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Irregular varieties with geometric genus one, theta divisors, and fake tori

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We study the Albanese image of a compact Kähler manifold whose geometric genus is one. In particular, we prove that if the Albanese map is not surjective, then the manifold maps surjectively onto an ample divisor in some abelian variety, and in many cases the ample divisor is a theta divisor. With a further natural assumption on the topology of the manifold, we prove that the manifold is an algebraic fiber space over a genus two curve. Finally we apply these results to study the geometry of a compact Kähler manifold which has the same Hodge numbers as those of an abelian variety of the same dimension.

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1. Introduction

Kawamata proved in [9] that if X is a smooth projective variety with Kodaira dimension $\kappa(X) = 0$, then the Albanese morphism $a_X : X \to A_X$ is an algebraic fiber space. An effective version of this result was obtained in [7]. For instance, the author proved that if $p_g(X) = P_2(X) = 1$, a_X is an algebraic fiber space. Pareschi, Popa and Schnell recently prove the same criterion for compact Kähler manifolds in [13].

On the other hand, if we only assume that $p_g(X) = 1$, a_X is not necessarily surjective. In this article we will show that, if $p_g(X) = 1$ and a_X is not surjective, the Albanese image is closely related to the geometry of theta divisors.

Theorem 1.1. *Let X be a compact Kähler manifold with $p_g(X) = 1$. Then*

(1) $\dim a_X(X) \geq \frac{1}{2} \dim A_X$*;*
(2) *a_X is not surjective if and only if there exists a quotient map of abelian varieties $A_X \to B$ such that the image of the induced map $f\colon X \to B$ is an ample divisor D of B.*

In the statement of (2), the quotient $A_X \to B$ is deduced from generic vanishing theory of Green–Lazarsfeld. The dual $\operatorname{Pic}^0 B \subset \operatorname{Pic}^0 A_X$ is actually the neutral component of the intersection of certain cohomological support loci of K_X.

In a special case when A_X is simple, we have:

Theorem 1.2. *Let X be a compact Kähler manifold with $p_g(X) = 1$. Assume that a_X is not surjective and A_X is simple. Then $a_X(X) := D$ is an ample divisor of A_X. Moreover, if D is smooth in codimension 1, then D is a theta divisor of A_X and a_X is a fibration onto D.*

One ingredient of the proof of Theorem 1.1 is the decomposition theorem introduced in [3] and proved in general by Pareschi, Popa, and Schnell in [13]. By the decomposition theorem, the "positive" part of $a_{X*}\omega_X$ comes from algebraic varieties and this allows us to reduce the statement to the algebraic setting.

Hacon and Pardini proved in [6] that $\dim a_X(X) \geq \frac{1}{2} \dim A_X$ if X is of maximal Albanese dimension and $p_g(X) = 1$. Hence Theorem 1.1 (1) is a natural generalization of Hacon and Pardini's theorem. The "if" part of (2) is clear. If there exists a surjective morphism from $X \to D$, the induced morphism $g : X \to D \hookrightarrow B$ factors through $a_X : X \to A_X$. Then a_X is not surjective. The "only if" part can then be proved using the idea in Pareschi's characterization of theta divisors (see [11]). In Section 3 and 4, we will see much more precise structures of $a_X(X)$ and why it should be related to theta divisors.

With a further assumption on the second Betti cohomology, we have a very strong conclusion:

Theorem 1.3. *Let X be a compact Kähler manifold with $p_g(X) = 1$. Then the pull-back map $a_X^* : H^2(A_X, \mathbb{Q}) \to H^2(X, \mathbb{Q})$ is not injective if and only if there exists a fibration $\varphi : X \to C$ to a smooth projective curve C of genus 2.*

The "if" part is again clear. The fibration f induces a fibration $A_X \to JC$. Since C is a curve, $H^2(JC, \mathbb{Q}) \to H^2(C, \mathbb{Q})$ is not injective. Hence $H^2(A_X, \mathbb{Q}) \to H^2(X, \mathbb{Q})$ is also not injective. The genus 2 curve appears because the assumption on the cohomology implies that the Lefschetz hyperplane theorem fails in this case, which forces the divisor D in Theorem 1.1 to admit a fibration onto a genus 2 curve.

A more careful analysis shows the following.

Corollary 1.4. *Let X be a compact Kähler manifold with $p_g(X) = 1$. Then the de Rham fundamental group $\pi_1(X) \otimes \mathbb{Q}$ of X is isomorphic to a product of $\mathbb{Q}^{2r} \times (\pi_1(C) \otimes \mathbb{Q})^i$, where C is a smooth curve of genus 2. To be more precise,* $\dim(Ker(H^2(A_X, \mathbb{Q}) = \Lambda^2 H^1(X, \mathbb{Q}) \to H^2(X, \mathbb{Q})))$ *is divisible by 5 and the number i appearing above is* $\frac{1}{5}\dim(Ker(H^2(A_X, \mathbb{Q}) = \Lambda^2 H^1(X, \mathbb{Q}) \to H^2(X, \mathbb{Q}))), 2r + 2i = b_1(X)$*. The number i is less or equal to the codimension of the the Albanese image of X.*

The de Rham fundamental group is the $\mathbb{Q}$-unipotent completion of the topological fundamental group (cf. [1]). Campana showed that a resolution of singularities of the Albanese image of X computes the de Rham fundamental group. In Corollary 1.4, we actually show that the Albanese image of X can be decomposed, in some sense, to a product of sub-torus of A_X, some smooth projective curves of genus 2, and some ample divisors of dimension ≥ 2 of abelian subvarieties of A_X. Hence we can read off from the de Rham fundamental group the numbers of factors of genus 2 curves and the number 5 comes from the dimension of kernels $H^2(JC, \mathbb{Q}) \to H^2(C, \mathbb{Q})$, where C is a smooth projective curve of genus 2.

The motivation to study the Albanese image of irregular varieties with geometric genus one comes from an explicit geometry question.

Catanese showed that a compact Kähler manifold, whose integral cohomology ring is isomorphic to that of a torus, is actually a complex torus.

In [4], the authors study projective varieties X with mild singularities, whose rational cohomology rings are isomorphic to those of complex tori. These varieties are called rational cohomology tori. The Albanese morphism of a rational cohomology torus is finite and is often an abelian cover of the Albanese variety.

It is then natural to ask what can we say about the general structure of X if we further loosen the condition on cohomology rings. The condition that $\dim H^i(X, \mathbb{Q}) = \dim H^i(A_X, \mathbb{Q})$ is too weak to say anything interesting. Indeed, by blowing-up subvarieties on $\mathbb{P}^m$-bundles over curves, we can construct many varieties verifying this condition.

On the other hand, Betti cohomology of smooth projective varieties carries Hodge structures, which usually inherit information about the complex structure of X. John Christian Ottem asked:

Question 1.5. Let X be a compact Kähler manifold. Assume that $h^{p,q}(X) = h^{p,q}(A_X)$ for all p and q. Is X a rational cohomology torus?

Note that the above question is equivalent to ask whether the Albanese morphism a_X is generically finite under the assumption of Hodge numbers. If so, the pull-back $a_X^* : H^*(A_X, \mathbb{Q}) \to H^*(X, \mathbb{Q})$ would be an isomorphism and hence X is a rational cohomology torus.

The answer to Ottem's question is negative. A counter-example is described in [4, Example 1.7], which is an elliptic curve fibration over a genus 2 curve. We will see that, despite the counterexamples, there are strong restrictions on the structure of $a_X : X \to A_X$.

Definition 1.6. Let X be a compact Kähler manifold. We say that X is a fake torus if the Hodge numbers of X are the same as those of a complex torus of the same dimension and the Albanese morphism a_X is not generically finite.

The following results are direct applications of Theorem 1.3.

Corollary 1.7. *Let X be a fake torus. There exists a fibration $f : X \to C$ to a smooth projective curve C of genus 2. In particular, the fundamental group of a fake torus is non-abelian.*

Proof. By definition of fake torus, $p_g(X) = 1$ and a_X is not surjective. Hence any Kähler class on X does not come from A_X. Moreover, since $\dim H^2(A_X, \mathbb{Q}) = \dim H^2(X, \mathbb{Q})$, the pull-back $a_X^* : H^2(A_X, \mathbb{C}) \to H^2(X, \mathbb{C})$ is not injective. Then Theorem 1.3 implies that there exists a fibration $f : X \to C$. Then we have a surjective map $\pi_1(X) \twoheadrightarrow \pi_1(C)$. Hence $\pi_1(X)$ is not abelian. □

Corollary 1.8. *Let X be a compact Kähler manifold, whose Hodge numbers are the same as those of a complex torus of the same dimension. Then either $\pi_1(X) \otimes \mathbb{Q} \simeq \mathbb{Q}^{2n}$ and X is a rational cohomology torus or $\dim_{\mathbb{Q}} \pi_1(X) \otimes \mathbb{Q} = \infty$ and X is a fake torus.*

Proof. If a_X is surjective, then $a_X^* : H^\cdot(A_X, \mathbb{Q}) \to H^\cdot(X, \mathbb{Q})$ is injective and hence is an isomorphism by assumption on Hodge numbers. Then X is a rational cohomology torus. Otherwise, X is a fake torus by definition.

When a_X is surjective, by [1, Corollary 2.18], $\pi_1(X) \otimes \mathbb{Q} \simeq \pi_1(A_X) \otimes \mathbb{Q} = \mathbb{Q}^{2n}$. When a_X is not surjective, X is a fake torus. Then by the above corollary, we have a surjective map $\pi_1(X) \otimes \mathbb{Q} \twoheadrightarrow \pi_1(C) \otimes \mathbb{Q}$, where C is a smooth projective curve of genus 2. Hence $\dim_{\mathbb{Q}} \pi_1(X) \otimes \mathbb{Q} = \infty$. □

We have a good understanding on the structure of $a_X(X)$ for a fake torus X, thanks to the theory of generic vanishing. However, the fiber of a_X is poorly understood. That is the reason that we don't have a picture of the general structure of fake tori. Nevertheless, when $\dim a_X(X) = \dim X - 1$, we have the following general result.

Theorem 1.9. *Let X be a fake torus of dimension n. If $\dim a_X(X) = n - 1$, then X is not of general type.*

Moreover, we can also describe explicitly fake tori in low dimensions. Here is the result in dimension two.

Proposition 1.10. *Let X be a fake torus of dimension 2. Then X is a minimal projective surface with $\kappa(X) = 1$. Furthermore, there exists a finite abelian group G acting faithfully on an elliptic curve E and on a smooth projective curve D of genus ≥ 3 such that $E/G \simeq \mathbb{P}^1$, $D/G = C$ is a smooth curve of genus 2, and X is isomorphic to the diagonal quotient $(D \times E)/G$.*

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2. Notations and preliminaries

2.1. Subvarieties of general type

A compact Kähler manifold is always connected and a variety is always supposed to be reduced and irreducible.

A subvariety of a torus is called of general type if any of its desingularization is a smooth projective variety of general type. Ueno ([20, Theorem 10.9]) proved that a subvariety of a complex torus is not of general type if and only if it is fibred by sub-torus. More precisely, given a subvariety Z of a complex torus B. Let K be the maximal subtorus of B such that $K + Z = Z$ and denoted $B^\flat = B/K$. Then there is $Z^\flat \subset B^\flat$ such that $Z^\flat$ is of general type and $Z \to Z^\flat$ is fibred by K. We call $Z^\flat$ (resp. $Z^\flat \subset B^\flat$) the *κ-reduction of Z* (resp. of $Z \subset B$). We call K the κ-kernel of Z. Notice that if Z is of general type, then clearly $Z^\flat = Z$. Hence one has $(Z^\flat)^\flat = Z^\flat$ in general.

Let X be a compact Kähler manifold. We denote by $Y \subset A_X$ the image of the Albanese morphism of X. In sequel, we will fix the following notation:

$$
\begin{tikzcd}
X \arrow[d, "g"] \arrow[dr, "a_X"] \arrow[dd, bend right=60, "f"'] & \\
Y \arrow[r, hook] \arrow[d, "h"] & A_X \arrow[d, "p"] \\
Z \arrow[r, hook] & B,
\end{tikzcd}
\tag{1}
$$

where $Z \subset B$ is the κ-reduction of $Y \subset A_X$. In this setting, if Y is of general type, then h and p are respectively isomorphisms of Y and A_X.

For any torus A, we will denote by $\widehat{A} = \mathrm{Pic}^0(A)$ the dual abelian variety.

We adapt the following notations. Assume that $Z \hookrightarrow A$ is a subvariety (possibly of general type) of an abelian variety and assume that $\widehat{B}$ is an abelian subvariety of $\widehat{A}$. Let B be a quotient abelian variety of A and Z_B denote the image of Z in B. The κ-reduction of Z_B, denoted $Z_B^\flat$, is called the *κ-reduction of Z with respect to B*. We will need the following easy lemma.

