hilb}^P(X/S) \to S$ over S and is called the Hilbert scheme. It has the following universal property.

There exists a closed subscheme $\mathcal{Z}$ in the fiber product $X \times_S \mathrm{Hilb}^P(X/S)$, which is called the *universal family*, satisfying the following conditions:

(1) The first projection $p_1 : \mathcal{Z} \to X$ induces an isomorphism on every fiber $p_2^{-1}(t)$ of the second projection $p_2 : \mathcal{Z} \to \mathrm{Hilb}^P(X/S)$ to a closed subscheme of $X_{g(t)}$, whose Hilbert polynomial is $P(m)$.

(2) For any S-scheme $T \to S$ and any closed subscheme Z_T of $X \times_S T$ such that the Hilbert polynomial of every fiber of the second projection $Z_T \to T$ is $P(m)$, there exists a unique morphism $T \to \mathrm{Hilb}^P(X/S)$ such that $\mathcal{Z} \times_{\mathrm{Hilb}^P(X/S)} T = Z_T$.

Note that a family with constant Hilbert polynomial is automatically flat. By taking disjoint union for all polynomials, we denote $\mathrm{Hilb}(X/S) = \coprod_P \mathrm{Hilb}^P(X/S)$.

The moduli space of morphisms is defined to be the moduli space of graphs of morphisms. Let $X \to S$ and $Y \to S$ be projective S-scheme such that X is flat over S, and take $G \subset X_s \times Y_s$ to be the graph of a morphism between fibers $g : X_s \to Y_s$. Fix a relatively ample sheaf H on $X \times_S Y$, take $P(m) = \chi(G, mH)$. Consider the Hilbert scheme $\mathrm{Hilb}^P(X \times_S Y/S)$, and take $\pi : \bar{\mathcal{G}} \to \mathrm{Hilb}^P(X \times_S Y/S)$ to be the universal family. Then the set of points in $\mathrm{Hilb}^P(X \times_S Y/S)$ whose fiber in the universal family is a graph of a morphism between fibers of X and Y is an open subset. In fact, a closed subscheme G' of $X_{s'} \times Y_{s'}$ is the graph of a morphism $X_{s'} \to Y_{s'}$ if and only if the first projection $p_1 : G' \to X_{s'}$ is an isomorphism, therefore being a graph is an open condition. This open subset is denoted by $\mathrm{Hom}_S^P(X, Y)$ and called the *moduli space of morphisms*.

The theory of infinitesimal deformation is very useful when studying the structure of Hilbert schemes. For example, let us assume that X is a smooth projective algebraic variety over a field k, and Z is a smooth closed subvariety. Then Z determines a point $[Z] \in \mathrm{Hilb}(X/k) = \mathrm{Hilb}(X)$. Then the *Zariski tangent space* $T_{\mathrm{Hilb}(X),[Z]} = (\mathfrak{m}_{[Z]}/\mathfrak{m}_{[Z]}^2)^*$ of $[Z]$ is isomorphic to $H^0(Z, N_{Z/X})$, the contangent bundle of $Z \subset X$. Here $\mathfrak{m}_{[Z]} \subset \mathcal{O}_{\mathrm{Hilb}(X),[Z]}$ is the maximal ideal of the local ring. On the other hand, the *obstruction space* is $H^1(Z, N_{Z/X})$. That is, the completion of $\mathrm{Hilb}(X)$ along $[Z]$ can be represented by $h^1(Z, N_{Z/X})$ equations in the completion of $h^0(Z, N_{Z/X})$-dimensional affine space along the origin. Therefore we have the inequality

$$\dim_{[Z]} \mathrm{Hilb}(X) \geq h^0(Z, N_{Z/X}) - h^1(Z, N_{Z/X}).$$

This can be also applied to moduli spaces of morphisms. Consider the deformation of a morphism between smooth projective algebraic varieties $g : X \to Y$, the cotangent bundle G is given by $N_{G/X \times Y} \cong p_2^* T_Y$. Here T_Y is the tangent bundle of Y and $p_2 : G \to Y$ is the second projection. Therefore we have the inequality

$$\dim_{[g]} \mathrm{Hom}_k(X, Y) \geq h^0(X, g^* T_Y) - h^1(X, g^* T_Y).$$

{existence of RC}

2.7.2 The bend-and-break method

Theorem 2.7.2 ([106]). *Let X be a normal projective algebraic variety of dimension n over an algebraically closed field of arbitrary characteristic. Take C be a curve on X contained in the smooth locus of X, fix a point P on C and take an ample divisor H on X. Suppose that C is not a rational curve and $(K_X \cdot C) < 0$. Then there exists a rational curve L on X passing through P satisfying*

$$(H \cdot L) \leq \frac{2n(H \cdot C)}{(-K_X \cdot C)}.$$

Here note that C and L might have singularities, and L might pass though singularities of X.

Proof. Firstly let us consider the case that the characteristic p of k is positive. The point is that by using Frobenius morphisms, we can make the degree of the curve sufficiently high while keep the genus unchanged.

Take the normalization $\nu : C' \to C$ and denote g to be the genus of C'. By assumption, $g > 0$. Take the m-th power of the Frobenius morphism $f' : C'_q \to C'$ where $q = p^m$. Here f' is the morphism defined over k defined by taking q-th power of coordinates, which exists only in positive characteristics. The genus of C'_q is again g. Take $f : C'_q \to X$ to be the composition morphism.

Since $(K_X \cdot C) < 0$, we can take $q = p^m$ sufficiently large such that the following inequality holds:

$$b = \llcorner \frac{q(-K_X \cdot C) - 1}{n} \lrcorner + 1 - g > 0.$$

Take b distinct points $P_1, \dots, P_b$ on C'_q, denote $B = \sum_{i=1}^{b} P_i$. Consider the deformation of the morphism $f : C'_q \to X$ fixing B. As the deformation of f is the deformation of G, by fixing B means that the graph contains $(P_i, f(P_i))$ for each i. The moduli space of such deformations $\mathrm{Hom}_k(C'_q, X; B)$ is a closed subscheme of $\mathrm{Hom}_k(C'_q, X)$.

We can compute the dimension of $\mathrm{Hom}_k(C'_q, X; B)$ by infinitesimal deformation theory. The Zariski tangent space of $\mathrm{Hom}_k(C'_q, X)$ at $[f]$ is isomorphic to $H^0(C'_q, f^*T_X)$, and the Zariski tangent space of its closed subscheme $\mathrm{Hom}_k(C'_q, X; B)$ is isomorphic to $H^0(C'_q, f^*T_X \otimes \mathcal{O}_{C'_q}(-B))$. The obstruction space is $H^1(C'_q, f^*T_X \otimes \mathcal{O}_{C'_q}(-B))$ instead of $H^1(C'_q, f^*T_X)$. Therefore, by dimension counting,

$$\begin{aligned}
&\dim_{[f]} \mathrm{Hom}_k(C'_q, X; B) \geq \chi(C'_q, f^*T_X \otimes \mathcal{O}_{C'_q}(-B)) \\
&= \deg_{C'_q}(f^*T_X \otimes \mathcal{O}_{C'_q}(-B)) + n(1-g) \\
&= q(-K_X \cdot C) - nb + n(1-g) \geq 1.
\end{aligned}$$

The first equality is given by the Riemann–Roch formula.

Therefore, there exists a non-trivial deformation family $F : C'_q \times T \to X$ of f fixing B parametrizing by a smooth affine algebraic curve T. Here T has a base point t_0 such that $F(P, t_0) = f(P)$ for all $P \in C'_q$, and also $F(P_i, t) = f(P_i)$ for all $1 \leq i \leq b$ and all $t \in T$. On the other hand, since $g > 0$ and $b > 0$, the morphism $C'_q \to C$ has no deformation. Therefore, the image of F is not contained in C, that is, $F(C'_q \times T) \not\subset C$.

Compactify the affine curve T into a smooth projective algebraic curve $\bar{T}$. We can extend F to a birational map $C'_q \times \bar{T} \dashrightarrow X$. Resolving this birational map by a sequence of blowing ups on points of indeterminacy, we can get a birational morphism $\mu : Y \to C'_q \times \bar{T}$ and a morphism $h = F \circ \mu : Y \to X$. Here μ is obtained by repeatedly blowing up points on the smooth projective surface $C'_q \times \bar{T}$. In each step of this procedure, if the image of the center of the blowing up in $C'_q \times \bar{T}$ is on $T_i = P_i \times \bar{T}$ $(i = 1, \ldots, b)$, we denote the exceptional divisor to be $\bar{E}_{i,j}$ $(j = 1, \ldots, n_i)$. Denote the total transforms of all such exceptional divisors on Y to be $E_{i,j}$ $(j = 1, \ldots, n_i)$. Take $T_0 = P \times \bar{T}$ for a general point P on C'_q and take T'_i $(i = 0, \ldots, b)$ to be the strict transform of T_i on Y. Since P is general, μ is isomorphic over T_0, and we have the linear equivalence

$$T'_i \sim T_0 - \sum_{j=1}^{n_i} \epsilon_{i,j} E_{i,j}$$

for $i = 1, \ldots, b$. Here $\epsilon_{i,j} = 1$ or 0 depending on whether $\bar{E}_{i,j}$ intersects the strict transform of T_i or not.

Take $C_0 = C'_q \times t_0 \subset Y$, since the morphism $C_0 \to C$ is of degree q,

$$(h^* H \cdot C_0) = q(H \cdot C).$$

Also $(T_0 \cdot C_0) = 1$. Since $\mathrm{NS}(Y)$ is generated by C_0, T_0 and exceptional divisors of μ, there exist integers c and $e_{i,j}$ such that

$$h^* H \equiv c C_0 + q(H \cdot C) T_0 - \sum_{i,j} e_{i,j} E_{i,j} + E.$$

Here the support of E consists of exceptional divisors whose image is not on T_i. Since $h^* H$ is nef, $c \geq 0$ and $e_{i,j} \geq 0$.

Since $\dim h(Y) = 2$, $(h^* H)^2 > 0$. Note that

$$(h^* H)^2 = 2cq(H \cdot C) + \sum_{i,j} e_{i,j}^2 (E_{i,j})^2 + E^2.$$

Since $(E^2) \leq 0$,

$$2cq(H \cdot C) - \sum_{i,j} \epsilon_{i,j}^2 e_{i,j}^2 \geq 2cq(H \cdot C) - \sum_{i,j} e_{i,j}^2 > 0.$$

Also for every i,

$$c - \sum_{j=1}^{n_i} \epsilon_{i,j} e_{i,j} = (h^* H \cdot T'_i) = 0.$$

Therefore,

$$2q(H \cdot C) \sum_{i,j} \epsilon_{i,j} e_{i,j} > b \sum_{i,j} \epsilon_{i,j}^2 e_{i,j}^2.$$

This implies that there exists i_0 and j_0 such that $\epsilon_{i_0,j_0} = 1$ and

$$2q(H \cdot C) > b \epsilon_{i_0,j_0} e_{i_0,j_0} > 0,$$

which means that

$$0 < (h^* H \cdot E_{i_0,j_0}) = e_{i_0,j_0} < \frac{2q(H \cdot C)}{b}.$$

Hence there exists an irreducible component L' of E_{i_0,j_0}, such that $L = h(L')$ is a rational curve, $P_{i_0} \in L$, and

$$(H \cdot L) < \frac{2q(H \cdot C)}{b}.$$

Recall that $q = p^m$, and by the definition of b, we have

$$\lim_{m \to \infty} \frac{2q(H \cdot C)}{b} = \frac{2n(H \cdot C)}{(-K_X \cdot C)},$$

so by taking m sufficiently large, we have

$$(H \cdot L) \leq \frac{2n(H \cdot C)}{(-K_X \cdot C)}.$$

Here note that the left hand side is always an integer.

We have shown that for the images of any b points on C'_q, there exists a rational curve L passing though one of them and $(H \cdot L)$ satisfies the required inequality. Next we use this to show that for any point $P \in C$, there exists a rational curve L such that $P \in L$ and $(H \cdot L)$ satisfies the required inequality.

In the Hilbert scheme $\mathrm{Hilb}(X)$, the set of points corresponding to all rational curves is a locally closed subset. This is because for a family of curves, genus is lower semicontinuous. Moreover, if we only consider all rational curves of degree (i.e. the intersection number with H) bounded from above by a constant number, then the set is a closed subset of finite type. What we proved is that there exists an irreducible locally closed subset $Z \subset \mathrm{Hilb}(X)$, such that if we take $\mathcal{U}_Z \subset X \times Z$ to be the restriction of the universal family $\mathcal{U} \subset X \times \mathrm{Hilb}(X)$ on Z, then the fibers of the second projection $p_2 : \mathcal{U}_Z \to Z$ are rational curves on X of degree bounded by $2n(H \cdot C)/(-K_X \cdot C)$, and the image of the first projection $p_1(\mathcal{U}_Z)$ contains a non-empty open subset of C. Take $\bar{Z}$ to be the closure of Z in $\mathrm{Hilb}(X)$, and take $\mathcal{U}_{\bar{Z}} \subset X \times \bar{Z}$ to be the restriction of the universal family. Then all irreducible components of the fibers of the second projection $p_2 : \mathcal{U}_{\bar{Z}} \to \bar{Z}$ are rational curves, and the image of the first projection $p_1(\mathcal{U}_{\bar{Z}})$ contains C. Therefore, there exists a rational curve passing though any fixed point on C with degree bounded by $2n(H \cdot C)/(-K_X \cdot C)$.

We can construct rational curves on algebraic varieties defined over a field of characteristic 0 by lift the above result to characteristic 0. The proof essentially uses the property of Hilbert schemes again.

All given data as X, H, C can be described by finitely many polynomials in finitely many invariables with finitely many coefficients in k. By adding those coefficients to $\mathbf{Z}$, we can construct a finitely generated $\mathbf{Z}$-algebra R satisfying the following conditions.

(1) There exists a projective morphism $X_R \to \operatorname{Spec} R$ such that all the geometric fibers X_t are normal, and the generic geometric fiber $X_{\bar{\eta}}$ is isomorphic to X. Here for a geometric point t of $\operatorname{Spec} R$, we denote X_t to be the fiber over t.

(2) There exists an ample Cartier divisor H_R on X_R whose restriction on $X_{\bar{\eta}}$ is H.

(3) There exists a closed subscheme C_R of X_R such that for any geometric point t of $\operatorname{Spec} R$, the fiber C_t is an irreducible algebraic curve on X_t contained in the smooth locus of X_t and not a rational curve.

Here note that all conditions on fibers are open conditions, so we can make localization to remove bad fibers.

Consider the universal family on the Hilbert scheme

$$\mathcal{U} \subset X_R \times_{\operatorname{Spec} R} \operatorname{Hilb}(X_R/\operatorname{Spec} R).$$

Then there exists a locally closed subset of finite type $Z_R \subset \operatorname{Hilb}(X_R/\operatorname{Spec} R)$ satisfying the following: for any geometric point t of $\operatorname{Spec} R$, the set of points in $\operatorname{Hilb}(X_R/\operatorname{Spec} R)$ corresponding to rational curves L on X_t such that

$$(H_t \cdot L) \leq \frac{2n(H \cdot C)}{(-K_X \cdot C)}$$

coincides with Z_t. As the right hand side is a constant, the degree of L is bounded from above uniformly.

Take the closure $\bar{Z}_R$ in $\operatorname{Hilb}(X_R/\operatorname{Spec} R)$ and take the restriction of the universal family $\mathcal{U}_{\bar{Z}_R} \subset X_R \times_{\operatorname{Spec} R} \bar{Z}_R$. The any irreducible component of any geometric fiber of the second projection $p_2 : \mathcal{U}_{\bar{Z}_R} \to \bar{Z}_R$ is a rational curve with degree bounded from above.

As the residue field of a geometric point is of positive characteristic, the image of the first projection $p_1(\mathcal{U}_{\bar{Z}_t})$ contains C_t. Since $\bar{Z}_R$ is a closed subscheme of finite type, it follows that $C_R \subset p_1(\mathcal{U}_{\bar{Z}_R})$. In particular, $C_{\bar{\eta}} \subset p_1(\mathcal{U}_{\bar{Z}_{\bar{\eta}}})$. This finishes the proof. □

The argument in the proof is by deforming the curve until its limit breaks up with a piece (irreducible component) of rational curve, which is called the *bend and break method*.

2.8 Length of extremal rays

{section 2.8}

In this section we define the "length" of an extremal ray, and shows that it is bounded by a constant depending only on the dimension. This theorem also contains the claim that extremal rays are generated by rational curves, which is essential for many boundedness results and termination results.

As the proof uses the existence theorem of rational curves proved in the previous section, it is based on algebraic geometry in positive characteristics. In addition to this, we use the vanishing theorem which is specific in characteristic 0. This theorem was also used to prove the discreteness of extremal rays in the cone theorem (Step 5').

For an extremal ray R of a morphism $f : (X, B) \to S$, the minimal value of the intersection numbers $-((K_X + B) \cdot C)$ for all irreducible curves C whose classes are contained in R is called the *length* of R.

Firstly, we begin with generalizing the vanishing theorem for complex analytic varieties.

{vanishing complex versio

Theorem 2.8.1 ([122, Theorem 3.7])**.** *Let $f : X \to S$ be a projective surjective morphism from a complex manifold to a complex variety, B an $\mathbf{R}$-divisor with normal crossing support and coefficients in $(0, 1)$, and D a Cartier divisor on X. Assume that $D - (K_X + B)$ is relatively nef and relatively big. Then $R^p f_*(\mathcal{O}_X(D)) = 0$ for any $p > 0$.*

The theorem is proved by generalize the Kodaira vainshing for compact complex manifolds to *weakly* 1-*complete* complex manifolds. A complex manifold is said to be *weakly* 1-*complete* if there exists a plurisubharmonic C^∞-function ϕ such that $X_c = \{x \in X \mid \phi(x) \le c\}$ is compact for all $c \in \mathbf{R}$. For a positive line bundle L on a weakly 1-complete complex manifold X, $H^p(X, K_X + L) = 0$ for all $p > 0$ ([119], [120]), the same as the Kodaira vanishing theorem.

{thm exceptional RC}

Theorem 2.8.2. *Let (X, B) be a KLT pair and $f : X \to Y$ be a projective birational morphism to a normal algebraic variety. Assume that $-(K_X + B)$ is f-ample. Take E to be any irreducible component of $\mathrm{Exc}(f)$, denote $n = \dim E - \dim f(E)$. Then the set $\{C_t\}$ of all rational curves C_t, such that C_t is contracted by f and $0 > ((K_X + B) \cdot C_t) > -2n$, covers E, that is, $\bigcup_t C_t = E$.*

Proof. For a flat family of curves whose general fibers are rational curves, any irreducible component of its special fiber is again a rational curve. Therefore it suffices to show that, passing though a general point of E, there exists a rational curve contracted by f and satisfies the required inequality. Replacing Y by an affine open subset intersecting $f(E)$ and cutting Y by general hyperplanes, we may assume that $f(E)$ is a point.

We need the following lemma.

Lemma 2.8.3. *Take $\nu : E' \to E$ to be the normalization and take an f-ample divisor H on X. Then*

$$(H^{n-1} \cdot (K_X + B) \cdot E) > ((\nu^* H)^{n-1} \cdot K_{E'}).$$

Proof. We may assume that H is very ample. Cutting by hyperplanes in $|H|$ for $n - 1$ times, we get $C \subset X_0$ from the restriction of $E \subset X$. Since $\dim E = n$, $\dim C = 1$. Denote $B_0 = B|_{X_0}$ and $\nu^{-1}(C) = C'$.

Since $K_{X_0} = (K_X + (n-1)H)|_{X_0}$ and $K_{C'} = (K_{E'} + (n-1)\nu^*H)|_{C'}$, if the required inequality fails, then $((K_{X_0} + B_0)\cdot C) \leq \deg K_{C'}$. Then we can take a Cartier divisor A_0 on C, such that $((K_{X_0} + B_0)\cdot C) \leq \deg A_0$ and $H^0(C', K_{C'} - \nu^* A_0) \neq 0$. By the trace map we have $H^0(C, \omega_C(-A_0)) \neq 0$. Here ω_C is the canonical sheaf of C.

On the other hand, since C is 1-dimensional, we can take a sufficiently small analytic neighborhood $V \subset Y$ of $f(C)$ and denote $U = f^{-1}(V) \cap X_0$, such that there exists a Cartier divisor A on U where $A_0 = A|_C$ and the support of A does not intersect with irreducible components of $\mathrm{Exc}(f|_U)$ other than C. Since $((K_{X_0}+B_0)\cdot C) \leq \deg A_0$, $A-(K_{X_0}+B_0)$ is relatively nef for $f : U \to V$.

By Theorem 2.8.1, $R^1 f_*(\mathcal{O}_U(A)) = 0$. Therefore $H^1(C, A_0) = 0$, and $H^0(C, \omega_C(-A_0)) = 0$ by the Serre duality, which is a contradiction. □

Go back to the proof of the theorem. If $n = 1$, then by $\deg K_{E'} < ((K_X + B)\cdot E) < 0$, it is easy to see that $E' \cong \mathbf{P}^1$ and $-2 < ((K_X+B)\cdot E)$. Moreover, by the vanishing theorem, $R^1 f_*\mathcal{O}_X = 0$, which implies that $E \cong \mathbf{P}^1$.

Suppose that $n > 1$. By taking the degree of H sufficiently large, we may assume that C is not a rational curve. By the lemma, $(K_{E'}\cdot C') < ((K_X + B)\cdot C) < 0$, we can apply Theorem 2.7.2 to $C' \subset E'$. Note that $M = -\nu^*(K_X + B)$ is ample on E', so by Theorem 2.7.2, passing through any point on C', there exists a rational curve L' satisfying $(M\cdot L') \leq 2n(M\cdot C')/(-K_{E'}\cdot C') < 2n$. $L = \nu(L')$ is the rational curve we are looking for. □

{length of extremal rays}

Corollary 2.8.4. *Let (X, B) be a $\mathbf{Q}$-factorial KLT pair, $f : X \to S$ a projective morphism. Take a (K_X+B)-negative extremal ray R in $\overline{\mathrm{NE}}(X/S)$. Take E to be the exceptional set of the corresponding contraction morphism h and denote $n = \dim E - \dim h(E)$. Here $E = X$ if h is a Mori fiber space. Then E is covered by rational curves L such that L are contracted by h and $-((K_X + B)\cdot L) < 2n$ (resp. $\leq 2n$) if $E \neq X$ (resp. $E = X$).*

Proof. If $E \neq X$, this is Theorem 2.8.2. If $E = X$, this is by Theorem 2.7.2. □

2.9 Divisorial Zariski decomposition

{section DZD}

In algebraic surface theory, the intersection theory of divisors is a very powerful tool. Since the intersection number is a symmetric bi-linear form, the Zariski decomposition theory can be developed in a strong form. In higher dimensional algebraic geometry, it is difficult or impossible to develop a strong Zariski decomposition theory, but if restricted to codimension 1, the "divisorial Zariski decomposition" can be easily constructed, and is sufficiently useful.

Definition 2.9.1. Let $f : X \to S$ be a projective morphism from a **Q**-factorial normal algebraic variety to a quasi-porjective algebraic variety, D a relatively pseudo-effective **R**-divisor, and H a relatively ample divisor. If

$$N = \lim_{t \downarrow 0} \inf\{D' \mid D + tH \equiv_S D' \geq 0\}$$

is a well-defined **R**-divisor[2.9.0.1], then we can define the relative *divisorial Zariski decomposition* $D = P + N$ of D over S by taking $P = D - N$. Here P is called the *numerically movable part*, and N is called the *numerically fixed part*.

If $D = P$, then D is called *numerically movable*. The cone consisting of numerical equivalence classes of all numerically movable **R**-divisors is denoted by $\overline{\mathrm{Mov}}(X/S) \subset N^1(X/S)$, and called the *numerically movable cone*.

Let us give more explanation about the definition. Fixing H and a positive number t, since $[D+tH] \in \mathrm{Big}(X/S)$, $D+tH$ is numerically equivalent to an effective **R**. Therefore, the effective **R**-divisor

$$N_t = \inf\{D' \mid D + tH \equiv_S D' \geq 0\}$$

can be defined. Here the inf of **R**-divisors is defined by taking the inf of coefficients of each component. Since H is numerically free, we know that $N_{t'} \geq N_t$ if $t' \leq t$. But here we should be careful that by taking limit $N = \lim_{t \downarrow 0} N_t$, the coefficients of N may go to infinity. An example given by Lesieutre [101] shows that this could happen. Therefore the the relative divisorial Zariski decomposition can be defined only if N is an **R**-divisor, that is, non of its coefficients is infinity. But nevertheless we know the existence of the relative divisorial Zariski decomposition in the following cases:

Lemma 2.9.2 ([123, Lemma III.4.3])**.** *The relative divisorial Zariski decomposition of a relatively pseudo-effective* **R***-divisor D exists if one of the following holds:*

(1) $S = \mathrm{Spec}\, k$ *is a point.*

(2) D *is relatively numerically equivalent to an effective* **R***-divisor.*

(3) $\mathrm{Supp}(D)$ *does not dominates* S*.*

(4) $\mathrm{codim}\, f(V) < 2$ *for every component* V *of* $\mathrm{Supp}(D)$*.*

If $\dim X = 2$, the divisorial Zariski decomposition and the classical Zariski decomposition coincide ([64]).

[2.9.0.1] I added more details here, because the relative divisorial Zariski decomposition does not always exist

Lemma 2.9.3. *Assume that the relative divisorial Zariski decomposition $D = P + N$ exists. Then*

(1) The number of irreducible components of N is bounded by $\rho(X/S)$.

(2) P is relatively pseudo-effective.

(3) N and P is independent of the choice of the relatively ample divisor H.

Proof. (1) The number of irreducible components of N_t is bounded by the number of numerically linearly independent **R**-divisors, which is $\rho(X/S)$.

(2) P is relatively pseudo-effective just because

$$P = \lim_{t \to 0}(D + tH - N_t),$$

where $D + tH - N_t$ is relatively pseudo-effective.

(3) For another relatively ample divisor H', there exist positive integers m, m' such that $mH - H'$ and $m'H' - H$ are both relatively ample. Then it is easy to show that N is independent of the choice of H. □

Lemma 2.9.4. *(1) The numerically movable cone $\overline{\mathrm{Mov}}(X/S)$ is a closed cone, and we have the following inclusions*

$$\overline{\mathrm{Amp}}(X/S) \subset \overline{\mathrm{Mov}}(X/S) \subset \overline{\mathrm{Eff}}(X/S).$$

*(2) Let $\alpha : X \dashrightarrow Y$ be a birational map between **Q**-factorial normal algebraic varieties over a quasi-porjective algebraic variety S. Assume that α is isomorphic in codimension 1, then the natural map $\alpha_* : N^1(X/S) \to N^1(Y/S)$ induces a bijective map $\alpha_*(\overline{\mathrm{Mov}}(X/S)) = \overline{\mathrm{Mov}}(Y/S)$.*

Proof. (1) Let D be a relatively pseudo-effective **R**-divisor and H a relatively ample divisor. If for any $t > 0$ $D + tH \in \overline{\mathrm{Mov}}(X/S)$, then it is easy to see that $N_t = 0$ for any $t > 0$, which implies that $D \in \overline{\mathrm{Mov}}(X/S)$. So the numerically movable cone is closed.

If D is relatively nef, then $D + tH$ is relatively ample and hence the nef cone is contained in the numerically movable cone.

