HASSE PRINCIPLE FOR THREE CLASSES OF VARIETIES OVER GLOBAL FUNCTION FIELDS

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Abstract
We give a geometric proof that the Hasse principle holds for the following varieties defined over global function fields: smooth quadric hypersurfaces, smooth cubic hypersurfaces of dimension at least 4 *in characteristic at least* 7*, and smooth complete intersections of two quadrics, which are of dimension at least* 3*, in odd characteristics.*

Contents

1. Introduction . . . 3349
2. Semistable models over global function fields . . . 3353
3. Asymptotically canonical sequence of spaces of sections . . . 3377
4. Hasse principle for quadrics . . . 3385
5. Hasse principle for cubics . . . 3388
6. Hasse principle and weak approximation for complete intersection of two quadrics . . . 3398
Appendices . . . 3411
A. Weak approximation for cubic hypersurfaces defined over function fields of curves . . . 3411
B. Fundamental group of rationally connected fibrations . . . 3416
References . . . 3422

1. Introduction

Given a variety X defined over a nonalgebraically closed field K, a fundamental question is to find necessary and sufficient conditions for X to have a K-rational point. When the field K is a global field (i.e., a number field or the function field of a curve defined over a finite field), we have a natural inclusion of the set $X(K)$ of rational points of X into the set $X(\mathbb{A})$ of adelic points of X. A classical result

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of Hasse–Minkowski says that if X is a smooth quadric hypersurface, then $X(K)$ is nonempty if and only if $X(\mathbb{A})$ is nonempty.

We say that a smooth projective variety defined over a global field *satisfies the Hasse principle* if the condition that $X(\mathbb{A})$ is nonempty implies that $X(K)$ is nonempty. The above-mentioned result of Hasse–Minkowski can be rephrased as saying that a smooth quadric hypersurface satisfies the Hasse principle. A natural question is to look for other varieties which satisfy the Hasse principle over global fields.

The Hasse principle fails in general. For (separably) rationally connected varieties, Colliot-Thélène conjectured that the Brauer–Manin obstruction is the only obstruction to the Hasse principle (see [9, p. 233] and see [7, p. 174] for the case of number fields).

In this article, we prove the following results.

THEOREM 1.1
The Hasse principle holds for the following varieties:
(1) *all smooth quadric hypersurfaces of positive dimension defined over global function fields;*
(2) *all smooth cubic hypersurfaces in $\mathbb{P}^n, n \geq 5$, defined over global function fields of characteristic at least 7;*
(3) *all smooth complete intersections of two quadric hypersurfaces in $\mathbb{P}^n, n \geq 5$, defined over global function fields of odd characteristic.*

As mentioned at the beginning of this article, the case of quadric hypersurfaces is well known and due to Hasse and Minkowski. So the main interest in presenting a different proof in the current article is the fact that one can find a geometric proof of this number-theoretic result, the method of which also works in other situations. We also use this proof as a warm-up for the later, more technical proofs, since the argument contains the essential key ideas and yet one is not overwhelmed by the technicalities appearing in the other two cases.

Browning and Vishe [4] use the circle method to prove the Hasse principle and weak approximation for smooth cubic hypersurfaces in $\mathbb{P}^n, n \geq 7$, defined over $\mathbb{F}_q(t)$ of characteristic at least 5. The dimension of the cubic hypersurfaces considered in this article is lower, and the global function field considered is not necessarily $\mathbb{F}_q(t)$.

We would also like to mention the following two related results on the Hasse principle for cubics. Colliot-Thélène [7, Théorèmes 3.1, 3.2] proved that if the field $\mathbb{F}_q$ has no cubic root of unity and the characteristic is not 3, then the diagonal cubic hypersurface $a_0X_0^3 + a_1X_1^3 + \cdots + a_4X_4^3 = 0, a_i \in \mathbb{F}_q(B)$ in $\mathbb{P}^4$ satisfies the Hasse principle, where B is a smooth projective curve. Colliot-Thélène and Swinnerton-

Dyer [12] proved that the Hasse principle holds for families of cubic surfaces defined by $f + tg = 0 \subset \mathbb{P}^3 \times \mathbb{A}^1$.

