代数几何1
期中考试试卷

注意事项：

1. 本次考试为**闭卷考试**，共5道题目，满分100分。
2. 考试中不得使用参考资料，笔记，可以使用电脑阅读试卷题目，但不可以使用电脑阅读或者搜索考试相关资料。
3. 答题时间**为2021年10月27日（周四）10点至11月3日（周四）10点**之间任何**连续的4个小时**。
4. 本次考试由考生**自行决定**考试时间地点。考试记时从考生阅读考试题目开始。考生需要在**2021年11月3日上课**时提交试题解答。过期未交则本次考试成绩为0。
5. **同时提交平时作业（30道题）**。扫描可以表明完成题目数量的一页纸，将扫描件以email形式发给助教。**Email 题目格式： 学号 姓名 XX道代数几何作业题目。**
6. 本次考试无人现场监督。考生在考试过程中应遵守北京大学所有关于闭卷考试的规定。如有违反，本次考试成绩为0。
7. 请考生**将答题纸装订完整，解答誊写清楚**，如有遗失或者解答无法辨认的情况，后果自负。考生所提交的答案以任课教师和助教所见纸质解答版本为准。
8. **如果不能做出一般情况下问题的解答，可以适当增加合理条件。如果只有思路而不能完整解答，可以明确写出思路及遇到困难。在此情况下会酌情给分。如果认为题目有误，可以简要说明题目中的错误，更改题目条件并解答。**
9. **如果发生在关键步骤以“显然有”等类似词语试图蒙混过关的情况，或者经批卷老师认定所写内容只是罗列结果而且与问题不相关的情况，相关题目可能会倒扣分，题目最终得分不排除为负分的可能。**
10. **原则上本次考试所有题目均可以使用课本的定理解决。如果需要使用课本没有的代数几何结果，需要给予证明。如果需要使用课本没有的交换代数结果，需要完整陈述该结果。**

QUIZ 1: ALGEBRAIC GEOMETRY I, FALL 2022

To be finished in consecutive 4 hours.

Problem 1

(1) (10 points) Let X be a k-scheme. Prove

$$\begin{aligned}&\mathrm{Hom}_{\text{k-Sch}}(\mathrm{Spec}\ k[\epsilon]/\epsilon^2, X)\\ \cong&\{(x,v)|x\in X, \kappa(x)\cong k, v\in Hom_k(m/m^2,k)\},\\ &\text{where } m\subset \mathcal{O}_{x,X} \text{ is the maximal ideal}, \kappa(x)=\mathcal{O}_{x,X}/m.\end{aligned}$$

(2) (5 points) Prove that this description uniquely characterizes the scheme Spec $k[\epsilon]/\epsilon^2$ among all k-schemes.

Problem 2

(1) (10 points) Let $X = \mathrm{Spec}\ A$ be an affine scheme and $Z \subset X$ be a closed subset. Prove that for any finite number of points $x_1, \dots, x_n$ (not necessarily closed), there is an element f of A such that $D(f)$ is disjoint from Z and contains all the points $x_1, \dots, x_n$.

(2) (5 points) Translate this into a statement about rings and ideals.

Hint: use induction. Even though the geometric form and algebraic form are completely equivalent to each other, you are encouraged to think and prove this in the geometric form.

The algebraic theorem is usually called **prime avoidance**. **If you want to reduce question 1 to this theorem, you have to prove this algebraic theorem as well.**

Problem 3 (20 points) Let X, Y be schemes over S. Assume that X is reduced, Y is separated over S. Let U be an open dense subset of X. Prove that $Hom_S(X,Y) \to Hom_S(U,Y)$ is injective.

Problem 4 (20 points) Let $f : X \to Y$ be a generically finite, dominant, finite type morphism between integral schemes. Prove that there is an open subscheme $V \subset Y$ such that $f : f^{-1}(V) \to V$ is a finite morphism.

Problem 5 Let L/K be a finite field extension and X a scheme over L. Define a functor

$$Res_{L/K}(X) : \{\text{finite type scheme over } K\}^{\mathrm{op}} \to \{set\}$$

sending a finite type affine K-scheme T to $\mathrm{Hom}_{L-\mathrm{scheme}}(T \times_{\mathrm{Spec}\ K} \mathrm{Spec}\ L, X)$ and morphisms as you could imagine.

(1) (10 points) Prove that if $X = \mathbb{A}^1_L$, then this functor is representable.

(2) (10 points) Prove that if X is a finite type affine L-scheme, $Res_{L/K}(X)$ is representable.

(3) (10 points) Let X be a group scheme over L (i.e. a group object in the category of L schemes, or an L- scheme such that all the group operations are L morphisms). Prove that if $Res_{L/K}(X)$ is representable, it is a group scheme over K. If $X = SL(n)_L$, the special linear group scheme over L. What is the Zariski tangent space of $Res_{L/K}(X)$ at the identity?