(2) Take projective birational morphisms $p : Z \to X$ and $q : Z \to Y$ from a common normal algebraic variety Z such that $\alpha = q \circ p^{-1}$. For any **R**-divisors D, D' on X, if $D \equiv_S D'$, then $q_*p^*D \equiv_S q_*p^*D'$. Note that $\alpha_* = q_* \circ p^*$.

Take relatively ample divisors H_X and H_Y on X and Y such that $H_Y - \alpha_* H_X$ is relatively ample. It is easy to see that if $\inf\{D' \mid D + tH_X \equiv_S D' \geq 0\} = 0$, then $\inf\{D'' \mid \alpha_* D + tH_Y \equiv_S D'' \geq 0\} = 0$, which means that the image of a numerically movable divisor is numerically movable. □

Remark 2.9.5. If $\dim X = 2$, then being numerically movable is equivalent to being nef. Hence in this case the numerically movable cone coincides with the nef cone, and the divisorial Zariski decomposition is the classical Zariski decomposition.

For a pair (X, B), the divisors that should be contracted in order to get a minimal model can be determined by the divisorial Zariski decomposition of $K_X + B$:

{thm E in N}

Theorem 2.9.6. *Let (X, B) be a $\mathbf{Q}$-factorial DLT pair and $f : X \to S$ a projective morphism to a quasi-projective variety. Assume that there exists a minimal model $\alpha : (X, B) \dashrightarrow (Y, C)$ with induced projective morphism $g : Y \to S$. Then the divisorial Zariski decomposition $K_X + B = P + N$ over S exists. Moreover, let E be a prime divisor on X, then E is contracted by α (that is, $\alpha_* E = 0$) if and only if E is a component of N.*

Proof. Note that $K_X + B$ is relatively pseudo-effective since it has a minimal model, hence we can consider the divisorial Zariski decomposition.

Take projective birational morphisms $p : Z \to X$ and $q : Z \to Y$ from a common normal algebraic variety Z such that $\alpha = q \circ p^{-1}$. By assumption, the *discrepancy* $G = p^*(K_X + B) - q^*(K_Y + C)$ is effective, and E is contracted by α if and only if $p_*^{-1} E$ is a component of G.

Take a relatively ample divisor H' on Y, and a relatively ample divisor H on X such that $H - p_* q^* H'$ is relatively ample. For any $t > 0$, since $K_Y + C + tH'$ is relatively ample and

$$K_X + B + tH = p_* q^*(K_Y + C + tH') + t(H - p_* q^* H') + p_* G,$$

we have

$$\inf\{D' \mid K_X + B + tH \equiv_S D' \geq 0\} \leq p_* G.$$

Therefore, N is well-defined and $N \leq p_* G$.

To finish the proof, we will show that $N \geq p_* G$.

If $K_X + B + tH \equiv_S D' \geq 0$, then $\alpha_* D' \equiv_S K_Y + C + t\alpha_* H$, and

$$\begin{aligned} p^* D' - q^* \alpha_* D' &\equiv_S p^*(K_X + B + tH) - q^*(K_Y + C + t\alpha_* H) \\ &= G + t(p^*(H) - q^*(\alpha_* H)). \end{aligned}$$

Note that both sides are exceptional divisors over Y, so they are actually equal by the negativity lemma. Therefore,

$$p^* D' \geq G + t(p^*(H) - q^*(\alpha_* H)).$$

Taking the limit when $t \to 0$, we can see that $N \geq p_* G$. □

Remark 2.9.7. (1) If $\dim X = 2$, contracting all those divisors in N, or in other words contracting all (-1)-curves will produce a minimal model. If $\dim X \geq 3$, then the situation becomes much more complicated because the geometry in codimension 2 or higher is involved.

(2) The Zariski decomposition of a divisor D on an algebraic surface is discovered by Zariski [153] during the study of the section ring $\bigoplus_{m=0}^{\infty} H^0(X, mD)$ of D. In particular, if we consider the Zariski decomposition of the canonical divisor, then the numerically movable part coincides with the pullback of the canonical divisor on the minimal model. In this sense, we can say that the Zariski decomposition of canonical divisors is equivalent to the minimal model theory.

Generalizing this idea, the log version of existence of minimal models in dimension 2 can be proved as an application of the Zariski decomposition ([54]). Moreover, [31] generalized the Zariski decomposition to pseudo-effective divisors.

In dimension 2, the intersection theory of divisors is available so that we can use the general theory of symmetric bilinear forms to define the Zariski decomposition, but this is not the case in dimension 3 and higher. So in [64], the divisorial Zariski decomposition was defined only for big divisors using the limit of linear systems. [123] pushed this forward and generalized the definition to pseudo-effective divisors. In [15], the fixed part was defined using $\mathbf{R}$-linear equivalence. Here the definition was simplified by replacing numerical equivalence with $\mathbf{R}$-linear equivalence.

Similar to the case of dimension 2, if the numerically movable part is nef, then in fact we can get a minimal model. In order to deal with problems caused by subsets of dimension 2 or higher, we need to replace X by blowing ups. Although this approach to the minimal models is not successful, it might be helpful for understanding the problem. In this book, we use flips instead of blowing ups to deal with subsets of dimension 2 or higher.

In addition, there is also an analytical approach to the analytical Zariski decomposition, which has played a certain role ([147]).

If the numerically movable part is not 0, then we can make many global sections by adding a little positivity:

{Nakayama-Zariski}

Theorem 2.9.8 (Nakayama [123]). *Let D be a pseudo-effective $\mathbf{R}$-divisor on a normal projective $\mathbf{Q}$-factorial algebraic variety X. Take $D = P + N$ to be the divisorial Zariski decomposition. If $P \not\equiv 0$, then there exists an ample divisor H, such that the function in positive integer m satisfies*

$$\lim_{m \to \infty} \dim H^0(X, \llcorner mD \lrcorner + H) = \infty.$$

Proof. Since N is effective, we may assume that $D = P$. Consider the *numerical base locus*

$$\mathrm{NBs}(D) = \lim_{t \downarrow 0} (\bigcap \{\mathrm{Supp}(D') \mid D + tH \equiv D' \geq 0\}).$$

Since $N = 0$, $\mathrm{NBs}(D)$ has no components of codimension 1. Also, since this is a limit of an increasing sequence of closed subsets, it is a union of at most countably many subvarieties of codimension at least 2. Therefore, we may take a very general smooth curve $C \subset X$ by cutting by very general hyperplanes such that $C \cap \mathrm{NBs}(D) = \emptyset$. Since $D \not\equiv 0$, $(D \cdot C) > 0$.

Fix an ample divisor H, take $L_m = \llcorner mD \lrcorner + H$. Note that $\deg_C \llcorner mD \lrcorner = (mD \cdot C) - ((mD - \llcorner mD \lrcorner) \cdot C$ can be arbitrarily large if m is sufficiently large, since $(D \cdot C) > 0$ and $((mD - \llcorner mD \lrcorner) \cdot C$ is bounded. Hence $H^0(C, L_m|_C)$ can be arbitrarily large. We will show that if m is sufficiently large, then the natural map $H^0(X, L_m) \to H^0(C, L_m|_C)$ is surjective.

Note that C is contained in the smooth locus of X, consider $g : Y \to X$ to be the blowing up along C, and denote E to be the exceptional divisor. For any $t > 0$, there exists an effective $\mathbf{R}$-divisor $D_m \equiv mD + tH$ such that its support does not contain C, and (Y, g^*D_m) is KLT in a neighborhood of E.

Note that

$$\begin{aligned} & g^*L_m - E - (K_Y + g^*D_m) \\ & = g^*(\llcorner mD \lrcorner + H - (K_X + D_m)) - (n-1)E \\ & \equiv g^*((1-t)H - (mD - \llcorner mD \lrcorner) - K_X) - (n-1)E. \end{aligned}$$

Here, $n = \dim X$. Note that we may take H sufficiently large comparing to components of K_X, E, D, and t sufficiently small, such that this divisor is ample.

By the Nadal vanishing theorem,

$$H^1(Y, I(Y, g^*D_m) \otimes \mathcal{O}_Y(g^*L_m - E)) = 0.$$

By assumption,

$$E \cap \mathrm{Supp}(\mathcal{O}_Y / I(Y, g^*D_m)) = \emptyset,$$

hence the natural map

$$H^0(Y, g^*L_m) \to H^0(E, (g^*L_m)|_E)$$

is surjective. This proves the claim. □

Conversely, if the function $\dim H^0(X, \llcorner mD \lrcorner + H)$ of positive integer m is bounded, then we say that the *numerical Kodaira dimension* of D is 0, which is denoted by $\nu(X, D) = 0$. In general, we define the numerical Kodaira dimension as the following:

Definition 2.9.9. The *numerical Kodaira dimension* $\nu(X, D)$ of an $\mathbf{R}$-divisor D is defined to be the minimal integer ν satisfying the following property ([123]): for any fixed H, there exists a positive real number c, such that for any positive integer m,

$$\dim H^0(X, \llcorner mD \lrcorner + H) \leq cm^{\nu}.$$

If D is not pseudo-effective, then we denote $\nu(X, D) = -\infty$.

This definition corresponds to the definition of the Kodaira dimension $\kappa(X, D)$, which is just the minimal integer κ satisfying that there exists a positive real number c, such that for any positive integer m

$$\dim H^0(X, \llcorner mD \lrcorner) \leq cm^\kappa.$$

2.10 Polyhedral decomposition of cone of divisors

A *polytope* in a real vector space is the convex closure of finitely many points. It is called a *rational polytope* if all the vertices are points with rational numbers as coordinates (rational points). In this section, we consider polyhedral decompositions of cone of divisors with respect to minimal models or canonical models and their applications. A line is an important example of a polytope, and the MMP with scaling is related to the decomposition of this polytope.

When changing the coefficients b_i in the log canonical divisor $K_X + \sum_i b_i B_i$, the corresponding canonical model changes. This phenomenon is the similar to that quotient spaces change according to polarizations in *geometric invariant theory (= GIT)*.

2.10.1 Rationality of sections of nef cones

Applying the length of extremal rays, we can show that the sections of nef cones are rational polytopes:

{Shokurov letterVII}

Theorem 2.10.1 (Shokurov [136])**.** *Let X be a $\mathbf{Q}$-factorial normal algebraic variety, $f : X \to S$ a projective morphism, and $B_1, \ldots, B_t$ effective $\mathbf{Q}$-divisors. Assume that (X, B_i) is $\overline{KLT}$ for all i. Take P be the smallest convex closed subset containing all B_i in the real vector space of $\mathbf{R}$-divisors on X, denote $N = \{B' \in P \mid K_X + B'$ is relatively nef$\}$. Take $\{R_j\}$ to be the set of all extremal rays R such that there exists a point $B' \in P$ such that $((K_X + B') \cdot R) < 0$. Take $H_j = \{B'' \in P \mid ((K_X + B'') \cdot R_j) = 0\}$ to be the rational hyperplane section of P determined by R_j. The the following statements hold:*

(1) For any interior point x in P, take U to be a sufficiently small neighborhood, then it intersects only finitely many rational hyperplanes H_j.

(2) N is a rational polytope.

Proof. (1) Assume, to the contrary, that any neighborhood U of x intersects infinitely many distinct H_j. Then there exists a rational line in the smallest real linear space containing P passing through a sufficiently small neighborhood U of x with the following property: $L \cap U$ is an open subset of the rational closed interval $L \cap P = [B, C]$ intersecting infinitely many H_j at distinct points. Denote $L \cap H_j = (1-t_j)B + t_jC$ and take $t_0 \in (0,1)$ be a limit point of $\{t_j\}$.

By construction, either $((K_X+B)\cdot R_j) < 0$ or $((K_X+C)\cdot R_j) < 0$ holds. By the length of extremal rays, we can take a rational curve l_j generating R_j such that either

$$0 < (-(K_X+B)\cdot l_j) \leq 2b$$

or

$$0 < (-(K_X+C)\cdot l_j) \leq 2b.$$

Here b is the maximal dimension of fibers of f.

Applying the Diophantine approximation theorem to t_0, there is a sufficiently large positive integer q and a rational number p/q such that $|t_0 - p/q| < 1/q^2$. Here we allow $t_0 = p/q$. On the other hand, there exists a positive integer m such that $m(K_X+B)$ and $m(K_X+C)$ are both Cartier. Therefore, the absolute value $|((K_X+(1-p/q)B+p/qC)\cdot l_j)|$ is either 0 or at least $1/mq$.

If $p/q \neq t_j$ for some j, then $((K_X+(1-p/q)B+p/qC)\cdot l_j) \neq 0$, otherwise by $((K_X+(1-t_j)B+t_jC)\cdot l_j) = 0$, we have $((K_X+B)\cdot l_j) = ((K_X+C)\cdot l_j) = 0$, a contradiction. Hence $|((K_X+(1-p/q)B+p/qC)\cdot l_j)| \geq 1/mq$. Moreover, we can take j sufficiently large such that $|t_j - p/q| < 2/q^2$, then the absolute value of the slope of the function $((K_X+(1-x)B+xC)\cdot l_j)$ is at least $q/2m$. This contradicts the fact that $(-(K_X+B)\cdot l_j)$ or $(-(K_X+C)\cdot l_j)$ is bounded.

(2) By the cone theorem, the nef cone N is the intersection of inner sides of hyperplanes H_j. Therefore, by (1), N is a rational polytope in the interior of P. We only need to investigate the neighborhood of the boundary of P.

Take L to be any rational linear subspace contained in the smallest real linear space containing P, we will prove that $N \cap L$ is a rational polytope by induction on $\dim L$. If $P \subset L$, then this is the statement of the theorem. Take P_L to be the smallest face of P containing $L \cap P$. We may replace P by P_L and assume that $P = P_L$, that is, L contains an interior point of P.

If $\dim L = 1$, then $N \cap L$ is a point or a closed interval. Every endpoint is a rational point: this is clear if it is on the boundary of P, and by (1) if it is an interior point of P.

Now assume that $\dim L > 1$. For any face P' of P, $N \cap P' \cap L$ is a rational polytope by induction hypothesis. Since N is locally a rational polytope near interior points of P, it suffices to show that $N \cap L$ is locally a rational polytope near every vertex B of $N \cap P' \cap L$.

Take any rational line $L' \subset L$ passing though B and containing an interior point of P, write $P \cap L' = [B, C]$. Then $N \cap L' = [B, (1-t_0)B + t_0C]$ for some $t_0 \in [0,1]$. Here t_0 is a rational number by (1). If $t_0 \neq 0, 1$, then $(1-t_0)B+t_0C$ is an interior point of P, and there exists an index j such that $L' \cap H_j = \{(1-t_0)B+t_0C\}$. Take a positive integer m such that $m(K_X+B)$ is Cartier. Since $(m(K_X+B)\cdot l_j) > 0$ and it is an integer, by the argument of (1), there exists a constant $c > 0$ depending only on B but not L' such that $t_0 \geq c$. Therefore, there exists a sufficiently small neighborhood U of B such that $N \cap L \cap U$ is a cone with vertex B.

Take a general rational hyperplane M sufficiently near to B, then $N \cap L \cap M$ is a rational polytope by induction hypothesis, hence $N \cap U$ is a cone over a rational polytope. This finishes the proof. □

Remark 2.10.2. In this theorem, the section of the nef cone is a rational polytope since finitely many divisors are fixed in the beginning. In general this statement is not true for $N^1(X/S)$ since there are infinitely many divisors. For example, the surface of the nef cone of an Abelian variety is defined by $(D^n) = 0$, which is not linear.

2.10.2 Polyhedral decomposition according to canonical models

For a given pair (X, B), its minimal model is not unique in general, but its canonical model is unique. Therefore, we first consider the decomposition according to canonical models:

{poly decomposition 1}

Theorem 2.10.3 (Polyhedral decomposition 1 ([136], [79]))**.** *Let X be a $\mathbf{Q}$-factorial normal algebraic variety, $f : X \to S$ a projective morphism to a quasi-projective variety, and $B_1, \ldots, B_t$ effective $\mathbf{R}$-divisors such that (X, B_i) is $\overline{KLT}$ for all i. Take V to be the affine subspace generated by all B_i in the real vector space of divisors. Take P' to be the polytope generated by all B_i. Consider the following convex closed subset of P':*

$$P = \{B = \sum_i b_i B_i \in P' \mid [K_X + B] \in \overline{\text{Eff}}(X/S)\}.$$

Assume the following conditions:

- *For each point $B \in P$, there exists a minimal model $\alpha : (X, B) \dashrightarrow (Y, C)$ and a canonical model $g : Y \to Z$ of $f : (X, B) \to S$.*
- *For each point $B \in P$, there exists a polytope $P'_B \subset V$ containing B as an interior point in the topology of V, such that if denote*

  $$P_B = \{B' \in P'_B \cap P' \mid [K_Y + \alpha_* B'] \in \overline{\text{Eff}}(Y/Z)\},$$

  then for any $B' \in P_B$, the morphism $g : (Y, \alpha_* B') \to Z$ admits a minimal model and a canonical model.

Then there exists a finite disjoint decomposition

$$P = \coprod_{j=1}^{s} P_j$$

and rational maps $\beta_j : X \dashrightarrow Z_j$ satisfying the following properties:

(1) $B \in P_j$ if and only if β_j gives the canonical model of $f : (X, B) \to S$.

(2) The closures $\bar{P}_j$ of P_j are unions of polytopes. In particular, P is a polytope.

(3) If $P_j \cap \bar{P}_{j'} \neq \emptyset$, then there exists a morphism $f_{jj'} : Z_{j'} \to Z_j$ such that $\beta_j = f_{jj'} \circ \beta_{j'}$.

Here note that P_j is not necessarily connected.

Proof. Firstly, note that (X, B) is $\overline{\text{KLT}}$ for any $B \in P'$, therefore we can use the framework of the minimal model theory. We prove the theorem by induction on $\dim V$. If $\dim V = 0$, then the statement is trivial. Assume that $\dim V \geq 1$. Fix any point $B \in P$. Take the minimal model $\alpha : (X, B) \dashrightarrow (Y, C)$ and the canonical model $g : (Y, C) \to Z$. There exists an $\mathbf{R}$-Cartier divisor H on Z relatively ample over S such that $K_Y + C = g^*H$. We can take P'_B sufficiently small, such that for any $B' \in P'_B \cap P$, $K_X + B'$ is negative with respect to α.

For any $B' \in P_B$, take a minimal model $\alpha' : (Y, \alpha_* B') \dashrightarrow (Y', C')$ and the canonical model $g' : (Y', C') \to Z'$ of $g : (Y, \alpha_* B') \to Z$. Take $h : Z' \to Z$ to be the natural morphism. There exists an $\mathbf{R}$-Cartier divisor H' on Z relatively ample over Z such that $K_{Y'} + C' = (g')^* H'$. Take a sufficiently small real number δ such that $(1 - \delta)h^*H + \delta H'$ is relatively ample over S. Take $B'' = (1 - \delta)B + \delta B'$ and $C'' = (1 - \delta)\alpha_* C + \delta C'$, then the negativity still holds, so $\alpha' \circ \alpha : (X, B'') \dashrightarrow (Y', C'')$ is a minimal model of $f : (X, B'') \to S$ and $g' : (Y', C'') \to Z'$ is the canonical model.

We remark that such δ does not depend on B', but only depends only on B (and H). Since H is ample over Z, we may take a sufficiently small $\epsilon > 0$ such that $(H \cdot \Gamma_Z) > \epsilon$ for any relative curve Γ_Z on Z. Then we may take $\delta = \epsilon/(2\epsilon + 4 \dim X)$. In fact, we will show that $K_{Y'} + (1 - 2\delta)\alpha_* C + 2\delta C'$ is relatively nef over S, which implies that $(1 - 2\delta)h^*H + 2\delta H'$ is relatively nef over S, and therefore $(1 - \delta)h^*H + \delta H'$ is relatively ample over S. Assume, to the contrary, that $K_{Y'} + (1 - 2\delta)\alpha_* C + 2\delta C'$ is not relatively nef over S, then there exists a negative extremal ray R, which is also a $(K_{Y'} + C')$-negative extremal ray since $K_{Y'} + \alpha_* C$ is relatively nef over S. By the length of extremal ray, R is generated by a rational curve Γ such that $((K_{Y'} + C') \cdot \Gamma) \geq -2 \dim X$. Note that C is not contacted over Z as $K_{Y'} + C'$ is nef over Z, therefore it is easy to compute that

$$((K_{Y'} + (1 - 2\delta)\alpha_* C + 2\delta C') \cdot \Gamma) \geq 0,$$

a contradiction.

Therefore, to summarize, if we take P'_B sufficiently small, the for any $B' \in P_B$, (Y', C'') and Z' are minimal model and canonical model for both $f : (X, B') \to S$ and $g : (Y, \alpha_* B') \to Z$. In particular, $P_B = P'_B \cap P$.

Also we can see that they are minimal model and canonical model for $f : (X, (1-t)B + tB') \to S$ for any $0 < t \leq 1$.

The boundary $\partial(P'_B \cap P')$ of $P'_B \cap P'$ (as a subset of V) is a finite union of $(\dim V - 1)$-dimensional polytopes $(\partial(P'_B \cap P'))_k$. Note that $K_Y + C$ is relatively numerically trivial over Z, hence $\partial P_B \subset \partial(P'_B \cap P')$. We can apply induction hypothesis to $(\partial(P'_B \cap P'))_k$ and $(Y, C) \to Z$, here to check the second condition, we use the second condition on X and the fact that X and Y have the same minimal model and canonical model for divisors in P_B. Then this implies that there is a decomposition of ∂P_B into finitely many polytopes corresponding to canonical models of $(Y, \alpha_* B') \to Z$ for $B' \in \partial P_B$. Therefore, P_B is decomposed into cones over these polytopes with vertex at B, which correspond to canonical models of $(Y, \alpha_* B') \to Z$ for $B' \in P_B$. Since P' is compact, it can be covered by finitely many such P'_B, and the first two statement are proved. For the third conclusion, just take $B \in P_j \cap \bar{P}_{j'}$ and it is clear from the above argument. □

2.10.3 Polyhedral decomposition according to minimal models

Next we consider the decomposition according to minimal models:

{poly decomposition 2}

Theorem 2.10.4 (Polyhedral decomposition 2 ([136], [79]))**.** *Keep the assumption in Theorem 2.10.3. Then for each P_j, there is finite disjoint decomposition*

$$P_j = \coprod_{k=1}^{t} Q_{j,k}$$

satisfying the following properties: fix a birational map $\alpha : X \dashrightarrow Y$ such that

$$Q = \{B \in P \mid \alpha \text{ is a minimal model of } f : (X, B) \to S\}$$

is non-empty, then

(1) Q is locally closed, whose closure is a polytope.

(2) There exists an index j such that $Q \subset \bar{P}_j$.

(3) If $Q \cap P_j \neq \emptyset$ for some j, then there exists k such that $Q \cap P_j = Q_{j,k}$.

(4) The closure of $\bar{Q}_{j,k}$ is a polytope for any j, k.

{remark mm no unique}

Remark 2.10.5. For any fixed j, k, it is possible that there are infinitely many α such that $Q \cap P_j = Q_{j,k}$. For example, for a pair (X, B) satisfying $K_X + B \equiv_S 0$, there might be infinitely birational maps α inducing minimal models (Example 2.10.7).

Proof. (1) Q is determined by cutting the pullback of the nef cone $\overline{\text{Amp}}(Y/S)$ by finitely many linear inequalities given by negativity of log canonical divisors. The nef cone is a closed polytope, and the inequalities are open conditions, hence we get the conclusion.

(2) It is easy to see that if $B, B' \in Q$, then $tB + (1-t)B' \in Q$ for any $t \in [0,1]$. Hence Q is a convex set. Take a relative interior point $B \in Q$, take $g : Y \to Z$ to be the canonical model of $(Y, \alpha_* B)$. Then $[\alpha_*(K_X + B)] \in g^* \text{Amp}(Z/S)$ and $g^* \overline{\text{Amp}}(Z/S)$ is a face of $\overline{\text{Amp}}(Y/S)$. For any $B' \in Q$, since $[\alpha_*(K_X + B')] \in \overline{\text{Amp}}(Y/S)$ and Q is convex, we have $[\alpha_*(K_X + B')] \in g^* \overline{\text{Amp}}(Z/S)$. Moreover, if B' is another relative interior point, then $[\alpha_*(K_X + B')] \in g^* \text{Amp}(Z/S)$. Hence if we take P_j to be the subset corresponding to the canonical model $g \circ \alpha$, then $Q \subset \bar{P}_j$.

(3) Given two birational maps $\alpha_i : X \dashrightarrow Y_i$ $(i = 1, 2)$ with corresponding subsets $\emptyset \neq Q_i \subset P$. Assume that there exist morphisms $g_i : Y_i \to Z$ such that $\beta = g_1 \circ \alpha_1 = g_2 \circ \alpha_2$ corresponds to some P_j. Consider the birational map $\gamma : Y_1 \dashrightarrow Y_2$ determined by $\alpha_2 = \gamma \circ \alpha_1$. We claim that if γ is isomorphic in codimension 1, then $Q_1 \cap P_j = Q_2 \cap P_j$. In fact, take a point $B \in Q_1 \cap P_j$, we can write $K_{Y_1} + \alpha_{1*} B \equiv g_1^* H$ for a relatively ample $\mathbf{R}$-divisor H on Z. Since γ is isomorphic in codimension 1, $K_{Y_2} + \alpha_{2*} B \equiv g_2^* H$. Therefore $B \in Q_2 \cap P_j$. In particular, if $Q_1 \cap Q_2 \cap P_j \neq \emptyset$, then the minimal models corresponding to a point $B \in Q_1 \cap Q_2 \cap P_j$ are isomorphic in codimension 1, and therefore $Q_1 \cap P_j = Q_2 \cap P_j$.

Hence by the above argument, we get a disjoint decomposition

$$P_j = \coprod_{\alpha} (Q \cap P_j)$$

where α runs over all birational contractions $\alpha : X \dashrightarrow Y$, and $Q \cap P_j$ depends only on divisors contracted by α.

Take $B_{j,l}$ to be vertices of $\bar{P}_j$ and take $\{E_m\}$ to be the set of prime divisors appearing in the numerical fixed part of some $K_X + B_{j,l}$ (note that the divisorial Zariski decompositions over S exist due to the existence of canonical models). Note that $\{E_m\}$ is a finite set and contains all prime divisors appearing in the numerical fixed part of $K_X + B$ for any $B \in P_j$. So by Theorem 2.9.6, there are finitely many possibilities for the set of prime divisors contracted by α, and hence the decomposition of P_j is finite.

(4) Since $\bar{P}_j$ is a union of polytopes and $\bar{Q}$ is a polytope, $\bar{Q}_{j,k}$ is a polytope. Here we remark that $Q_{j,k}$ and $\bar{Q}_{j,k}$ are convex. □

Corollary 2.10.6. *In Theorems 2.10.3 and 2.10.4, if all B_i are $\mathbf{Q}$-divisors, then P, P', $\bar{Q}_{j,k}$ are all rational polytopes, and $\bar{P}_j$ is a union of rational polytopes.*

Proof. As in the proof, $\bar{Q}$ is determined by cutting the pullback of the nef cone of the minimal model by finitely many linear inequalities with rational coefficients. As the nef cone is a rational polytope, $\bar{Q}$ is also a rational polytope.

Fix a P_j, denote P'_j to be the set of interior points of P_j as a subset of V. Then a point in P'_j is contained in some $Q \subset \bar{P}_j$ and hence $\bar{P}'_j$ is the union of such $\bar{Q}$, which is a union of rational polytopes. On the other hand, $P_j \setminus \bar{P}'_j$ is contained in a union of faces of rational polytopes, so we may replace P by those faces and show that the closure of $P_j \setminus \bar{P}'_j$ is a union of rational polytopes. Therefore, $\bar{P}_j$ is a union of rational polytopes and P is a rational polytope.