Lemma 2.1. *Let $Z \hookrightarrow A$ be a subvariety of an abelian variety, $\widehat{B}_2 \subset \widehat{B}_1 \subset \widehat{A}$ be abelian subvarieties, and $Z_{B_i} \hookrightarrow B_i$ be the κ-reduction with respect to B_i. Then there is induced commutative diagram with surjective vertical morphisms*

$$
\begin{tikzcd}
Z_{B_1}^\flat \arrow[r, hook] \arrow[d] & B_1^\flat \arrow[d] \\
Z_{B_2}^\flat \arrow[r, hook] & B_2^\flat.
\end{tikzcd}
\tag{2}
$$

In particular, the torus $\widehat{B_2^\flat}$ is a subtorus of $\widehat{B_1^\flat}$.

Lemma 2.2. *Let $Z \hookrightarrow B$ be a subvariety of general type. For $\widehat{B}_1$ and $\widehat{B}_2$ two abelian subvarieties of $\widehat{B}$. Let $\widehat{B_{12}}$ be the neutral component of $\widehat{B}_1 \cap \widehat{B}_2$ and let $\widehat{B_2^1}$ be the neutral component of $\widehat{B_1^\flat} \cap \widehat{B}_2$. Then $(B_2^1)^\flat = (B_{12})^\flat$.*

Proof. It is clear that $\widehat{B_2^1} \subset \widehat{B_{12}}$ and hence $\widehat{(B_2^1)^\flat} \subset \widehat{(B_{12})^\flat}$ by Lemma 2.1. Moreover, since $\widehat{(B_{12})^\flat} \subset \widehat{B_1^\flat}$ and $\widehat{(B_{12})^\flat} \subset \widehat{B_2^\flat} \subset \widehat{B_2}$, one has $\widehat{(B_{12})^\flat} \subset \widehat{B_2^1}$. Hence

$$
\widehat{(B_{12})^\flat} = \widehat{(B_{12})^{\flat}}^{\flat} \subset \widehat{(B_2^1)^\flat}.
$$

This completes the proof. $\square$

2.2. Hodge type sheaves

It is well-known that coherent sheaves on a projective varieties with origin from Hodge theory usually carry nice positive properties. We now recall the definition of some positive properties.

Let $\mathscr{F}$ be a coherent sheaf of an abelian variety A, the cohomological support loci of $\mathscr{F}$ is the closed subset

$$V^i(\mathscr{F}) := \{P \in \mathrm{Pic}^0(A) \mid \dim H^i(\mathscr{F} \otimes P) > 0\}$$

of $\widehat{A}$. Moreover, we will also consider the jump loci

$$V^i_k(\mathscr{F}) := \{P \in \mathrm{Pic}^0(A) \mid \dim H^i(\mathscr{F} \otimes P) \geq k\}.$$

We say that $\mathscr{F}$ is a GV-sheaf if $\mathrm{codim}_{\widehat{A}} V^i(\mathscr{F}) \geq i$ for each $i > 0$ and we say that $\mathscr{F}$ is M-regular if $\mathrm{codim}_{\widehat{A}} V^i(\mathscr{F}) > i$ for each $i > 0$.

Let $\mathscr{F}$ be a torsion-free coherent sheaf of a smooth projective variety X. We say that $\mathscr{F}$ is weakly positive (see also [15, Definition 2.1]) if there exists a Zariski open subset U of X such that for every ample divisor H of X, every integer $\alpha > 0$, there exists an integer $\beta > 0$ such that $\hat{S}^{\alpha\beta}\mathscr{F} \otimes H^{\beta}$ is generated by global sections over U, where $\hat{S}^{\alpha\beta}\mathscr{F}$ stands for the reflexive hull of $S^{\alpha\beta}\mathscr{F}$.

In the following, we often consider a torsion-free coherent sheaf $\mathscr{F}$ on a subvariety $i : Z \hookrightarrow A$ (we usually do not distinguish $\mathscr{F}$ and $i_*\mathscr{F}$) satisfying the following properties:

(P1) $\mathscr{F}$ is a GV sheaf on A;
(P2) for all $i, k \geq 0$, $V^i(\mathscr{F})$ and $V^i_k(\mathscr{F})$ are union of torsion translated abelian subvarieties of $\mathrm{Pic}^0(A)$;
(P3) let $g : A \to B$ be a morphism between abelian varieties, let Z_B be the image of Z, and let $r = \dim Z - \dim Z_B$, then

$$\mathbf{R}g_*(\mathscr{F} \otimes Q) = \bigoplus_{0 \leq j \leq r} R^j g_*(\mathscr{F} \otimes Q)[-j] \in D^b(B),$$

for any torsion line bundle $Q \in \mathrm{Pic}^0(A)$;
(P4) moreover, $R^j g_*(\mathscr{F} \otimes Q)$ is either 0 or is a torsion-free GV sheaf on Z_B.

We call a torsion-free coherent sheaf on Z satisfying Properties (1), (2), (3) and (4) a *Hodge sheaf on A supported on Z*.

Lemma 2.3. *Let $\mathscr{F}$ be a Hodge sheaf on A supported on Z.*

(1) *If for some $i > 0$, $V^i(\mathscr{F})$ has a component $P_0 + \widehat{B}_0$ of codimension i, where P_0 is a torsion line bundle and $\widehat{B}_0$ is an abelian subvariety. Let K be the kernel of $A \to B_0$, then $Z + K = Z$. In particular, if Z is of general type, $\mathscr{F}$ is M-regular.*

(2) *Let $g : A \to B$ be a morphism between abelian varieties and let $Q \in \mathrm{Pic}^0(A)$ a torsion line bundle. If $R^j g_*(\mathscr{F} \otimes Q) \neq 0$ for some $j \geq 0$, $R^j g_*(\mathscr{F} \otimes Q)$ is a Hodge sheaf on B supported on Z_B.*

(3) *Let $\mathscr{F}'$ be a direct summand of $\mathscr{F}$, then $\mathscr{F}'$ is also a Hodge sheaf.*

Proof. The proof of (1) is standard, see for instance [8, Lemma 1.1]. (3) is also clear.

For (2), let $\mathscr{Q} := R^j g_*(\mathscr{F} \otimes Q)$. Then $\mathscr{Q}$ is a GV sheaf on Z_B by $(P4)$. By $(P3)$, we know that, for any $j \geq 0$ and $Q' \in \mathrm{Pic}^0(B)$,

$$h^j(A, \mathscr{F} \otimes Q \otimes g^* Q') = \sum_{s+t=j} h^s(B, R^t g_*(\mathscr{F} \otimes Q) \otimes Q').$$

Since all $V^j_k(\mathscr{F})$ are unions of torsion translated abelian subvarieties of $\mathrm{Pic}^0(A)$, all $V^s_k(\mathscr{Q})$ are union of torsion translated abelian subvarieties of $\mathrm{Pic}^0(B)$. Let $g' : B \to B'$ be a morphism between abelian varieties and let $f = g' \circ g$. Then

$$\mathbf{R} f_*(\mathscr{F} \otimes Q) = \mathbf{R} g'_* \mathbf{R} g_*(\mathscr{F} \otimes Q) = \mathbf{R} g'_*\big(\bigoplus_j R^j g_*\big(\mathscr{F} \otimes Q)[-j]\big).$$

We then conclude that $\mathscr{Q}$ satisfies $(P3)$ and $(P4)$ by the same argument as in [10, Theorem 3.4]. $\square$

We call a Hodge sheaf on A supported on Z a *strong Hodge sheaf* if it satisfies furthermore the following two properties.

(P5) For any morphism $g : A \to B$ between abelian varieties and for any $Q \in \mathrm{Pic}^0(A)$, let $\epsilon : Z'_B \to Z_B$ be a desingularization, then there exists a torsion-free coherent sheaf $\mathscr{F}_Q$ on Z'_B such that $\mathbf{R}\epsilon_* \mathscr{F}_Q = g_*(\mathscr{F} \otimes Q)$ and $\mathscr{F}_Q \otimes \omega^{-1}_{Z'_B}$ is weakly positive on Z'_B;

(P6) Let $g : A \to B$ be as above. For $b \in Z_B$ general, denote $j : Z_b \hookrightarrow K$ the fibers of Z and A over b. Then $\mathscr{F} \mid_{Z_b}$ satisfies $(P1) - (P5)$.

Remark 2.4. It is clear that $\mathscr{F} \mid_{Z_b}$ is a strong Hodge sheaf on K supported on Z_b.

Lemma 2.5. *Let $f : X \to A$ be a morphism from a compact Kähler manifold to an abelian variety. Let $\mathscr{F} = R^j f_* \omega_X$ for some $j \geq 0$. Then $\mathscr{F}$ is a strong Hodge sheaf on A supported on $f(X)$.*

Proof. First of all, [13, Theorem A] implies the property $(P1)$ for $\mathscr{F}$.

The property $(P3)$ was proved by Kollár in [10] when X is projective and was proved by Saito in general (see [16] and [17], or [13, Theorem 14.2]).

Combining $(P2)$ with the work of Green–Lazarsfeld [5], we know that all cohomological support loci $V^i(\mathscr{F})$ are translated abelian subvarieties of $\mathrm{Pic}^0(A)$. The fact that $V^i(\mathscr{F})$

always contains a torsion point is first proved by Wang in [22], see also [13, Corollary 17.1].

Let $Z \to X$ be the étale cover induced by the torsion line bundle Q and let $h : Z \to B$ be the composition of morphisms. Then $R^k g_*(\mathscr{F} \otimes Q)$ is a direct summand of $R^{j+k} h_* \omega_Z$ and hence we have $(P4)$.

For $(P5)$, we consider a birational modification $\pi : X' \to X$ between compact Kähler manifolds such that the composition of morphisms $f \circ \pi : X' \to Z_B$ factors through ϵ as follows: $X' \xrightarrow{f'} Z \xrightarrow{g'} Z'_B \xrightarrow{\epsilon} Z_B$. Then by Saito's decomposition, we have $g_*(R^j f_* \omega_X \otimes Q) = \epsilon_*(g'_* R^j f'_* \omega_{X'})$. Note that $g'_* R^j f'_* \omega_{X'}$ is a direct summand of $R^j(g' \circ f')_* \omega_{X'}$ by Saito's decomposition. Moreover, $R^j(g' \circ f')_* \omega_{X'/Z'_B}$ is weakly positive (see Popa [14, Theorem 10.4] or Schnell [18, Theorem 1.4]). Hence $g'_* R^j f'_* \omega_{X'/Z'_B}$ is also weakly positive.

Finally, by base change, $\mathscr{F} \mid_{Z_b} = R^j f_{b*} \omega_{X_b}$, where $f_b : X_b \to K$ is the induced morphism between fibers. Hence we also have $(P6)$. □

Remark 2.6. More generally, the properties $(P1) - (P6)$ are satisfied by certain coherent sheaves which are graded pieces of the underlying D-modules of mixed Hodge modules.

Let $M = (\mathscr{M}, F_\cdot \mathscr{M}, M_{\mathbb{Q}})$ be a polarizable Hodge module on an abelian variety A. Then for each $k \in \mathbb{Z}$, the coherent sheaf $gr_k^F \mathscr{M}$ satisfies $(P1)$ and $(P2)$ (see [13]). Moreover, let p be the smallest number such that $F_p \mathscr{M} \neq 0$ and let $S(\mathscr{M}) := F_p \mathscr{M}$. Then Saito ([17]) showed that $S(\mathscr{M})$ satisfies $(P3)$ and $(P4)$. By Popa and Schnell's result on weakly positive properties of $S(\mathscr{M})$, it is also easy to show that $S(\mathscr{M})$ satisfies $(P5)$ and $(P6)$.

3. Hodge sheaf $\mathscr{F}$ supported on Z with $\chi(\mathscr{F}) = 1$

In this section, we will prove Theorem 1.1. We first prove a general but technical result on the structure of Z and Theorem 1.1 is a direct consequence.

The following lemma is essentially due to Pareschi (see [11]).