Using decent and fibration methods, Colliot-Thélène, Sansuc, and Swinnerton-Dyer [10], [11] proved that the Hasse principle holds for complete intersections of two quadrics in $\mathbb{P}^n, n \geq 8$, over number fields. Their methods should prove a similar statement (i.e., for complete intersections of two quadrics in $\mathbb{P}^n, n \geq 8$) over global function fields of odd characteristic. In this article, we are able to prove the result for complete intersections of two quadrics in $\mathbb{P}^n, n \geq 5$. Note that when n is 4 (i.e., a del Pezzo surface of degree 4), there are Brauer–Manin obstructions to the Hasse principle.

We should remark that global function fields $\mathbb{F}_q(B)$ are C_2-fields, which implies that a complete intersection of degree $d_1, \dots, d_c$ in $\mathbb{P}^n$ with $\sum_{i=1}^c d_i^2 \leq n$ always has a rational point. That is, the Hasse principle trivially holds in this degree range. In particular, a quadric hypersurface (resp., a cubic hypersurface, or a complete intersection of two quadrics) in $\mathbb{P}^n$ satisfies the Hasse principle if n is at least 4 (resp., 9 or 8). Therefore, Theorem 1.1 is new for cubic hypersurfaces and complete intersections of two quadrics in the low-dimensional case.

We would like to point out that the approach to the Hasse principle in the article is geometric, as opposed to the analytic or algebraic techniques used in establishing previously mentioned results. Once we know that the Hasse principle holds for smooth complete intersections of two quadrics in $\mathbb{P}^n, n \geq 5$, it is very easy to deduce weak approximation results by using a geometric argument as in [10].

THEOREM 1.2

Smooth complete intersections of two quadric hypersurfaces in $\mathbb{P}^n$, $n \geq 5$, defined over global function fields of odd characteristic satisfy weak approximation.

The more difficult problem of weak approximation on cubic hypersurfaces and a del Pezzo surface of degree 4 will be discussed in a subsequent paper.

Our approach to the above theorems is geometric in nature. Any variety defined over $\mathbb{F}_q(B)$ admits (nonuniquely) a model over B, $\pi : \mathcal{X} \to B$, and rational points correspond to sections of π. Thus we need to establish the existence of a section under the assumption that there are formal sections everywhere locally, which turns the problem into finding a rational point of the moduli space of sections over $\mathbb{F}_q$. There are geometric conditions which would guarantee the existence of a rational point. But these conditions are usually difficult to check. When the base field is the function field of a complex curve instead of a finite field (thus the variety is defined over the function field of a complex surface), this line of argument is encoded in the theory of rational simple connectedness (see [19]), the technical core of which is to check rational connectedness of the moduli space by using very twisting surfaces.

In some sense, this article is an application of the general philosophy of rational simple connectedness in the nonrationally simply connected case.

The basic observation of this article is that to prove the existence of a section under the assumptions of the main theorem, it suffices to prove a much weaker statement, namely, the existence of a geometrically irreducible component of the space of sections (see Lemma 3.12). To find such a component, we use a slight variant of an argument of de Jong, He, and Starr [19]. This argument produces a sequence of irreducible components of spaces of sections over $\bar{\mathbb{F}}_q$ which becomes Galois invariant when the degree is large enough. These are discussed in Section 3. One subtle point in our variation of their construction is that in our case the family of lines through a general point could be reducible. So a monodromy argument is necessary to show that we get a well-defined sequence of irreducible components. We deal with these problems in Appendix B, using some results of Kollár [25] on the fundamental group of rationally connected varieties. This sequence is called an *Abel sequence* in [19], as it is related to the Abel–Jacobi map to the Picard variety of the base curve. Here we use a different name since no Abel–Jacobi map is involved. Several interesting arithmetic questions seem to be related to this sequence (see questions (1), (2), (3) at the end of Section 3.1).