$\bar{Q}_{j,k}$ is the intersection of a rational polytope and a union of rational polytopes, hence is a rational polytope. □

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Example 2.10.7. Consider a general hypersurface X in $\mathbf{P}^2 \times \mathbf{P}^1 \times \mathbf{P}^1$ of type $(3, 2, 2)$. X is a smooth projective 3-dimensional algebraic variety with $K_X \sim 0$. We consider the polyhedral decomposition of pseudo-effective cone of this example, in which there are infinitely many rational polytopes. This is also an example such that the quotient of birational automorphism group by the biregular automorphism group $\mathrm{Bir}(X)/\mathrm{Aut}(X)$ is an infinite group.

Denote P_1, P_2, P_3 by the projective spaces in the fiber product, take L_i to be the pullback of hyperplanes H_i by the projection $p_i : X \to P_i$ $(i = 1, 2, 3)$. L_1, L_2, L_3 is a basis of $N_1(X)$. The nef cone $\overline{\mathrm{Amp}}(X)$ is the simple cone generated by L_1, L_2, L_3.

The projection $p_i : X \to P_i$ corresponds to the extremal ray $\langle L_i \rangle$. Here $\langle\rangle$ means the generated cone. A general fiber of p_1 is an elliptic curve, and that of p_2, p_3 is a K3 surface. The projection $p_{ij} : X \to P_i \times P_j$ corresponds to the face $\langle L_i, L_j \rangle$. A general fiber of p_{23} is an elliptic curve, and that of p_{12}, p_{13} is a set of 2 points. Taking a stein factorization, q_{12}, q_{13} are small contractions.

Express the equation of X by $f(x, y)z_0^2 + g(x, y)z_0z_1 + h(x, y)z_1^2 = 0$. Here, $[x_0 : x_1 : x_2]$, $[y_0 : y_1]$, $[z_0 : z_1]$ are homogeneous coordinates of $P_1, P_2.P_3$, f, g, h are homogenous polynomials of degree 3 for x_0, x_1, x_2 and of degree 2 for y_0, y_1. The exceptional locus of $q_{12} : X \to Y_{12}$ is defined by $f = y = h = 0$, which consists of 54 $\mathbf{P}^1$.

As $p_{12} : X \to P_1 \times P_2$ gives a degree 2 extension of function fields, X as a birational automorphism induced by the Galois group $\mathbf{Z}/(2)$, which is a birational map $\alpha : X \dashrightarrow X$ exchanging 2 points in general fibers of p_{12}, and given by $(x, y, [z_0 : z_1]) \mapsto (x, y, [hz_1 : fz_0])$. This birational map is non-trivial but preserves the equation of X and q_{12}. Note that

$$\begin{aligned} \alpha^* L_1 &= L_1, \\ \alpha^* L_2 &= L_2, \\ \alpha^* L_3 &= 3L_1 + 2L_2 - L_3. \end{aligned}$$

Here to distinguish with X, we denote $\alpha : X_0 \dashrightarrow X_1$. We consider $(X_1, 0)$ as a non-trivial minimal model of $(X_0, 0)$.

For $p_{13} : X \to P_1 \times P_3$, we can similarly define $\beta : X_0 \dashrightarrow X_{-1}$. Note that $\beta^* L_1 = L_1$, $\beta^* L_2 = 3L_1 - L_2 + 2L_3$, $\beta^* L_3 = L_3$.

Note that α^2 and β^2 are identity, α and β are not commutative. For each $n \in \mathbf{Z}$, we inductively define birational maps $\alpha_n : X_0 \dashrightarrow X_n$ by $\alpha \circ \alpha_n = \alpha_{-n+1}$, $\beta \circ \alpha_n = \alpha_{-n-1}$. If we take $M_k = \frac{3}{2}(k^2 + k)L_1 + (k+1)L_2 - kL_3$, then

$$\alpha_n^* L_1 = L_1,$$
$$\alpha_n^* L_2 = \begin{cases} M_{2m} & n = 2m; \\ M_{2m} & n = 2m+1, \end{cases}$$
$$\alpha_n^* L_3 = \begin{cases} M_{2m-1} & n = 2m; \\ M_{2m+1} & n = 2m+1. \end{cases}$$

So the image of the nef cone $\alpha_n^* \overline{\mathrm{Amp}}(X_n)$ is generated by L_1, M_{n-1}, M_n, which is different from each other for each n. So we get a subgroup $\mathbf{Z}/(2) * \mathbf{Z}/(2) \subset \mathrm{Bir}(X)$ of the birational automorphism group.

The pseudo-effective cone are decomposed into nef cones:

$$\overline{\mathrm{Eff}}(X) = \bigcup_{n \in \mathbf{Z}} \alpha_n^* \overline{\mathrm{Amp}}(X_n).$$

In fact, the right hand side is generated by L_1 and M_k, and note that the limit of the ray generated by M_k is L_1 as $|k| \to \infty$, so it is easy to see that any divisor D outside this cone, we can find a big divisor D' such that the segment $[D, D']$ interests the face generated by M_{n-1}, M_n for some n, but divisors on this face is not big, which implies that D can not be pseudo-effective. Moreover, since L_1 and M_k are all effective, we know that $\overline{\mathrm{Eff}}(X) = \mathrm{Eff}(X)$.

This cone is decomposed into infinitely many rational polytopes, and each of them corresponds to a minimal model of X. The reason that infinitely many cones appear is because the finite-dimensional space of divisor classes is the projection of the space of all divisors, which is of infinite dimension.

2.10.4 Applications of polyhedral decompositions

The polyhedral decomposition theorem plays an important role in the proof of the existence of the minimal models in the next chapter. Here we introduce other applications as the finiteness of crepant blowing ups, the termination of MMP with scaling, the fact that birational minimal models are connected by flops, and the generalization of MMP with scaling under weaker conditions.

For a KLT pair (X, B), a **crepant blowing up** of (X, B) is a projective birational morphism $g : (Y, C) \to (X, B)$ from a $\mathbf{Q}$-factorial KLT pair such that $g^*(K_X + B) = K_Y + C$. In particular, if (Y, C) admits no crepant blowing up other than automorphisms, then it is called a **maximal crepant blowing up**.

As an application of [15], we can get the following corollary by the argument in [64]:

{cbu}

Corollary 2.10.8 (*Crepant blowing up*)**.** *For a KLT pair (X, B), there exists a maximal crepant blowing up for (X, B). Moreover, the set of crepant blowing ups of (X, B) is finite up to isomorphisms.*

Proof. Take a very log resolution $f : \tilde{Y} \to (X, B)$, write $f^*(K_X + B) = K_{\tilde{Y}} + \tilde{C}$. Write $\tilde{C} = \tilde{C}^+ - \tilde{C}^-$ into positive part and negative part. Take a minimal model $g : (Y, C) \to (X, B)$ of $f : (\tilde{Y}, \tilde{C}^+) \to X$. Since $[K_{\tilde{Y}} + \tilde{C}^+] = \tilde{C}^- \in N^1(Y/X)$, all irreducible components of $\tilde{C}^-$ are contracted by f, and the set of divisors contracted by $\alpha : \tilde{Y} \dashrightarrow Y$ induced by the minimal model coincides with the support of $\tilde{C}^-$. That is, the set of exceptional divisors of g coincides with the set of exceptional divisors of f with non-negative coefficients in $\tilde{C}$. As f is a very log resolution, any blowing up of $\tilde{Y}$ does not create new prime divisors in the latter set, hence g is a maximal crepant blowing up.

Since g is birational, for any divisor D on Y, there exists an effective divisor D' on Y such that $D \equiv_X D'$. For any sufficiently small $\epsilon > 0$, $(Y, C + \epsilon D')$ is KLT. By [15], there exists a minimal model over X, and the canonical model exists by the base point free theorem. Hence by the polyhedral decomposition theorem, there exists a decomposition in a neighborhood of origin of $N^1(Y/X)$ corresponding to the canonical models. Taking cones of those polyhedrons, we get a decomposition of $N^1(Y/X)$ into polyhedral cones.

For any maximal maximal crepant blowing up$g' : (Y', C') \to X$, the set of exceptional divisors of g' coincides with the set of exceptional divisors of f with non-negative coefficients in $\tilde{C}$ as discrete valuations on $k(X)$. In fact, if this is not the case, we can take a common very log resolution and a minimal model over Y' as above to create a non-trivial crepant blowing up of Y'. Therefore, Y and Y' are isomorphic in codimension 1. The image of $\overline{\mathrm{Amp}}(Y'/X)$ under $N^1(Y'/X) \to N^1(Y/X)$ coincides with one of the above polyhedral cones. Hence there are only finitely many maximal crepant blowing ups.

For a crepant blowing up $g'' : (Y'', C'') \to X$, we can take a maximal crepant blowing up (Y', C') of (Y'', C''), which is also a maximal crepant blowing up (Y', C') of (X, B), and the nef cone $\overline{\mathrm{Amp}}(Y''/X'')$ corresponds to a face of $\overline{\mathrm{Amp}}(Y'/X')$. Hence crepant blowing ups are finite. □

Assuming the existence of minimal models and canonical models, we can show the termination of flips in MMP with scaling. Note that if there exists a sequence of flips that terminates, then it implies the existence of minimal models, but be aware that this is different with that any sequence of flips terminates.

{termination of MMP with s

Corollary 2.10.9 (*Termination of MMP with scaling*)**.** *Let $f : (X, B) \to S$ be a projective morphism from a* **Q***-factorial KLT pair. Consider the MMP with scaling of H. Here $(X, B + H)$ is KLT, $[K_X + B] \in \overline{\mathrm{Eff}}(X/S)$, and $[K_X + B + H] \in \mathrm{Big}(X/S) \cap \overline{\mathrm{Amp}}(X/S)$. Assume that there exists a minimal model and canonical model for (X, B). Then this MMP with scaling terminates.*

Proof. Take $X = X_0$ and denote $\alpha_i : X_i \dashrightarrow X_{i+1}$ to be each step of the MMP. Since there are only finitely many divisorial contractions, after removing finitely many steps, we may assume that α_i are all flips.

Since $K_X + B + H$ is relatively big and $K_X + B$ is relatively pseudo-effective, for any $1 \geq t > 0$, $K_{X_i} + B + tH$ is relatively big, hence its minimal model exists, by the existence of minimal models. Moreover, by the base point free theorem, its canonical model exists. By assumption, minimal model and canonical model exist if $t = 0$. We may apply the polyhedral decomposition theorem to the segment $[B, B + H]$, and get a decomposition of finitely many interval P_j. To simplify the notation, we denote $B + tH$ by t and consider the decomposition on $[0, 1]$. Take

$$t_i = \min\{t \in \mathbf{R} \mid K_{X_i} + B + tH \text{ is relatively nef}\},$$
$$t'_i = \max\{t \in \mathbf{R} \mid K_{X_i} + B + tH \text{ is relatively nef}\}.$$

In other words, the interval Q_i in which $X \dashrightarrow X_i$ gives a minimal model is just $[t_i, t'_i]$. Recall that for the extremal ray corresponding to α_i, we have $((K_{X_i} + B) \cdot R) < 0$, $((K_{X_{i+1}} + B) \cdot R) > 0$, $((K_{X_i} + B + t_iH) \cdot R) = ((K_{X_{i+1}} + B + t_iH) \cdot R) = 0$. In particular, $t_i = t'_{i+1}$.

Assume that there are infinitely many distinct intervals Q_i, then there exists an interval P_j in which t_i is an interior point, take $\beta : X \dashrightarrow Y$ to be the corresponding canonical model. We can find a $Q_{i'}$ $(i' \leq i)$ such that there exists $t > t_i$ in $P_j \cap Q_i$ and a $Q_{i''}$ $(i'' > i)$ such that there exists $t' < t_i$ in $P_j \cap Q_{i''}$. In this case, there exist morphisms $g_{i'} : X_{i'} \to Y$ and $g_{i''} : X_{i''} \to Y$. By construction, there exists a **R**-divisor H on Y such that $K_{X_i} + B + tH = g_i^* H$, $K_{X_{i'}} + B + tH = g_{i'}^* H$, but the former one is relatively nef while the latter one is not, a contradiction. Therefore, there are only finitely many distinct intervals Q_i.

Then we consider the case $t_i = t_{i+1} > 0$. In this case, take (Y, C) to be the common canonical model of $(X_i, B + t_iH)$ and $(X_{i+1}, B + t_iH)$. Since $K_{X_i} + B + t_iH$ is relatively nef and relatively big over S, $g_i : (X_i, B + t_iH) \to (Y, C)$ is a crepant blowing up. Then by Corollary 2.10.8, such g_i is finite, that is, there exists no infinite sequence $0 < t_i = t_{i+1} = t_{i+2} = \cdots$. In summary, there exists some n such that $t_n = 0$, which means that the MMP terminates. □

For a given pair, minimal models, if exist, are not unique in general. However, we can show that minimal models are connected by elementary birational maps so-called "flops".

A birational map $\alpha : (X, B) \dashrightarrow (Y, C)$ between two $\mathbf{Q}$-factorial DLT pairs is called a *flop* if there exists projective birational morphisms $f : (X, B) \to (Z, D)$, $g : (Y, C) \to (Z, D)$ to a third pair satisfying the following:

(1) $\alpha = g^{-1} \circ f$.

(2) f, g are isomorphic in codimension 1.

(3) $\rho(X/Z) = \rho(Y/Z) = 1$.

(4) $f^*(K_Z + D) = K_X + B$, $g^*(K_Z + D) = K_Y + C$.

The definition is the same as flips except for condition (4). Different from a flip, we require that the levels of canonical divisors are preserved.

{flop decomposition}

Corollary 2.10.10 (*Flop decomposition*)**.** *Let $f : (X, B) \to S$ be a projective morphism from a KLT pair. Assume that it admits a minimal model and a canonical model. Then any two minimal models $\alpha_i : (X, B) \dashrightarrow (Y_i, C_i)$ ($i = 1, 2$) are connected by a sequence of flops.*

Proof. By Lemma 2.5.12, Y_i are isomorphic in codimension 1, and has the same canonical model. Take $g_i : Y_i \to Z$ to be the morphism to the canonical model. Take a general ample effective $\mathbf{Q}$-divisor H_i on Y_i. After replacing H_i by ϵH_i for some sufficiently small $\epsilon > 0$, we may assume that $(Y_1, C_1 + H_2)$ is KLT. Here we use the same notation for strict transforms of divisors. Then we can run a $(K_{Y_1} + C_1 + H_2)$-MMP over Z with scaling of an ample divisor, and reach a canonical model Y' such that $K_{Y'} + C_1 + H_2$ is ample over Z. As $K_{Y_2} + C_2 + H_2$ is ample over Z since $K_{Y_i} + C_i$ is numerically trivial over Z, it is clear that $Y' = Y_2$. As Y_i are isomorphic in codimension 1 and Y_2 is $\mathbf{Q}$-factorial, Y_2 is also a minimal model of $(Y_1, C_1 + H_2)$ over Z, and the MMP is a sequence of flips, which is also a sequence of flops with respect to (Y_1, C_1). □

Remark 2.10.11. In [78], the same result is proved without assuming the existence of canonical models.

Applying the polyhedral decomposition theorem, we can generalized the MMP with scaling under weaker assumption:

{Birkar measure}

Corollary 2.10.12 ([13]). *Let $f : X \to S$ be a projective morphism from a $\mathbf{Q}$-factorial normal algebraic variety, B, C two effective $\mathbf{R}$-divisors. Assume that $(X, B + C)$ is $\overline{KLT}$ and $K_X + B + C$ is relatively nef, and $K_X + B$ is not relatively nef. Take*

$$t_0 = \min\{t \mid K_X + B + tC \text{ is relatively nef}\}.$$

Then then there exists a (K_X+B)-negative extremal ray R such that $((K_X + B + t_0C) \cdot R) = 0$.

Proof. Take effective $\mathbf{Q}$-divisors $B_1, \dots, B_s$ such that (X, B_i) is $\overline{\text{KLT}}$ and the spanned rational polytope P contains $B, B + C$. By Theorem 2.10.1, $N = \{B' \in P \mid K_X + B' \text{ is relatively nef}\}$ is a rational polytope.

Consider all $(K_X + B)$-negative extremal rays R_k, and take l_k to be a curve generating R_k with $((K_X + B) \cdot l_k) \geq -2 \dim X$. Take real number t_k determined by $((K_X + B + t_kC) \cdot R_k) = 0$, then $\sup_k t_k = t_0$. Assume, to the contrary that $t_k < t_0$ for all k, we will show that there is no infinite sequence $\{t_k\}$ converging to t_0.

Note that we may take rational points B'_i in P and real numbers $b_i > 0$ $(i = 1, \dots, u)$, such that $\sum b_i = 1$ and $B = \sum b_i B'_i$. Moreover, (X, B'_i) is $\overline{\text{KLT}}$ and $((K_X + B'_i) \cdot l_k) \geq -2 \dim X$ for all i, k. Since N is a rational polytope, there exist rational points C_j in N and real numbers $c_j > 0$ $(j = 1, \dots, v)$ such that $\sum c_j = 1$ and $B + t_0C = \sum c_jC_j$.

Take a positive integer m such that mK_X, mB'_i and mC_j are all Cartier. Then we have integers m_{ik}, n_{jk} as the following

$$m_{ik} = (m(K_X + B'_i) \cdot l_k) \geq -2m \dim X;$$
$$n_{jk} = (m(K_X + C_j) \cdot l_k) \geq 0.$$

Moreover, since $\sum m_{ik}b_i < 0$, the possible values of m_{ik} are finite.

Since $K_X + B + t_kC = (1 - t_k/t_0)(K_X + B) + t_k/t_0(K_X + B + t_0C)$, we have $(1 - t_k/t_0) \sum_i b_im_{ik} + t_k/t_0 \sum_j c_jn_{jk} = 0$. Therefore

$$1 - t_0/t_k = \frac{\sum_j c_jn_{jk}}{\sum_i b_im_{ik}}$$

which is in a discrete set in $\mathbf{R}$, and the conclusion is proved. □

2.11 Multiplier ideal sheaves

{section mis}

In this section, we give the algebraic definition of a multiplier ideal sheaf and introduce the Nadel vanishing theorem. The theory of multiplier ideal sheaves is a basic tool in the L^2-theory in complex analysis and multiplier ideal sheaves are defined for line bundles with metrics. Here we only consider the case when metrics are defined algebraically. Also we consider the so-called adjoint ideal sheaf which is the log version of the multiplier ideal sheaf.

2.11.1 Multiplier ideal sheaves

It is classical in complex analysis to investigate functions which are not L^2 by multiplying functions to make them L^2, but it has been found in recent years that the multiplier ideal sheaf consisting of all multiplier functions is very useful in algebraic geometry.

Definition 2.11.1. Let X be a normal algebraic variety and B an effective $\mathbf{R}$-divisor. Assume that K_X+B is $\mathbf{R}$-Cartier, then the *multiplier ideal sheaf* $I(X,B)$ is defined as the following. Take a log resolution $f: Y \to (X,B)$, write $f^*(K_X+B) = K_Y + C$, then

$$I(X,B) = f_*(\mathcal{O}_Y(\ulcorner -C \urcorner)).$$

{prop mis}

Proposition 2.11.2. *(1) The multiplier ideal sheaf $I(X,B)$ is a non-zero coherent ideal sheaf, and it does not depend on the choice of log resolutions.*

(2) $R^p f_*(\mathcal{O}_Y(\ulcorner -C \urcorner)) = 0$ for any $p>0$.

(3) The cosupport of $I(X,B)$, or the support of $\mathcal{O}_X/I(X,B)$, coincides with the non-KLT locus of (X,B). Therefore, $I(X,B) = \mathcal{O}_X$ if and only if (X,B) is KLT.

Proof. (1) Since the irreducible components of C with negative coefficients are contracted by f, $I(X,B)$ is a coherent subsheaf of $\mathcal{O}_X$.

Take $f_1 : Y_1 \to X$ to be another log resolution. By the desingularization theorem, there exists a log resolution dominating both f and f_1. So we only need to consider the case that f_1 dominates f, that is, there exists a morphism $g: Y_1 \to Y$ such that $f_1 = f \circ g$. Write $f_1^*(K_X+B) = K_{Y_1} + C_1$. Moreover, we may assume that g is a permissible blowing up. It suffices to show that

$$g_*\mathcal{O}_{Y_1}(\ulcorner -C_1 \urcorner) = \mathcal{O}_Y(\ulcorner -C \urcorner),$$

which is easy to check for a permissible blowing up.

(2) As $-C-K_Y$ is relatively numerically trivial over X and f is birational, $-C-K_Y$ is relatively nef and relatively big over X. Then we can apply the vanishing theorem to get the conclusion.

(3) Write $C = C^+ - C^-$ where C^+, C^- are effective $\mathbf{R}$-divisors with no common components. Then as in the proof of Lemma 1.11.9, by (2) we know that the natural map

$$\mathcal{O}_X \simeq f_*\mathcal{O}_Y \to f_*(\mathcal{O}_{\llcorner C^+ \lrcorner}(\ulcorner C^- \urcorner))$$

is surjective. Hence $f_*\mathcal{O}_{\llcorner C^+ \lrcorner} \simeq f_*(\mathcal{O}_{\llcorner C^+ \lrcorner}(\ulcorner C^- \urcorner))$. On the other hand, $\mathcal{O}_X/I(X,B) \simeq f_*(\mathcal{O}_{\llcorner C^+ \lrcorner}(\ulcorner C^- \urcorner))$ and the support of $f_*\mathcal{O}_{\llcorner C^+ \lrcorner}$ is exactly the non-KLT locus of (X,B). □

The fact (2) in the above proposition seems to be a reason why multiplier ideal sheaves are useful.

Example 2.11.3. If X is smooth and the support of B is normal crossing, then $I(X, B) = \mathcal{O}_X(\ulcorner -B \urcorner)$.

We will need the following lemma in the next section:

{global section mis}

Lemma 2.11.4. *Let (X, B) be a KLT pair, B' an effective $\mathbf{R}$-Cartier divisor, L a line bundle, and s a global section of L. Assume that K_X is $\mathbf{Q}$-Cartier and*

$$B' - B \le \operatorname{div}(s).$$

Then

$$s \in H^0(X, L \otimes I(X, B')).$$

Proof. Take a log resolution $f : Y \to (X, B + B')$, and write $f^*(K_X + B) = K_Y + C$, $f^*(K_X + B') = K_Y + C'$. Note that

$$C' - C = f^*(B' - B) \le f^*\operatorname{div}(s).$$

Then

$$I(X, B') = f_*\mathcal{O}_Y(\ulcorner -C' \urcorner) \supset f_*\mathcal{O}_Y(\ulcorner -C \urcorner + f^*\operatorname{div}(s)) = \mathcal{O}_X(-\operatorname{div}(s)).$$

Here we used the projection formula and the fact that $I(X, B) = \mathcal{O}_X$. □

The Nadel vanishing theorem is a basic tool in the proof of the extension theorem in the next section. Here, if we only consider algebraic multiplier ideal sheaves, then the Nadel vanishing theorem is an easy consequence of the Kawamata–Viehweg vanishing theorem:

{Nadel}

Theorem 2.11.5 (*Nadel vanishing theorem*)**.** *Let X be a normal algebraic variety and B an effective $\mathbf{R}$-divisor, such that $K_X + B$ is $\mathbf{R}$-Cartier. Let $f : X \to S$ be a projective morphism and D a Cartier divisor. Assume that $D - (K_X + B)$ is relatively nef and relatively big over S. Then*

$$R^p f_*(\mathcal{O}_X(D) \otimes I(X, B)) = 0$$

for any $p > 0$.

Proof. Take a log resolution $g : (Y, C) \to (X, B)$, then $g^*D - C - K_Y$ is relatively nef and relatively big over X and over S. Therefore

$$R^p g_*(\mathcal{O}_Y(g^*D + \ulcorner -C \urcorner)) = 0,$$
$$R^p (f \circ g)_*(\mathcal{O}_Y(g^*D + \ulcorner -C \urcorner)) = 0$$

for any $p > 0$. The conclusion follows from the spectral sequence

$$E_2^{p,q} = R^p f_* R^q g_*(\mathcal{O}_Y(g^*D + \ulcorner -C \urcorner)) \Rightarrow R^{p+q}(f \circ g)_*(\mathcal{O}_Y(g^*D + \ulcorner -C \urcorner))$$

and

$$g_*(\mathcal{O}_Y(g^*D + \ulcorner -C \urcorner)) = \mathcal{O}_X(D) \otimes I(X, B).$$

□

Here we remark that it is important to assume that D is Cartier in the theorem.

For reference, we define analytic multiplier ideal sheaves. Let X be a smooth complex manifold and L a line bundle on X. A *singular Hermitian metric* h on L is a Hermitian metric of the form $h = h_0 e^{-\phi}$ where ϕ is a locally L^1 function and h_0 is a C^∞ Hermitian metric. The curvature of h can be defined similarly as curvature of usual Hermitian metrics and it is a real current of type $(1,1)$. The the *multiplier ideal sheaf* $I = I(L,h)$ is defined by

$$\Gamma(U,I) = \{p \in \Gamma(U,\mathcal{O}_X) \mid pe^{-\phi} \text{ is locally } L^2\}.$$

As h is singular, regular functions are not necessarily L^2 integrable. The name "multiplier" is clear from the definition. It can be shown that I is an analytic coherent ideal sheaf.

Example 2.11.6. Let g_i $(i = 1,\dots,r)$ be regular functions on a complex manifold X, take divisors $B_i = \operatorname{div}(g_i)$. Take an $\mathbf{R}$-divisor $B = \sum_i b_i B_i$ where b_i are positive real numbers. Define a singular Hermitian metric h on the trivial line bundle $\mathcal{O}_X$ as

$$h = \sum_i |g_i|^{-2b_i}.$$

In this case, the algebraic multiplier ideal sheaf coincides with the analytic multiplier ideal sheaf: $I(X,B) = I(\mathcal{O}_X,h)$.

Of course, analytic multiplier ideal sheaves are more general than algebraic multiplier ideal sheaves considered in this book. For example, singular Hermitian metrics appearing in (algebraic) Hodge theory are known to be different from algebraic ones[2.11.1.1].

The following theorem is the original form of the Nadel vanishing theorem. As the metric h is not necessarily induced by a divisor, it is more general than the algebro-geometric version.

Theorem 2.11.7. *Let X be a compact complex smooth manifold and L a line bundle admitting a singular Hermitian metric h. Denote I to be the corresponding multiplier ideal sheaf. Assume that the curvature of h is semi-positive and strictly positive at some point of X*[2.11.1.2]. *Then $H^p(X,\mathcal{O}_X(K_X+L)\otimes I) = 0$ for any $p > 0$.*

2.11.2 Adjoint ideal sheaves

Next we define adjoint ideal sheaves as a variant of multiplier ideal sheaves. Adjoint ideal sheaves are defined in algebraic geometry, and there is no

[2.11.1.1] this sentence needs to be corrected
[2.11.1.2] Any reference?

natural analogue in complex analysis. The reason is that the logarithmic differential form dz/z is not L^2. This definition is natural when considering residue map and doing induction on dimensions.