Lemma 3.1. *Let $Z \hookrightarrow B$ be a subvariety of general type. Let $\mathscr{F}$ be a torsion-free sheaf on Z such that $\mathscr{F}$ is M-regular on B and $\chi(B, \mathscr{F}) = 1$. Then $V^1(\mathscr{F}) \neq \emptyset$. Moreover, for a component W of $V^1(\mathscr{F})$, if $\operatorname{codim}_{\widehat{B}} W = j + 1 \geq 2$, then W is indeed a component of $V^j(\mathscr{F})$.*

Proof. Indeed, we can apply the argument of Pareschi in the proof of [11, Theorem 5.1]. Denote by $\mathbf{R}\Phi : \mathrm{D}(B) \to \mathrm{D}(\widehat{B})$ and $\mathbf{R}\Psi : \mathrm{D}(\widehat{B}) \to \mathrm{D}(B)$ the Fourier–Mukai functors induced by the normalized Poincaré line bundles $\mathscr{P}$ on $B \times \widehat{B}$. Let $\mathbf{R}\Delta(\mathscr{F}) := \mathbf{R}\mathscr{H}om(\mathscr{F}, \mathscr{O}_B) \in \mathrm{D}(B)$. Then Pareschi and Popa ([12, Corollary 3.2]) proved that $\mathbf{R}\Phi(\mathbf{R}\Delta(\mathscr{F}))[-g] = \mathbf{R}^g\Phi(\mathbf{R}\Delta(\mathscr{F}))$ is a torsion-free coherent sheaf on $\widehat{B}$ of rank equal to $\chi(B, \mathscr{F})$, i.e. 1. Hence, we can write $\mathbf{R}\Phi(\mathbf{R}\Delta(\mathscr{F}))[-g] = L \otimes \mathscr{I}_{\mathscr{Z}}$, where L is a line bundle on $\widehat{B}$ and $\mathscr{I}_{\mathscr{Z}} \hookrightarrow \mathscr{O}_{\widehat{B}}$ is an ideal sheaf of a subscheme $\mathscr{Z}$ of of $\widehat{B}$. On the other hand, by Mukai's formula, we know that $(-1)^*\mathbf{R}\Psi(L \otimes \mathscr{I}_{\mathscr{Z}}) = \mathbf{R}\Delta(\mathscr{F})$, whose support is also Z. Hence $\mathscr{I}_{\mathscr{Z}}$ is a proper subsheaf of $\mathscr{O}_{\widehat{B}}$ otherwise the support of $(-1)^*\mathbf{R}\Psi(L\otimes\mathscr{I}_{\mathscr{Z}})$ is a union of translated abelian subvarieties.

Moreover, we know that $(-1_{\widehat{B}})^*\mathbf{R}^j\Phi(\mathscr{F}) \simeq \mathscr{E}xt^j(L\otimes\mathscr{I}_{\mathscr{Z}}, \mathscr{O}_{\widehat{B}})$ (see for instance [11, Proposition 1.6]). For any $j\geq 1$, $\mathrm{Supp}\big(\mathscr{E}xt^j(L\otimes\mathscr{I}_{\mathscr{Z}}, \mathscr{O}_{\widehat{B}})\big)\subset\mathscr{Z}$. Hence for any $j\geq 1$, $\mathrm{Supp}(\mathbf{R}^j\Phi(\mathscr{F}))\subset 0_{\widehat{B}}-\mathscr{Z}$. By cohomology and base-change, $V^1(\mathscr{F})\subset 0_{\widehat{B}}-\mathscr{Z}$.

Let W be a component of $V^1(\mathscr{F})$ of codimension $j+1\geq 2$. Then the support of $\mathscr{E}xt^j(L\otimes\mathscr{I}_{\mathscr{Z}}, \mathscr{O}_{\widehat{B}})\simeq\mathscr{E}xt^{j+1}(L\otimes\mathscr{O}_{\mathscr{Z}}, \mathscr{O}_{\widehat{B}})$ has $0_{\widehat{B}}-W$ as a component. Hence $V^j(\mathscr{F})$ has a component W of codimension $j+1, j>0$. $\square$

Let Z be a subvariety of general type of an abelian variety B of codimension s. If $s=1$, Z is an ample divisor of B. The following lemma deals with the higher codimension cases.

Lemma 3.2. *Let $Z\hookrightarrow B$ be a subvariety of general type of codimension $s\geq 1$. Assume that there exists a Hodge sheaf $\mathscr{F}$ supported on Z with $\chi(Z,\mathscr{F})=1$. For any component $T_i=P_i+\widehat{B}_i$ of $V^1(\mathscr{F})$, where P_i is a torsion line bundle and $\widehat{B}_i$ is an abelian subvariety of $\widehat{B}$, let $Z_i:=Z_{B_i}$ and $Z_i^\flat:=Z_{B_i}^\flat$ be the κ-reduction of Z_i. Then $\mathrm{codim}_{B_i}Z_i=\mathrm{codim}_{B_i^\flat}Z_i^\flat=s-1$.*

Proof. Since $\mathscr{F}$ is a Hodge sheaf, by definition, we know that each component T_i of $V^1(\mathscr{F})$ can be written as $P_i+\widehat{B}_i$, where P_i is a torsion line bundle and $\widehat{B}_i$ is an abelian subvariety of $\widehat{B}$.

We consider the commutative diagram

$$\begin{tikzcd}
Z \arrow[r, hook] \arrow[d, "h_i"] \arrow[dd, bend right=60, "h_i^\flat"'] & B \arrow[d, "p_i"'] \arrow[dd, bend left=60, "p_i^\flat"] \\
Z_i \arrow[r, hook] \arrow[d] & B_i \arrow[d] \\
Z_i^\flat \arrow[r, hook] & B_i^\flat,
\end{tikzcd} \tag{3}$$

where both the fibers of $Z_i\to Z_i^\flat$ and $B_i\to B_i^\flat$ are translates of abelian variety $R_{B_i}=\ker(B_i\to B_i^\flat)$.

By construction, $\mathrm{codim}_{B_i}Z_i=\mathrm{codim}_{B_i^\flat}Z_i^\flat$, we just need to show that $\mathrm{codim}_{B_i}Z_i=s-1$.

Assume that $\mathrm{codim}_{\widehat{B}}\widehat{B}_i=j+1$. Then, by Lemma 3.1, $P_i+\widehat{B}_i$ is a component of $V^j(\mathscr{F})$, by $(P3)$, we know that for any $Q\in\widehat{B}_i$,

$$0\neq H^j(Z,\mathscr{F}\otimes P_i\otimes Q)=\sum_{l+k=j}H^l(Z_i, R^kh_{i*}(\mathscr{F}\otimes P_i)\otimes Q).$$

By $(P4)$, all $R^k h_{i*}(\mathscr{F}\otimes P_i)$ are GV-sheaves on Z_i. Thus for a general Q the right hand side has a single term $H^0(Z_i, R^j h_{i*}(\mathscr{F}\otimes P_i)\otimes Q)$. We then conclude that $R^j h_{i*}(\mathscr{F}\otimes P_i)$ is a non-trivial torsion-free sheaf on Z_i. By $(P3)$, a general fiber of h_i has dimension at least j. Moreover, Z is of general type, thus a general fiber Z_t of h_i must be a divisor of B_t. Hence $\dim Z_i = \dim Z - j$, $\dim B_i = \dim B - j - 1$, and $\operatorname{codim}_{B_i} Z_i = \operatorname{codim}_B Z - 1 = s - 1$. $\square$

For the inductive purpose, we need the following more refined statement.

Lemma 3.3. *Keep the assumptions of Lemma 3.2. We then consider the diagram (3) with $i = 1$. Let $Q \in \widehat{B}$ be a general torsion point (in particular, $Q \notin V^1(\mathscr{F})$) and consider the sheaf $\mathscr{F}_Q := h^\flat_{1*}(\mathscr{F}\otimes Q)$.*

Then the map

$$\widehat{B}_k \mapsto \widehat{B^1_k}: \ = \text{the neutral component of } \widehat{B}_k \cap \widehat{B^\flat_1}$$

induces an bijection between the following sets:

$$S_1 := \{\widehat{B}_k \mid T_k = P_k + \widehat{B}_k \textit{ is a component of } V^1(\mathscr{F}) \textit{ and } q(\widehat{B}_k) = \widehat{B}/\widehat{B^\flat_1}\},$$

and

$$S_2 := \{\widehat{B}'_k \mid T'_k = P'_k + \widehat{B}'_k \textit{ is a component of } V^1(\mathscr{F}_Q)\},$$

where $q : \widehat{B} \to \widehat{B}/\widehat{B^\flat_1}$ is the natural quotient.

Proof. We have $h^0(Z^\flat_1, \mathscr{F}_Q) = h^0(Z, \mathscr{F}\otimes Q) = \chi(Z, \mathscr{F}\otimes Q) = \chi(Z, \mathscr{F}) = 1$, where the second equality uses the assumption that Q is general. Since $Z^\flat_1$ is of general type and by Lemma 2.3, $\mathscr{F}_Q$ is a Hodge sheaf on $B^\flat_1$ supported on $Z^\flat_1$, $\mathscr{F}_Q$ is M-regular. By upper-semi-continuity of cohomology, $h^0(Z^\flat_1, \mathscr{F}_Q\otimes Q') = 1$ for $Q' \in \operatorname{Pic}^0(B^\flat_1)$ general. Hence, $\chi(Z^\flat_1, \mathscr{F}_Q) = h^0(Z^\flat_1, \mathscr{F}_Q\otimes Q') = 1$.

Since $Q \notin V^1(\mathscr{F})$, we know that $R^1 h^\flat_{1*}(\mathscr{F}\otimes Q) = 0$. Otherwise, $R^1 h^\flat_{1*}(\mathscr{F}\otimes Q)$ is a M-regular sheaf on $Z^\flat_1$ since $Z^\flat_1$ is of general type. Then $V^0(R^1 h^\flat_{1*}(\mathscr{F}\otimes Q)) = \widehat{B^\flat_1}$ and by $(P3)$, $Q \in V^1(\mathscr{F})$, which is a contradiction. Hence,

$$h^1(Z, \mathscr{F}\otimes Q\otimes {h^\flat_1}^* P) = h^1(Z^\flat_1, \mathscr{F}_Q\otimes P),$$

for any $P \in \widehat{B^\flat_1}$.

Consider the sequence

$$\widehat{B^\flat_1} \xhookrightarrow{{h^\flat_1}^*} \widehat{B} \overset{q}{\twoheadrightarrow} \widehat{B}/\widehat{B^\flat_1}\ .$$

We then conclude that $V^1(\mathscr{F}_Q) + Q = q^{-1}(q(Q)) \cap V^1(\mathscr{F})$. As $Q \in \widehat{B_1^\flat}$ is general, we have the bijection $S_2 \to S_1$ described above. $\square$

Lemma 3.4. *Let $Z \hookrightarrow B$ be a subvariety of general type of codimension $s \geq 2$. Assume that Z generates B and there exists a Hodge sheaf $\mathscr{F}$ supported on Z with $\chi(Z, \mathscr{F}) = 1$. For any two components $\widehat{T}_i = P_i + \widehat{B}_i$ and $\widehat{T}_j = P_j + \widehat{B}_j$, if $\widehat{B}_i + \widehat{B}_j = \widehat{B}$, then $\widehat{B_i^\flat} + \widehat{B_j^\flat} = \widehat{B}$.*

Proof. We denote by $\widehat{B_{ij}}$ the neutral component of $\widehat{B}_i \cap \widehat{B}_j$ and let Z_{ij} be the image of $Z \hookrightarrow B \to B_{ij}$. Then, the induced morphism $B \to B_i \times_{B_{ij}} B_j$ is an isogeny since $\widehat{B}_i + \widehat{B}_j = \widehat{B}$. We have the following commutative diagram:

$$\begin{tikzcd} Z \arrow[r, hook] \arrow[d] & B \arrow[d, "\text{isogeny}"] \\ Z_i \times_{Z_{ij}} Z_j \arrow[r, hook] & B_i \times_{B_{ij}} B_j. \end{tikzcd}$$

Hence Z is an irreducible component of an étale cover of $Z_i \times_{Z_{ij}} Z_j$.

Let R_i (resp. R_{ij}) be the κ-kernel of Z_i (resp. Z_{ij}). Since Z is of general type, a general fiber of $Z \to Z_j$ is of general type, and hence the general fiber of $Z_i \to Z_{ij}$ is also of general type.

Then the composition of morphisms $R_i \to B_i \to B_{ij}$ is an isogeny onto its image R'_i and $R'_i + Z_{ij} = Z_{ij}$. We denote by B'_{ij} the quotient B_{ij}/R'_i and $Z'_{ij} = Z_{ij}/R'_i$. Note that $B_i^\flat = B_i/R_i$. Hence

$$\begin{aligned} \dim B_i^\flat + \dim B_j - \dim B'_{ij} &= \dim B_i - \dim R_i + \dim B_j - (\dim B_{ij} - \dim R'_i) \\ &= \dim B_i + \dim B_j - \dim B_{ij} = \dim B. \end{aligned}$$

It follows that the natural surjective morphism $B \to B_i^\flat \times_{B'_{ij}} B_j$ is again an isogeny and Z is an irreducible component of the inverse image under this isogeny of $Z_i^\flat \times_{Z'_{ij}} Z_j$. Hence $\widehat{B_i^\flat} + \widehat{B}_j = \widehat{B}$.

We apply the same argument to B_j and $B_i^\flat$. We then conclude as before that the morphism $R_j \to B'_{ij}$ is also an isogeny onto its image. In particular, $\widehat{B_i^\flat} + \widehat{B_j^\flat} = \widehat{B}$. $\square$

Proposition 3.5. *Let $Z \hookrightarrow B$ be a subvariety of general type of codimension $s \geq 2$. Assume that Z generates B and there exists a Hodge sheaf $\mathscr{F}$ supported on Z with $\chi(Z, \mathscr{F}) = 1$. Then, there exist at least s components $T_i = P_i + \widehat{B}_i$ of $V^1(\mathscr{F})$, $1 \leq i \leq s$, such that $\widehat{B}_i + \widehat{B}_j = \widehat{B}$ for any $i \neq j$. We call them a collection of essential components of $V^1(\mathscr{F})$.*

Proof. Note that $\operatorname{codim}_{B_1^\flat} Z_1^\flat = s-1$ by Lemma 3.2, $Z_1^\flat$ generates $B_1^\flat$, and $\chi(Z_1^\flat, \mathscr{F}_Q) = 1$ for the Hodge sheaf $\mathscr{F}_Q$ as in Lemma 3.3.