As usual, singularities cause problems in deforming sections of the family. The condition that one has a formal section everywhere is used to analyze singularities. Combining this with a result of Kollár, which describes a semistable integral model, we can have some control of the singularities. This is done in Section 2. The argument is straightforward once the theory of semistable models is established. However, the computation is quite tedious. The readers who trust the author's computation can simply take a look at Corollaries 2.12, 2.20, and 2.27 and proceed to later sections.

The main theorems are proved in Sections 4, 5, and 6. The case of quadric hypersurfaces is the simplest. We recommend reading this case (see Section 4) first to get a general idea of the proof. The main argument is to construct a ruled surface containing two given sections (again following an idea of [19]). However, in our case, we cannot use a chain of lines that does the job as in [19]. Families of higher degree curves are necessary for the construction. As a result, we have to be very careful about places of bad reduction and the degeneration of the family of rulings of the ruled surface. This constitutes most of the technical arguments in these sections.

In Appendix A, we show how to modify the argument of [29] to prove weak approximation of cubic hypersurfaces defined over function fields of curves defined over an algebraically closed field of characteristic at least 7. This result is used in Section 5.

Finally, we make a remark about some references on rational curves on complete intersections. Rational curves, especially low-degree rational curves on Fano complete intersections, play an important role in the proofs in Sections 3, 4, 5, and 6. A canonical reference that contains a lot more than what is needed to understand the proof is Kollár's book (see [23], especially Chapter V). Another nice introduction is by Debarre [17]. Some research articles about Fano schemes of lines include [2], [3], and [18].

2. Semistable models over global function fields

In this section, we first review the theory of semistable models of hypersurfaces defined over global function fields by Kollár [24] and then generalize the theory to the case of complete intersections of two quadrics. After this, we discuss in detail the singularities appearing in the semistable model under the assumption that there is a (formal) section. This part is crucial for the proof of the Hasse principle.

Let S be a discrete valuation ring with K the quotient field, let t be a generator of the maximal ideal, and let $k = S/(t)$ be the residue field. Given a polynomial $f \in S[X_0, \dots, X_n]$, we write f_k and f_K as the image of f in $k[X_0, \dots, X_n]$ and $K[X_0, \dots, X_n]$ under the natural quotient map and inclusion.

2.1. Semistable models

We first review the case of hypersurfaces treated in [24].

Definition 2.1

A *weight system* W on $S[X_0, \dots, X_n]$ consists of an $(n+1)$-tuple of nonnegative integers $w = (w_0, \dots, w_n)$ and an element $a = (a_{ij}) \in \mathrm{SL}_{n+1}(S)$. The numbers $(w_0, \dots, w_n)$ are called *weights* of the weight system. We write $F(t^W \cdot X) = F(t^{w_0} \sum a_{0j} X_j, \dots, t^{w_n} \sum a_{nj} X_j)$ for $F \in S[X_0, \dots, X_n]$. The *multiplicity* of F at W, denoted by $\mathrm{mult}_W F$, is defined as the minimum of the valuations on the coefficients of $F(t^W \cdot X)$.

The family of degree d hypersurfaces defined over Spec S by $F \in S[X_0, \dots, X_n]$ is *semistable* if for any weight system W, we have $\mathrm{mult}_W F \leq \frac{d \sum w_i}{n+1}$. Otherwise it is called *non-semistable*.

Remark 2.2

The elements in $\mathrm{SL}_{n+1}(S)$ act on the family by isomorphisms. The important part of the definition is the weights, which measure the singularity of the total space and the central fiber along some linear space. So later in the paper, we will simply say "a weight system whose weights are $(w_0, \dots, w_n)$," with the understanding that one can easily find a suitable element $a \in \mathrm{SL}_{n+1}(S)$ (usually the identity element or the element which makes the linear space defined by the vanishing of some coordinates) that will make the statement true.

Remark 2.3
When the weight system has weights of the form $w_0 = \cdots = w_i = 1, w_{i+1} = \cdots = w_n = 0$, the multiplicity $\mathrm{mult}_W F$ is the same as the multiplicity of the model along the linear space defined by $X_0 = \cdots = X_i = 0$. Thus being semistable is closely related to the singularities of the family.

The following theorem is from Kollár [24].