Definition 2.11.8. Let X be a normal algebraic variety and B an effective $\mathbf{R}$-divisor. Assume that $K_X + B$ is $\mathbf{R}$-Cartier. Assume that there exists an irreducible component Z in B with coefficient 1. Then the *adjoint ideal sheaf* $I_Z(X, B)$ is defined as the following. Take a log resolution $f : Y \to (X, B)$, write $f^*(K_X + B) = K_Y + C$ and $W = f_*^{-1}Z$, then

$$I_Z(X, B) = f_*(\mathcal{O}_Y(\ulcorner -C \urcorner + W)).$$

The adjoint ideal sheaf measures how far the pair (X, B) is from being PLT. Fix an irreducible component Z in B with coefficient 1, then the set of points on Z, in a neighborhood of which (X, B) is PLT, is a closed subset of Z. It is called the *non-PLT locus* of (X, B) with respect to Z.

{prop ais}

Proposition 2.11.9. *(1) The adjoint ideal sheaf $I_Z(X, B)$ is a non-zero coherent ideal sheaf, and it does not depend on the choice of log resolutions.*

(2) $R^p f_*(\mathcal{O}_Y(\ulcorner -C \urcorner + W)) = 0$ for any $p > 0$.

(3) The intersection of Z and the support of $\mathcal{O}_X/I_Z(X, B)$ coincides with the non-PLT locus of (X, B) with respect to Z. In particular, $I_Z(X, B) = \mathcal{O}_X$ in a neighborhood Z if and only if (X, B) is PLT in a neighborhood Z.

Proof. The proof is the same as that of Proposition 2.11.2.

(1) Given another log resolution $f_1 : Y_1 \to X$, we may assume that there exists a morphism $g : Y_1 \to Y$ such that $f_1 = f \circ g$. Write $f_1^*(K_X + B) = K_{Y_1} + C_1$ and $W_1 = f_{1*}^{-1}Z$. It suffices to show that

$$g_*\mathcal{O}_{Y_1}(\ulcorner -C_1 \urcorner + W_1) = \mathcal{O}_Y(\ulcorner -C \urcorner + W).$$

This is easy to check.

(2) Note that $-C + W - (K_Y + W)$ is relatively nef and relatively big over X, and its restriction to W is again relatively nef and relatively big over Z.

(3) Note that, in a neighborhood Z, the intersection of Z with the image of the negative part of $\ulcorner -C \urcorner$ is exactly the non-PLT locus of (X, B) with respect to Z. □

The relation of multiplier ideal sheaves and adjoint ideal sheaves is as the following:

{mis and ais}

Lemma 2.11.10. *Let X be a normal algebraic variety and B an effective $\mathbf{R}$-divisor. Assume that $K_X + B$ is $\mathbf{R}$-Cartier. Assume that there exists an irreducible component Z in B with coefficient 1. Assume that Z is normal and write $(K_X + B)|_Z = K_Z + B_Z$. Then there is a short exact sequence:*

$$0 \to I(X, B) \to I_Z(X, B) \to I(Z, B_Z) \to 0.$$

Therefore, $I_Z(X, B)\mathcal{O}_Z = I(Z, B_Z)$.

Proof. Write $(K_Y + C)|_W = K_W + C_W$ where $C_W = (C - W)|_W$. Denote $f_Z = f|_Z$, then $f_Z^*(K_Z + B_Z) = K_W + C_W$. We get the short exact sequence from the exact sequence

$$0 \to \mathcal{O}_Y(\ulcorner -C \urcorner) \to \mathcal{O}_Y(\ulcorner -C \urcorner + W) \to \mathcal{O}_W(\ulcorner -C_W \urcorner) \to 0$$

and $R^1 f_* \mathcal{O}_Y(\ulcorner -C \urcorner) = 0$. The last statement follows from $I(X, B) \subset \mathcal{O}_X(-Z)$. □

We can extend the Nadel vanishing theorem to adjoint ideal sheaves:

Theorem 2.11.11. *Let X be a normal algebraic variety and B an effective $\mathbf{R}$-divisor. Assume that $K_X + B$ is $\mathbf{R}$-Cartier. Assume that there exists an irreducible component Z in B with coefficient 1. Let $f : X \to S$ be a projective morphism and D a Cartier divisor. Assume that $D - (K_X + B)$ is relatively nef and relatively big over S and $(D - (K_X + B))|_Z$ is relatively nef and relatively big over $f(Z)$. Then*

$$R^p f_*(\mathcal{O}_X(D) \otimes I_Z(X, B)) = 0$$

for any $p > 0$.

Proof. The proof is similar to that of Theorem 2.11.5. If Z is normal, then this is a consequence of Theorem 2.11.5 by using the exact sequence in Lemma 2.11.10. □

Let us define a special case of logarithmic multiplier ideal sheaf, which is a general version of adjoint ideal sheaf:

Definition 2.11.12. Let (X, B) be a DLT pair consisting of a normal algebraic variety X and an $\mathbf{R}$-divisor B on X. Let L be a linear system of divisors and m a positive integer. Take $Z = \llcorner B \lrcorner$, which is not necessarily irreducible. Take a general element $G \in L$, assume that it does not contain LC centers of (X, B). Then the *logarithmic multiplier ideal sheaf* $I_Z(X, B + L/m)$ is defined as the following. Take a log resolution $f : Y \to X$ of $(X, B + G)$ in strong sense, which is isomorphic over the generic point of each LC center of (X, B) and resolves the base locus of L. Write $f^*(K_X + B) = K_Y + C$, $f^*G = P + N$, $W = f_*^{-1}Z$. Here P is a general element of the movable part of f^*L and N is the fixed part. By construction, P is free. Then we define

$$I_Z(X, B + L/m) = f_*(\mathcal{O}_Y(\ulcorner -C - N/m \urcorner + W)).$$

{mis3}

Lemma 2.11.13. *(1) The logarithmic adjoint ideal sheaf $I_Z(X, B + L/m)$ is a non-zero coherent ideal sheaf, and it does not depend on the choice of log resolutions.*

(2) $R^p f_*(\mathcal{O}_Y(\ulcorner -C - N/m \urcorner + W)) = 0$ for any $p > 0$.

Proof. (1) Given another log resolution $f_1 : Y_1 \to X$, we may assume that there exists a morphism $g : Y_1 \to Y$ such that $f_1 = f \circ g$. Write $f_1^*(K_X + B) = K_{Y_1} + C_1$, $f_1^* D = P_1 + N_1$, $W_1 = f_{1*}^{-1} Z$. It suffices to show that

$$g_*(\mathcal{O}_{Y_1}(\ulcorner -C_1 - N_1/m \urcorner + W_1)) = \mathcal{O}_Y(\ulcorner -C - N/m \urcorner + W),$$

which is easy to check.

(2) Note that $-C - N/m + W - (K_Y + W) \equiv_X P/m$ is relatively nef and relatively big over X, also its restriction on each LC center of (Y, W) is again relatively nef and relatively big. The conclusion follows from applying the vanishing theorem inductively. □

We can prove the Nadel vanishing theorem for logarithmic adjoint ideal sheaves:

{Nadel I_Z}

Theorem 2.11.14. *Let (X, B) be a DLT pair, L a linear system of divisors, m a positive integer, D a Cartier divisor, and $f : X \to S$ a projective morphism to an affine variety. Take $Z = \llcorner B \lrcorner$ Assume the following:*

(1) A general element $G \in L$ does not contain LC centers of (X, B).

(2) $D - (K_X + B + G/m)$ and its restriction to each LC center are relatively nef and relatively big over S or the image of the center in S, respectively.

Then

$$H^p(X, I_Z(X, B + L/m) \otimes \mathcal{O}_X(D)) = 0$$

for any $p > 0$.

Proof. The proof is similar to that of Theorem 2.11.5. We leave the details to the readers for exercise. □

In order to simultaneously investigate linear systems induced by multiples of a divisor, we define asymptotic multiplier ideal sheaves. They play important roles in the proof of extension theorems.

Definition 2.11.15. Let (X, B) be a DLT pair. Let L_m ($m \in \mathbf{Z}_{>0}$) be a sequence of linear systems of divisors satisfying $L_m + L_{m'} \subset L_{m+m'}$, that is, $D + D' \in L_{m+m'}$ if $D \in L_m$, $D' \in L_{m'}$. Take $Z = \llcorner B \lrcorner$. Assume that there exists m such that a general element $D \in L_m$ does not contain LC centers of (X, B). Then define the *asymptotic multiplier ideal sheaf* to be

$$I_Z(X, B + \{L_m/m\}) = \bigcup_{m>0} I_Z(X, B + L_m/m).$$

Remark 2.11.16. By assumption, $I_Z(X, B+L_m/m) \subset I_Z(X, B+L_{m'}/m')$ if if $m|m'$. By the Noetherian property, the right hand side which is a union of infinitely many ideals is actually obtained by a sufficiently large and sufficiently divisible m. However such m can not be determined priorly. This is one advantage of asymptotic multiplier ideal sheaves.

The following lemma uses the vanishing theorem to show global generation of sheaves, which gives a corollary we will use in the next section. For ample sheaves the same statement is difficult to prove, but for very ample sheaves it is easy. We use the so-called *Castelnuovo–Mumford regularity* method:

{F globally generated}

Lemma 2.11.17. *Let X be an n-dimensional quasi-projective algebraic variety, $\mathcal{O}_X(1)$ a very ample invertible sheaf, and $\mathcal{F}$ a coherent sheaf. Assume that*

$$H^p(X, \mathcal{F} \otimes \mathcal{O}_X(m)) = 0$$

for any $m \in \mathbf{Z}_{\geq 0}$ and any $p \in \mathbf{Z}_{>0}$. Then $\mathcal{F} \otimes \mathcal{O}_X(n)$ is generated by global sections.

Proof. The proof is by induction on n. We may assume that $n > 0$. Fix any point $x \in X$. Take $\mathcal{F}_0 = H^0_{\{x\}}(\mathcal{F})$ to be the subsheaf of $\mathcal{F}$ containing all local sections whose supports are x, then the quotient sheaf $\mathcal{F}_1 = \mathcal{F}/\mathcal{F}_0$ has no local section whose support is x. Consider the exact sequence

$$0 \to \mathcal{F}_0 \to \mathcal{F} \to \mathcal{F}_1 \to 0.$$

Since $H^1(\mathcal{F}_0) = 0$ by dimension reason, $H^0(\mathcal{F}) \to H^0(\mathcal{F}_1)$ is surjective. Therefore, if $\mathcal{F}_1 \otimes \mathcal{O}_X(n)$ is generated by global sections at x, then so is $\mathcal{F} \otimes \mathcal{O}_X(n)$. So we may assume in the beginning that $\mathcal{F}$ has no local section whose support is x.

Take a general global section s of $\mathcal{O}_X(1)$ that vanishes at x. Take X' to be the corresponding hyperplane passing through x. Take $\mathcal{O}_{X'}(1) = \mathcal{O}_X(1) \otimes \mathcal{O}_{X'}$, $\mathcal{F}' = \mathcal{F} \otimes \mathcal{O}_{X'}(1)$. Since 0 is the only section of $\mathcal{F}$ that becomes 0 after multiplying s, we get an exact sequence

$$0 \to \mathcal{F} \to \mathcal{F} \otimes \mathcal{O}_X(1) \to \mathcal{F}' \to 0.$$

Hence

$$H^p(X', \mathcal{F}' \otimes \mathcal{O}_{X'}(m)) = 0$$

for any $m \geq 0$ and any $p > 0$. By induction hypothesis, $\mathcal{F}' \otimes \mathcal{O}_{X'}(n-1)$ is generated by global sections. Since $H^1(X, \mathcal{F} \otimes \mathcal{O}_X(n-1)) = 0$, $H^0(X, \mathcal{F} \otimes \mathcal{O}_X(n)) \to H^0(X, \mathcal{F}' \otimes \mathcal{O}_{X'}(n-1))$ is surjective, and hence $\mathcal{F} \otimes \mathcal{O}_X(n)$ is generated by global sections at x. $\square$

{generated}

Corollary 2.11.18. *Keep the assumptions in Theorem 2.11.14. Take a very ample divisor H on X and denote* $\dim X = n$*. Then*

$$I_Z(X, B + L/m) \otimes \mathcal{O}_X(D + nH)$$

is generated by global sections.

Proof. This follows directly from Theorem 2.11.14 and Lemma 2.11.17. □

2.12 Extension theorems

In this section, we prove extension theorems for pluri-log-canonical forms.

2.12.1 Extension theorems 1

There are many variants of extension theorems. The following form due to Hacon–McKernan and Takayama is a key point in the proof of existence of flips.

{ext thm 1}

Theorem 2.12.1 (*Extension theorem 1, [37], [146]*)**.** *Let (X, B) be a PLT pair where X is a smooth algebraic variety and B is a* **Q***-divisor with normal crossing support. Let $f : X \to S$ be a projective morphism to an affine algebraic variety. Fix a positive integer m_0 such that $D = m_0(K_X + B)$ is an integral divisor. Assume that $Y = \llcorner B \lrcorner$ is irreducible. Assume the following conditions.*

(1) There exists an ample **Q***-divisor A*[2.12.1.1] *and an effective* **Q***-divisor E whose support does not contain Y, such that*

$$B = A + E + Y.$$

(2) There exists a positive integer m_1 such that the support of a general element $G \in |m_1 D|$ does not contain any LC center of $(X, \ulcorner B \urcorner)$, that is, does not contain any irreducible component of intersections of irreducible components of B.

Then the restriction map

$$H^0(X, mD) \to H^0(Y, mD|_Y)$$

is surjective for any positive integer m.

Remark 2.12.2. (1) In condition (1), it is an equation of **Q**-divisors, not just an equivalence.[2.12.1.2]

[2.12.1.1] here no need to assume A is effective

[2.12.1.2] this sentence should be removed as A is not effective, equivalence is ok.

(2) The proof of the extension theorem described below is extremely technical, which is not just something that can be reached by calculating carefully.

(3) Trying to relaxing the assumptions of this theorem is an important question which may have many interesting applications.

Proof. The proof follows by the following Propositions 2.12.3 and 2.12.7. □

Firstly, we use the usual multiplier ideal sheaves to reduce the problem to the extension problem for a sequence of slightly bigger divisors:

{Step 1}

Proposition 2.12.3. *Under condition (1) of Theorem 2.12.1, assume further that there exists an effective divisor F whose support does not contain Y such that for any sufficiently large positive integer l, the image of the natural homomorphism*

$$H^0(X, lD + F) \to H^0(Y, (lD + F)|_Y)$$

contains the image of

$$H^0(Y, lD|_Y) \to H^0(Y, (lD + F)|_Y).$$

Then

$$H^0(X, D) \to H^0(Y, D|_Y)$$

is surjective.

Proof. Take a sufficiently small positive rational number ϵ, take $E' = (1 - \epsilon)(B - Y) + \epsilon E$, we may assume that $(X, Y + E')$ is PLT since (X, B) is PLT. Note that we can write $B - Y = \epsilon A + E'$, hence we may assume that $(X, Y + E)$ is PLT in the beginning after replacing E by E'.

Take any $s \in H^0(Y, D|_Y)$, take $D' = \mathrm{div}(s)$. By assumption, for a sufficiently large and sufficiently divisible positive integer l, there exists $G_l \in H^0(X, lD + F)$ such that

$$G_l|_Y = lD' + F|_Y.$$

Here note that this is an equality of divisors, not just a linear equivalence. Take

$$B' = \frac{m_0 - 1}{lm_0} G_l + Y + E,$$

and consider the multiplier ideal sheaf $I = I(X, B')$. Note that

$$\begin{aligned}
&D - K_X - B' \\
&= m_0(K_X + B) - K_X - B' \\
&\sim_{\mathbf{Q}} (m_0 - 1)(K_X + B) + B - \frac{m_0 - 1}{m_0} D - \frac{m_0 - 1}{lm_0} F - Y - E \\
&= A - \frac{m_0 - 1}{lm_0} F
\end{aligned}$$

is ample if l is sufficiently large. Therefore, by the Nadel vanishing theorem,

$$H^1(X, \mathcal{I}(X, B') \otimes \mathcal{O}_X(D)) = 0.$$

Take

$$C' = (B' - Y)|_Y = \frac{m_0 - 1}{m_0} D' + \frac{m_0 - 1}{l m_0} F|_Y + E|_Y,$$

then we have the following exact sequence

$$0 \to \mathcal{I}(X, B') \to \mathcal{I}_Y(X, B') \to \mathcal{I}(Y, C') \to 0.$$

Hence the restriction map

$$H^0(X, \mathcal{I}_Y(X, B') \otimes \mathcal{O}_X(D)) \to H^0(Y, \mathcal{I}(Y, C') \otimes \mathcal{O}_Y(D|_Y))$$

is surjective. On the other hand, $(X, Y + E)$ is PLT, hence

$$(Y, \frac{m_0 - 1}{l m_0} F|_Y + E|_Y)$$

is KLT if l is sufficiently large. Note that

$$C' - \frac{m_0 - 1}{l m_0} F|_Y - E|_Y \leq D'.$$

Hence by Lemma 2.11.4,

$$s \in H^0(Y, \mathcal{I}(Y, C') \otimes \mathcal{O}_Y(D|_Y)).$$

Therefore s can be extend to a global section of $H^0(X, D)$. □

Let us forget the situation of the theorem for a moment and use the following notation in the following two lemmas. Let X be a smooth algebraic variety, B a normal crossing divisor, Y an irreducible component of B, and D another divisor. Let $f : X \to S$ be a projective morphism to an affine algebraic variety. Here all coefficients of B are taken to be 1 (in the situation of the theorem $\ulcorner B \urcorner$ corresponds to B here). Take $C = (B - Y)|_Y$. Assume that there exists a positive integer m_1 such that the support of a general element $G \in |m_1 D|$ does not contain any LC center of (X, B).

Consider the following two series of linear systems on Y:

$$L^0_m = |H^0(Y, mD|_Y)|,$$
$$L^1_m = |\mathrm{Im}(H^0(X, mD) \to H^0(Y, mD|_Y))|.$$

Here $|\quad|$ denotes the linear system of the corresponding linear space. Then we can define the corresponding asymptotic multiplier ideal sheaves

$$\mathcal{J}^0_C(Y, D|_Y) = \mathcal{I}_C(Y, C + \{L^0_m/m\}),$$
$$\mathcal{J}^1_C(Y, D|_Y) = \mathcal{I}_C(Y, C + \{L^1_m/m\}).$$

As $L^1_m \subset L^0_m$, $J^1_C(Y, D|_Y) \subset J^0_C(Y, D|_Y)$. In the case $C = 0$, we simply write $J^0(Y, D|_Y)$, $J^1(Y, D|_Y)$.

We compare the set of global sections and the set of extendable global sections as m goes to infinity. The next two lemmas prove inclusion relations in two directions.

{inclusion 1}

Lemma 2.12.4.

$$H^0(Y, D|_Y) = H^0(Y, J^0_C(Y, D|_Y) \otimes \mathcal{O}_Y(D|_Y));$$
$$\mathrm{Im}(H^0(X, D) \to H^0(Y, D|_Y)) \subset H^0(Y, J^1_C(Y, D|_Y) \otimes \mathcal{O}_Y(D|_Y)).$$

Proof. We only show the second one. The first one is similar but easier. Take a general element $G \in |m_1 D|$ whose support does not contain any LC center of (X, B). Take a log resolution $g : X' \to (X, B + G)$, write $g^*(K_X + B) = K_{X'} + B'$, $Y' = g_*^{-1}Y$, $(K_{X'} + B')|_{Y'} = K_{Y'} + C'$, $g^*G = P + N$. Here we may assume that P is free and N is the fixed part, and $(B')^+ = g_*^{-1}B$. Then

$$\begin{aligned}
&\mathrm{Im}(H^0(X, D) \to H^0(Y, D|_Y)) \\
&\subset H^0(Y', \mathcal{O}_{Y'}(g^*D|_Y + \llcorner -N/m_1|_Y \lrcorner)) \\
&\subset H^0(Y', \mathcal{O}_{Y'}(g^*D|_Y + \ulcorner -C' - N/m_1|_Y \urcorner + (C')^+)) \\
&\subset H^0(Y, J^1_C(Y, D|_Y) \otimes \mathcal{O}_Y(D|_Y)).
\end{aligned}$$

Here the first inclusion is by the fact that $\mathrm{Fix}|g^*D| \geq \ulcorner N/m_1 \urcorner$. All spaces are viewed as subspaces of $H^0(Y, D|_Y)$ under certain maps. □

{inclusion 2}

Lemma 2.12.5. *Assume the following conditions:*

(1) There exists an ample $\mathbf{Q}$*-divisor* A' *and an effective* $\mathbf{Q}$*-divisor* E' *such that* $D = A' + E'$*.*

(2) Assume that there exists a positive integer m'_1 *such that the support of a general element* $G' \in |m'_1 E'|$ *does not contain any LC center of* (X, B)*.*

Then

$$\begin{aligned}
&H^0(Y, J^1_C(Y, D|_Y) \otimes \mathcal{O}_Y(D|_Y + K_Y + C)) \\
&\subset \mathrm{Im}(H^0(X, D + K_X + B) \to H^0(Y, D|_Y + K_Y + C)).
\end{aligned}$$

Proof. Take a sufficiently large and sufficiently divisible m which obtains $J^1_C(Y, D|_Y)$, that is, $J^1_C(Y, D|_Y) = I_C(Y, C + L^1_m/m)$. For a general element $D_m \in |mD|$, take a log resolution $g : X' \to (X, B + D_m + G')$ in strong sense, write $g^*(K_X + B) = K_{X'} + B'$, $Y' = g_*^{-1}Y$, $(K_{X'} + B')|_{Y'} = K_{Y'} + C'$, $g^*D_m = P + N$. Here we may assume that P is free and N is the fixed part, and $(B')^+ = g_*^{-1}B$. Then $(B')^+$ has no common component with exceptional divisors of g, N, and g^*E'. Take an effective $\mathbf{Q}$-divisor F supported on exceptional divisors of g such that $g^*A' - F$ is ample. Then for any sufficiently small positive number ϵ,

$$g^*D - (1-\epsilon)N/m - \epsilon(g^*E' + F) \sim_{\mathbf{Q}} (1-\epsilon)P/m + \epsilon(g^*A' - F)$$

is ample. Since A' is ample, $N/m \le g^*E'$ for any sufficiently divisible m. Therefore, for a sufficiently small ϵ,

$$\ulcorner g^*D - (1-\epsilon)N/m - \epsilon(g^*E' + F)\urcorner = g^*D - \llcorner N/m \lrcorner.$$

By the vanishing theorem,

$$H^1(X', K_{X'} + (B')^+ - Y' + g^*D - \llcorner N/m \lrcorner) = 0.$$

Hence

$$\begin{aligned} &H^0(X', K_{X'} + (B')^+ + g^*D - \llcorner N/m \lrcorner) \\ &\to H^0(Y', K_{Y'} + (C')^+ + g^*D|_{Y'} - \llcorner N/m|_{Y'} \lrcorner) \end{aligned}$$

is surjective. On the other hand,

$$H^0(X', K_{X'} + (B')^+ + g^*D - \llcorner N/m \lrcorner) \subset H^0(X, D + K_X + B)$$

and

$$\begin{aligned} &H^0(Y', K_{Y'} + (C')^+ + g^*D|_{Y'} - \llcorner N/m|_{Y'} \lrcorner) \\ &= H^0(Y, J^1_C(Y, D|_Y) \otimes \mathcal{O}_Y(K_Y + C + D|_Y)), \end{aligned}$$

this proves the conclusion. □

The following lemma is the key of the proof of the extension theorem:

{inclusion 3}

Lemma 2.12.6. *Let (X, B) be a PLT pair where X is a smooth algebraic variety of dimension n and B is a $\mathbf{Q}$-divisor with normal crossing support. Let $f : X \to S$ be a projective morphism to an affine algebraic variety. Fix a positive integer m_0 such that $D = m_0(K_X + B)$ is an integral divisor. Fix a very ample divisor H on X and take $M = nH$. Assume the following conditions:*

(1) H is sufficiently ample comparing to B and D (this condition will be clarified in the proof).

(2) There exists a positive integer m_1 such that the support of a general element $G \in |m_1 D|$ does not contain any LC center of $(X, \ulcorner B \urcorner)$.

Then

(1) The inclusion

$$J^0(Y,(mD+H)|_Y) \subset J^1_{\ulcorner C \urcorner}(Y,(mD+H+M)|_Y)$$

holds for any non-negative integer m.

(2) The inclusion

$$\begin{aligned} &H^0(Y, J^0(Y,(mD+H)|_Y) \otimes \mathcal{O}_Y((mD+H+M)|_Y)) \\ &\subset \operatorname{Im}(H^0(X, mD+H+M) \to H^0(Y,(mD+H+M)|_Y)) \end{aligned}$$

holds for any non-negative integer m.

Proof. (1) We will prove by induction on m. If $m=0$, then both sides are $\mathcal{O}_Y$. Let us prove the conclusion for the case $m+1$ assuming the case m.

Define the increasing sequence of integral divisors

$$Y \leq B^{[1]} \leq \cdots \leq B^{[m_0]} = \ulcorner B \urcorner$$

by

$$\sum_{k=1}^{m_0} B^{[k]} = m_0 B.$$

Take $D_k = K_X + B^{[k]}$, $D_{\leq k} = \sum_{s=1}^k D_s$, $C^{[k]} = (B^{[k]} - Y)|_Y$. Also denote $D_{\leq 0} = 0$, $B^{[m_0+1]} = \ulcorner B \urcorner$. Note that $D = D_{\leq m_0}$.

Here we clarify the assumption on H: for any $0 \leq k \leq m_0$, (1-a) $D_{\leq k} + H + M$ is free, (1-b) $D_{\leq k} + H - K_X - Y$ is ample. Note that such condition does not depend on m.

We will prove the claim that

$$J^0(Y,(mD+H)|_Y) \subset J^1_{C^{[k+1]}}(Y,(mD+D_{\leq k}+H+M)|_Y)$$

by induction on $0 \leq k \leq m_0$. Note that the right hand side is well-defined by assumption (1-a) on H.

Once the claim is proves, take $k = m_0$, then

$$\begin{aligned} &J^0(Y,((m+1)D+H)|_Y) \subset J^0(Y,(mD+H)|_Y) \\ &\subset J^1_{\ulcorner C \urcorner}(Y,((m+1)D+H+M)|_Y), \end{aligned}$$

which proves the conclusion for the case $m+1$ and finishes the proof of (1).