We run induction on s. When $s = 2$, we know that $V^1(\mathscr{F}_Q) \neq \emptyset$, hence by the correspondence in Lemma 3.3, there exists T_2 such that $q(\widehat{B}_2) = \widehat{B}/\widehat{B_1^\flat}$. Then $\widehat{B}_1 + \widehat{B}_2 = \widehat{B}$.

When $s \geq 3$, by induction, there exist $s-1$ essential components of $V^1(\mathscr{F}_Q)$ corresponds to subvarieties: $\widehat{B_2^1}, \ldots, \widehat{B_s^1}$ such that $\widehat{B_i^1} + \widehat{B_j^1} = \widehat{B_1^\flat}$ for any $2 \leq i \neq j \leq s$.

By the bijection in Lemma 3.3, there exist correspondingly $s-1$ components of $V^1(\mathscr{F})$: $T_2 = P_2 + \widehat{B}_2, \ldots, \widehat{T}_s = P_s + \widehat{B}_s$. Since $q(\widehat{B}_i) = \widehat{B}/\widehat{B_1^\flat}$ for each $2 \leq i \leq s$, $\widehat{B}_i + \widehat{B_1^\flat} = \widehat{B}$ and hence $\widehat{B}_i + \widehat{B}_1 = \widehat{B}$. Moreover, as $\widehat{B_i^1} + \widehat{B_j^1} = \widehat{B_1^\flat}$, one has $\widehat{B}_i + \widehat{B}_j = \widehat{B}$ for $2 \leq i < j \leq s$. $\square$

Theorem 3.6. *Let $Z \hookrightarrow B$ be a subvariety of general type of codimension $s \geq 2$. Assume that Z generates B and that there exists a Hodge sheaf $\mathscr{F}$ supported on Z with $\chi(Z, \mathscr{F}) = 1$. Fix a collection of essential components $T_i = P_i + \widehat{B}_i$ of $V^1(\mathscr{F})$, $1 \leq i \leq s$. Let $\widehat{U_i} = \big(\bigcap_{j\neq i} \widehat{B_j}\big)_0$ be the neutral component of $\big(\bigcap_{j\neq i} \widehat{B_j}\big)$. Let $K_i := U_i^\flat$ (cf. the notations in Section 2). Then we have the followings:*

(1) $\sum_{1\leq j\leq s} \widehat{K}_j = \widehat{B}$ *and* $\sum_{j\neq i} \widehat{K}_j = \widehat{B_i^\flat}$ *for each i;*
(2) *the image of the composition of morphisms $Z \hookrightarrow B \twoheadrightarrow K_i$ is an ample divisor D_i of K_i for each i.*

Proof. We will prove by induction on $s \geq 2$. If $s = 2$, $U_1 = B_2$ and $U_2 = B_1$. Hence $K_1 = B_2^\flat$ and $K_2 = B_1^\flat$ then we are done.

We assume that the statement of Theorem 3.6 holds when the codimension of the subvariety in abelian variety is at most $s-1$. As in Lemma 3.3, we consider $\mathscr{F}_Q^i = h_{i*}^\flat(\mathscr{F} \otimes Q)$, for a general torsion $Q \in \widehat{B}$. Then $\mathscr{F}_Q^i$ is a Hodge sheaf on $Z_i^\flat$ with $\chi(Z_i^\flat, \mathscr{F}_Q^i) = 1$.

Indeed, for any i, by the bijection in Lemma 3.3, each T_j for $j \neq i$ corresponds to a component $T_j^i = P_j^i + \widehat{B_j^i}$ of $V^1(\mathscr{F}_Q^i)$. Since $\widehat{B}_j + \widehat{B}_k = \widehat{B}$ for any $j \neq k$, we have $\widehat{B}_j^i + \widehat{B}_k^i = \widehat{B_i^\flat}$, for any i, j, k pairwise distinct.

Since $\operatorname{codim}_{B_i^\flat} Z_i^\flat = s-1$, by induction, for each $t \neq i$, let

$$\widehat{U_t^i} := \Big(\bigcap_{j\neq i,t} (\widehat{B_j^i})\Big)_0,$$

and $K_t^i = (U_t^i)^\flat$. Then by induction hypothesis, one has

(1′) $\sum_{t\neq i} \widehat{K_t^i} = \widehat{B_i^\flat}$;
(2′) the image of the composition of morphisms $Z \hookrightarrow B \twoheadrightarrow K_t^i$ is an ample divisor.

On the other hand, we have

$$\widehat{U_t^i} = \Big(\bigcap_{j\neq i,t} \widehat{B_j^i}\Big)_0 = \Big(\big(\bigcap_{j\neq i,t} \widehat{B_j}\big) \cap \widehat{B}_i^\flat\Big)_0.$$

Since $\widehat{U_t} = \big(\bigcap_{j\neq t} \widehat{B_j}\big)_0 = \big((\bigcap_{j\neq i,t} \widehat{B_j}) \cap \widehat{B}_i\big)_0$, by Lemma 2.2, we have

$$K_t^i = (U_t^i)^\flat = U_t^\flat = K_t.$$

Hence, by (1′), $\sum_{j\neq i} \widehat{K}_j = B_i^\flat$ for each $j \neq i$. Then $\sum_{1\leq j\leq s} \widehat{K}_j = \widehat{B}$. We deduce (2) from (2′). $\square$

Remark 3.7. We have actually proved that Z is an irreducible component of an étale cover of certain fibre product of the ample divisors D_i of K_i, $1 \leq i \leq s$.

In general, we can not expect that $\dim(\widehat{K}_i \cap \widehat{B_i^\flat}) = 0$ for $1 \leq i \leq s$ and that Z is an étale cover of the product of these D_i. However, we have this nice picture in some special cases.

Lemma 3.8. *Under the assumption of Theorem 3.6, if* $\dim K_i = 2$, *then* $\dim(\widehat{K}_i \cap \widehat{B_i^\flat}) = 0$ *and hence we have a commutative diagram*

$$\begin{tikzcd} Z \arrow[r, hook] \arrow[d] & B \arrow[d, "\rho", "\text{isogeny}"'] \\ D_i \times Z_i \arrow[r, hook] & K_i \times B_i. \end{tikzcd}$$

Proof. We know that $\widehat{K}_i + \widehat{B_i^\flat} = \widehat{B}$ by Theorem 3.6 and $\operatorname{codim}_{\widehat{B}} \widehat{B}_i \geq 2$. If $\dim \widehat{K}_i = \dim K_i = 2$, then $\widehat{B_i^\flat} = \widehat{B}_i$ and $\dim(\widehat{K}_i \cap \widehat{B_i^\flat}) = 0$. $\square$

Corollary 3.9. *Under the assumption of Theorem 3.6, we have* $2\dim Z \geq \dim B$.

Assume that $2\dim Z = \dim B$. *Pick* $s = \operatorname{codim}_B Z$ *essential components* $\widehat{T}_i$ *of* $V^1(\mathscr{F})$. *For each* $\widehat{K}_i$ *defined in Theorem 3.6, we have* $\dim \widehat{K}_i = 2$ *and we have a commutative diagram:*

$$\begin{tikzcd} Z \arrow[r, hook] \arrow[d] & B \arrow[d, "\rho", "\text{isogeny}"'] \\ D_1 \times \cdots \times D_s \arrow[r, hook] & K_1 \times \cdots \times K_s. \end{tikzcd} \tag{4}$$

Proof. We prove by induction on s. If $s = 1$, since $V^1(\mathscr{F}) \neq \emptyset$ and a codimension-$(s+1)$ component of $V^1(\mathscr{F})$ is a component of $V^s(\mathscr{F})$, we conclude that B is an abelian surface and we are done.

In general, we take $\widehat{T}_1 = P_1 + \widehat{B}_1$ a component of $V^1(\mathscr{F})$. Assume that $\dim Z_1^\flat = n - s$ and $\dim B_1^\flat = \dim Z - s - 1$ for $s \geq 1$. As for $Q \in \widehat{B}$ general, $\mathscr{F}_Q^1 = h_{1*}^\flat(\mathscr{F} \otimes Q)$ is a Hodge sheaf on $Z_1^\flat$ with $\chi(Z_1^\flat, \mathscr{F}_Q^1) = 1$. By induction, one has

$$\dim Z_1^\flat \geq \operatorname{codim}_{B_1^\flat} Z_1^\flat = \operatorname{codim}_{B_1} Z_1 = s-1,$$

where the last equality follows from Lemma 3.2. Since $\dim Z > \dim Z_1^\flat$, one has $\dim Z \geq s$ immediately.

Assume that $\dim B = 2s$. For the s components $\widehat{T_i} = P_i + \widehat{B}_i$ of $V^1(\mathscr{F})$, by the same argument as above, we see that $\dim Z_i^\flat = s-1$ and $\dim B_i^\flat = \dim B_i = 2s-2$. Hence, by induction, each $Z_i^\flat$ has the structure as in (4). Hence $\dim K_i = 2$ and $\dim(\widehat{K}_i \cap \widehat{B}_i) = 0$ for each $1 \leq i \leq s$. We then have the diagram (4). □

Corollary 3.10. *Use the same assumptions of Theorem 3.6 and furthermore assume that B has k simple factors. Then $s = \operatorname{codim}_B Z \leq k$. If $s = \operatorname{codim}_B Z = k$, we have a commutative diagram:*

$$\begin{tikzcd} Z \arrow[r, hook] \arrow[d] & B \arrow[d, "\rho", "\text{isogeny}"'] \\ D_1 \times \cdots \times D_s \arrow[r, hook] & K_1 \times \cdots \times K_s, \end{tikzcd} \tag{5}$$

where $D_i \hookrightarrow K_i$ is an ample divisor for each $1 \leq i \leq s$. In particular Z is an irreducible component of $\rho^{-1}(D_1 \times \cdots \times D_s)$.

Proof. We argue by induction on k. If $k = 1$, all components of $V^1(\mathscr{F})$ are isolated points. By Lemma 3.2, Z is a divisor of B. In general, if $s > 2$, we consider $Z_1^\flat \hookrightarrow B_1^\flat$ with the Hodge sheaf $\mathscr{F}_Q^1$ such that $\chi(Z_1^\flat, \mathscr{F}_Q^1) = 1$. Note that $\operatorname{codim}_{B_1^\flat} Z_1^\flat = s-1$ and $B_1^\flat$ has at most $k-1$ simple factors. Hence by induction, $s \leq k$.

If the equality holds, each K_i is a simple abelian variety and $\dim(\widehat{K}_i \cap \sum_{j\neq i} \widehat{K}_j) = 0$. Then the natural morphism $\rho : B \to K_1 \times \cdots \times K_s$ is an isogeny and we have (5). □

We then finish the proof of Theorem 1.1.

Proof. Let X be a compact Kähler manifold with $p_g = 1$. We consider the diagram (1). Note that $\mathscr{F} = f_*\omega_X$ is a Hodge sheaf on B supported on Z with $h^0(Z, f_*\omega_X) = p_g(X) = 1$. Being a Hodge sheaf supported on Z, we have $\chi(Z, \mathscr{F}) \geq 1$. On the other hand, by semi-continuity, $h^0(Z, \mathscr{F} \otimes Q) \leq 1$, for $Q \in \widehat{A}$ general. As $\mathscr{F}$ is a GV-sheaf, $\chi(Z, \mathscr{F}) = \chi(Z, \mathscr{F} \otimes Q) = h^0(Z, \mathscr{F} \otimes Q) \leq 1$. Hence $\chi(Z\mathscr{F}) = 1$. Then Theorem 1.1 follows easily from Theorem 3.6 and Corollary 3.9. □

4. Strong Hodge sheaf $\mathscr{F}$ supported on Z with $\chi(\mathscr{F}) = 1$ and theta divisors

The main goal of this section is to study the following problem, which is a generalization of question asked by Pareschi (see [8, Question 4.6]).

Question 4.1. Let $Z \hookrightarrow B$ be a subvariety of an abelian variety and Z generates B. Assume that Z is of general type and there exists a *strong Hodge sheaf* $\mathscr{F}$ on Z such that $\chi(Z, \mathscr{F}) = 1$. Then does there exist theta divisors Θ_i, $1 \leq i \leq m$ and a birational morphism $t : Z' := \Theta_1 \times \cdots \times \Theta_m \to Z$ such that $\mathscr{F} = t_*(\omega_{Z'} \otimes Q)$ for some torsion line bundle Q on Z'?