THEOREM 2.4 ([24, Theorem 5.1])
Given a degree d hypersurface $f \in K[X_0, \ldots, X_n]$ which defines a semistable hypersurface in $\mathbb{P}^n_K$ for the action of $\mathrm{SL}_{n+1}(K)$ on $\mathbb{P}(H^0(\mathbb{P}^n, \mathcal{O}(d)))$ in the sense of geometric invariant theory (GIT), there is a semistable model $F \in S[X_0, \ldots, X_n]$ such that $F_K = 0$ defines a hypersurface isomorphic to $f = 0$. In particular, this holds for smooth hypersurfaces.

We remark that semistable hypersurfaces exist. In fact, the classical theory of discriminant shows that a smooth hypersurface is semistable. Here we briefly discuss the proof.

Let $f \in K[X_0, \ldots, X_n]$ be a homogeneous polynomial of degree d. Assume that the hypersurface defined by $f = 0$ is semistable (in the sense of GIT) over K. By the definition of semistability in GIT, there is a nonconstant homogeneous $\mathrm{SL}_{n+1}(K)$-invariant polynomial I on the coefficients of degree d homogeneous polynomials in $K[X_0, \ldots, X_n]$, such that the value of I on the coefficients of f is a nonzero element in K. We can find a polynomial $F \in S[X_0, \ldots, X_n]$ such that $F = 0$ defines a flat family of hypersurfaces of degree d over $\operatorname{Spec} S$ and over K, and the hypersurface defined by $F_K = 0$ is in the same $\mathrm{SL}_{n+1}(K)$ orbit as the hypersurface defined by $f = 0$. We may assume that the polynomial I has coefficients in S, which can always be achieved by clearing the denominators. So when we apply the function I to the coefficients of $F \in S[X_0, \ldots, X_n]$ we get a nonzero element of S, which is in the ideal (t^e) for some $e \geq 0$. In the following, we write the value of the polynomial I on the coefficients of a homogeneous degree d polynomial G as $I(G(X_0, \ldots, X_n))$. Assume that F is not semistable with respect to a weight system W. Then we perform the change of coordinates $X_i = t^{w_i} \sum a_{ij} Y_j$,

$$F(X_0, \ldots, X_n) = F(t^W \cdot Y) = t^{\mathrm{mult}_W F} F'(Y_0, \ldots, Y_n).$$

Let r be the homogeneous degree of the function I. Lemma 4.5 of [24] gives the following:

$$I\big(F'(X_0, \ldots, X_n)\big) = t^{r(-\mathrm{mult}_W F + \frac{d \sum w_i}{n+1})} I\big(F(X_0, \ldots, X_n)\big).$$

In other words, if F is not semistable with respect to the weight system W, we may find another model defined by F' such that the valuation of $I(F') \in S$ is strictly smaller (and remains nonnegative). By this calculation, a semistable model is nothing but a model which minimizes the valuation of $I(F)$ among all the models defined by $F = 0$. We therefore have the following observation which will be used several times.

LEMMA 2.5
Let $F_S \in S[X_0, \dots, X_n]$ be a semistable family. Assume that there is a weight system W such that $\mathrm{mult}_W F_S$ *equals to* $\frac{d \sum w_i}{n+1}$. *Then* $F'_S(X_0, \dots, X_n) = t^{-\mathrm{mult}_W F_S} F_S(t^W \cdot X)$ *is also a semistable family.*

The generalization to the case of complete intersections of two quadrics is easy. We first introduce some notation. We use multi-index notation. So the monomial $X_0^{j_0} \dots X_n^{j_n}, \sum_{i=0}^n j_i = d$ is abbreviated as X_J. We use these monomials as a basis of the vector space of homogeneous polynomials of degree d. The wedge products $\{X_J \wedge X_{J'}, J \neq J'\}$ form a basis of the vector space $\wedge^2 H^0(\mathbb{P}^n, \mathcal{O}(d))$. Let $F_S, G_S \in S[X_0, \dots, X_n]$ be two homogeneous polynomials of degree d. The pencil spanned by them corresponds to a point of the Grassmannian $G(2, H^0(\mathbb{P}^n, \mathcal{O}(d)))$. We expand the wedge product $F_S \wedge G_S$ in terms of the basis $X_J \wedge X_{J'}, J \neq J'$

$$F_S \wedge G_S = \sum a_{JJ'} X_J \wedge X_{J'}.$$

The coefficients $a_{JJ'}$ are the homogeneous coordinates under the Plücker embedding of the Grassmannian $G(2, H^0(\mathbb{P}^n, \mathcal{O}(d)))$ into $\mathbb{P}(\wedge^2 H^0(\mathbb{P}^n, \mathcal{O}(d)))$.