If $k=0$, by induction hypothesis,

$$J^0(Y,(mD+H)|_Y) \subset J^1_{\ulcorner C \urcorner}(Y,(mD+H+M)|_Y) \subset J^1_{C^{[1]}}(Y,(mD+H+M)|_Y).$$

Assume that the claim holds for $k-1$, then we have the following 3 inclusions:

$$\begin{aligned} &H^0(Y, J^0(Y,(mD+H)|_Y) \otimes \mathcal{O}_Y((mD+D_{\leq k}+H+M)|_Y)) \\ &\subset H^0(Y, J^1_{C^{[k]}}(Y,(mD+D_{\leq k-1}+H+M)|_Y) \otimes \mathcal{O}_Y((mD+D_{\leq k}+H+M)|_Y)) \\ &\subset \operatorname{Im}(H^0(X, mD+D_{\leq k}+H+M) \to H^0(Y,(mD+D_{\leq k}+H+M)|_Y)) \\ &\subset H^0(Y, J^1_{C^{[k+1]}}(Y,(mD+D_{\leq k}+H+M)|_Y) \otimes \mathcal{O}_Y((mD+D_{\leq k}+H+M)|_Y)). \end{aligned}$$

Here the first inclusion is by induction hypothesis, the second is by Lemma 2.12.5, and the third is by Lemma 2.12.4. Note that

$$mD + D_{\leq k} + H - (K_Y + (mD + H)|_Y) = (D_{\leq k} + H - K_X - Y)|_Y$$

is ample by assumption (1-b) on H, hence by Corollary 2.11.18,

$$J^0(Y, (mD + H)|_Y) \otimes \mathcal{O}_Y((mD + D_{\leq k} + H + M)|_Y)$$

is generated by global sections. To summarize, we showed that

$$J^0(Y, (mD + H)|_Y) \subset J^1_{C^{[k+1]}}(Y, (mD + D_{\leq k} + H + M)|_Y).$$

(2) When $m = 0$ this is clear. For $m > 0$, using above inclusions for $m - 1$ and $k = m_0$, we have

$$\begin{aligned}
&H^0(Y, J^0(Y, (mD + H)|_Y) \otimes \mathcal{O}_Y((mD + H + M)|_Y)) \\
&\subset H^0(Y, J^0(Y, ((m-1)D + H)|_Y) \otimes \mathcal{O}_Y((mD + H + M)|_Y)) \\
&\subset \operatorname{Im}(H^0(X, mD + H + M) \to H^0(Y, (mD + H + M)|_Y)).
\end{aligned}$$

□

{Step 2}

Proposition 2.12.7. *Under condition (2) of Theorem 2.12.1, there exists a very ample divisor F, such that for any sufficiently large positive integer m, the image of the restriction map*

$$H^0(X, mD + F) \to H^0(Y, (mD + F)|_Y)$$

contains the image of $H^0(Y, mD|_Y)$.

Proof. By Lemma 2.12.6

$$\begin{aligned}
&H^0(Y, mD|_Y) \subset H^0(Y, (mD + H)|_Y) \\
&= H^0(J^0(Y, (mD + H)|_Y) \otimes \mathcal{O}_Y((mD + H)|_Y)) \\
&\subset H^0(J^0(Y, (mD + H)|_Y) \otimes \mathcal{O}_Y((mD + H + M)|_Y)) \\
&\subset \operatorname{Im}(H^0(X, mD + H + M) \to H^0(Y, (mD + H + M)|_Y)).
\end{aligned}$$

So we may just take $F = H + M$. □

2.12.2 Extension theorems 2

There are various versions of the extension theorem. The following theorem is close to the original form of the extension theorem. This theorem has many important corollaries such as the deformation invariance of plurigenera and canonical singularities, but will not be used in subsequent sections.

{ext thm 2}

Theorem 2.12.8 (*Extension theorem* 2)**.** *Let (X, B) be a PLT pair where X is a smooth algebraic variety and B is a $\mathbf{Q}$-divisor with normal crossing support. Let $f : X \to S$ be a projective morphism to an affine algebraic variety. Fix an integer $m_0 \geq 2$ such that $D = m_0(K_X + B)$ is an integral divisor. Assume that $Y = \llcorner B \lrcorner$ is irreducible. Assume the following conditions.*

(1) There exists an ample $\mathbf{Q}$-divisor A and an effective $\mathbf{Q}$-divisor E whose support does not contain Y, such that

$$K_X + B = A + E.$$

(2) There exists a positive integer m_1 such that the support of a general element $G \in |m_1 D|$ does not contain any LC center of $(X, \ulcorner B \urcorner)$.

Then the homomorphism

$$H^0(X, mD) \to H^0(Y, mD|_Y)$$

is surjective for any positive integer m.

Remark 2.12.9. (1) If taking $B = Y$ in Theorem 2.12.8, then it is a theorem in [77]. Theorem 2.12.1 is a generalization of this theorem.

(2) For a sufficiently large and sufficiently divisible positive integer m and a general element $G \in |m(K_X + B)|$, replacing B by $B' = B + \epsilon G$ for a sufficiently small ϵ and taking a log resolution, we are in a similar situation as Theorem 2.12.1. But it is easy to see that the conditions of Theorem 2.12.1 are not satisfied, because $\ulcorner B' \urcorner$ has more irreducible components than $\ulcorner B \urcorner$. So Theorem 2.12.8 is not a corollary of Theorem 2.12.1.

Proof. The proof as basically the same as that of Theorem 2.12.1. Just modify Proposition 2.12.3 by taking

$$B' = \frac{m_0 - 1 - \epsilon}{l m_0} G_l + B + \epsilon E$$

for some sufficiently small ϵ. We omit the details. □

An important corollary of Theorem 2.12.8 is the following theorem on deformation invariance of plurigenera:

Corollary 2.12.10 (Siu [141])**.** *Let $f : X \to S$ be a smooth projective morphism between smooth algebraic varieties. Assume that the fiber $X_\eta = f^{-1}(\eta)$ over the generic point $\eta \in S$ is of general type. Then for any positive integer m, the plurigenus $\dim H^0(X_s, mK_{X_s})$ of a fiber $X_s = f^{-1}(s)$ is independent of the choice of $s \in S$.*

Proof. Note that the case $m = 1$ is classical. We may assume that S is a smooth affine curve. Fix any point $s \in S$. Fix an effective ample divisor A on X and take $A_\eta = A|_{X_\eta}$. Since X_η is of general type, there exists a sufficiently large positive integer m_1 and an effective divisor E_η such that $m_1 K_{X_\eta} \sim A_\eta + E_\eta$. Taking the closure, there exists an effective divisor E which does not contain X_s such that $m_1 K_X \sim A + E$.

Applying Theorem 2.12.8 to $B = Y = X_s$, for any integer $m \geq 2$,

$$H^0(X, m(K_X + X_s)) \to H^0(X_s, mK_{X_s})$$

is surjective. Then the conclusion follows from upper semicontinuity. □

The following theorem stating that flat deformations of canonical singularities are again canonical singularities is important in the study of moduli spaces of algebraic varieties:

Corollary 2.12.11 ([76]). *Let $f : X \to S$ be a flat morphism from an algebraic variety X to a smooth affine curve. Fix $x \in X$, $s = f(x) \in S$. Assume that the fiber $X_s = f^{-1}(s)$ over s has at worst canonical singularities at x. Then X has at worst canonical singularities at x. In particular, there exists a neighborhood $U \subset X$ of x such that for any $s' \in S$, $X_{s'} \cap U$ has at worst canonical singularities.*

Proof. Replacing X by a sufficiently small affine neighborhood of x, we may assume that X_s has at worst canonical singularities. Take a log resolution $g : X' \to (X, X_s)$, denote $B = Y$ to be the strict transform of X_s. Since X_s is normal, we may assume that X is also normal if X is sufficiently small. Since X_s has at worst canonical singularities, there exists a positive integer m such that mK_{X_s} is Cartier and the natural map

$$H^0(Y, mK_Y) \to H^0(X_s, mK_{X_s})$$

is isomorphic. Applying Theorem 2.12.8 to g, we have

$$H^0(X', m(K_{X'} + Y)) \to H^0(Y, mK_Y)$$

is surjective. Therefore,

$$H^0(X', m(K_{X'} + Y)) \to H^0(X, m(K_X + X_s)) \to H^0(X_s, mK_{X_s})$$

is surjective. So if X is sufficiently small, a nowhere vanishing section of mK_{X_s} extends to a nowhere vanishing section of $m(K_X + X_s)$ and a global section of $m(K_{X'} + Y)$. This implies that $m(K_X + X_s)$ is Cartier and $m(K_{X'} + Y) \geq g^*(m(K_X + X_s))$. Since $Y \leq g^* X_s$, X has at worst canonical singularities. □

Remark 2.12.12. The technique in the proof of the extension theorem was originally developed by Siu in the proof of the deformation invariance of plurigenera ([141]). Later [76] proved the deformation invariance of canonical singularities by an algebraic interpretation of Sius argument (see also [77], [123]). Here instead of considering limits of metrics in complex analysis, asymptotic multiplier ideal sheaves are introduced. By the Noetherian property, an asymptotic multiplier ideal sheaf is actually obtained at a finite stage, but we can not tell at which stage it will be obtained, so it is helpful for proving certain finiteness theorem. However this method also has its limitation as it can not reflect infinite limits as analytic multiplier ideal sheaves. The extension theorem introduced in this section was proved by the log version of this method ([37], [146]). After this, Siu proved the deformation invariance of plurigenera without assuming bigness of canonical divisors ([142]). The algebraic interpretation of this result is still not known. It seems that an algebraic interpretation of infinite limits is necessary.

Chapter 3

The finite generation theorem

{Chapter 3}

In this chapter we prove the finite generation of canonical rings. Firstly, for algebraic varieties of general type, we prove the existence of minimal models by induction on dimensions, then we use the semi-positivity theorem for algebraic fiber spaces to reduce the problem to algebraic varieties of general type.

3.1 Setting of the inductive proof

In BCHM [15], it turns out that for MMP with scaling, the induction on dimensions goes well under the assumption that the boundary contains an ample divisor. To be more precise, we should put the following conditions on (X, B) and f. We will simply call it the *BCHM condition* in this book.

(1) X is an n-dimensional $\mathbf{Q}$-factorial normal algebraic variety, B is an effective $\mathbf{R}$-divisor on X, $f : X \to S$ is a projective morphism to a quasi-projective variety.

(2) (X, B) is DLT.

(3) There exists a relatively ample effective $\mathbf{R}$-divisor A[3.1.0.1] over S and an effective $\mathbf{R}$-divisor E, such that $B = A + E + \llcorner B \lrcorner$.

For $f : (X, B) \to S$ satisfying the BCHM condition, we will show the following theorems:

- (Existence of flips) For any small contraction of $K_X + B$, the flip exists.
- (Existence of PL flips) For any small contraction of $K_X + B$ with respect to which $-\llcorner B \lrcorner$ is relatively ample, the flip exists.

[3.1.0.1] here A need to be effective

- (Existence of minimal models) If $K_X + B$ is relatively pseudo-effective, then there exists a minimal model of $f : (X, B) \to S$.
- (Finiteness of minimal models) Suppose that P is a polytope spanned by effective **R**-divisors such that for any $B' \in P$, $f : (X, B') \to S$ satisfies the BCHM condition. Then there exist finitely many rational maps $g_k : X \dashrightarrow Y_k$, such that for any $B' \in P$ with $K_X + B$ relatively pseudo-effective, there exists a minimal model of $f : (X, B') \to S$, and any minimal model of $f : (X, B') \to S$ coincides with one of g_k.
- (Termination of MMP with scaling) Assume further that $f : (X, B') \to S$ satisfies the BCHM condition for some $B' \geq B$ and assume that $K_X + B'$ is relatively nef. Then the MMP on $f : (X, B) \to S$ with scaling of $B' - B$ terminates at finitely many steps.
- (Special termination of MMP with scaling) Assume further that $f : (X, B') \to S$ satisfies the BCHM condition for some $B' \geq B$ and assume that $K_X + B'$ is relatively nef. Then the MMP on $f : (X, B) \to S$ with scaling of $B' - B$ is isomorphic in a neighborhood of $\llcorner B \lrcorner$ after finitely many steps.
- (Non-vanishing theorem) If $K_X + B$ is relatively pseudo-effective, then there exists an effective **R**-divisor D such that $D \equiv_S K_X + B$.

Remark 3.1.1. (1) The existence of PL flips and the special termination of MMP with scaling are special cases of the existence of flips and the termination of MMP with scaling.

(2) The existence of flips is a special case of the existence of minimal models. In fact, a flip is the relative canonical model of a relative minimal model.

(3) The statement of the finiteness of minimal models includes the existence of minimal models.

Remark 3.1.2. The finite generation theorem, which is the main purpose of this book, is obtained by showing the existence of minimal models for KLT pairs with $K_X + B$ big. However, the bigness of $K_X + B$ is not preserved if restricted on the boundary. On the other hand, the BCHM condition, considering DLT pairs with boundaries containing ample divisors, is preserved if restricted on the boundary. If (X, B) is KLT and $K_X + B$ is big, after replacing B, we may assume that B contains an ample divisor. This condition is preserved in the process of MMP, so the induction on dimensions works well in this situation.

Firstly, we modify the BCHM condition for KLT pairs, and show that the DLT version and the KLT version can be used appropriately according to the situations. The KTL version BCHM condition is the following:

(1) X is an n-dimensional $\mathbf{Q}$-factorial normal algebraic variety, B is an effective $\mathbf{R}$-divisor on X, $f : X \to S$ is a projective morphism to a quasi-projective variety.

(2) (X, B) is KLT.

(3) B is relatively big over S.

Lemma 3.1.3. *For each statement of the existence of flips, the existence of minimal models, the finiteness of minimal models, the termination of MMP with scaling, and the non-vanishing theorem, the DLT version holds if and only if the KLT version holds.*

Proof. Let us explain how to exchange DLT and KLT conditions.

Let $f : (X, B) \to S$ be a morphism satisfying the DLT version BCHM condition. Then there exists a sufficiently small positive real number t such that $A + t\llcorner B \lrcorner$ is relatively ample. Take a general effective relatively ample $\mathbf{R}$-divisor $A_1 \equiv A + tZ$, denote $B' = A_1 + E + (1-t)\llcorner B \lrcorner$, then (X, B') is KLT. So $B \equiv B'$ and it satisfies the KLT version BCHM condition.

Conversely, let $f : (X, B) \to S$ be a morphism satisfying the KLT version BCHM condition. As B is relatively big, there exists an effective relatively ample $\mathbf{R}$-divisor A and an effective $\mathbf{R}$-divisor E such that $B \equiv A + E$. For a sufficiently small positive real number t, denote $B' = (1-t)B + tA + tE$, then $(X, (1-t)B + tA + tE)$ is KLT. So $B \equiv B'$ and it satisfies the DLT version BCHM condition. □

Remark 3.1.4. Another advantage of the BCHM condition is that it is preserved by MMP in the following sense. For the KLT version, this is simply because that KLT condition and the bigness of B are both preserved by MMP. For the DLT version, consider a pair $(X, A + E + \llcorner B \lrcorner)$ satisfying the BCHM condition, suppose that $\alpha : X \dashrightarrow X'$ is obtained by several steps of MMP and $(X', A' + E' + \llcorner B' \lrcorner)$ is the induced pair. Pick a general relatively ample effective $\mathbf{R}$-divisor A'_1 on X' and take a sufficiently small positive real number ϵ such that $(X, A + \epsilon\alpha_*^{-1}A'_1 + E + \llcorner B \lrcorner)$ is still DLT and $A - \epsilon\alpha_*^{-1}A'$ is relatively ample. Take a general effective $\mathbf{R}$-divisor $A_2 \sim_{\mathbf{R}} A - \epsilon\alpha_*^{-1}A'$ such that $(X, A_2 + \epsilon\alpha_*^{-1}A'_1 + E + \llcorner B \lrcorner)$ is DLT, then $(X', B'_1 := \epsilon A'_1 + \alpha_* A_2 + E' + \llcorner B' \lrcorner)$ is DLT and satisfies the BCHM condition. On the other hand, $B'_1 \sim_{\mathbf{R}} B'$, hence it suffices to consider (X', B'_1) instead.

Remark 3.1.5. For the existence of PL flips and the special termination of MMP with scaling, the KLT version makes no sense. It is natural to run MMP within the KLT category, but the point to extend to the DLT category is that, for DLT pairs we can consider the restriction on the integral part $\llcorner B \lrcorner$, which opens the gate of induction on dimensions.

We are going to prove the following claims under the DLT version BCHM condition. Combining all these claims, all theorems are proved by induction on dimensions.

(a) (Theorem 3.3.1) The existence of flips in dimension $n-1$ and the termination of MMP with scaling in dimension $n-1$ imply the special termination of MMP with scaling in dimension n.

(b) (Theorem 3.4.1) The existence of PL flips in dimension n, the special termination of MMP with scaling in dimension n, and the non-vanishing theorem for a pair (X, B) in dimension n imply the existence of minimal models for the (X, B) in dimension n.

(c) (Theorem 3.4.1, see also Theorem 3.6.5) The existence of PL flips in dimension n and the special termination of MMP with scaling in dimension n imply the existence of flips in dimension n.

(d) (Theorem 3.2.1) The existence and finiteness of minimal models in dimension $n-1$ imply the existence of PL flips in dimension n.

(e) (Theorem 3.4.6[3.1.0.2]) The existence of minimal models in dimension n implies the finiteness of minimal models in dimension n.

(f) (Corollary 2.10.9) The finiteness of minimal models in dimension n implies the termination of MMP with scaling in dimension n.

(g) (Theorem 3.5.1) The existence of PL flips in dimension n, the special termination of MMP with scaling in dimension n, the existence and finiteness of minimal models for pairs (X, B) with $K_X + B$ relatively big in dimension n imply the non-vanishing theorem in dimension n.

Remark 3.1.6. If $K_X + B$ is relatively big, then the non-vanishing theorem automatically holds. Therefore, (c) is a special case of (b). (c) is originally proved by Shokurov, which is the origin of the induction method on dimensions.

In (d), Hacon and McKernan showed the existence of flips in all dimensions by induction on dimensions. Before that, the only proof of existence of flips used classification of singularities, which is only available in dimension 3. The proof of (d) needs the extension theorem of pluri-canonical forms. The base point free theorem is also an extension theorem for pluri-canonical forms, but here we need to use a much more powerful extension theorem.

[3.1.0.2] this theorem for (e) is added in translation

3.2 PL flips

In this section, we introduce the proof of the existence of PL flips due to Hacon and McKernan ([40]). Recall that a PL contraction $f : (X, B) \to S$ is a small contraction from a $\mathbf{Q}$-factorial DLT pair, such that $-P$ is f-ample for some irreducible component P of $\llcorner B \lrcorner$.

{PL flip exists}

Theorem 3.2.1 (*The existence of PL flips*)**.** *Let $f : (X, B) \to S$ be an n-dimensional PL contraction. Suppose that, under the BCHM condition, the existence and finiteness of minimal models in dimension $n-1$ hold, then the flip of f exists.*

This theorem is a pillar and an opportunity for recent great redevelopment of the minimal model theory.

The existence of flips is a special case of the finite generation of canonical rings, but we need to prove it first in order to establish the minimal model program, and then prove the finite generation theorem.

Let us describe the sketch of the proof. A PL flip is the flip of a PL contraction $f : (X, B) \to S$. We may assume that S is affine, (X, B) is PLT, and $Y = \llcorner B \lrcorner$ is irreducible. We may assume that B is a $\mathbf{Q}$-divisor after perturbing the coefficients.

Since f is small, the restriction $f|_Y$ is also a birational morphism, and the BCHM condition holds on Y.

In order to show the existence of the flip, it suffices to show that $R(X/S, K_X + B)$ is finitely generated. By the sub-adjunction formular, $(K_X + B)|_Y = K_Y + B_Y$ and (Y, B_Y) is KLT. Since $\dim Y = n - 1$, $R(Y/S, K_Y + B_Y)$ is finitely generated by induction hypothesis.

If the natural homomorphism $R(X/S, K_X + B) \to R(Y/S, K_Y + B_Y)$ is surjective, then we can finish the proof. However, as $K_X + B$ is negative with respect to f, we cannot establish the vanishing of higher cohomologies, so this is in general not surjective. In other words, pluri-canonical forms on Y are not necessarily extendable to X. Therefore, a key point in the proof is to determine the set of pluri-canonical forms on Y that are extendable to X. In order to do this, we will make full use of $(n-1)$-dimensional MMP and the vanishing theorem for multiplier ideal sheaves.

Fix a positive integer m_0 such that $m_0 B$ is an integral divisor. Denote $(K_X + B)|_Y = K_Y + B_Y$. For a positive integer m, the restriction map

$$H^0(X, mm_0(K_X + B)) \to H^0(Y, mm_0(K_Y + B_Y))$$

is not surjective in general.

Applying the extension theorem, we can identify the image of this map with the space of pluri-canonical maps $H^0(Y', m(K_{Y'} + B_{Y',m}))$ of another pair $(Y', B_{Y',m})$ different from (Y, B_Y). Here the point is that the new variety Y' can be chosen independently of m by the extension theorem.

By blowing up X, we can resolve the base locus of $|mm_0(K_X + B)|$. However, this cannot be done independently of m, so as m increases, we get a tower of blowing ups over X, that is, an inverse system of algebraic varieties. This inverse system is equivalent to Shokurov's *b-divisor* ([137]) in the divisor level, which is hard to handle. This concept is similar to *Zariski's Riemann surface.*

However, by applying the extension theorem, b-divisors are not needed. Instead of an infinite tower of varieties, we consider an infinite sequence of $\mathbb{Q}$-divisors $B_{Y',m}$ on a fixed variety Y'. In general the limit $B_{Y'}$ of $B_{Y',m}$ is an $\mathbb{R}$-divisor, this is one reason that we must formulate the MMP for $\mathbb{R}$-divisors.

If applying MMP to divisors on Y' and using the existence and finiteness of minimal models, this limit can be obtained within finite steps, and there exists a positive integer m such that $B_{Y'} = B_{Y',m}$.

3.2.1 Restriction of canonical rings to divisors

Firstly we show the following lemma.

{bourbaki}

Lemma 3.2.2. *Let $R = \bigoplus_{m=0}^{\infty} R_m$ be a sheaf of graded $\mathcal{O}_S$-rings such that $R_0 = \mathcal{O}_S$.*

(1) R is a finitely generated graded $\mathcal{O}_S$-algebra if and only if the ideal $R_+ = \bigoplus_{m>0} R_m$ is a finitely generated R-module.

(2) If R is a finitely generated graded $\mathcal{O}_S$-algebra, then the sub-algebra

$$R^{(m_1)} = \bigoplus_{m=0}^{\infty} R_{mm_1}$$

is a finitely generated graded $\mathcal{O}_Y$-algebra. Here m_1 is any fixed positive integer. Moreover, the converse holds if R is a domain.

Proof. (1) The homogenous generators of graded $\mathcal{O}_S$-algebra R are the same with the homogenous generators of the ideal R_+.

(2) Suppose that R is finitely generated, and take $x_1, \dots, x_t$ to be homogenous generators. Then R is generated by $\prod_{i=1}^{t} x_i^{d_i}$ $(0 \le d_i < m_1)$ as an $R^{(m_1)}$-module. Therefore, R is a finitely generated $R^{(m_1)}$-module. Since R_+ is a direct sum of $R^{(m_1)}$-modules $M_j = \bigoplus_{m=0}^{\infty} R_{j+mm_1}$ $(1 \le j \le m_1)$ and is a finitely generated R-module, it is a finitely generated $R^{(m_1)}$-module. In particular, $R_+^{(m_1)} = M_{m_1}$ is a finitely generated $R^{(m_1)}$-module. Hence $R^{(m_1)}$ is a finitely generated graded $\mathcal{O}_S$-algebra.

Conversely, suppose that $R^{(m_1)}$ is finitely generated and R is a domain. Then $R_+^{(m_1)}$ is a finitely generated $R^{(m_1)}$-module where $R^{(m_1)}$ is a Noetherian ring. If for some j, $M_j \neq 0$, then $x^{m_1-1} : M_j \to R_+^{(m_1)}$ is injective for

$0 \neq x \in M_j$, which implies that M_j is a finitely generated $R^{(m_1)}$-module. Therefore, R_+ is a finitely generated $R^{(m_1)}$-module, which is also a finitely generated R-module. □

For a PL contraction $f : (X, B) \to S$, the flip exists is equivalent to that

$$R = \bigoplus_{m=0}^{\infty} f_*\mathcal{O}_X(\llcorner m(K_X + B)\lrcorner)$$

is finitely generated over $\mathcal{O}_S$. Fix a positive integer m_0 such that $m_0 B$ is an integral divisor, the latter is equivalent to the finite generation of

$$R^{(m_0)} = \bigoplus_{m=0}^{\infty} f_*\mathcal{O}_X(mm_0(K_X + B)).$$

Since $-Y$ is f-ample, it is proportional to $-(K_X+B)$, hence this is equivalent to the finite generation of

$$R' = \bigoplus_{m=0}^{\infty} f_*\mathcal{O}_X(mY).$$

Lemma 3.2.3. *For any positive integer m, the reflexive sheaves $\mathcal{O}_X(mY)$ and $\mathcal{O}_Y(mY)$ on X and Y are well-defined, and satisfy the exact sequence*

$$0 \to \mathcal{O}_X((m-1)Y) \to \mathcal{O}_X(mY) \to \mathcal{O}_Y(mY) \to 0.$$

Proof. Note that Y is just $\mathbf{Q}$-Cartier, so these sheaves are not invertible in general. For any point $x \in X$, take a sufficiently small neighborhood X_x, and take the index 1 cover $\pi_x : \tilde{X}_x \to X_x$ of Y. Then $\tilde{Y}_x = \pi_x^{-1}(Y)$ is Cartier and we can define invertible sheaves $\mathcal{O}_{\tilde{X}_x}(m\tilde{Y}_x)$ and $\mathcal{O}_{\tilde{Y}_x}(m\tilde{Y}_x)$ satisfying the exact sequence

$$0 \to \mathcal{O}_{\tilde{X}_x}((m-1)\tilde{Y}_x) \to \mathcal{O}_{\tilde{X}_x}(m\tilde{Y}_x) \to \mathcal{O}_{\tilde{Y}_x}(m\tilde{Y}_x) \to 0.$$

Here the first homomorphism is defined by multiplying $s \in \Gamma(\tilde{X}_x, \mathcal{O}_{\tilde{X}_x}(\tilde{Y}_x))$. Take the invariant parts with respect to $\mathrm{Gal}(\tilde{X}_x/X_x)$, we get the required exact sequence. □

Since $-Y$ is f-ample, in the exact sequence

$$0 \to f_*\mathcal{O}_X((m-1)Y) \to f_*\mathcal{O}_X(mY) \to f_*\mathcal{O}_Y(mY),$$

the last homomorphism is not surjective in general.

Lemma 3.2.4. *If the restriction algebra*

$$R'_Y = \bigoplus_{m=0}^{\infty} \mathrm{Im}(f_*\mathcal{O}_X(mY) \to f_*\mathcal{O}_Y(mY)).$$

is finitely generated, then R' is finitely generated.

Proof. Take $s_1, \dots, s_k \in R'$ to be the generators of R'_Y on X, then R' is generated by $s, s_1, \dots, s_k$. □

Applying Lemma 3.2.2 again, we can reduce the problem to the finite generation of

$$R_Y = \bigoplus_{m=0}^{\infty} \mathrm{Im}(f_*\mathcal{O}_X(mm_0(K_X+B)) \to f_*\mathcal{O}_Y(mm_0(K_Y+B_Y))).$$

Here $K_Y + B_Y = (K_X+B)|_Y$. The restriction map

$$H^0(X, mm_0(K_X+B)) \to H^0(Y, mm_0(K_Y+B_Y))$$

is not surjective. The idea is to replace this image by the space of pluricanonical forms with respect to a new boundary smaller than B_Y.

Firstly we construct a tower of log resolutions:

{restricted1}

Proposition 3.2.5 ([40, Lemma 6.4])**.** *Let $f : (X, B) \to S$ be a small contraction from a $\mathbf{Q}$-factorial PLT pair. Assume that S is affine, $Y = \llcorner B \lrcorner$ is irreducible, $-Y$ is f-ample, and B is a $\mathbf{Q}$-divisor. Take a positive integer m_0 such that m_0B is an integral divisor. Then for any positive integer m, there exists a log resolution $\mu_m : X_m \to X$ and a $\mathbf{Q}$-divisor B'_m on X_m satisfying the following conditions:*

(1) Write $\mu_m^*(K_X+B) = K_{X_m} + B_m$, $Y_m = \mu_{m*}^{-1}Y$, then the irreducible components of $B_m^+ - Y_m$ are disjoint. Here B_m^+ is the positive part of B_m, that is, we can write $B_m = B_m^+ - B_m^-$ where B_m^+, B_m^- are effective divisors without common components.