The main results in [8] state that we have a positive answer to the above question in two special cases:

(1) $\mathscr{F}$ is the pushforward of the canonical sheaf of a desingularization of Z and Z is smooth in codimension 1;
(2) $\dim Z = \frac{1}{2} \dim B$ and $\mathscr{F} = f_*\omega_X$, where $f : X \to Z$ is a surjective morphism from a smooth projective variety X to Z.

Results in Section 3 provide further evidences for a positive answer and we will prove it in some other cases.

Lemma 4.2. *Assume that Z is a smooth projective curve of genus at least 2 and $\mathscr{F}$ is a torsion-free sheaf on Z such that $\mathscr{F} \otimes \omega_Z^{-1}$ is weakly positive. Then $\chi(Z, \mathscr{F}) \geq 1$. If $\chi(Z, \mathscr{F}) = 1$, then Z is a smooth projective curve of genus 2 and $\mathscr{F} = \omega_Z \otimes Q$ for some line bundle $Q \in Pic^0(Z)$. If $\mathscr{F}$ is a Hodge sheaf and $h^0(C, \mathscr{F}) = 1$, then Q is a torsion line bundle.*

Proof. Since $\mathscr{F} \otimes \omega_Z^{-1}$ is weakly positive on the smooth projective curve Z, it is nef. In particular $\deg \mathscr{F} \geq r(2g - 2)$, where r is the rank of $\mathscr{F}$. By the Riemann–Roch formula, $\chi(C, \mathscr{F}) \geq r(g - 1) \geq 1$. If equality holds, $r = 1, g = 2, \deg \mathscr{F} = 2$.

If $\mathscr{F}$ is a Hodge sheaf, then the cohomology support loci is a union of torsion translates of abelian subvarieties of $J(C)$. So Q has to be a torsion line bundle. $\square$

Theorem 4.3. *Under the assumption of Question (4.1), assume moreover that $\dim B = 2 \dim Z$, then Question (4.1) has an affirmative answer.*

Proof. We apply Corollary (3.10) and diagram (4). Let C_i be the normalization of D_i and Z' the connected component of $(C_1 \times \cdots \times C_n) \times_{(K_1 \times \cdots \times K_n)} B$ which dominates Z. We then have

$$\begin{tikzcd}
Z' \arrow[r, "\tau"] \arrow[d, "\pi"'] & Z \arrow[r, hook] \arrow[d] & B \arrow[d, "\text{isogeny}"', "\rho"] \\
C_1 \times \cdots \times C_n \arrow[r] \arrow[d] & D_1 \times \cdots \times D_n \arrow[r, hook] \arrow[d] & K_1 \times \cdots \times K_n \arrow[d] \\
C_1 \arrow[r, "\tau_1"] & D_1 \arrow[r, hook] & K_1,
\end{tikzcd}$$

where π is an étale morphism and τ is a desingularization.

We prove by induction on $\dim Z$. If $\dim Z = 1$, then there exists $\mathscr{F}'$ on Z' such that $\mathbf{R}\tau_*\mathscr{F}' = \mathscr{F}$ and $\mathscr{F}' \otimes \omega_{Z'}^{-1}$ is weakly positive. By Lemma 4.2, we conclude the proof.

We then assume that Theorem 4.3 holds in dimension $< n$. Take $Q \in \widehat{B}$ a general torsion point, for the morphism $t_1 : Z \to D_1$, let $\mathscr{Q}_Q = t_{1*}(\mathscr{F} \otimes Q)$. Then $\mathscr{Q}_Q$ is a strong Hodge sheaf on K_1 supported on D_1. Moreover, $h^0(D_1, \mathscr{Q}_Q) = 1$ and hence $\chi(D_1, \mathscr{Q}_Q) = 1$. By $(P5)$, there exists $\mathscr{Q}'_Q$ on C_1 such that $\mathbf{R}\tau_{1*}\mathscr{Q}'_Q = \mathscr{Q}_Q$ and $\mathscr{Q}'_Q \otimes \omega_{C_1}^{-1}$ is weakly positive. Then by Lemma 4.2, C_1 is a smooth curve of genus 2 and $\mathscr{Q}_Q = \tau_{1*}(\omega_{C_1} \otimes Q_1)$ for a torsion line bundle $Q_1 \in \mathrm{Pic}^0(C_1)$. In particular, $\mathrm{rank}\,\mathscr{Q}_Q = 1$. Similarly, we see that each C_i has genus 2.

Let F be a general fiber of t_1 and let B_F the correspondingly fiber of $B \to K_1$. Then $\mathscr{F}'_Q := (\mathscr{F} \otimes Q) \mid_F$ is a strong Hodge sheaf on B_F supported on F such that $h^0(F, \mathscr{F}'_Q) = 1$. By induction, F is birational to a product of $n-1$ genus 2 curves. Then the natural morphism $F \to D_2 \times \cdots \times D_s$ is birational and so is the morphism $Z \to D_1 \times \cdots \times D_s$. Hence π is an isomorphism.

It remains to show that $\mathscr{F}'$ is isomorphic to the canonical bundle of Z' twisted by a torsion line bundle. Note that $\mathscr{F}'$ restricted to each factor C_i is of such form by Lemma 4.2 and we conclude by the see-saw principle. □

Theorem 4.4. *Let Z be a subvariety of general type of an abelian variety B of codimension s. Assume that there exists a strong Hodge sheaf $\mathscr{F}$ on B supported on Z such that $\chi(Z, \mathscr{F}) = 1$, Z is smooth in codimension 1 and B has s simple factors. Then Question 4.1 has an affirmative answer.*

We first prove the divisorial case and then apply Theorem 3.6 to conclude the proof for the general case.

4.1. Divisorial case

In this subsection, we assume that B is a simple abelian variey of dimension g, Z is an irreducible ample divisor smooth in codimension 1 and there exists a strong Hodge sheaf $\mathscr{F}$ on B such that $\chi(Z, \mathscr{F}) = 1$. Note that in this case Z is normal.

We aim to prove Theorem 4.3 in this case. Our argument follows closely that of Pareschi in [11, Theorem 5.1].

Let $\rho : X \to Z$ be a desingularization and by definition of stong Hodge sheaf, there exists a torsion-free sheaf $\mathscr{F}'$ on X such that $\mathbf{R}\rho_*\mathscr{F}' = \rho_*\mathscr{F}' = \mathscr{F}$ and $\mathscr{F}' \otimes \omega_X^{-1}$ is weakly positive. Since Z is normal, the composition of morphism $X \xrightarrow{\rho} Z \hookrightarrow B$ is primitive, namely the pull-back $\widehat{B} \to \mathrm{Pic}^0(X)$ is injective (in other words, any étale over of Z induced by an étale cover of B remains irreducible).

Let $\mathbf{R}\Delta_B(\mathscr{F}) = \mathbf{R}\mathscr{H}om(\mathscr{F}, \mathscr{O}_B)$. Since $\mathscr{F}$ is M-regular on B and $\chi(\mathscr{F}) = 1$,

$$\mathbf{R}\Phi_{\mathscr{P}_B}(\mathbf{R}\Delta_B(\mathscr{F})) = (L \otimes \mathscr{I}_V)[-g],$$

where V is a subscheme of $\widehat{B}$ of dimension zero and L is a line bundle on $\widehat{B}$. Since B is simple, L is either ample, anti-ample, or trivial. We will see soon that L is ample.

By Fourier–Mukai equivalence,

$$\mathbf{R}\Delta_B(\mathscr{F}) \simeq (-1_B)^*\mathbf{R}\Psi_{\mathscr{P}_B}(L \otimes \mathscr{I}_V), \tag{6}$$

and both are supported on B.

We then consider the short exact sequence on $\widehat{B}$:

$$0 \to L \otimes \mathscr{I}_V \to L \to L\,|_V \to 0,$$

and apply the functor $\mathbf{R}\Psi_{\mathscr{P}_B}$:

$$0 \to \mathbf{R}^0\Psi_{\mathscr{P}_B}(L \otimes \mathscr{I}_V) \to \mathbf{R}^0\Psi_{\mathscr{P}_B}(L) \to \mathbf{R}^0\Psi_{\mathscr{P}_B}(L\,|_V) \to \mathbf{R}^1\Psi_{\mathscr{P}_B}(L \otimes \mathscr{I}_V) \to \mathbf{R}^1\Psi(L).$$

From this long exact sequence, we know that L has sections otherwise $\mathbf{R}^0\Psi_{\mathscr{P}_B}(L\,|_V) \to \mathbf{R}^1\Psi_{\mathscr{P}_B}(L \otimes \mathscr{I}_V)$ is an injection and $\mathbf{R}\Delta_B(\mathscr{F})$ cannot be supported on B. If L is trivial, then $\mathbf{R}^0\Psi_{\mathscr{P}_B}(L \otimes \mathscr{I}_V) = 0$, $\mathbf{R}^0\Psi_{\mathscr{P}_B}(L)$ is the skyscraper sheaf supported at $0 \in B$, and $\mathbf{R}^0\Psi_{\mathscr{P}_B}(L\,|_V)$ is (always) locally free. Thus this does not happen either. That is, L is ample and $\mathbf{R}^i\Psi_{\mathscr{P}_B}(L \otimes \mathscr{I}_V) \simeq \mathbf{R}^i\Psi_{\mathscr{P}_B}(L) = 0$ for $i > 1$. We can write the previous long exact sequence more explicitly as

$$0 \to \mathbf{R}^0\Psi_{\mathscr{P}_B}(L \otimes \mathscr{I}_V) \to \mathbf{R}^0\Psi_{\mathscr{P}_B}(L) \to \mathbf{R}^0\Psi_{\mathscr{P}_B}(L\,|_V) \to \mathbf{R}^1\Psi_{\mathscr{P}_B}(L \otimes \mathscr{I}_V) \to 0.$$

Since $\mathbf{R}^0\Delta_B(\mathscr{F}) = 0$, we have $\mathbf{R}\Psi_{\mathscr{P}_B}(L \otimes \mathscr{I}_V) = \mathbf{R}^1\Psi_{\mathscr{P}_B}(L \otimes \mathscr{I}_V)[-1]$ and

$$0 \to \mathbf{R}^0\Psi_{\mathscr{P}_B}(L) \to \mathbf{R}^0\Psi_{\mathscr{P}_B}(L\,|_V) \to \mathbf{R}^1\Psi_{\mathscr{P}_B}(L \otimes \mathscr{I}_V) \to 0. \tag{7}$$

Taking the duality of (6), we see that $\mathscr{F} \simeq \mathscr{E}xt^1((-1_B)^*\mathbf{R}^1\Psi_{\mathscr{P}_B}(L \otimes \mathscr{I}_V), \mathscr{O}_{\widehat{B}})$. Apply the functor $\mathbf{R}\Delta_B$ to (7), we have:

$$0 \to \mathbf{R}^0\Psi_{\mathscr{P}_B}(L\,|_V)^\vee \to \mathbf{R}^0\Psi_{\mathscr{P}_B}(L)^\vee \to \mathscr{F} \to 0.$$

We know that $W := \mathbf{R}^0\Psi_{\mathscr{P}_B}(L\,|_V)^\vee$ is a flat vector bundle and $\mathbf{R}^0\Psi_{\mathscr{P}_B}(L)^\vee$ is an ample vector bundle on B.

Let $\psi_L : \widehat{B} \to B$ be the isogeny induced by L. Assume that $h^0(\widehat{B}, L) = k \geq 1$. It follows that $\deg \psi_L = k^2$. Moreover, we know that $\psi_L^*\mathbf{R}^0\Psi_{\mathscr{P}_B}(L)^\vee = L^{\oplus k}$. Let $\widetilde{W} = \psi_L^*W$ and $\widetilde{\mathscr{F}} = \psi_L^*\mathscr{F}$. As a consequence,

$$0 \to \widetilde{W} \to L^{\oplus k} \to \widetilde{\mathscr{F}} \to 0. \tag{8}$$

We compute

$$\begin{aligned} ch(\widetilde{\mathscr{F}}) &= ch(L^{\oplus k}) - ch(\widetilde{W}) \\ &\equiv k \sum_{m \geq 1} \frac{1}{m!} L^m, \end{aligned}$$

where $\equiv$ means algebraic equivalence between algebraic cycles on $\widehat{B}$.