Definition 2.6
Let $F_S, G_S \in S[X_0, \dots, X_n]$ be two homogeneous polynomials of degree d, and let W be a weight system on $S[X_0, \dots, X_n]$. The *multiplicity* of the pencil $\lambda F_S + \mu G_S$ at the weight system W, denoted by $\mathrm{mult}_W(F_S, G_S)$, is the minimum of the valuations on all the coefficients $a_{JJ'} \in S$ of all the terms $a_{JJ'} X_J \wedge X_{J'}$ in $F_S(t^W \cdot X) \wedge G_S(t^W \cdot X)$. This multiplicity only depends on the pencil, not on F_S and G_S.

We say the pencil is *semistable* if $\mathrm{mult}_W(F_S, G_S) \leq \frac{2d(\sum w_i)}{n+1}$. Otherwise it is called *non-semistable*.

Note that $F'_S = t^{-\mathrm{mult}_W F_S} F_S(t^W \cdot X)$ and $G'_S = t^{-\mathrm{mult}_W G_S} G_S(t^W \cdot X)$ belong to $S[X_0, \dots, X_n]$. Therefore, the valuations on the coefficients of

$$F_S(t^W \cdot X) \wedge G_S(t^W \cdot X) = t^{\mathrm{mult}_W F_S + \mathrm{mult}_W G_S} F'_S \wedge G'_S$$

are at least $\mathrm{mult}_W F_S + \mathrm{mult}_W G_S$. Thus we always have

$$\mathrm{mult}_W(F_S, G_S) \geq \mathrm{mult}_W F_S + \mathrm{mult}_W G_S$$

for any weight system W. This is a strict inequality if and only if the wedge product of the reductions modulo t of $F_S(t^W \cdot X)$ and $G_S(t^W \cdot X)$ is zero, which is the same as saying that these two reductions are proportional to each other.

THEOREM 2.7

Given a pencil of degree d polynomials $\lambda F_S + \mu G_S$, if the complete intersection over K defined by $F_K = G_K = 0$ is GIT semistable for the action of $\mathrm{SL}_{n+1}(K)$ on $\mathbb{P}(\wedge^2 H^0(\mathbb{P}^n, \mathcal{O}(d)))$ (in particular, if it is smooth), then there is a semistable model of the pencil $\lambda F_S + \mu G_S$.

Proof

If the pencil spanned by (F_S, G_S) is not semistable with respect to a weight system W, then set $F'_S = t^{-\mathrm{mult}_W F_S} F_S(t^W \cdot X)$ and $G'_S = t^{-\mathrm{mult}_W G_S} G_S(t^W \cdot X)$. If $\mathrm{mult}_W(F_S, G_S) = \mathrm{mult}_W F_S + \mathrm{mult}_W G_S$, we replace F_S and G_S with this new pair (F'_S, G'_S). If $\mathrm{mult}_W(F_S, G_S) > \mathrm{mult}_W F_S + \mathrm{mult}_W G_S$, then F'_k and G'_k are proportional. After multiplying by units in S, we may write

$$F' \equiv \tilde{F}'_k + t^u H_1 \mod t^{u+1},$$
$$G' \equiv a\tilde{F}'_k + t^v H_2 \mod t^{v+1}, \quad u, v \geq 1, a \in k^*,$$

where $\tilde{F}'_k$ is a lift of F'_k to a polynomial defined over S, and H_1, H_2 are polynomials with coefficients in S whose reductions modulo t are not proportional to F'_k. Without loss of generality, we may assume that $u \geq v$. Thus

$$\mathrm{mult}_W(F'_S, G'_S) = v = \mathrm{mult}_W(F_S, G_S) - \mathrm{mult}_W F_S - \mathrm{mult}_W G_S.$$

We replace the pair (F_S, G_S) with $G''_S = t^{-v}(G'_S - aF'_S)$ and $F''_S = F'_S$.