(2) $mm_0B'_m$ is an integral divisor with $Y_m \le B'_m \le B_m^+$.

(3) A general element in $|mm_0(K_{X_m} + B'_m)|$ does not contain LC centers of $(X_m, \ulcorner B'_m \urcorner)$.

(4) The natural map

$$\begin{aligned} &H^0(X_m, mm_0(K_{X_m}+B'_m)) \to H^0(X_m, mm_0(K_{X_m}+B_m^+)) \\ &\cong H^0(X, mm_0(K_X+B)) \end{aligned}$$

is bijective.

(5) Y_m is isomorphic to a fixed variety Y', and μ_m induces a fixed morphism $\mu_Y : Y' \to Y$.

(6) Write $(K_{X_m} + B'_m)|_{Y_m} = K_{Y'} + B_{Y',m}$, then $B_{Y',m}$ satisfy the following convexity on m:

$$m_1 B_{Y',m_1} + m_2 B_{Y',m_2} \le (m_1+m_2) B_{Y',m_1+m_2}.$$

(7) *The limit* $B_{Y'} = \lim_{m\to\infty} B_{Y',m}$ *exists as an* $\mathbf{R}$*-divisor, and* $(Y', B_{Y'})$ *is KLT.*

Proof. Take a log resolution $\mu : (X', B') \to (X, B)$, write $\mu^*(K_X + B) = K_{X'} + B'$, $Y' = \mu_*^{-1}Y$, we may assume that the irreducible components of $(B')^+ - Y'$ are disjoint. We will construct μ_m by blowing up X'. To make the notation simpler, we use the same notation as X' to denote the variety after blowing up.

Fix m and a general element $D \in |mm_0(K_X + B)|$. As $f|_Y$ is birational, D does not contain Y. D induces an element $D' \in |mm_0(K_{X'} + (B')^+)|$. If D' and $(B')^+$ have common components, we replace them by $D' - \min\{D', mm_0(B')^+\}$ and $(B')^+ - \min\{D'/mm_0, (B')^+\}$. Here note that $\min\{D', mm_0(B')^+\}$ is contained in the fixed part of the linear system, hence $D' \in |mm_0(K_{X'} + (B')^+)|$ is preserved. The new D' and $(B')^+$ have no common components.

Next we show that, after replacing X' by blowing ups and replace D', we may assume that D' contains no LC centers of $(X', \ulcorner(B')^+\urcorner)$. By the construction of μ, LC centers of $(X', \ulcorner(B')^+\urcorner)$ are irreducible components of $(B')^+$ and irreducible components of $Y' \cap ((B')^+ - Y')$. The former are already handled, consider the case that D' contains an irreducible component Z of $Y' \cap ((B')^+ - Y')$. In this case we blow up X' along Z, and keep blowing up X' until the irreducible components of $(B')^+ - Y'$ are disjoint. Subtract the common part of new D' and $(B')^+$ as above. Note that in this process we only blowing up prime divisors on Y', hence Y' is not changed. On the other hand, in this process, at least one coefficient of $((B')^+ - Y')|_{Y'}$ decreases. Hence repeating this process finitely times, eventually D' contains no irreducible components of $Y' \cap ((B')^+ - Y')$.

In this way we constructed a very log resolution $\mu_m : X_m \to X$. Take $B'_m = (B')^+$. Here recall that $(B')^+$ is obtained by subtracting redundant irreducible components. From the construction, (1), (2), (3) directly follow. (4) follows as we only subtract fixed part of D' from B_m^+ and B_m^- is μ_m-exceptional. (5) follows since in the beginning $\mu : X' \to X$ does not depend on m and Y' keeps unchanged. (6) follows from the natural homomorphism

$$\begin{aligned} &H^0(X, m_0m_1(K_X + B)) \otimes H^0(X, m_0m_2(K_X + B)) \\ &\to H^0(X, m_0(m_1 + m_2)(K_X + B)). \end{aligned}$$

(7) follows since $B_{Y',m} \le (B_m^+ - Y_m)|_{Y_m}$ and $(Y_m, (B_m^+ - Y_m)|_{Y_m})$ is KLT, where $(B_m^+ - Y_m)|_{Y_m}$ is a divisor on Y' indenpendent of m. □

Next we apply the extension theorem:

{restricted2}

Theorem 3.2.6. *Under the setting of Proposition 3.2.5, the restriction map*

$$H^0(X_m, lmm_0(K_{X_m} + B'_m)) \to H^0(Y', lmm_0(K_{Y'} + B_{Y',m}))$$

is surjective for any positive integer l*.*

Proof. Let us check the conditions of the extension theorem. Firstly, $\llcorner B'_m \lrcorner = Y'$ and (X_m, B'_m) is PLT.

1. Since $f \circ \mu$ is an isomorphism on the generic points of X_m and Y', we can write $B'_m - Y' = A + E$ where A is an ample $\mathbf{Q}$-divisor and E is an effective $\mathbf{Q}$-divisor whose support does not contain Y.

2. D_m does not contain LC centers of $(X_m, \ulcorner B'_m \urcorner)$. □

3.2.2 The existence of PL flips

In this subsection, we prove the the existence of PL flips.

Let us recall some symbols. For a divisor D on a normal algebraic variety X, its *fixed part* $\mathrm{Fix}(D)$ and *movable part* $\mathrm{Mov}(D)$ are defined as the following:

$$\begin{aligned}
&|D| = \{D' \mid D \sim D' \geq 0\},\\
&\mathrm{Fix}(D) = \inf |D|,\\
&\mathrm{Mov}(D) = D - \mathrm{Fix}(D).
\end{aligned}$$

Here the infimum of divisors is defined by taking infimum of coefficients of each component. Note that if X is not projective, then the complete linear system $|D|$ is not necessarily finite dimensional, but the fixed part is well-defined as a divisor.

Proof of Theorem 3.2.1. By the finiteness of minimal models, for sufficiently large m, pairs $(Y', B_{Y',m})$ and $(Y, B_{Y'})$ have the same canonical model Z. Here $f|_{Y'} : Y' \to S$ is birational to its image, so the KLT version BCHM condition automatically holds.

By a log resolution $\nu : Y'' \to (Y', B_{Y'})$, we may assume that the induced map $\nu : Y'' \to Z$ is a morphism over S. We may replace Y' by Y'' in the following.

For positive integers m, l, denote

$$\begin{aligned}
&\tilde{P}_m = \mathrm{Mov}(mm_0(K_{X_m} + B_m^+)) = \mathrm{Mov}(mm_0(K_{X_m} + B'_m)),\\
&P_m = \tilde{P}_m|_{Y'} = \mathrm{Mov}(mm_0(K_{Y'} + B_{Y',m})),\\
&\tilde{P}_{l,m} = \mathrm{Mov}(lmm_0(K_{X_m} + B'_m)),\\
&P_{l,m} = \tilde{P}_{l,m}|_{Y'} = \mathrm{Mov}(lmm_0(K_{Y'} + B_{Y',m})),
\end{aligned}$$

according to Theorem 3.2.6.

Since Y' dominates Z which is the canonical model of $(Y', B_{Y'_m})$, there exists a positive integer l_m such that $P_{l_m,m}$ is free. Since $\tilde{P}_{1,m} = \tilde{P}_m$, $P_{1,m} = P_m$. We have

$$l_m P_m = l_m P_{1,m} \leq P_{l_m,m} \leq P_{l_m m}.$$

Since

$$(K_{X_m} + B_m)|_{Y'} = K_{Y'} + \tilde{B}_{Y'} = \mu_Y^*(K_Y + B_Y),$$

we get

$$P_{m_1} + P_{m_2} \le P_{m_1+m_2} \le (m_1+m_2)(K_{Y'} + (\tilde{B}_{Y'})^+)$$

hence the limit $P = \lim_{m\to\infty} P_m/m$ defines an $\mathbb{R}$-divisor on Y'. Note that $P_{l_m,m}/l_m m$ is the pullback of the log canonical divisor of the canonical model of $(Y', B_{Y'_m})$ and

$$\lim_{m\to\infty} \frac{P_m}{m} = \lim_{m\to\infty} \frac{P_{l_m,m}}{l_m m},$$

hence P is the pullback of the log canonical divisor of the canonical model of $(Y', B_{Y'})$. Therefore P is semi-ample, that is, it is a sum of free divisors with positive real number coefficients.

In the following, we will show that there exists a positive integer m_1 such that $P = P_{m_1}/m_1$ and P_{m_1} is free. If this is proved, then for any positive integer l, $lP_{m_1} = P_{lm_1}$, hence

$$\begin{aligned} &\bigoplus_{l\ge 0} H^0(Y', lP_{m_1}) \\ &\cong \bigoplus_{l\ge 0} \mathrm{Im}(H^0(X, lm_0m_1(K_X+B)) \to H^0(Y, lm_0m_1(K_Y+B_Y))) \end{aligned}$$

is finitely generated, and the proof is finished.

Lemma 3.2.7. *For any positive integers m, m',*

$$\mathrm{Mov}(\ulcorner \frac{m'P_{l_m,m}}{l_m m} - \tilde{B}_{Y'} \urcorner) \le P_{m'}.$$

Therefore, after taking limit,

$$\mathrm{Mov}(\ulcorner m'P - \tilde{B}_{Y'} \urcorner) \le P_{m'}.$$

Proof. Take a general effective $\mathbf{Q}$-divisor $D \equiv m'\tilde{P}_{l_m,m}/l_m m$. That is, take a sufficiently large and sufficiently divisible positive integer N, take a general element in $|Nm'\tilde{P}_{l_m,m}/l_m m|$, and divide it by N to get D. Also take a general effective $\mathbf{Q}$-divisor

$$D' \equiv \ulcorner \frac{m'\tilde{P}_{l_m,m}}{l_m m} - B_m \urcorner - (\frac{m'\tilde{P}_{l_m,m}}{l_m m} - B_m).$$

Take J to be the multiplier ideal sheaf of $(X_m, D + D')$. Since $P_{l_m,m} = \tilde{P}_{l_m,m}|_{Y_m}$ is free, Y_m does not intersect the support of $\mathcal{O}_{X_m}/J$. As

$$\ulcorner \frac{m'\tilde{P}_{l_m,m}}{l_m m} - B_m \urcorner - (K_{X_m} + D + D') \equiv -\mu^*(K_X + B)$$

is relatively nef and relatively big,

$$H^1(X_m, J(\ulcorner \frac{m'\tilde{P}_{l_m,m}}{l_m m} - B_m \urcorner)). = 0$$

Hence

$$H^0(X_m, \ulcorner \frac{m'\tilde{P}_{l_m,m}}{l_m m} - (B_m - Y_m)\urcorner) \to H^0(Y', \ulcorner \frac{m'P_{l_m,m}}{l_m m} - \tilde{B}_{Y'}\urcorner)$$

is surjective. On the other hand, since

$$\mu_{m*}(\ulcorner \frac{m'\tilde{P}_{l_m,m}}{l_m m} - (B_m - Y_m)\urcorner) \le m'm_0(K_X + B),$$

we get

$$\mathrm{Mov}(\ulcorner \frac{m'\tilde{P}_{l_m,m}}{l_m m} - (B_m - Y_m)\urcorner) \le \tilde{P}_{m'}.$$

This proves the former statement. For the latter one, just take the limit. □

Now go back to the proof of the existence of PL flips. Firstly suppose that P is a $\mathbb{Q}$-divisor. In this case, there exists a positive integer m_1 such that $m_1 P$ is Cartier and free. Then

$$m_1 P \le \mathrm{Mov}(\ulcorner m_1 P - \tilde{B}_{Y'}\urcorner) \le P_{m_1} \le m_1 P.$$

Here the first equality follows as $\ulcorner -\tilde{B}_{Y'}\urcorner \ge 0$. Hence $m_1 P = P_{m_1}$.

Finally, suppose that P is not a $\mathbb{Q}$-divisor. We can use positive real numbers p_j and free Cartier divisors L_j to express $P = \sum_j p_j L_j$. Take an effective divisor M containing supports of all L_j, and take a sufficiently small real number $\epsilon > 0$ such that $\llcorner \tilde{B}_{Y'} + \epsilon M \lrcorner \le 0$. Suppose that at least one p_j is not rational, the there exists a positive integer m and a free Cartier divisor L such that

$$mP - \epsilon M < L < mP + \epsilon M$$

and $L \not\le mP$. Then

$$L \le \ulcorner mP + \epsilon M - \tilde{B}_{Y'} - \epsilon M\urcorner,$$

which implies that

$$L \le \mathrm{Mov}(\ulcorner mP - \tilde{B}_{Y'}\urcorner) \le P_m \le mP,$$

a contradiction. □

Remark 3.2.8. (1) In this way, the existence of flips can be proved in any dimension. One should be reminded that the proof of the existence of flips in dimension 3 was very difficult [109]. So this is a big success of the formulation using log pairs. It can be seen that in the above argument, the base point free theorem plays an important role in the background.

(2) P is equivalent to the positive part of the Zariski decomposition. In [63], assuming the existence of Zariski decomposition in the sense that the positive part is nef, even if the positive part might be an $\mathbf{R}$-divisor, it can be shown that the canonical ring is finitely generated and the positive part is in fact a $\mathbf{Q}$-divisor. The technique in that proof might be applied here, but the proof introduced here uses the idea of "*saturation*" due to Shokurov ([137]).

3.3 The special termination

The special termination is a key point for the induction on dimensions.

DLT pairs are suitable for inductive argument on dimensions, as an irreducible component in the boundary with coefficient 1 determines a DLT pair of one dimension lower. The special termination theorem by Shokurov is basic in the discussion of the termination of MMP by induction on dimensions. In this section, we show the special termination of MMP with scaling.

The log version, as the generalization of the non-log version, should be more complicated originally. For example, log terminal singularities are more complicated than terminal singularities. On the other hand, the log version has more freedom. We can perturb coefficients. Also if there is an irreducible component in the boundary with coefficient 1, we can use the subadjunction formula to get a DLT pair of one dimension lower. In addition, for a fixed algebraic variety X, if the coefficients of B increases, then the condition that (X, B) is log terminal gets stronger, and the singularities of X gets better.

Firstly let us recall the statement of the special termination of MMP with scaling: Suppose that $f : (X, B) \to S$ and $f : (X, B') \to S$ satisfy the BCHM condition. Assume that $B' \geq B$ and $K_X + B'$ is relatively nef. Run MMP on $f : (X, B) \to S$ with scaling of $B' - B$, starting from $(X_0, B_0) = (X, B)$, we get an infinite sequence of flips

$$\alpha_m : (X_m, B_m) \dashrightarrow (X_{m+1}, B_{m+1}), \qquad m = 0, 1, 2, \ldots.$$

Here $B_{m+1} = \alpha_{m*} B_m$. Then there exists a positive integer m_0, such that for any $m \geq m_0$, α_m is isomorphic in a neighborhood of $\llcorner B_m \lrcorner$. That is,

$$\mathrm{Exc}(\alpha_m) \cap \mathrm{Supp} \llcorner B_m \lrcorner = \emptyset.$$

{special termination thm}

Theorem 3.3.1 (*Special termination theorem*)**.** *Under the BCHM condition, suppose that the existence of flips in dimension $n-1$ and the termination of MMP with scaling in dimension $n-1$ hold, then the special termination of MMP with scaling in dimension n holds.*

Remark 3.3.2. (1) The special termination was originally proved by Shokurov [135]. It opened the gate of proving the existence of minimal models by induction on dimensions.

(2) As we will discuss later, the assumption above implies the existence of $\mathbf{Q}$-factorialization of KLT pairs in dimension $n-1$ (see Corollary 3.6.9). In fact, a $\mathbf{Q}$-factorialization is a minimal model of certain birational morphism. Assuming the existence of flips in dimension $n-1$ and the termination of MMP with scaling in dimension $n-1$, the existence of minimal models in dimension $n-1$ follows.

Proof. Fix any irreducible component Z_1 of $Z = \llcorner B \lrcorner$. Given an MMP with scaling consisting of flips, it suffices to show that the MMP is isomorphic in a neighborhood of Z_1 after finitely many steps.

Recall that we can write $B = A + E + Z$ and $B' = A + E' + Z'$ where $Z' = \llcorner B' \lrcorner$. Here we can pick the same A since $B \leq B'$. Take a sufficiently small real number t such that $A + t(Z - Z_1)$ is relatively ample. Take a sufficiently general effective $\mathbf{R}$-divisor $A_1 \equiv A + t(Z - Z_1)$. Then $(X, A_1 + E + (1-t)(Z - Z_1) + Z_1)$ is PLT and $(X, A_1' + E' + Z' - t(Z - Z_1))$ is DLT. As $B \equiv A_1 + E + (1-t)(Z - Z_1) + Z_1$, $B' \equiv A_1' + E' + Z' - t(Z - Z_1)$, After replacing B and B', we may assume that (X, B) is PLT in the beginning. Denote $Z_m = \llcorner B_m \lrcorner$.

The sequence of flips of (X, B) induces birational maps $\alpha_m' : Z_m \dashrightarrow Z_{m+1}$. Note that α_m' can contract divisors on Z_m and also extract new divisors on Z_{m+1}.

The set of coefficients of B is a finite set $\mathrm{Coeff}(B) \subset [0, 1]$. So $\mathrm{Coeff}(B_m) = \mathrm{Coeff}(B)$ is a fixed set. On the other hand, by the sub-adjunction formula $(K_{X_m} + B_m)|_{Z_m} = K_{Z_m} + B_{Z_m}$ we get an $\mathbf{R}$-divisor B_{Z_m} on Z_m, the set $\mathrm{Coeff}(B_{Z_m})$ might depend on m.

Define the set $\Sigma \subset [0, 1]$ as the following:

$$\Sigma = \{x \in [0,1] \mid x = 1 - \frac{1}{r} + \sum_i \frac{r_i b_i}{r}, b_i \in \mathrm{Coeff}(B), r \in \mathbf{Z}_{>0}, r_i \in \mathbf{Z}_{\geq 0}\}.$$

Then by the sub-adjunction formula, $\mathrm{Coeff}(B_{Z_m}) \subset \Sigma$ for any m.

Lemma 3.3.3. *For any $\epsilon > 0$, The set $\Sigma \cap [0, 1-\epsilon]$ is finite.*

Proof. Consider $x = 1 - \frac{1}{r} + \sum_i \frac{r_i b_i}{r} \in \Sigma$. Since b_1 is in a finite set, if $x \leq 1 - \epsilon$, then it is easy to see that there are only finitely many possible values for r and r_i. □

For any positive number m, take a common resolution $g : Y \to X_m$ and $g' : Y \to X_{m+1}$ of (X_m, B_m) and (X_{m+1}, B_{m+1}) in strong sense. Write $K_Y + C = g^*(K_{X_m} + B_m)$ and $K_Y + C' = (g')^*(K_{X_{m+1}} + B_{m+1})$. Write $C = \sum c_i C_i$, $C' = \sum c_i' C_i$ into prime divisors, then $c_i \geq c_i'$ for all C_i, and

$c_i > c'_i$ for divisors supported over the exceptional locus of $X_m \dashrightarrow X_{m+1}$ (cf. Theorem 2.5.6).

Take $Z' \subset Y$ to be the common strict transform of Z_m and Z_{m+1}, take $\bar{C}_i = C_i \cap Z'$, then $K_{Z_m} + B_{Z_m} = g_*(K_{Z'} + \sum c_i\bar{C}_i)$, $K_{Z_{m+1}} + B_{Z_{m+1}} = g'_*(K_{Z'} + \sum c'_i\bar{C}_i)$. Here the sum runs over all $C_i \neq Z'$. As (Z_m, B_{Z_m}) and $Z_{m+1}, B_{Z_{m+1}})$ are KLT, we may assume that g is a very log resolution of (Z_m, B_{Z_m}) and $(Z_{m+1}, B_{Z_{m+1}})$.

For each m, consider the number

$$d_m = \sum_{a\in\Sigma} \#\{c_i \mid c_i > a\}.$$

Here we consider all coefficients appearing in $\sum c_i\bar{C}_i$. Note that d_m is a well-defined non-negative number since the sum only considers finitely many a as $c_i < 1$, and does not depend on the choice of very log resolutions since we only consider non-negative coefficients. As $c_i \geq c'_i$, we know that $d_m \geq d_{m+1}$. If Z_{m+1} contains a divisor P which is not a divisor on Z_m, then P comes from some $\bar{C}_i$ where C_i is supported over the exceptional locus of $X_m \dashrightarrow X_{m+1}$. So this means that $c_i > c'_i \in \Sigma$, and in this case we have $d_m > d_{m+1}$.

In this way, we can see that $\alpha'_m : Z_m \dashrightarrow Z_{m+1}$ is surjective in codimension one after deleting finitely many steps, hence it is isomorphic in codimension one after deleting finitely many steps.

By the sub-adjunction formula, $(K_X+B)|_Z = K_Z+B_Z$. As A is general, the pair (Z, B_Z) is KLT and $B_Z \geq A|_Z$. However Z is not necessarily $\mathbf{Q}$-factorial. So we take $h : \tilde{Z} \to Z$ to be a $\mathbf{Q}$-factorialization.

We will show that if pulling backing the original MMP on $f : (X, B) \to S$ to $\tilde{Z}$, then we get an MMP on $\tilde{Z}$, and hence the original MMP terminates in a neighborhood of Z. However, each step of the MMP on $f : (X, B) \to S$ corresponds to a composition of several steps of the MMP on $\tilde{Z}$, so we need to explain this in more details.

Suppose that the flip $\alpha_m : (X_m, B_m) \dashrightarrow (X_{m+1}, B_{m+1})$ is the composition of small birational maps $\phi_m : X_m \to S_m$ and $\phi_m^+ : X_{m+1} \to S_m$, denote $t_m = \min\{t \mid K_{X_m} + B_m + t_m(B'_m - B_m)$ is nef over $S\}$, then $K_{X_m} + B_m + t_m(B'_m - B_m)$ is numerically trivial over S_m.

Consider the induced map $\alpha'_m : Z_m \dashrightarrow Z_{m+1}$, suppose that we have a birational modification $h_m : \tilde{Z}_m \to Z_m$ such that $\tilde{Z}_m$ is $\mathbf{Q}$-factorial and $K_{\tilde{Z}_m} + \tilde{B}_{Z,m} = h_m^*(K_{Z_m} + B_{Z_m})$, where $\tilde{B}_{Z,m} \geq 0$. Then $(\tilde{Z}_m, \tilde{B}_{Z,m})$ is $\mathbf{Q}$-factorial and KLT, and $\phi_m \circ h_m : (\tilde{Z}_m, \tilde{B}_{Z,m}) \to S_m$ satisfies the BCHM condition as $\tilde{Z}_m$ is birational to its image on S_m. So, by induction hypothesis, we can run an MMP on $\phi_m \circ h_m : (\tilde{Z}_m, \tilde{B}_{Z,m}) \to S_m$ with scaling of a general relatively ample divisor, which terminates to a minimal model $(\tilde{Z}_{m+1}, \tilde{B}_{Z,m+1}) \to S_m$. Since $K_{Z_{m+1}} + B_{Z,m+1}$ is ample over S_m, it is the canonical model of $\tilde{Z}_{m+1}$ and induces $h_{m+1} : \tilde{Z}_{m+1} \to Z_{m+1}$.

In this way, for each m, we can inductively construct a sequence of MMP on $(\tilde{Z}_m, \tilde{B}_{Z,m})$ over S_m:

$$\tilde{Z}_m = Z_{m,0} \dashrightarrow Z_{m,1} \dashrightarrow \cdots \dashrightarrow Z_{m,l} = \tilde{Z}_{m+1}.$$

Since $K_{X_m} + B_m + t_m(B'_m - B_m)$ is numerically trivial over S_m, the induced divisors $K_{Z_{m,i}} + B_{m,i} + t_m(B'_{m,i} - B_{m,i})$ are all numerically trivial over S_m. Hence it is easy to check that the sequence

$$\tilde{Z} = \tilde{Z}_0 = Z_{0,0} \dashrightarrow Z_{0,1} \dashrightarrow \cdots \dashrightarrow Z_{0,l} = \tilde{Z}_1 = Z_{1,0} \dashrightarrow Z_{1,1} \dashrightarrow \ldots$$

is in fact an MMP on $\tilde{f} : (\tilde{Z}, \tilde{B}) \to S$ with scaling of $\tilde{B}' - \tilde{B}$, where $\tilde{f} = f \circ h$, $\tilde{B} = h_*^{-1}(B-Z)|_Z$, $\tilde{B}' = h_*^{-1}(B'-Z)|_Z$. Here $B_{m,i}, B'_{m,i}$ are strict transforms of $\tilde{B}, \tilde{B}'$. By inductive hypothesis, this MMP terminates. This means that, after finitely many steps, $\tilde{Z}_m$ does not change, which means that $K_{\tilde{Z}_m} + \tilde{B}_{Z,m}$ is nef over S_m, and then $K_{Z_m} + B_{Z_m}$ is nef over S_m. On the other hand, $-(K_{Z_m} + B_{Z_m})$ is ample over S_m, so $Z_m \to S_m$ does not contract any curve on Z_m. Similarly, $Z_{m+1} \to S_m$ does not contract any curve on Z_{m+1}. If Z_m intersects $\mathrm{Exc}\alpha_m$, then Z_m, as a divisor on X_m, is ample over S_m. Hence $-Z_{m+1}$ is ample over S_m, which contradicts to the fact that $Z_{m+1} \to S_m$ does not contract any curve on Z_{m+1}. This implies that the original MMP terminates in a neighborhood of Z. □

eral special termination}

Remark 3.3.4. Without assuming the BCHM condition, we have the following special termination: "suppose that the existence of flips in dimension $n-1$ and the termination of MMP in dimension $n-1$ hold, then the special termination in dimension n holds." That is, given a projective morphism $f : (X, B) \to S$ from a DLT pair, for an infinite sequence of flips over S starting from $(X_0, B_0) = (X, B)$:

$$\alpha_m : (X_m, B_m) \dashrightarrow (X_{m+1}, B_{m+1}), \qquad m = 0, 1, 2, \ldots$$

where $B_{m+1} = \alpha_{m*} B_m$, there exists a positive integer m_0 such that for any $m \geq m_0$, α_m is isomorphic in a neighborhood of $\llcorner B_m \lrcorner$. We will not use this fact in this book, please refer to [28].

3.4 The existence and the finiteness of minimal models

In this section, we show the existence of minimal models by induction on dimensions.

{existence of mm}

Theorem 3.4.1 (*Existence of minimal models*)**.** *Under the BCHM condition, suppose that the existence of PL flips in dimension n, the special termination of MMP with scaling in dimension n, and the non-vanishing theorem in dimension n hold, then the existence of minimal models in dimension n holds.*

xistence and nv}

Remark 3.4.2. As can be seen below, the BCHM condition is not essential in this proof. To be more precise, we can prove the following claim:

Let (X, B) be a KLT pair consisting of an n-dimensional $\mathbf{Q}$-factorial algebraic variety and an effective $\mathbf{R}$-divisor, $f : X \to S$ a projective morphism to a quasi-projective algebraic variety. Assume the following:

1. (Existence of PL flips) For any n-dimensional $\mathbf{Q}$-factorial pair (X', B') with a PL contraction, the flip always exists.

2. (Special termination) Any MMP with scaling starting from an n-dimensional $\mathbf{Q}$-factorial pair (X', B') is isomorphic in neighborhoods of strict transforms of $\llcorner B' \lrcorner$.

3. (Non-vanishing theorem) For the given pair (X, B), there exists an effective $\mathbf{R}$-divisor D such that $K_X + B \equiv_S D$.