On the other hand, since $Z \to B$ is primitive, the induced étale cover $\widetilde{X} := X \times_A \widehat{A}$ is irreducible and so is $\widetilde{Z} := Z \times_A \widehat{A}$. Let $\rho' : \widetilde{X} \to \widetilde{Z}$ be the induced desingularization and $\widetilde{\mathscr{F}'}$ be the pullback of $\mathscr{F}$ on $\widetilde{X}$. We denote

$$i : \widetilde{X} \xrightarrow{\rho'} \widetilde{Z} \xrightarrow{j} \widehat{B} \ .$$

Thus

$$\mathbf{R}i_* \widetilde{\mathscr{F}'} = i_* \widetilde{\mathscr{F}'} = \widetilde{\mathscr{F}},$$

and $\widetilde{\mathscr{F}'} \otimes \omega_{\widetilde{X}}^{-1}$ is also weakly positive on $\widetilde{X}$. By Grothendieck–Riemann–Roch,

$$i_*(ch(\widetilde{\mathscr{F}'}) Td(\widetilde{X})) \equiv ch(\widetilde{\mathscr{F}}) \equiv k \sum_{m \geq 1} \frac{1}{m!} L^m.$$

Let $\operatorname{rank} \widetilde{\mathscr{F}} = \operatorname{rank} \widetilde{\mathscr{F}'} = k_1 \leq k$. Compute the degree 1 terms, we have $k_1 \widetilde{Z} \equiv kL$. Compute the degree 2 terms, we have

$$i_*\big(c_1(\widetilde{\mathscr{F}'}) - \frac{1}{2} k c_1(\omega_{\widetilde{X}})\big) \equiv \frac{1}{2} k L^2.$$

Since $\widetilde{\mathscr{F}'} \otimes \omega_{X'}^{-1}$ is weakly positive, $D := \det \widetilde{\mathscr{F}'} - k_1 K_{\widetilde{X}}$ is a psuedo-effective divisor on $\widetilde{X}$ (see for instance [21, Corollary 2.20]). Then $i_*(\frac{1}{2} k_1 K_{\widetilde{X}} + D) \equiv \frac{1}{2} k L^2$. Hence we write

$$j_*(\rho'_* K_{\widetilde{X}} + \rho'_* D') \equiv \frac{k}{k_1} L^2,$$

where D' is a pseudo-effective $\mathbb{Q}$-divisor on $\widetilde{X}$.

Since $\widetilde{Z}$ is normal, we have $\rho'_* K_{\widetilde{X}} = K_{\widetilde{Z}} = \mathscr{O}_{\widehat{B}}(\widetilde{Z}) \mid_{\widetilde{Z}}$ and hence $[j_* \rho'_* (K_{\widetilde{X}})] = (\frac{k}{k_1})^2 [L]^2 \in H^4(X, \mathbb{Q})$. As D is pseudo-effective, we see immediately that $k = k_1$ and $\mathscr{O}_{\widehat{B}}(\widetilde{Z})$ is algebraically equivalent to L. We note moreover that $\rho'_* \omega_{\widetilde{X}} = \omega_{\widetilde{Z}} \otimes \mathscr{I}$ for some ideal sheaf $\mathscr{I}$. Hence, for a general $Q \in \operatorname{Pic}^0(\widehat{B})$,

$$\chi(\widetilde{X}, \omega_{\widetilde{X}}) = h^0(\widetilde{X}, \omega_{\widetilde{X}} \otimes i^* Q) = h^0(\widetilde{Z}, \rho'_* \omega_{\widetilde{X}} \otimes Q) \leq h^0(\widetilde{Z}, \mathscr{O}_{\widehat{B}}(\widetilde{Z}) \mid_{\widetilde{Z}} \otimes Q) = k.$$

On the other hand, $\widetilde{X} \to X$ is an étale cover of degree k^2. Thus, $k = 1$, $\varphi_L : \widehat{A} \to A$ is an isomorphism and $Z \simeq \widetilde{Z}$ is a theta divisor. By (8), $\mathscr{F} = \omega_Z \otimes Q$ for some torsion line bundle $Q \in \widehat{B}$.

Remark 4.5. Instead of assuming that Z is smooth in codimension 1, we can conclude by a similar argument by simply assuming that $\rho_* K_X \equiv M \mid_D$ for some line bundle M on B.

4.2. General case

We consider the commutative diagram (5) in Corollary 3.10 and argue by induction on s.

If $Z \hookrightarrow B$ is not primitive, we can take an étale cover of $B' \to B$ such that, for an irreducible component Z' of $Z \times_B B'$, $Z' \hookrightarrow B'$ is primitive and Z' is birational to Z. Hence we will assume that $Z \hookrightarrow B$ is primitive.

Since D_i is the image of the natural morphism $Z \to K_i$, $D_i \hookrightarrow K_i$ is also primitive. Hence, $\rho^{-1}(D_1 \times \cdots \times D_s)$ is irreducible and $Z \simeq \rho^{-1}(D_1 \times \cdots \times D_s)$. Since Z is smooth in codimension 1, each D_i is smooth in codimension 1. Moreover each D_i is an ample divisor of the simple abelian variety K_i. We denote by $p : Z \to D_1$ the natural morphism. Then for $Q \in \mathrm{Pic}^0(B)$ general, we have

$$\chi(D_1, p_*(\mathscr{F} \otimes Q)) = h^0(D_1, p_*(\mathscr{F} \otimes Q)) = h^0(Z, \mathscr{F} \otimes Q) = 1.$$

By (4.1), each D_i is a theta divisor and the sheaf $p_*(\mathscr{F} \otimes Q)$ has rank 1. Then for a general fiber F of p, F is a subvariety of general type of $B_1 := \ker(B \to K_1)$ and is smooth in codimension 1. Note that $(\mathscr{F} \otimes Q) \mid_F$ is a strong Hodge sheaf on B_1 supported on F. Since $p_*(\mathscr{F} \otimes Q)$ has rank 1, $h^0(F, (\mathscr{F} \otimes Q) \mid_F) = 1$. It then follows that $\chi(F, \mathscr{F} \otimes Q \mid_F) = 1$.

By induction, F is birational to a product of theta divisors. Consider the induced morphisms

$$\begin{tikzcd} F \arrow[r, hook] \arrow[d, "\pi_F"] & B_1 \arrow[d, "\pi"] \\ D_2 \times \cdots \times D_s \arrow[r, hook] & K_2 \times \cdots \times K_s. \end{tikzcd}$$

Since π is an isogeny, we see immediately that π is an isomorphism and π_F is also an isomorphism. Thus $Z \simeq D_1 \times \cdots \times D_s$. Moreover, $\mathscr{F}$ is a torsion-free rank 1 sheaf and by induction, $\mathscr{F} \mid_{D_1 \times y} \simeq \omega_{D_1} \otimes Q_1$ for all $y \in D_2 \times \cdots \times D_s$, where Q_1 is a fixed torsion line bundle on D_1 and $\mathscr{F} \mid_{x \times D_2 \times \cdots \times D_s} \simeq \omega_{D_2 \times \cdots \times D_s} \otimes Q_2$ for all $x \in D_1$, where Q_2 is a fixed torsion line bundle on $D_2 \times \cdots \times D_s$. Indeed, Q_1 and Q_2 can be read from the cohomological support loci of $\mathscr{F}$. We then conclude that $\mathscr{F} = \omega_Z \otimes (Q_1 \boxtimes Q_2)$.

4.3. Proof of Theorem 1.2

Theorem 1.2 is a direct corollary of the proof of Theorem 4.4. We have already proved that all the assertions except the last one that the Albanese map is a fibration. To prove that the Albanese map is a fibration, simply note that in the first part of the proof, we have already established that $a_{X*}(\omega_X)$ has rank 1. Then the Albanese map has to be a fibration.

5. Fibrations over genus 2 curves

In this section, we take into considerations the map between the second Betti cohomology.

We first assume Theorem 5.1 and complete the proof of Theorem 1.3.

Proof. We use the commutative diagram (1).

We first claim that the restriction map $H^2(B,\mathbb{Q}) \to H^2(Z_X,\mathbb{Q})$ is not injective. Otherwise, since $Y = Z_X \times_B A_X$ and Z generates B, the map $H^2(A_X,\mathbb{Q}) \to H^2(Y,\mathbb{Q})$ is also injective. Moreover, since g is a surjective, the pull-back $g^* : H^i(Y,\mathbb{Q}) \to H^i(X,\mathbb{Q})$ induces an injective map $\mathrm{Gr}_i^W H^i(Y,\mathbb{Q}) \to H^i(X,\mathbb{Q})$, where $W^{\cdot}$ is the weight filtration on $H^*(Y,\mathbb{Q})$. Thus, $a_X^* : H^2(A_X,\mathbb{Q}) \to H^2(X,\mathbb{Q})$ is injective and hence is a contradiction.

Thus, the restriction map $H^2(B,\mathbb{Q}) \to H^2(Z_X,\mathbb{Q})$ is not injective and Z_X generates B. By Theorem 5.1 below, we have a fibration $h : \overline{Z}_X \to C$, where $\overline{Z}_X$ is the normalization of Z_X and C is a smooth projective curve of genus 2. Since $f : X \to Z_X$ factors through the normalization of Z_X, we then have a fibration $\varphi : X \to C$. $\square$

Theorem 5.1. *Let $Z \hookrightarrow B$ be a subvariety of general type generating B and let $\overline{Z}$ be the normalization of Z. Let $\mathscr{F}$ be a strong Hodge sheaf on Z. Assume that $\chi(Z,\mathscr{F}) = 1$ and the restriction map $H^2(B,\mathbb{Q}) \to H^2(Z,\mathbb{Q})$ is not injective. Then there exists a fibration $h : \overline{Z} \to C$ to a smooth projective genus 2 curve C.*

Proof. We argue by induction on $\operatorname{codim}_B Z$. If $\operatorname{codim}_B Z = 1$, Z is an ample divisor of B. By Lefschetz hyperplane theorem, the restriction map $H^i(B,\mathbb{Q}) \to H^i(Z,\mathbb{Q})$ is injective for all $0 \le i \le \dim Z$. Hence $\dim Z = 1$. By Lemma 4.2, $\overline{Z}$ is a smooth projective curve of genus 2.

In the following assume that $\operatorname{codim}_B Z = k \ge 2$ and that Theorem 5.1 holds for subvarieties of B whose codimension is less than k.

Pick two components $\widehat{T}_1 = P_1 + \widehat{B}_1$ and $\widehat{T}_2 = P_2 + \widehat{B}_2$ of $V^1(\mathscr{F})$ such that $\widehat{B}_1 + \widehat{B}_2 = \widehat{B}$. Consider the morphisms $h_1^\flat : Z \to Z_1^\flat$ and $h_2 : Z \to Z_2^\flat$ as in Lemma 3.2.

Claim: *Either*

$$\varphi_1 : H^2(B_1^\flat,\mathbb{Q}) \to H^2(Z_1^\flat,\mathbb{Q})$$

is not injective or

$$\varphi_2 : H^2(B_2^\flat,\mathbb{Q}) \to H^2(Z_2^\flat,\mathbb{Q})$$

is not injective.

We argue by contradiction. Assume that both φ_1 and φ_2 are injective. Let $\widehat{K}$ be the neutral component of $\widehat{B_1^\flat} \cap \widehat{B_2^\flat}$. Then the induced morphism $B \to B_1^\flat \times_K B_2^\flat$ is an isogeny. We also take $B_1^\flat \to K_1'$ and $B_2^\flat \to K_2'$ be quotients with connected fibers such that the indcued morphisms $B_i^\flat \to K_i' \times K$ are isogenies for $i = 1, 2$.

Note that

$$H^2(B,\mathbb{Q}) = \big(H^2(B_1^\flat,\mathbb{Q}) + H^2(B_2^\flat,\mathbb{Q})\big) \oplus \big(H^1(K_1',\mathbb{Q}) \wedge H^1(K_2',\mathbb{Q})\big).$$

Let $0 \neq \alpha = w + v$ in the kernel of the restriction map $\varphi : H^2(B,\mathbb{Q}) \to H^2(Z,\mathbb{Q})$, where $w \in H^2(B_1^\flat,\mathbb{Q}) + H^2(B_2^\flat,\mathbb{Q})$ and $v \in H^1(K_1',\mathbb{Q}) \wedge H^1(K_2',\mathbb{Q})$. Moreover we can write $H^2(B_i^\flat,\mathbb{Q}) = W_i \oplus H^2(K,\mathbb{Q})$, where $W_i = H^2(K_i',\mathbb{Q}) \oplus \big(H^1(K_i',\mathbb{Q}) \wedge H^1(K,\mathbb{Q})\big)$. Then

$$H^2(B_1^\flat,\mathbb{Q}) + H^2(B_2^\flat,\mathbb{Q}) = W_1 \oplus W_2 \oplus H^2(K,\mathbb{Q}),$$

and we suppose that $w = w_1 + w_2 + w_3$, where $w_i \in W_i$ for $i = 1, 2$ and $w_3 \in H^2(K,\mathbb{Q})$.

We then take a smooth models Z' of Z, Z_i' of $Z_i^\flat$ for $i = 1, 2$ and consider the maps

$$\begin{tikzcd} Z' \arrow[r, "\rho"] \arrow[d, "h_i'"] & B \arrow[r] \arrow[d, "p_i"] & K_{3-i}' \\ Z_i' \arrow[r, "\rho_i"] & B_i^\flat. & \end{tikzcd}$$

Since φ_i is injective and the Hodge structures on $H^2(B_i^\flat,\mathbb{Q})$ is pure, $\rho_i^* : H^2(B_i^\flat,\mathbb{Q}) \to H^2(Z_i',\mathbb{Q})$ is also injective, for $i = 1, 2$. Note that α is also in the kernel of $\rho^* : H^2(B,\mathbb{Q}) \to H^2(Z',\mathbb{Q})$.