Note that F_k and G_k are proportional if and only if the multiplicity of the pencil (F_S, G_S) is at least 1 with respect to a weight system W whose weights are $(0, \ldots, 0)$; equivalently, the pair is not semistable with respect to the weight system whose weights are $(0, \ldots, 0)$.

Now we show that this process will eventually produce a semistable model. The idea is the same as Kollár's argument for the case of a single polynomial. Let I be a nonconstant $\mathrm{SL}_{n+1}(K)$-invariant homogeneous polynomial of degree r in the coordinates $a_{JJ'}$. We may assume that I has coefficients in S (by clearing the denominators) and that $I(F_S \wedge G_S)$ is nonzero in K and lies in the ideal $(t^e) \subset S$ for some nonnegative integer e. It suffices to show that in each step we decrease the number e.

Let W be a weight system with weights $(w_0, \ldots, w_n)$ and $a = (a_{ij}) \in SL_{n+1}(S)$. We have

$$F_S(t^W \cdot X) = t^{\mathrm{mult}_W F_S} F'_S, \qquad G_S(t^W \cdot X) = t^{\mathrm{mult}_W G_S} G'_S,$$

and

$$F_S(t^W \cdot X) \wedge G_S(t^W \cdot X) = t^{\mathrm{mult}_W F_S + \mathrm{mult}_W G_S} F'_S \wedge G'_S.$$

Thus $I(F_S(t^W \cdot X) \wedge G_S(t^W \cdot X)) = t^{r(\mathrm{mult}_W F_S + \mathrm{mult}_W G_S)} I(F'_S \wedge G'_S)$ by homogeneity. On the other hand, we have

$$\begin{aligned} I\big(F_S(t^W \cdot X) \wedge G_S(t^W \cdot X)\big) &= I\big(t^{\frac{d\sum w_i}{n+1}} F_S(t^{W'} \cdot X) \wedge t^{\frac{d\sum w_i}{n+1}} G_S(t^{W'} \cdot X)\big) \\ &= t^{r\frac{2d\sum w_i}{n+1}} I\big(F_S(t^{W'} \cdot X) \wedge G_S(t^{W'} \cdot X)\big) \\ &= t^{r\frac{2d\sum w_i}{n+1}} I(F_S \wedge G_S), \end{aligned}$$

where W' is an integral weight system over a ramified base change of degree $n+1$ with local uniformizer $t^{\frac{1}{n+1}}$, whose weights are $((n+1)w_0 - \sum_i w_i, \dots, (n+1)w_n - \sum_i w_i)$, and $a \in \mathrm{SL}_{n+1}(S)$ is the same element appearing in the weight system W,

$$\begin{aligned} F_S(t^{W'} \cdot X) &= F_S\Big(t^{w_0 - \frac{\sum_i w_i}{n+1}} \sum a_{0j} X_j, \dots, t^{w_n - \frac{\sum_i w_i}{n+1}} \sum a_{nj} X_j\Big), \\ G_S(t^{W'} \cdot X) &= G_S\Big(t^{w_0 - \frac{\sum_i w_i}{n+1}} \sum a_{0j} X_j, \dots, t^{w_n - \frac{\sum_i w_i}{n+1}} \sum a_{nj} X_j\Big). \end{aligned}$$

For these two equations to make sense, we have to work over a ramified base change and add the $(n+1)$th root $t^{\frac{1}{n+1}}$ of t, which does not matter for the purpose of computing the value over the original base S. The last equality follows from the $\mathrm{SL}_{n+1}(K)$-invariance (and thus $\mathrm{SL}_{n+1}(K')$-invariance for any field extension K' of K) of I. One should compare the above formula with the one in [24, Lemma 4.5].