The there exists a minimal model of the given morphism $f : (X, B) \to S$.

The existence of PL flips holds as a result of this chapter, the special termination can be proved assuming lower dimensional minimal model theory (Remark 3.3.4). Therefore, if one want to try to prove the existence of minimal models without the BCHM condition, the non-vanishing theorem is a key point.

Here we will follow the proof of Birkar [13] which modifies that of [15]. Firstly recall the definition of minimal models:

Definition 3.4.3. Let (X, B) be a $\mathbf{Q}$-factorial DLT pair and $f : X \to S$ a projective morphism to a quasi-projective algebraic variety. A *minimal model* of $f : (X, B) \to S$ is given by another $\mathbf{Q}$-factorial DLT pair (Y, C) over S with a birational map $\alpha : (X, B) \dashrightarrow (Y, C)$ satisfying the following:

(1) α is surjective in codimension 1, $C = \alpha_* B$.

(2) $K_Y + C$ is relatively nef over S.

(3) If we take a normal algebraic variety Z with birational projective morphisms $p : Z \to X$, $q : Z \to Y$ such that $\alpha = q \circ p^{-1}$, then the *discrepancy divisor* $G = p^*(K_X + B) - q^*(K_Y + C)$ is effective, and the support of $p_* G$ contains all prime divisors contracted by α.

If in the third condition we only assume $G \geq 0$, then it is called a *weak minimal model.*

Remark 3.4.4. Minimal models obtained by MMP satisfy the following condition stronger than (3): denote $\mathrm{Exc}(\alpha)$ to be the *exceptional set* of α, that is, the complement of the maximal open subset on which α is an isomorphism, then the support of G coincides with $p^{-1}(\mathrm{Exc}(\alpha))$. Condition (3) only focuses on the phenomenon in codimension 1, but it is in fact sufficient.

Proof. Since we only assume the existence of PL flips, we need to be careful on running MMP, that is, we can run the MMP as long as each small contraction is a PL contraction. On the other hand, in order to show the termination of certain MMP, the idea is to adjust the boundary B and apply the special termination. Therefore, it is necessary to consider DLT pairs by increasing the boundary B, instead of KLT pairs.

By assumption, $K_X + B \equiv_S D$ for some effective $\mathbf{R}$-divisor D.

Step 0. We reduce to the case that X is smooth and the support of $B + D$ is of normal crossing.

After replacing B, we may assume that (X, B) is KLT and satisfies the BCHM condition and $B \geq A$ where A is a general effective relatively ample $\mathbf{R}$-divisor. Take a log resolution $g : X' \to (X, B + D)$, we can construct an effective $\mathbf{R}$-divisor B' such that (X', B') is KLT and satisfies the BCHM condition,

$$E = K_{X'} + B' - g^*(K_X + B)$$

is effective and its support coincides with the support of the exceptional set of g. Then a minimal model of $f \circ g : (X', B') \to S$ is also a minimal model of $f : (X, B) \to S$.

In fact, take $\alpha : (X', B') \dashrightarrow (X'', B'')$ to be a minimal model, since E is contained in the numerically fixed part of $K_{X'} + B'$ over S, it is contracted by α by Theorem 2.9.6, that is $\alpha_* E = 0$. Hence $\alpha \circ g^{-1} : X \dashrightarrow X''$ is surjective in codimension 1, and the negativity can be checked easily. So it is a minimal model of $f : (X, B) \to S$.

Also, note that $B' \geq g_*^{-1} A$ is still relatively big as A is general. Hence $f \circ g : (X', B') \to S$ satisfies the KLT version BCHM condition. After replacing $f : (X, B) \to S$ by $f \circ g : (X', B') \to S$, we may assume that X is smooth and the support of $B + D$ is of normal crossing in the beginning. Note that we do not assume that (X, B) is KLT, but we assume that $f : (X, B) \to S$ satisfies the BCHM condition, and $B \geq A$ a general effective relatively ample $\mathbf{R}$-divisor, in particular, A has no common component with D.

Step 1. Write $B = \sum b_i D_i$ and $D = \sum d_i D_i$ by distinct prime divisors D_i. We will do induction on

$$\theta = \theta(X, B, D) = \#\{i \mid b_i \neq 1, d_i \neq 0\}.$$

If $D = 0$, then $f : (X, B) \to S$ is already minimal, hence we assume that $D \neq 0$.

Suppose that $\theta = 0$. Take a general effective relatively ample $\mathbf{R}$-divisor H such that $K_X + B + H$ is relatively nef and $(X, B + H)$ is DLT, we can run an MMP with scaling of H. Here since $\theta = 0$, the support of D is contained in $\llcorner B \lrcorner$, hence in each step the contracted curves are contained in the support of D and each small contraction is a PL contraction. By the existence of PL flips and the special termination of MMP with scaling, this MMP works and terminates.

Next, suppose that $\theta > 0$. Take

$$t = \min\{t' \in \mathbf{R}_{>0} \mid \mathrm{Supp}(\llcorner B \lrcorner) \neq \mathrm{Supp}(\llcorner B + t'D \lrcorner)\}.$$

Take $B + C = \sum \min\{b_i + td_i, 1\}D_i$, where C is effective. Then $K_X + B + C \equiv_S D + C$ and $\mathrm{Supp}(D+C) = \mathrm{Supp}\, D$. Note that t is the smallest number making $\theta(X, B + C, D + C) < \theta(X, B, D)$.

Consider the morphism $f : (X, B+C) \to S$. Here we remind that (X, B) satisfies the BCHM condition and $B \geq A$, which implies that $f : (X, B + C) \to S$ still satisfies the BCHM condition. Since $\theta(X, B+C, D)$ is smaller, by the induction hypothesis, there exists a minimal model $f' : X' \to S$, a birational morphism $\alpha : (X, B + C) \dashrightarrow (X', B' + C')$ over S such that $K_{X'}+B'+C'$ is relatively nef. Here $B' = \alpha_* B$, $C' = \alpha_* C$. Denote $D' = \alpha_* D$, then $D' \equiv_S K_{X'} + B'$.

We may run an MMP on $f' : (X', B') \to S$ with scaling of C'. As (X, B) and $(X, B+C)$ satisfy the BCHM condition, (X', B') and $(X', B'+C')$ also satisfy the BCHM condition.

By construction, there exists an effective $\mathbf{R}$-divisor E' whose support is contained in $\llcorner B' \lrcorner$, such that $K_{X'} + B' + C' + E' = K_{X'} + B' + tD' \equiv_S (1 + t)(K_{X'} + B')$. For any extremal ray R in this MMP, we have $((K_{X'} + B') \cdot R) < 0$ and $((K_{X'} + B' + C') \cdot R) \geq 0$, hence $(E \cdot R) < 0$. Therefore, any contracted curve is contained in $\llcorner B' \lrcorner$ and each small contraction is a PL contraction. Here we keep the same symbol to denote strict transforms in the MMP. By the existence of PL flips and the special termination of MMP with scaling, this MMP works and terminates to a minimal model $f'' : X'' \to S$ with a birational map $\beta : (X', B') \dashrightarrow (X'', B'')$ over S.

The composition map $\beta \circ \alpha : X \dashrightarrow X''$ is surjective in codimension 1 and $K_{X''} + B''$ is relatively nef. However, $\beta \circ \alpha$ might not satisfy the negativity for $K_X + B$, so we need to discuss more.

Step 2. Consider the following set:

$$I = \{s \in [0, 1] \mid f : (X, B + sC) \to S \text{ has a minimal model}\}.$$

Note that $1 \in I$, and our goal is to show that $0 \in I$. We will show that if $s \in I$ and $s > 0$, then there exists a sufficiently small $\epsilon > 0$ such that $s' \in I$ if $0 \leq s - s' \leq \epsilon$ by modifying the argument in Step 1.

Take $\alpha_s : (X, B + sC) \dashrightarrow (X'_s, B'_s + sC'_s)$ to be a minimal model of $(X, B + sC)$ with natural morphism $f'_s : X'_s \to S$. As negativity is an open condition, if ϵ is sufficiently small, $K_X + B + s'C$ is negative with respect to α_s.

We can run an MMP on $f'_s : (X'_s, B'_s + s'C'_s) \to S$ with scaling of $(s-s')C'_s$. As before, extremal rays in this MMP intersects negatively on an $\mathbf{R}$-divisor supported in $\llcorner B'_s \lrcorner$. Hence by the existence of PL flips and the special termination of MMP with scaling, this MMP works and terminates to a minimal model $\beta'_s : (X'_s, B'_s + s'C'_s) \dashrightarrow (X''_{s'}, B''_{s'} + s'C''_{s'})$ with morphism $f''_{s'} : X''_{s'} \to S$. The composition map $\beta_{s'} \circ \alpha_s : X \dashrightarrow X''_{s'}$ is surjective in codimension 1 and $K_{X''_s} + B''_s + s'C''_{s'}$ is relatively nef. Moreover, $K_X + B + s'C$ satisfies negativity, hence $s' \in I$.

Step 3. Take $s_0 = \inf I$. Note that $s_0 < 1$. We will show that $s_0 \in I$, then by Step 2, $s_0 = 0 \in I$. In order to show this, firstly we construct a weak minimal model.

Take a strictly decreasing sequence $I \ni s_k \to s_0$, take minimal models $\alpha_k : (X, B + s_k C) \dashrightarrow (X'_k, B'_k + s_k C'_k)$ with morphisms $f'_k : X'_k \to S$.

We can run an MMP on $f'_k : (X'_k, B'_k + s_0 C'_k) \to S$ with scaling of $(s_k - s_0)C'_k$. As before, by the existence of PL flips and the special termination of MMP with scaling, this MMP works and terminates to a minimal model $\beta_k : (X'_k, B'_k + s_0 C'_k) \dashrightarrow (X''_k, B''_k + s_0 C''_k)$ with morphism $f''_k : X''_k \to S$.

The divisors contracted by $\beta_k \circ \alpha_k$ are all irreducible components of D, hence after replacing $\{s_k\}$ by a subsequence, we may assume that the contracted divisors do not depend on k. Then X''_k are all isomorphic in codimension 1. Since $K_{X''_k} + B''_k + s_0 C''_k$ is relatively nef, they are crepant to each other. That is, their pullbacks coincide on a common resolution. Therefore, the discrepancy divisor of $K_X + B + s_0 C$ is independent of k.

Take an arbitrary prime divisor P over X. Denote $a_k \geq 0$ to be the coefficient of P in the discrepancy divisor of $K_X + B + s_k C$ with respect to α_k, and $b_k \geq 0$ to be the coefficient of P in the discrepancy divisor of $K_{X'_k} + B'_k + s_0 C'_k$ with respect to β_k. Denote a_P to be the coefficient of P in the discrepancy divisor of $K_X + B + s_0 C$ with respect to $\beta_k \circ \alpha_k$, then $a_P = \lim_{k \to \infty}(a_k + b_k) \geq 0$.

Therefore, for any k, $\gamma = \beta_k \circ \alpha_k$ gives a weak minimal model $(X''_k, B''_k + s_0 C''_k) = (X'', B'' + s_0 C'')$ of $(X, B + s_0 C)$. Recall that $(X, B + s_0 C)$ satisfies the BCHM condition, in the following argument, after changing the boundary, we may assume that $(X, B + s_0 C)$ is KLT, and hence $(X'', B'' + s_0 C'')$ is also KLT.

Step 4. If $a_P > 0$ holds for any prime divisor P on X contracted by γ, then γ is a minimal model, and $s_0 \in I$, which concludes the proof.

If $a_P = 0$ for some prime divisors contracted by γ, as $(X'', B'' + s_0 C'')$ is KLT, by the following lemma, we can take a "crepant extraction" of these divisors from $(X'', B'' + s_0 C'')$, and this gives a minimal model of $(X, B + s_0 C)$. □

Lemma 3.4.5 (*Crepant extraction*). *Let (X, B) be an n-dimensional quasi-projective DLT pair and P be a discrete valuation on the function field of X. Take a log resolution $f : Y \to (X, B)$ such that the center of P is a prime divisor E_P on Y, write $f^*(K_X + B) = K_Y + B_Y$, suppose that the coefficient of E_P in B_Y is in $[0, 1)$. Moreover, if the coefficient is 0, then we assume that the center of P on X is not contained in $\llcorner B \lrcorner$. Suppose the existence of PL flips in dimension n and the special termination of MMP with scaling in dimension n hold. Then there exists a projective birational morphism $g : (X', B') \to (X, B)$ from a $\mathbf{Q}$-factorial DLT pair such that*

(1) g is crepant: $g^*(K_X + B) = K_{X'} + B'$.

(2) The center of P is a prime divisor on X', and it is the only exceptional divisor of g.

Proof. Note that if we construct a $\mathbf{Q}$-factorial algebraic variety X' satisfying (2), then (1) automatically follows. After slightly modifying B by Lemma 2.1.7, we may assume that (X, B) is KLT while the assumption on the coefficient of E_P is preserved. Take E to be the sum of all exceptional divisors of f except for E_P, take $B'_Y = \max\{B_Y, E\}$. Then (Y, B'_Y) is a DLT pair and $K_Y + B'_Y - f^*(K_X + B)$ is effective with support $E = \llcorner B'_Y \lrcorner$. Hence by the existence of PL flips and the special termination of MMP with scaling, we can run an MMP on (Y, B'_Y) over X with scaling of a general ample divisor, which terminates to a minimal model $g : (X', B') \to X$. Note that $K_{X'} + B' - g^*(K_X + B)$ is effective and exceptional over X, which implies that it is 0 by the negativity lemma. In other words, all divisors in E are contracted by this MMP. On the other hand, since all extremal rays in this MMP intersect negatively along E, only divisors in E can be contracted. So the strict transform of E_P is the only exceptional divisor of g. □

In the end of this section, we show the finiteness of minimal models.

{finiteness of mm}

Theorem 3.4.6 (*Finiteness of minimal models*). *Under the BCHM condition, suppose that the existence of minimal models in dimension n holds, then the finiteness of minimal models in dimension n holds.*

Recall that in Theorem 2.10.3, we showed the finiteness of canonical models assuming the existence of minimal models and canonical models. On the other hand, in Remark 2.10.5 and Example 2.10.7, finiteness of minimal models does not hold in general. But if we assume the BCHM condition, then we can show the finiteness of minimal models, which in fact can be reduced to the finiteness of canonical models.

Proof. Fix $f : X \to S$. Suppose that P is a polytope spanned by effective $\mathbf{R}$-divisors such that for any $B \in P$, $f : (X, B) \to S$ satisfies the BCHM condition. We may assume further that for any $B \in P$, (X, B) is KLT and $B \geq A$ where A is a relatively ample effective $\mathbf{Q}$-divisor.

Take $H_1, \ldots, H_s$ to be relatively ample effective divisors whose classes form a basis of $N^1(X/S)$. Fix a sufficiently small real number $\epsilon > 0$, after changing A, we may assume that $A - \epsilon \sum_i H_i$ is effective and relatively ample. Consider a new polytope

$$Q' := \{B + \sum_i h_i H_i \mid B \in P, -\epsilon \leq h_i \leq \epsilon\}.$$

If taking ϵ is sufficiently small, we may assume that for any $B' \in Q'$, $f : (X, B') \to S$ satisfies the BCHM condition. Consider the polytope

$$Q := \{B' \in Q' \mid [K_X + B'] \in \overline{\mathrm{Eff}}(X/S)\}.$$

Hence by the existence of minimal models and Theorem 2.10.3, there are finitely many canonical models corresponding to Q. In order to finish the proof, we only need to show that every minimal model corresponding to P is a canonical model corresponding to Q.

Take any $B \in P$, and suppose $\alpha : (X, B) \dashrightarrow (Y, C)$ is a minimal model of $f : (X, B) \to S$. Take a relatively ample divisor H_Y on Y, take $H = \alpha_*^{-1} H_Y$. We can write $H \equiv_S \sum d_i H_i$ for some real numbers d_i. Take a sufficiently small real number $\delta > 0$, we may assume that $B + \delta \sum d_i H_i \in Q$ and $\alpha : (X, B + \delta \sum d_i H_i) \dashrightarrow (Y, C + \delta \sum d_i \alpha_* H_i)$ is a minimal model of $f : (X, B + \delta \sum d_i H_i) \to S$ as negativity is an open condition, this is also a canonical model since $K_Y + C + \delta \sum d_i \alpha_* H_i \equiv_S K_Y + C + \delta H_Y$ is ample over S. □

3.5 The non-vainishing theorem

Among a series of theorems deriving geometric consequences from numerical conditions, the non-vanishing theorem is one of the most difficult ones. It was prove in dimension 3 unconditionally ([103], [104]). Under the BCHM condition that the boundary is big, this difficult theorem can be proved by induction on dimensions.

Let us recall the statement of the non-vanishing theorem: $K_X + B$ is relatively pseudo-effective, then there exists an effective **R**-divisor D such that $D \equiv_S K_X + B$.

{NV thm}

Theorem 3.5.1 (*Non-vanishing theorem*)**.** *Under the BCHM condition, suppose that the existence of PL flips in dimension n, the special termination of MMP with scaling in dimension n, the existence and finiteness of minimal models for pairs (X, B) with $K_X + B$ relatively big in dimension n hold, then the non-vanishing theorem in dimension n holds.*

Remark 3.5.2. In the following proof, we actually prove a slightly stronger statement: " there exists an effective **R**-divisor D such that $D \sim_{\mathbf{R}} K_X + B$".

This is because we are going to derive the general result from the statement on generic fibers, but numerical equivalence does not work for this purpose.

On the other hand, by Theorem 3.4.1 and Remark 3.4.2, assuming the existence of PL flips in dimension n and the special termination of MMP with scaling in dimension n, then the non-vanishing up to numerical equivalence is sufficient to show the existence of a minimal model (Y, C). We can make (X, B) satisfying the KLT version BCHM condition, then (Y, C) is KLT and C is big, then by the base point free theorem, $K_Y + C$ is semiample, and in particular, it is $\mathbf{R}$-linearly equivalent to an effective $\mathbf{R}$-divisor. Pulling back to X, we know that $K_X + B$ is $\mathbf{R}$-linearly equivalent to an effective $\mathbf{R}$-divisor.

Proof. Step 0. We may assume that (X, B) is KLT and $B = 3A + E$ is big, where A is a general effective ample $\mathbf{R}$-divisor and E is an effective $\mathbf{R}$-divisor . Until Step 5, we suppose that S is a point.

Take a log resolution $g : X' \to (X, B)$, write $g^*(K_X + B) = K_{X'} + B'$, it suffices to show the theorem for $(X', (B')^+)$. Therefore, from the beginning, we may assume that X is smooth and the support of B is of normal crossing. Also we may assume that A is a $\mathbf{Q}$-divisor and kA is integral for a sufficiently large positive integer k.

Step 1. Consider the divisorial Zariski decomposition $K_X + B = P + N$. Firstly we consider the case $P \equiv 0$. In this case, $K_X + B \equiv N$. From the above remark, we can get the non-vanishing up to $\mathbf{R}$-linear equivalence.

Step 2. In the following, we assume that $P \not\equiv 0$. We will construct a PLT pair (Y, C) by increasing the boundary.

Since $K_X + B$ is pseudo-effective, by Theorem 2.9.8, after replacing k, for any sufficiently large m,

$$\dim H^0(X, \llcorner mk(K_X + B)\lrcorner + kA) > \binom{(k+1)n}{n}.$$

Fix a general smooth point x in X, denote m_x to be the maximal ideal of the local ring of this point, as $\mathrm{length}(\mathcal{O}_{X,x}/m_x^{kn+1}) = \binom{(k+1)n}{n}$, there exists an effective $\mathbf{R}$-divisor $G \sim_{\mathbf{R}} m(K_X + B) + A$ such that $\mathrm{mult}_x G > n$.

Then (X, G) is not LC at x. In fact, if consider the blowing up a x, and denote the exceptional divisor to be E_x, then the coefficient of E_x in the pullback of K_X+G is larger than 1. Take a log resolution $g : Y \to (X, B+G)$ such that E_x is a divisor on Y, take an effective $\mathbf{R}$-divisor F with sufficiently small coefficients whose support is the exceptional locus of g.

Take $B_t = 2A+(1-t/m)A+E+(t/m)G$. Note that $B_0 = B$, $B_m = E+G$, and $K_X + B_t \sim_{\mathbf{R}} (1 + t)(K_X + B)$.

Denote $g^*(K_X + B_t) = K_Y + \bar{C}_t$, take $C'_t = (\bar{C}_t)^+ + F$, $E_t = (\bar{C}_t)^- + F$. Here $^+, ^-$ are positive part and negative part. Then

$$K_Y + C'_t = g^*(K_X + B_t) + E_t.$$

Take the divisorial Zariski decomposition $K_Y + C'_t = P_t + N_t$, and take $C_t = (C'_t - N_t)^+$. By convexity, C_t is continuous for $t \in (0, m)$.

As $B_0 = B$, (Y, C_0) is KLT. On the other hand, as $B_m = E + G$ and x is a general point, the coefficient of E_x in N_m is not greater than that in F. Here we use the fact that

$$K_Y + C'_t = g^*(K_X + B_t) + E_t \equiv (1+t)g^*(K_X + B) + (\bar{C}_t)^- + F$$

and E_x is not in the support of $(\bar{C}_t)^-$. Therefore, (Y, C_m) is not LC. We can consider the *LC threshold*

$$t_0 = \max\{t \mid (Y, C_t) \text{ is LC}\}.$$

Take $C = C_{t_0}$. As the support of C is of normal crossing, (Y, C) is DLT.

By using the part of A contained in B_t to perturb the tie breaking, we may assume that (Y, C) has a unique LC center.

By the construction, $K_Y + C$ is pseudo-effective. Moreover, using the fact that $C \geq g_*^{-1}A$, we may further assume that C contains an ample divisor. Once we can show that $K_Y + C$ is numerically equivalent to an effective $\mathbf{R}$-divisor, then $K_X + B_{t_0}$ is numerically equivalent to an effective $\mathbf{R}$-divisor. Since $K_X + B_{t_0} \sim_{\mathbf{R}} (1+t_0)(K_X + B)$, we can conclude the proof.

Step 3. Replacing (X, B) by the PLT pair (Y, C), we may assume that X is smooth, $B = A + E + Z$ where A is an effective ample $\mathbf{Q}$-divisor, E is an effective $\mathbf{R}$-divisor, and $Z = \llcorner B \lrcorner$ is irreducible. By the sub-adjunction formula, $(K_X + B)|_Z = K_Z + B_Z$. By the construction in Step 2, Z is not contained in the numerically fixed part of $K_X + B$.

Take $\{E_i\}$ to be irreducible components of E, consider the vector $v = \sum e_i E_i$. For a sufficiently small real number $\epsilon > 0$, suppose that $|e_i| \leq \epsilon$, for a sufficiently small real number $t > 0$, take $B_{t,v} = B + t(v + A)$. As ϵ is sufficiently small, we may assume that $v + A$ is ample.

Since $K_X + B_{t,v}$ is big, by the assumption on the existence of minimal models, there exists a minimal model $\alpha_{t,v} : (X, B_{t,v}) \dashrightarrow (Y_{t,v}, C_{t,v})$.

Step 4. We will show that if ϵ and t are sufficiently small, the birational map $\alpha_{t,v}$ induces a birational map $\alpha_Z : Z \dashrightarrow W$ independent of the choice of t, v. Here note that Z is not contained in the numerically fixed part of $K_X + B_{t,v}$, hence is not contracted by $\alpha_{t,v}$.

Fix a sufficiently small $t_1 > 0$, take $0 \leq t < t_1$, consider the polytope

$$V_t = \{B + t'(v + A) \mid v = \sum e_i E_i, |e_i| \leq \epsilon, t \leq t' \leq t_1\}$$

in the linear space of divisors. Fixing $t > 0$, for any $B' \in V_t$, (X, B') is PLT and $K_X + B'$ is big. By the assumption on the existence and finiteness of minimal models, there are finitely many birational maps $\alpha_k : X \dashrightarrow X_k$, such that for any $B_{t',v} \in V_t$, $\alpha_{t',v}$ coincides with one of α_k. So by taking the limit $t \to 0$, there are at most countably many minimal models for boundaries in $V_0 \setminus \{B\}$.

Consider the restrictions of those birational maps on Z. If all $\alpha_k|_Z$ are the same, then we can finish this step. So we may assume that there are at least 2 different $\alpha_k|_Z$. Fix a birational map α_{k_1}. Consider the subset

$$Q_{k_1} = \{B' \in V_0 \mid \alpha_{k_1} \text{ is a weak minimal model of } (X, B')\}.$$

If (X, B) has a weak minimal model, then we can finish the proof. So we may assume that Q_{k_1} does not contain B, which implies that Q_{k_1} is a closed sub-polytope of $V_0 \setminus \{B\}$. Take V'_{k_1} to be the smallest polytope containing Q_{k_1} and B.

Replacing V_0 by V'_{k_1}, we can do the same argument on V'_{k_1} as above. If we could not get the conclusion of this step, then this process does not terminate, and we can get a decreasing sequence of polytopes $V_0 \supset V'_{k_1} \supset V'_{k_2} \supset \ldots$ with vertex B, where we can make proper choice of α_{k_i} in each step, such that $\alpha_{k_{i+1}}|_Z$ is different from $\alpha_{k_i}|_Z$ for each i. By the compactness, we can find a ray L starting from B such that L intersects all Q_{k_i}. Using this line, we can construct an MMP on (X, B) with scaling which consists of an infinite sequence of flips and is not isomorphic in a neighborhood of Z for infinitely many steps, which contradicts to the special termination of MMP with scaling.

Step 5. By using similar argument as in the proof of the base point free theorem, we will show certain extension theorem from Z to X by the vanishing theorem. By the sub-adjunction formula, we have

$$(K_{Y_{t,v}} + C_{t,v})|_W = K_W + C_{W,t,v},$$

denote $\lim_{t\to 0} C_{W,t,v} = C_W$, note that this limit does not depends on v. Since $K_W + C_{W,t,v}$ is nef, $K_W + C_W$ is also nef. Take $\bar{A} = \alpha_{Z*}(A|_Z)$.

Recall that $B = A + E + Z$ and $\{E_i\}$ are irreducible components of E. Consider P to be a sufficiently small rational polytope in the linear space spanned by E_i containing E such that if $E' \in P$, then $(X, A + E' + Z)$ is PLT. As $\bar{A}$ is big,

$$N_W = \{\bar{E}' = \alpha_{Z*}(E'|_Z) \mid E' \in P, K_W + \bar{A} + \bar{E}' \text{ is nef}\}$$

is a rational polytope in the linear space of divisors on W by the cone theorem. Its pullback

$$N = \{E' \in P \mid \bar{E}' = \alpha_{Z*}(E'|_Z) \in N_W\}$$

is a rational polytope containing E.

Take rational points $F_j \in N$ and real numbers $r_j > 0$ such that $\sum r_j = 1$ and $E = \sum r_j F_j$. Correspondingly we have $\mathbf{Q}$-divisor $B_j = A + F_j + Z$, such that $B = \sum r_j B_j$. Take $C_{W,j} = \bar{A} + \alpha_{Z*}(F_j|_Z)$, then $C_W = \sum r_j C_{W,j}$.

Taking $t > 0$ sufficiently small and F_j sufficiently close to E, donote

$$B_j = A + F_j + Z = B + tv_j,$$

we may assume that $B + t(v_j + A)$ satisfies conditions in Steps 3-4. Denote $Y_j = Y_{t,v_j}$ and $(\alpha_{t,v_j})_* B_j = C_j$, note that

$$(K_{Y_j} + C_j)|_W = K_W + C_{W,j}.$$

Denote $A_j = \alpha_{j*} A$, then $K_{Y_j} + C_j + tA_j$ is nef and big.