We take a ample class $l \in H^2(B,\mathbb{Q})$. Let s_i be the dimension of a general fiber of p_i. Then

$$0 = h_{i*}'\rho^*((w+v) \cup l^{2s_i}) = h_{i*}'\rho^*(w \cup l^{2s_i}) = M_i\rho_i^*(w_i + w_3),$$

for some positive number M_i. Since $\rho_i^* : H^2(B_i^\flat,\mathbb{Q}) \to H^2(Z_i',\mathbb{Q})$ is injective, we conclude that $w_i = 0$ for $i = 1, 2$, and 3. Thus $w = 0$.

Let Z_3 be the image of the morphisms $Z \hookrightarrow A \to K$. Since Z generates B, Z_i generates B_i. Hence for a general fiber F_i of $Z_i \to Z_3$, the natural map $H^1(K_i',\mathbb{Q}) \to H^1(F_i,\mathbb{Q})$ is injective. Let F be a general fiber of $Z \to Z_3$. Then we have natural morphisms $F \twoheadrightarrow F_1 \times F_2 \to K_1' \times K_2'$. Since the map $H^1(K_1',\mathbb{Q}) \wedge H^1(K_2',\mathbb{Q}) \to H^2(F_1 \times F_2,\mathbb{Q})$ is injective and the Hodge structure on $H^1(K_1',\mathbb{Q}) \wedge H^1(K_2',\mathbb{Q})$ is pure, we conclude that the map

$$H^1(K_1',\mathbb{Q}) \wedge H^1(K_2',\mathbb{Q}) \to H^2(F,\mathbb{Q})$$

is also injective. This map factors through $\varphi\mid_{H^1(K_1',\mathbb{Q})\wedge H^1(K_2',\mathbb{Q})}$: $H^1(K_1',\mathbb{Q})\wedge H^1(K_2',\mathbb{Q})\to H^2(Z,\mathbb{Q})$, hence $\varphi\mid_{H^1(K_1',\mathbb{Q})\wedge H^1(K_2',\mathbb{Q})}$ is also injective and $v=0$, which is a contradiction.

Conclusion: We may assume that φ_1 is not injective. Moreover, $Z_1^\flat\hookrightarrow B_1^\flat$ is a subvariety of general type and Z_1 generates B_1 and for $Q\in\widehat{B}$ a general torsion point, $\mathscr{F}_Q:=h^\flat_{1*}(\mathscr{F}\otimes Q)$ is a Hodge sheaf supported on $Z_1^\flat$ with $\chi(Z_1^\flat,\mathscr{F}_Q)=1$. Hence by induction, there exists a fibration $\overline{Z_1^\flat}\to C$ to a smooth projective genus 2 curve. Hence we have the induced fibration $h:\overline{Z}\to C$. $\square$

Theorem 5.2. *Under the assumption of Theorem 5.1, let $m=\dim\ker(H^2(B,\mathbb{Q})\to H^2(Z,\mathbb{Q}))$. Then m is divisible by 5. Let $m=5k$. Then there exists a fibration $\overline{Z}\to C_1\times\cdots\times C_k$, where C_i is a smooth projective curve of genus 2 for each $1\le i\le k$.*

Proof. Let $s=\operatorname{codim}_B Z$. Take components $\widehat{T}_i=P_i+\widehat{B}_i$, $1\le i\le s$, of $V^1(\mathscr{F})$ such that $\widehat{B}_i+\widehat{B}_j=\widehat{B}$ for all $i\neq j$. Then, as in the proof of 5.1, by induction on s, we actually show that for some K_i defined as in Theorem 3.6, the map $H^2(K_j,\mathbb{Q})\to H^2(D_j,\mathbb{Q})$ is not injective for some j. Since D_j is an ample divisor of K_j, we conclude that $\dim K_j=2$ and D_j is a curve. Then by Lemma 4.2, the normalization C_j of D_j is a smooth projective curve of genus 2. Moreover, by Lemma 3.8, we have a commutative diagram:

$$\begin{tikzcd}
\overline{Z} \arrow[r, "\text{normalization}"] \arrow[d, "\text{abelian etale cover}"'] & Z \arrow[r, hook] \arrow[d] & B \arrow[d, "\rho", "\text{isogeny}"'] \\
C_j\times\overline{Z_j} \arrow[r, "\text{normalization}"] & D_j\times Z_j \arrow[r, hook] & K_j\times B_j.
\end{tikzcd}$$

Since B is smooth, $W:=\ker(H^2(B,\mathbb{Q})\to H^2(Z,\mathbb{Q}))=\ker(H^2(B,\mathbb{Q})\to H^2(\overline{Z},\mathbb{Q}))$. Moreover, ρ is an isogeny, hence $W\simeq\ker(H^2(K_j\times B_j,\mathbb{Q})\to H^2(C_j\times\overline{Z_j},\mathbb{Q}))$. Since $H^2(K_j\times B_j,\mathbb{Q})=H^2(K_j,\mathbb{Q})\oplus\big(H^1(K_j,\mathbb{Q})\wedge H^1(B_j,\mathbb{Q})\big)\oplus H^2(B_j,\mathbb{Q})$. As Z generates B, we conclude that

$$W\simeq\ker(H^2(K_j,\mathbb{Q})\to H^2(C_j,\mathbb{Q}))\bigoplus\ker(H^2(B_j,\mathbb{Q})\to H^2(Z_j,\mathbb{Q})).$$

Hence $\dim W=5+\dim\ker(H^2(B_j,\mathbb{Q})\to H^2(Z_j,\mathbb{Q}))$. Since $Z_j\hookrightarrow B_j$ also satisfies the assumption of Theorem 5.1. We use induction to construct the morphism $\bar{Z}\to C_1\times\ldots\times C_k$. To see this map is a fibration, it suffices to show that a general fiber is connected. For this purpose, one only need to show that push forward of $\mathscr{F}$ has rank 1, or equivalently, the restriction of $\mathscr{F}$ to a general fiber is one dimensional global sections. One can prove this using induction. For the morphism $f_1:\bar{Z}\to C_1$, this is done in Lemma 4.2. Then we can work with a general fiber F of f_1, which is of general type, maps to the product $C_2\times\ldots\times C_k$ and carries a strong Hodge sheaf $\mathscr{F}|_F$ with $h^0(F,\mathscr{F})=1$. $\square$

With this observation it is very easy to prove Corollary 1.4.

Proof of Corollary 1.4. For a general discussion of the de Rham fundamental group, we refer the readers to [1]. For our purpose, it suffices to know that if the pull-back on cohomology of a morphism $f : X \to Y$ between smooth compact Kähler manifolds induces an isomorphism $f^* : H^1(Y, \mathbb{Q}) \to H^1(X, \mathbb{Q})$ and an injection $f^* : H^2(Y, \mathbb{Q}) \to H^2(X, \mathbb{Q})$, then f_* induces an isomorphism on de Rham fundamental groups. A direct consequence of this observation is a result of Campana that a resolution of singularities of the Albanese image of a compact Kähler manifold X computes the de Rham fundamental group $\pi_1(X) \otimes \mathbb{Q}$.

Let $5s = \dim\ker(H^2(A_X, \mathbb{Q}) \to H^2(Z_X, \mathbb{Q}))$. By Theorem 5.2 and its proof, we have a commutative diagram

$$\begin{tikzcd}
\overline{Z} \arrow[r, "\text{normalization}"] \arrow[d, "\text{abelian etale cover}"'] & Z \arrow[r, hook] \arrow[d] & B \arrow[d, "\text{isogeny}"', "\rho"] \\
C_1 \times \cdots \times C_s \times \overline{Z'} \arrow[r, "\text{normalization}"] & D_1 \times \cdots \times D_s \times Z' \arrow[r, hook] & K_1 \times \cdots \times K_i \times B',
\end{tikzcd}$$

where C_i is a smooth projective curve of genus 2 for each $1 \le i \le s$ and the map $H^2(B', \mathbb{Q}) \to H^2(Z', \mathbb{Q})$ is injective.

We apply the above observation to a resolution of singularities of Z and Z', $\widetilde{Z} \to C_1 \times \cdots \times C_s \times \widetilde{Z'}$. Since $\bar{Z}$ is an abelian étale cover of the product $C_1 \times \cdots \times C_s \times \widetilde{\bar{Z}}'$, the induced map is an isomorphism on H^1 and injective on H^2 (for the resolutions). Hence $\pi_1(X) \otimes \mathbb{Q} \simeq \pi_1(\widetilde{Z}) \otimes \mathbb{Q} \simeq (\pi_1(C_1) \otimes \mathbb{Q})^s \times (\pi_1(\widetilde{Z'}) \otimes \mathbb{Q}) \simeq (\pi_1(C_1) \otimes \mathbb{Q})^s \times (\pi_1(B') \otimes \mathbb{Q})$. $\square$

6. Fake tori

In this section, we will always assume that X is a fake torus of dimension n and consider the commutative diagram (1), with Z replaced by Z_X in the following, namely:

$$\begin{tikzcd}
X \arrow[d, "g"] \arrow[dr, "a_X"] \arrow[dd, bend right=60, "f"'] & \\
Y \arrow[r, hook] \arrow[d, "h"] & A_X \arrow[d, "p"] \\
Z_X \arrow[r, hook] & B,
\end{tikzcd}$$

Note that $Y = Z_X \times_B A_X$. We summarize what we know about Z_X. Let $s = \operatorname{codim}_{A_X} Y = \operatorname{codim}_B Z_X$. Note that since Y is the Albanese image of X, both $Y \hookrightarrow A_X$ and $Z_X \hookrightarrow B$ are primitive. By Theorem 5.1, Remark 5.2, and Corollary 4.3, we have:

(1) $\dim B \geq 2s$. If equality holds, $Z_X \simeq C_1 \times \cdots \times C_s$, where each C_i is a smooth projective curve of genus 2 and $f_*\omega_X = \omega_Z \otimes Q$ for some torsion line bundle $Q \in \widehat{B}$;
(2) if $s = 1$, then Z_X is a smooth projective curve of genus 2;
(3) if $s = 2$, then there exists a genus two curve C and an ample divisor $D \hookrightarrow K$ with a commutative diagram:

$$\begin{tikzcd} \overline{Z_X} \arrow[r, hook] \arrow[d] & B \arrow[d, "\rho", "\text{isogeny}"'] \\ C \times D \arrow[r, hook] & JC \times K. \end{tikzcd}$$

Here is a list of possibile Z_X in low dimensions.

Corollary 6.1.

1) *If $n = 2$ or 3, Z_X is always a curve of genus 2.*
2) *If $n = 4$, either $s = 1$ or $s = 2$ and then Z_X is isomorphic to a product $C_1 \times C_2$ of two smooth curves of genus 2.*
3) *If $n = 5$, either $s = 1$, or $s = 2$ and Z_X is isomorphic to a product $C_1 \times C_2$ of two smooth curves of genus 2 or is an étale cover of $C \times D$, where D is an ample divisor of an abelian 3 fold.*
4) *If $n = 6$, either $s = 1$, or $s = 2$ and Z_X is isomorphic to a product $C_1 \times C_2$ of two smooth curves of genus 2 or is an étale cover of $C \times D$ as in 3), or $s = 3$ and Z_X is isomorphic to a product $C_1 \times C_2 \times C_3$ of three smooth curves of genus 2.*

We now focus on the case $s = 1$.