Take q to be the common multiple of the denominators of coefficients of all F_j, since $K_W + C_{W,j}$ is nef and $C_{W,j}$ is big, by the effective base point free theorem, there exists a positive integer m independent of q, such that $|mq(K_W + C_{W,j})|$ is free. Consider the approximation of coefficients of E by those of F_j, the differences are bounded by order $\frac{1}{q^{1+\delta}}$ for some $\delta > 0$. As $F_j - E = tv_j$, if q is sufficiently large, then we can make tq sufficiently small. Note that

$$\begin{aligned} &mq(K_{Y_j} + C_j) - W \\ &= (mq-1)(K_{Y_j} + C_j + tA_j) + K_{Y_j} + \alpha_{j*}(1 - (mq-1)t)A + F_j), \end{aligned}$$

and we may assume that

$$(Y_j, \alpha_{j*}((1 - (mq-1)t)A + F_j))$$

is KLT, by the vanishing theorem,

$$H^1(Y_j, mq(K_{Y_j} + C_j) - W) = 0.$$

Hence

$$H^0(Y_j, mq(K_{Y_j} + C_j)) \to H^0(W, mq(K_W + C_{W,j}))$$

is surjective. So $H^0(Y_j, mq(K_{Y_j} + C_j)) \neq 0$. Recall that $(Y_j, C_j + tA_j)$ is a minimal model of $(X, B_j + tA)$, take a common resolution $p : X' \to X$ and $q : X' \to Y$, we know that $p^*(K_X + B_j + tA) \geq q^*(K_{Y_j} + C_j + tA_j)$. On the other hand, $p^*A \leq q^*A_j$ by the negativity lemma, hence $p^*(K_X + B_j) \geq q^*(K_{Y_j} + C_j)$. This implies that $H^0(X, mq(K_X + B_j)) \neq 0$. As $B = \sum r_j B_j$, there exists an effective $\mathbf{R}$-divisor D such that $K_X + B \sim_{\mathbf{R}} D$.

Step 6. Finally we consider the case that S is not a point. Restricting to the generic fiber X_η of f, from the above argument, there exists an effective $\mathbf{R}$-divisor D_η such that $K_{X_\eta} + B_\eta \sim_{\mathbf{R}} D_\eta$. That is, there exists real numbers r_i and rational functions h_i on X_η such that, $K_{X_\eta} + B_\eta - D_\eta = \sum r_i \mathrm{div}(h_i)$.

Denote D to be the closure of D_η on X. As h_i are also rational functions on X, $G = K_X + B - D - \sum r_i \mathrm{div}(h_i)$ defines an $\mathbf{R}$-divisor G on X. Note that G does not dominate S, so there exists an effective ample divisor H on S such that $f(\mathrm{Supp}(G)) \subset H$ and $f^*H + G \geq 0$. Hence $K_X + B \sim_{\mathbf{R},S} D + G + f^*H$ which proves the theorem. $\square$

Remark 3.5.3. (1) The non-vanishing theorem is to show the *weak effectivity*(numerically equivalent to an effective divisor) assuming the pseudo-effectivity. At the first glance the difference between effectivity and pseudo-effectivity looks easy, but in fact it is the root of difficulty and fun in the minimal model theory. It assert that some nature of mathematics is hiding in the boundary of the cone of big divisors. This is a statement of the type of the base point free theorem.

(2) If one wants to partially solve some conjectures in the minimal model theory, what immediately comes to mind is, for example, the case that $B = 0$, or the case that $K_X + B$ is big. However, such conditions are not compatible with the inductive argument. On the other hand, the condition that B can be written as $B = A + E$ works very well in the induction.

(3) If B is a $\mathbf{Q}$-divisor, then in the proof we can show that D can be also taken to be a $\mathbf{Q}$-divisor.

3.6 Summary

In summary, by complicated inductive arguments, all the theorems have been proved at the same time. In conclusion, we get the following theorem:

{existence of mm final}

Theorem 3.6.1 (*Existence of minimal models*)**.** *Let (X, B) be a $\mathbf{Q}$-factorial KLT pair and $f : X \to S$ a projective morphism to a quasi-projective variety. Assume the following conditions:*

(1) B is relatively big, that is, there exists a relatively ample $\mathbf{R}$-divisor A and an effective $\mathbf{R}$-divisor E such that $B = A + E$.

(2) $K_X + B$ is relatively pseudo-effective, that is, $[K_X + B] \in \overline{\mathrm{Eff}}(X/S)$.

Then there exists a minimal model of $f : (X, B) \to S$.

This theorem has many important corollaries. Firstly, combining with the base point free theorem, the following corollary directly follows:

Corollary 3.6.2. *Under the assumption of Theorem 3.6.1, assume further that B is a $\mathbf{Q}$-divisor. Then the canonical ring*

$$R(X/S, K_X + B) = \bigoplus_{m=0}^{\infty} f_*(\mathcal{O}_X(\llcorner m(K_X + B) \lrcorner))$$

is a finitely generated graded $\mathbf{O}_S$*-algebra.*

In the above result, the assumption that B is big is not necessary (see Theorem **??**).

We can also conclude the existence of minimal models when the log canonical divisor is big:

{existence of mm log big}

Corollary 3.6.3. *In Theorem 3.6.1, if we replace conditions (1), (2) with the following condition (3), we can get the same conclusion:*

(3) $K_X + B$ *is relatively big.*

Proof. By the assumption, there exists an effective $\mathbf{R}$-divisor B' such that $K_X + B \equiv_S B'$. For a sufficiently small $\epsilon > 0$, $(X, B + \epsilon B')$ is KLT and $B + \epsilon B'$ is relatively big. A minimal model of $f : (X, B + \epsilon B') \to S$ is also a minimal model of $f : (X, B) \to S$. □

Corollary 3.6.4. *Let* X *be a* $\mathbf{Q}$*-factorial normal algebraic variety with terminal singularities and* $f : X \to S$ *a projective morphism to a quasi-projective variety. Assume that* K_X *is relatively big. Then there exists a minimal model of* $f : X \to S$ *with* $\mathbf{Q}$*-factorial terminal singularities.*

Proof. This follows from Corollary 3.6.3. As $B = 0$ and X is terminal, the resulting minimal model is automatically terminal. □

As a special case of the existence of minimal models, we get the following theorem:

{Existence of flips}

Theorem 3.6.5 (*Existence of flips*)**.** *Let* (X, B) *be a* $\mathbf{Q}$*-factorial DLT pair and* $f : X \to S$ *a projective morphism to a quasi-projective variety. Then the flip of any small contraction of* $f : (X, B) \to S$ *always exists.*

If $K_X + B$ is not relatively pseudo-effective, then we can get a Mori fiber space unconditionally:

Theorem 3.6.6 (*Existence of Mori fiber spaces*)**.** *Let* (X, B) *be a* $\mathbf{Q}$*-factorial KLT pair and* $f : X \to S$ *a projective morphism to a quasi-projective variety. Assume that* $K_X + B$ *is not relatively pseudo-effective, that is,* $[K_X + B] \not\in \overline{\mathrm{Eff}}(X/S)$*. Then there exists a birational model of* f *admitting a Mori fiber space structure. More precisely, there exists a* $\mathbf{Q}$*-factorial KLT pair* (Y, C) *over* S *and a birational map* $\alpha : (X, B) \dashrightarrow (Y, C)$ *satisfying the following conditions:*

(1) α *is surjective in codimension* 1*,* $C = \alpha_* B$*.*

(2) If we take a normal algebraic variety Z *with birational projective morphisms* $p : Z \to X$*,* $q : Z \to Y$ *such that* $\alpha = q \circ p^{-1}$*, then* $G = p^*(K_X + B) - q^*(K_Y + C)$ *is effective, and the support of* $p_* G$ *contains all prime divisors contracted by* α*.*

(3) There exists a Mori fiber space $h : Y \to Z$ over S.

Note that we do not assume that B is relatively big.

Proof. Take a sufficiently general relatively ample effective **Q**-divisor H such that $(X, B + H)$ is KLT and $K_X + B + H$ is relatively ample. Take a sufficiently small $\epsilon > 0$ such that $K_X + B + \epsilon H$ is not relatively pseudo-effective. As $B + \epsilon H$ is relatively big, we can run an MMP on $(X, B + \epsilon H)$ with scaling of $(1 - \epsilon)H$, which terminates to a Mori fiber space. It is easy to see that this MMP is also an MMP on $(X, B + H)$. □

Remark 3.6.7. Currently we have proven strong partial results in the minimal model theory under the BCHM condition. It is expected that induction methods will be successful even if we drop the BCHM condition, and all problems in the minimal model theory can be settled in the near future.

As an auxiliary result, the following theorem were already proved in the process of the proof:

Theorem 3.6.8 (*Crepant blowing up* or *Crepant extraction*)**.** *Let (X, B) be a quasi-projective KLT pair. Take a very log resolution $f' : Y' \to X$ and write $(f')^*(K_X + B) = K_{Y'} + C'$. Choose a set of several f'-exceptional divisors on Y with non-negative coefficients in C'. Then there exists a* **Q***-factorial KLT pair (Y, C) and a projective birational morphism $g : (Y, C) \to (X, B)$, such that g is crepant, that is, $g^*(K_X + B) = K_Y + C$, and the set of exceptional divisor of g coincides with the chosen set.*

Proof. Take an effective **R**-divisor F on Y' whose support is the f'-exceptional divisors not contained in the chosen set. If F is sufficiently small, then $(Y, (C')^+ + F)$ is KLT. Then a minimal model of $f' : (Y, (C')^+ + F) \to X$ is the crepant blowing up we want. Here as f' is birational, every **R**-divisor is relatively big. □

As special cases, we get "**Q**-factorialization" and"**Q**-factorial terminalization":

{Q-factorialization}

Corollary 3.6.9 (**Q**-*factorialization*)**.** *Let (X, B) be a quasi-projective KLT pair. Then there exists a* **Q***-factorialization of X, that is, there exists a* **Q***-factorial normal algebraic variety Y and a projective birational morphism $g : Y \to X$ which is isomorphic in codimension* 1*.*

Proof. Take the chosen set to be the empty set. □

{Q-factorial terminalizat

Corollary 3.6.10 (**Q**-*factorial terminalization*)**.** *Let (X, B) be a quasi-projective KLT pair. Then there exists a* **Q***-factorial terminalization of (X, B), that is, there exists a* **Q***-factorial terminal pair (Y, C) and a projective birational crepant morphism $g : (Y, C) \to (X, B)$.*

Proof. Take the chosen set to be the set of all exceptional divisors with non-negative coefficients. □

Note that the **Q**-factorial terminalizations are maximal among all crepant blowing ups. In particular, if $B = 0$ and X has canonical singularities, then this is the crepant blowing up considered in [64], which is applied to proved the termination of flips inductively in [70].

Example 3.6.11. As a toric variety is $\overline{KLT}$, it admits a **Q**-factorialization.

Take a toric variety (X, B). That is, $T \subset X$ is a T-equivariant open immersion into a normal algebraic variety with a T-action where T is an algebraic torus, and $B = X \setminus T$ is a reduced divisor. Take $\Sigma = \{\sigma\}$ to be the corresponding fan. Take Σ' to be a fan with same vertices as Σ in which each σ is subdivided into simplicial cones. Take (X', B') to be the corresponding toric variety. In this case, X' is **Q**-factorial and there is a natural birational morphism $f : X' \to X$ isomorphic in codimension 1 such that $f^*(K_X + B) = K_{X'} + B'$. The choice of subdivision is not necessarily unique. If taking the subdivision appropriately, then f is projective and we get a **Q**-factorialization.

Similarly, we can prove the following result, which is useful for generalizing statements for KLT or DLT pairs to LC pairs:

{DLT blowup}

Corollary 3.6.12 (*DLT blow-up*)**.** *Let (X, B) be a quasi-projective LC pair. Then there exists a **Q**-factorial DLT pair (Y, C) and a projective birational crepant morphism $f : (Y, C) \to (X, B)$ such that exceptional divisors of f are contained in $\llcorner C \lrcorner$.*

Proof. Take a log resolution $f' : Y' \to (X, B)$, write $(f')^*(K_X + B) = K_{Y'} + C'$. As (X, B) is LC, the coefficients of C' are at most 1. Write $C'' = (f')^{-1}_* B + \mathrm{Exc}(f')$. Then $C'' - C'$ is effective and its support is the union of all exceptional divisors of f' with coefficients less than 1 in C'.

Take a general relatively ample effective divisor A and a sufficiently small real number $t > 0$, we can apply MMP to $f : (Y', C'' + tA) \to X$. As $K_{Y'} + C'' + tA = (f')^*(K_X + B) + (C'' - C') + tA$, if t is sufficiently small, the support of the numerically fixed part of $K_{Y'} + C'' + tA$ over X coincides with the support of $C'' - C'$. Therefore, the minimal model of $f : (Y', C'' + tA) \to X$ contracts all divisors in $C'' - C'$, which is a DLT blow-up. □

For a DLT blow-up, we can blowing up LC centers to get another DLT blow-up. So in general DLT blow-ups do not have maximality as **Q**-factorial terminalizations.

3.7 Algebraic fiber spaces

In this section, we introduce the weak semi-stable reduction theorem ([1]) and the semi-positivity theorem ([55]) for algebraic fiber spaces. We will just give outlines without proof. There is a relatively simple proof for the latter one [81].

Algebraic fiber spaces can be viewed as relative version of algebraic varieties. Birational equivalences between algebraic varieties are given by their function fields. The function fields of algebraic varieties are regular extensions of the base field. So birational equivalences between algebraic fiber spaces are given by regular extensions of function fields.

The weak semi-stable reduction theorem can be viewed as the desingularization theorem for algebraic fiber spaces. The semi-positivity theorem is an important consequence of Hodge theory. Both theorems are proved when the base field is of characteristic 0, and in positive characteristics the latter one has counterexample.

3.7.1 Algebraic fiber spaces and toroidal geometry

A finite extension L/K of fields is a *regular extension* if the following conditions are satisfied:

(1) (Separability) There exists a transcendence basis $t_1, \dots, t_n$ over K such that L is a separable algebraic extension of $K(t_1, \dots, t_n)$.

(2) (Relative algebraic closedness) The set of elements in L algebraic over K is K.

If K is an algebraically closed field, then the above 2 conditions automatically hold. In this case, there exists an algebraic variety X over K such that $L = K(X)$.

If K is a regular extension of an algebraically closed field k, then there exist algebraic varieties X, Y over k such that $L = k(X), K = k(Y)$ and a morphism $f : X \to Y$ satisfying the following conditions:

(1) f is dominant, that is, the generic point of X is mapped to that of Y.

(2) The geometric generic fiber of f is irreducible and reduced.

The morphism $f : X \to Y$ above is called an *algebraic fiber space*. As in this book we mainly interested in projective algebraic varieties, X and Y are usually assumed to be projective. We often work over base field of characteristic 0, in which case the separability of field extensions automatically holds.

Next we explain the language of toroidal geometry. A toroidal variety is a pair of variety and divisor locally isomorphic to a toric variety. Here "locally" means in the classical analytic topology or étale topology, and the base field is the complex number field.

A pair (X, B) consisting of a normal algebraic variety and a reduced divisor is called a *toroidal variety*, if for each point $x_i \in X$, there exists an analytic neighborhood U_i of x_i, a toric variety Y_i, a point $y_i \in Y_i$, and an analytic neighborhood V_i of $y_i \in Y_i$, such that there is an analytic isomorphism $(U_i, B \cap U_i, x_i) \cong (V_i, C_i \cap V_i, y_i)$. Here $C_i = Y_i \setminus T_i$ is the complement of the torus on Y_i. Here in addition we assume that irreducible components of B are normal, that is, we only consider toroidal varieties *without self-intersection*.

A pair (X, B) is called a *smooth toroidal variety*, if X is a smooth algebraic variety and B is a normal crossing divisor. A pair (X, B) is called a *quasi-smooth toroidal variety*, if locally it is a quotient of a smooth toroidal variety by a finite abelian group: for any point $x_i \in X$, there exists an analytic neighborhood $x_i \in U_i$, and a finite abelian group acting diagonally on an analytic neighborhood $\tilde{V}_i$ of a point $\tilde{y}_i$ on $\mathbf{C}^n$, such that there is an analytic isomorphism $(U_i, B \cap U_i, x_i) \cong (\tilde{V}_i/G_i, (\tilde{C} \cap \tilde{V}_i)/G_i, y_i)$. Here $\tilde{C}$ is the union of coordinate hyperplanes, and y_i is the image of $\tilde{y}_i$.

Remark 3.7.1. Similar to toric varieties, a toroidal variety is also associated with a fan ([85]). A toric variety is determined by its fan, the information of analytically local structure and global glueing is determined by the fan.

A toroidal variety is analytically locally isomorphic to a $\overline{KLT}$ pair, so it is also $\overline{KLT}$, and admits a $\mathbf{Q}$-factorialization. A toroidal variety is $\mathbf{Q}$-factorial if and only if the fan is simplicial, if and only if it is quasi-smooth.

For a toric variety (X, B), the sheaf $\Omega^1_X(\log B)$ of all *logaritmic differential forms*, that is, rational differential forms on X with at most logarithmic poles along B, is a locally free sheaf of rank $n = \dim X$. In fact, the extension of regular differential forms dz_i/z_i $(i = 1, \ldots, n)$ on the toruc $T = X \setminus B$ form a basis. Here z_i $(i = 1, \ldots, n)$ are coordinates of T. Hence for a toroidal variety (X, B), $\Omega^1_X(\log B)$ is also locally free.

Take $\Omega^{\bullet}_X(\log B)$ to be the wedge product algebra of $\Omega^1_X(\log B)$. Using the exterior derivative d on logaritmic differential forms, we can define the *log De Rham complex*

$$\begin{aligned}&\Omega^{\bullet}_X(\log B)\\ &= \{0 \to \mathcal{O}_X \to \Omega^1_X(\log B) \to \Omega^2_X(\log B) \to \cdots \to \Omega^n_X(\log B) \to 0\}.\end{aligned}$$

A dominant morphism $f : (X, B) \to (Y, C)$ between toroidal varieties is called a *toroidal morphism* if locally it is isomorphic to a toric morphism. If (X, B) and (Y, C) are quasi-smooth, this is equivalent to the following: for any point $x_i \in X$, there exist analytic neighborhoods $x_i \in U_i$and $y_i = f(x_i) \in U'_i$, and finite morphisms from open subsets of affine spaces π_i :

$\tilde{V}_i \to U_i$ and $\pi'_i : \tilde{V}'_i \to U'_i$ such that f is induced by $f_i : \tilde{V}_i \to \tilde{V}'_i$, where we may write

$$f_i^* w_j = \prod z_k^{c_{jk}}$$

for coordinates $(z_1, \dots, z_n)$ and $(w_1, \dots, w_m)$. Here $n = \dim X$, $m = \dim Y$, $\{c_{jk}\}$ are non-negative integers.

For a toroidal morphism $f : (X, B) \to (Y, C)$, the sheaf of *relative logarithmic differential forms* $\Omega^1_{X/Y}(\log)$ is defined by

$$\Omega^1_{X/Y}(\log) = \Omega^1_X(\log B)/f^*\Omega^1_Y(\log C).$$

It is a locally free sheaf on X of rank $\dim X - \dim Y$. In particular, if f is finite, then $f^*\Omega^1_Y(\log C) \cong \Omega^1_X(\log B)$ and $\Omega^1_{X/Y}(\log) \cong 0$.

We can similarly define the *relative log De Rham complex* $\Omega^\bullet_{X/Y}(\log)$. Denote $d = \dim X - \dim Y$, then

$$\Omega^d_{X/Y}(\log) \cong \mathcal{O}_X(K_X + B - f^*(K_Y + C)).$$

This is denoted by $\omega_{X/Y}(\log)$ and called the *relative log canonical sheaf*.

3.7.2 The weak semi-stable reduction theorem and the semi-positivity theorem

Assume that the base field is of characteristic 0.

The disingularization theorem is a fundamental theorem in birational geometry for algebraic varieties, while the “weak semi-stable reduction theorem” is a fundamental theorem in birational geometry for algebraic fiber spaces:

Theorem 3.7.2 (*Weak semistable reduction theorem* [1]). *Let $f_0 : X_0 \to Y_0$ be a surjective morphism between projective varieties with geometrically integral generic fiber, and $Z \subset X_0$ a closed proper subset. Then we can construct the following algebraic fiber space models:*

(1) **Well prepared model**: *There exists a quasi-smooth projective toroidal variety (X_1, B_1), a smooth projective toroidal variety (Y_1, C_1), a morphism $f_1 : X_1 \to Y_1$, and birational morphisms $g_1 : X_1 \to X_0$, $h_1 : Y_1 \to Y_0$ such that $f_0 \circ g_1 = h_1 \circ f_1$ with the following properties:*

(1-1) $g_1^{-1}(Z) \subset B_1$.

(1-2) $f_1 : (X_1, B_1) \to (Y_1, C_1)$ *is toroidal.*

(1-3) f_1 *is* equi-dimensional, *that is, every geometric fiber is of dimension* $\dim X_0 - \dim Y_0$.

(2) **Weakly semistable model**: *There exists a quasi-smooth projective toroidal variety* (X_2, B_2), *a smooth projective toroidal variety* (Y_2, C_2), *a morphism* $f_2 : X_2 \to Y_2$, *a Galois finite morphism* $h_2 : Y_2 \to Y_1$, *and a projective birational morphism* $\mu : X_2 \to (X_1 \times_{Y_1} Y_2)^\nu$ *isomorphic in codimension* 1 *with the following properties:*

(2-1) $g_2^{-1}(B_1) = B_2$, $h_2^{-1}(C_1) = C_2$.

(2-2) $f_2 : (X_2, B_2) \to (Y_2, C_2)$ *is toroidal.*

(2-3) f_2 *is equi-dimensional, and every geometric fiber is reduced.*

Here $(X_1 \times_{Y_1} Y_2)^\nu$ *is the normalization and* $g_2 : X_2 \to X_1$ *is the induced morphism. Moreover, by adding some reduced divisor to* C_1 *and replace* B_1 *accordingly, we may assume that* $g_2 : (X_2, B_2) \to (X_1, B_1)$ *and* $h_2 : (Y_2, C_2) \to (Y_1, C_1)$ *are toroidal.*

Remark 3.7.3. (1) A well prepared model is a birational model, but a weakly semistable model is not as there is a base change. The birational morphism μ is a $\mathbf{Q}$-factorialization.

(2) The reason that a weakly semistable model is called "weak" is that the ambient space X_2 is not necessarily smooth. If the base space Y_0 is of dimension 1, then there exists a semistable model in which X_2 is smooth. However, $X_2 \to (X_1 \times_{Y_1} Y_2)^\nu$ is not isomorphic in codimension 1, but a resolution of singularities ([85]). The base change h_2 is constructed by using Theorem 1.8.2.

A locally free sheaf F in a projective algebraic variety X is called *numerically semipositive* or *nef* if the tautological quotient invertible sheaf $\mathcal{O}_{\mathbf{P}_X(F)}(1)$ on $\mathbf{P}_X(F)$ is nef. The following semipositivity theorem represents the geometric property of algebraic fiber spaces:

Theorem 3.7.4 (*Semipositivity theorem*, [55])**.** *For a well prepared algebraic fiber space* $f : (X, B) \to (Y, C)$, *the following properties hold:*

(1) *For any integers* p, q, $R^q f_*(\Omega^p_{X/Y}(\log))$ *is a locally free sheaf on* Y.

(2) *For any integer* q, $R^q f_*(\omega_{X/Y}(\log))$ *is numerically semipositive.*

This result can be generalized by using the covering trick:

{subadjunctionTh2}

Theorem 3.7.5 ([75, Theorem 2])**.** *Let* $f : (X, \bar{B}) \to (Y, \bar{C})$ *be a well prepared algebraic fiber space and* B *an effective* $\mathbf{Q}$*-divisor whose support is contained in* $\bar{B}$ *with coefficients in* $[0, 1)$. *Assume that* $\kappa(X_y, (K_X + B)|_{X_y}) = 0$ *for a general fiber* X_y *of* f. *Write* $\bar{B} = \sum B_i$, $\bar{C} = \sum C_j$ *into irreducible components, write* $B = \sum b_i B_i$.

Assume that there exists a positive integer m *and an integral effective divisor* D *on* X *satisfying the following conditions, which determines effective* $\mathbf{Q}$*-divisors* M *and* C *on* Y*:*

1. *$m(K_X+B)$ is Cartier, $D \in |m(K_X+B)|$, the support of D is contained in $\bar{B}$. Write $D = \sum d_i B_i$.*

2. *M is the largest $\mathbf{Q}$-divisor on Y satisfying $f^*M \leq D$. Write $M = \sum m_j C_j$.*

3. *Take $B^0 := B - D/m + f^*(M/m) = \sum b_i^0 B_i$, $f^*C_j = \sum b_{ij} B_i$, and*

$$c_j = \max_i \{(b_i^0 + b_{ij} - 1)/b_{ij} \mid f(B_i) = C_j\}.$$

Take $C = \sum c_j C_j$.

Then (Y, C) is KLT and $L := M/m - (K_Y + C)$ is nef.

Proof. Step 1. From the construction, $K_X + B^0 \sim_{\mathbf{Q}} f^*M/m \sim_{\mathbf{Q}} f^*(L + K_Y + C)$. Take $B^1 = B^0 - f^*C$, then $K_X + B^1 \sim_{\mathbf{Q}} f^*(L + K_Y)$.

C is the smallest $\mathbf{Q}$-divisor satisfying $f^*(\bar{C} - C) \leq \bar{B} - B^0$. Also

$$\begin{aligned} f^*L &\sim_{\mathbf{Q}} K_X + B^0 - f^*(K_Y + C) \\ &= K_X + \bar{B} - f^*(K_Y + \bar{C}) - (\bar{B} - B^0) + f^*(\bar{C} - C). \end{aligned}$$

Therefore, L is the largest $\mathbf{Q}$-divisor satisfying

$$f^*L \leq K_X + \bar{B} - f^*(K_Y + \bar{C}).$$

□

3.8 The finite generation theorem

In this section, we prove the main theorem of this book: the finite generation theorem, that is, the canonical ring of any smooth algebraic variety is finitely generated. This can be reduced to the general type case as in BCHM using the semi-positivity theorem after simplifying the situation by the weak semi-stable reduction theorem. Here slightly generally, we introduce the proof for KLT pairs ([30]).

3.9 Generalizations of the minimal model theory

So far in this book, we established the minimal model theory for algebraic varieties over algebraically closed fields of characteristic zero. This result can be easily generalized to algebraic varieties admitting finite group actions or over algebraically non-closed fields by modifying certain results appropriately. Let us check these one by one.

3.9.1 The equivariant minimal model theory

Consider a pair (X, B) with morphism $f : X \to S$ admitting a finite group G action. That is, G acts on X, S, f is G-equivariant, and B is a G-invariant $\mathbf{R}$-divisor.

3.9.2 The MMP over algebraically non-sclosed fields

Consider the generalization to algebraically non-closed fields.

3.10 Remaining problems

3.10.1 The abundance conjecture

3.10.2 Case of numerical Kodaira dimension zero

3.10.3 Generalization to positive characteristics

3.11 Related topics

3.11.1 Boundedness results

3.11.2 Minimal log discrepancies

3.11.3 The Sarkisov program

3.11.4 Rationally connected varieties

3.11.5 The category of smooth algebraic varieties

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