Lemma 6.2. *If $s = 1$, then we write $Z_X = C$ a smooth curve of genus 2. We have $f_*(\omega_X) = \omega_C \otimes Q$ for some nontrivial torsion line bundle Q on C. Morevoer,*

1) *we have a decomposition:*

$$g_*\omega_X = h^*(\omega_C \otimes Q) \bigoplus_t (q_t^* \mathscr{Q}_t \otimes Q_t), \tag{9}$$

where for each t, $q_t : A_X \to T_t$ is a quotient of abelian varieties with connected fibers, $\mathscr{Q}_t$ is a M-regular sheaf on T_t, $Q_t \notin \widehat{T}_t$ is a non-trivial torsion line bundle;
2) *let $\widetilde{C} \to C$ be the cyclic étale cover induced by Q and let $\widetilde{X} = X \times_C \widetilde{C}$ be the induced étale cover of X, then $\widetilde{X}$ is of maximal Albanese dimension;*
3) *let F be a general fiber of f, then F is of maximal Albanese fibration and $p_g(F) = 1$ hence $q(F) = \dim F$.*

Proof. Note that by the main theorem in [13],

$$g_*\omega_X \simeq \bigoplus_t q_t^*\mathscr{F}_t \otimes Q_t,$$

where each $\mathscr{F}_t$ is an M-regular coherent sheaf supported on the complex torus T_t, each $q_t : A_X \to T_t$ is surjective with connected fibers, and each Q_t is a torision line bundle on A_X. Since $h^0(Y, g_*\omega_X) = 1$, there exists a unique t_0 such that $q_{t_0}^*\mathscr{F}_{t_0} \otimes Q_{t_0}$ has a non-trivial global section. Note that the natural morphism $h^*(h_*g_*\omega_X) = h^*(f_*(\omega_X)) = h^*(\omega_C \otimes Q) \to g_*\omega_X$ is injective. Since $h^0(Y, h^*(\omega_Z \otimes Q))$ is also 1, this natural injective morphism factors through an injective morphism

$$h^*(\omega_C \otimes Q) \to q_{t_0}^*\mathscr{F}_{t_0} \otimes Q_{t_0}.$$

Since $h^0(q_{t_0}^*\mathscr{F}_{t_0} \otimes Q_{t_0})$ is non-zero, the torsion sheaf Q_{t_0} is contained in $V^0(q_{t_0}^*\mathscr{F}_{t_0} \otimes Q_{t_0}) = \widehat{T}_{t_0}$ and we may write $q_{t_0}^*\mathscr{F}_{t_0} \otimes Q_{t_0} = q_{t_0}^*(\mathscr{F})$. Since $V^0(h^*(\omega_C \otimes Q)) = \widehat{B}$ is contained in $V^0(q_{t_0}^*\mathscr{F}_{t_0} \otimes Q_{t_0}) = \widehat{T}_{t_0}$, the morphism $p : A_X \to B$ factors through $q_{t_0} : A_X \to T_{t_0}$ and we have the injective morphism on T_{t_0}:

$$\varphi : q^*(\omega_C \otimes Q) \to \mathscr{F}_{t_0},$$

where $q : T_{t_0} \to B$ is the natural surjective morphism. Denote by $\mathscr{Q}$ the kernel of φ. Since $\mathscr{F}_{t_0}$ is M-regular and $q^*(\omega_X \otimes Q)$ is GV, we conclude that $\mathscr{Q}$ is also an M-regular coherent sheaf. On the other hand, $h^0(A_{t_0}, \mathscr{Q}) = 0$. Hence $\mathscr{Q} = 0$ and φ is an isomorphism. Therefore, we may write

$$g_*\omega_X = h^*(\omega_C \otimes Q) \bigoplus_t (q_t^*\mathscr{Q}_t \otimes Q_t), \tag{10}$$

where for each t, Q_t is a torsion line bundle on X. Since $h^0(Y, g_*\omega_X) = h^0(Y, h^*(\omega_Z \otimes Q)) = 1$, none of the Q_t's is contained in $\widehat{T}_t$.

Note that $f_*\omega_X = \omega_C \otimes Q$ is of rank 1. Hence $p_g(F) = \operatorname{rank} f_*\omega_X = 1$. On the other hand, let $\pi : \widetilde{C} \to C$ be the étale cover of C induced by the torsion line bundle Q and let $\widetilde{X}$ and $\widetilde{Y}$ be the induced étale covers $X \times_C \widetilde{C}$ and $Y \times_C \widetilde{C}$. We then consider the fibration $\widetilde{f} : \widetilde{X} \xrightarrow{\widetilde{g}} \widetilde{Y} \xrightarrow{\widetilde{h}} \widetilde{C}$. Let $g' : \widetilde{X} \to Y'$ be the Stein facorization of $\widetilde{g}$ and after birational modifications, we may suppose that Y' is smooth. By the first part, we know that $\widetilde{h}^*\omega_{\widetilde{C}}$ is a direct summand of $\widetilde{g}_*\omega_{\widetilde{X}}$. Hence $h^{n-1}(\widetilde{Y}, \widetilde{g}_*\omega_{\widetilde{X}}) > 0$. Thus, $h^{n-1}(Y', g'_*\omega_{\widetilde{X}}) > 0$.

By Kollár's splitting,

$$\begin{aligned} q(\widetilde{X}) = h^{n-1}(\widetilde{X}, \omega_{\widetilde{X}}) &= h^{n-1}(Y', g'_*\omega_{\widetilde{X}}) + h^{n-2}(Y', R^1g'_*\omega_{\widetilde{X}}) \\ &= h^{n-1}(Y', g'_*\omega_{\widetilde{X}}) + h^{n-2}(Y', \omega_{Y'}) = h^{n-1}(Y', g'_*\omega_{\widetilde{X}}) + q(Y'). \end{aligned}$$

Hence $q(\widetilde{X}) > q(Y')$. Since $g' : \widetilde{X} \to Y'$ is a fibration, Y' is of maximal Albanese dimension and $\dim Y' = \dim X - 1$, we conclude that $\widetilde{X}$ is of maximal Albanese dimension and hence so is F.

Since F is of maximal Albanese dimension and $p_g(F) = 1$, we know that $h^i(F, \mathscr{O}_F) = h^0(F, \Omega_F^i) = \binom{\dim F}{i}$ (see for instance [2, Proposition 6.1]). $\square$

Theorem 6.3. *Assume that X is a fake torus of dimesnion $n \geq 3$ with $\dim Y = n - 1$. Then*

(1) *let $\widetilde{X} = X \times_C \widetilde{C}$ defined as in 2) of Lemma 6.2, then $a_{\widetilde{X}}$ is a finite morphism onto its image;*
(2) *X is not of general type.*

Proof. Note that $Z = C$ is a smooth curve of genus 2 and $Y = C \times_{JC} A_X$ and F is of maximal Albanese dimension with $p_g(F) = 1$. We know from Lemma 6.2 that $q(\widetilde{X}) \geq q(\widetilde{Y}) + 1 = q(\widetilde{C}) + n - 1$. Moreover, F is also a general fiber of $\widetilde{X} \to \widetilde{C}$. By Lemma 6.2, $q(F) = n-1$. Hence $q(\widetilde{X}) - q(\widetilde{C}) \leq q(F) = n-1$. Thus, $q(\widetilde{X}) = q(\widetilde{C}) + n - 1$.

We now consider the induced fibration $\widetilde{f} : \widetilde{X} \to \widetilde{C}$ in Lemma 6.2 and the following commutative diagram:

$$\begin{tikzcd}
\widetilde{X} \arrow[r, "G"] \arrow[d] & X \arrow[d, "g"] \\
\widetilde{Y} \arrow[r, "G"] \arrow[d] & Y \arrow[d, "h"] \\
\widetilde{C} \arrow[r, "G"] & C.
\end{tikzcd}$$

Let G be the Galois group of the cover $\widetilde{C} \to C$. We define $\widetilde{M} := \widetilde{C} \times_{J\widetilde{C}} A_{\widetilde{X}}$ and let K be the neutral component of the kernel of $A_{\widetilde{X}} \to J\widetilde{C}$. We then have a natural generically finite morphism $\widetilde{X} \to \widetilde{M}$ and a surjective morphism $\widetilde{M} \to \widetilde{Y}$. Note that these morphisms are G-equivariant. Let $M = \widetilde{M}/G$. We then have the induced morphisms on the quotient: $g : X \xrightarrow{\rho} M \xrightarrow{\varphi} Y$.

We claim that $h^2(M, \mathbb{Q}) = h^2(X, \mathbb{Q})$.

Note that $H^2(M, \mathbb{Q}) = H^2(\widetilde{M}, \mathbb{Q})^G$ and

$$H^2(\widetilde{M}, \mathbb{Q}) = H^2(\widetilde{C}, \mathbb{Q}) \oplus \big(H^1(\widetilde{C}, \mathbb{Q})) \wedge H^1(K, \mathbb{Q})\big) \oplus H^2(K, \mathbb{Q}).$$

Let K' be the neutral component of the kernel of $A_X \to JC$, which is also a fiber of h. Then we have the quotient morphism $K \to K'$ by G. Hence $H^1(K, \mathbb{Q})^G \simeq H^1(K', \mathbb{Q})$ and there exists only one non-trivial character χ of G such that $H^1(K, \mathbb{Q})^\chi \neq 0$ and hence $\dim H^1(K, \mathbb{Q})^\chi = 2$.

Thus

$$\begin{aligned}H^2(\widetilde{M},\mathbb{C})^G &= H^2(\widetilde{C},\mathbb{C})^G \oplus \left(H^1(\widetilde{C},\mathbb{C}))^G \wedge H^1(K,\mathbb{C})^G\right) \oplus \left(H^1(\widetilde{C},\mathbb{C}))^{\chi^*} \wedge H^1(K,\mathbb{C})^{\chi}\right)\\ &\quad \oplus H^2(K,\mathbb{C})^G\\ &= H^2(C,\mathbb{C}) \oplus \left(H^1(C,\mathbb{C})) \wedge H^1(K',\mathbb{C})\right) \oplus \left(H^1(\widetilde{C},\mathbb{C}))^{\chi^*} \wedge H^1(K,\mathbb{C})^{\chi}\right)\\ &\quad \oplus H^2(K,\mathbb{C})^G.\end{aligned}$$

We also have

$$H^2(Y,\mathbb{Q}) = H^2(C,\mathbb{Q}) \oplus \left(H^1(C,\mathbb{Q})) \wedge H^1(K',\mathbb{Q})\right) \oplus H^2(K',\mathbb{Q}).$$

It is easy to see that $h^2(K,\mathbb{Q})^G = \dim(\wedge^2 H^1(K,\mathbb{Q}))^G = h^2(K',\mathbb{Q}) + 1$ and, for any non-trivial character ψ, $\dim H^1(\widetilde{C},\mathbb{Q})^{\psi} = 2$. Hence $h^2(M,\mathbb{Q}) = \dim H^2(\widetilde{M},\mathbb{Q})^G = h^2(Y,\mathbb{Q}) + 1 + 4 = h^2(X,\mathbb{Q})$.

Since $h^2(X,\mathbb{Q}) = h^2(M,\mathbb{Q})$ and both X and M are smooth projective varieties, the surjective morphism $\rho : X \to M$ is finite. Then so is the induced morphism on the étale covers $\widetilde{X} \to \widetilde{M}$. Hence $a_{\widetilde{X}}$ is finite onto its image. However, $\chi_{top}(\widetilde{X}) = \chi_{top}(X) = 0$. By [4, Theorem 1 of the appendix], $\widetilde{X}$ can not be of general type and neither can X. □

Proposition 6.4. *Let X be a fake torus of dimension 2. Then X is a minimal projective surface with $\kappa(X) = 1$. Furthermore, there exists a finite abelian group G acting faithfully on an elliptic curve E and on a smooth projective curve D of genus ≥ 3 such that $E/G \simeq \mathbb{P}^1$, $D/G = C$ is a smooth curve of genus 2, and X is isomorphic to the diagonal quotient $(D \times E)/G$.*

Proof. Let $\widetilde{X} \to X$ and $\widetilde{C} \to C$ be the étale covers induced by Q as in 2) of Lemma 6.2. Since $a_{\widetilde{X}}$ is finite and X is not of general type, we conclude that a general fiber of $\widetilde{X} \to \widetilde{C}$ is isogenous to the kernel $A_{\widetilde{X}} \to J\widetilde{C}$. Hence $f : X \to C$ is isotrivial with a smooth fiber isomorphic to an elliptic curve E. Moreover, by Kawamata's theorem ([9, Theorem 15]), we know that a fiber of $\widetilde{X} \to \widetilde{C}$ is either smooth or is a multiple of a smooth curve. Hence both $\widetilde{X} \to \widetilde{C}$ and $f : X \to C$ are quasi-bundles in the terminology of [19].

By the main result of [19], we conclude that there exists a Galois cover $D \to C$ with Galois group G such that $D \times_C X \simeq D \times E$. Moreover, $X \simeq (D \times E)/G$, where G acts faithfully on both factors and the action on the product is the diagonal action. Since $h^1(X,\mathscr{O}_X) = 2 = h^1(C,\mathscr{O}_C)$, we conclude that $E/G \simeq \mathbb{P}^1$.

On the other hand, any smooth surface isomorphic to $(D \times E)/G$ with D/G a smooth projective curve of genus 2 and $E/G \simeq \mathbb{P}^1$ is a fake torus of dimension 2. □

A fake torus of dimension 3 has Kodaira dimension 1 or 2. With some efforts, in both cases, we can prove a similar structural result as in the surface case. Here is a typical example.

Example 6.5. Let G be an abelian group acting faithfully on an elliptic curve E by translation. Let S be a smooth projective surface such that G acts faithfully on S and S/G is a fake torus of dimension 2. Then the diagonal quotient $(S \times E)/G$ is a fake torus in dimension 3.

When X is a fake torus of dimension 4 and $Y = Z = C_1 \times C_2$ is a product of two smooth curves of genus 2. We know that $f_*\omega_X = \omega_Z \otimes Q$. Hence f is a fibration and a general fiber F of f has $p_g(F) = 1$. Moreover, we can verify by Kollár's splitting and the Hodge diamond of X that $h^0(Z, R^1 f_*\omega_X) = 4$ and $h^1(Z, R^1 f_*\omega_X) = 2$. Hence F is an irregular surface. We do not know whether or not F is an abelian surface. In general, for a fake torus X, we do not have a systematic way to study the fibers of a_X.

Finally we conclude by the following questions:

Question 6.6.

(1) Does there exist fake tori of general type?
(2) Does there exist a fake torus X such that Z_X is not a product of genus 2 curves?

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