Combining everything, we have

$$I(F'_S \wedge G'_S) = t^{r(\frac{2d\sum w_i}{n+1} - \mathrm{mult}_W F_S - \mathrm{mult}_W G_S)} I(F_S \wedge G_S).$$

In the case $\mathrm{mult}_W(F_S, G_S) > \mathrm{mult}_W F_S + \mathrm{mult}_W G_S$, we also have

$$I(F''_S \wedge G''_S) = t^{-rv} I(F'_S \wedge G'_S).$$

Thus

$$I(F''_S \wedge G''_S) = t^{r(\frac{2d\sum w_i}{n+1} - \mathrm{mult}_W(F_S, G_S))} I(F_S \wedge G_S).$$

So the second step will decrease the exponent of t if the pair is not semistable with respect to the weight system W. □

Remark 2.8

It seems that the central fiber of a semistable model defined by $F_S = G_S = 0$ may not be a complete intersection, except in the case $d = 2$ (see Lemma 2.9). The reason is, we can only guarantee that F_S and G_S are nonproportional modulo t. So, a priori there is a possibility that even if we start with two polynomials whose reductions modulo t define a complete intersection, we cannot guarantee that we can keep this condition. More precisely, the author does not know if it is possible for the new pair F'_S and G'_S (or F''_S and G''_S) to have a common factor without being proportional modulo t after each step. Luckily, in the case $d = 2$, this will not happen, because a common factor, if one exists, will have to be a linear polynomial which will destabilize the pair as shown below.

LEMMA 2.9
If $d = 2$ and (F_S, G_S) is a semistable family of a pencil of two quadrics in $\mathbb{P}^n_S, n \geq 2$, then $F_S = G_S = 0$ defines a flat projective family of complete intersections of two quadrics over Spec S.

Proof
It suffices to show that the family $F_S = G_S = 0$ has constant fiber dimension. Since F_S and G_S are families of quadric hypersurfaces, the intersection has constant fiber dimension if and only if F_k and G_k do not contain a common linear factor over k and F_k and G_k are not proportional. If F_k and G_k have a common linear factor, which, without loss of generality, can be assumed to be X_0, then the family is not semistable with respect to a weight system whose weights are $(1, 0, \ldots, 0)$. If F_k and G_k are proportional, the pencil is not semistable with respect to the weight system $(0, \ldots, 0)$. □

Recall that for a semistable model and for any weight system, we always have the inequalities

$$\frac{2d \sum w_i}{n+1} \geq \operatorname{mult}_W(F_S, G_S) \geq \operatorname{mult}_W F_S + \operatorname{mult}_W G_S.$$

From this we deduce the following useful lemma.

LEMMA 2.10
Let (F_S, G_S) be a semistable family of a pencil of homogeneous polynomials of degree d. Assume that there is a weight system W such that

$$\operatorname{mult}_W F_S + \operatorname{mult}_W G_S = \frac{2d \sum w_i}{n+1}.$$

Then

$$F'_S(Y_0, \ldots, Y_n) = t^{-\operatorname{mult}_W F_S} F_S(t^W \cdot Y)$$

and

$$G'_S(Y_0, \dots, Y_n) = t^{-\mathrm{mult}_W G_S} G_S(t^W \cdot Y)$$

also define a semistable family.

2.2. Semistable models for quadrics

From now on we discuss the singularities of a semistable model over $\mathbb{F}_q[\![t]\!]$ (in the presence of a formal section). All of the results in the following three sections can be proved almost without change for any nonarchimedean local field.

We first discuss the easiest case: quadric hypersurfaces.

LEMMA 2.11

Let $\mathcal{X} \to \mathrm{Spec}\, \mathbb{F}_q[\![t]\!]$ be a semistable model of a smooth quadric hypersurface in $\mathbb{P}^n$, $n \geq 2$, defined over $\mathbb{F}_q(\!(t)\!)$.

(1) *$\mathcal{X}$ is regular at any $\mathbb{F}_q$-rational point in the central fiber $\mathcal{X}_0$.*

(2) *If there is a formal section $\widehat{s}$ of $\mathcal{X} \to \mathrm{Spec}\, \mathbb{F}_q[\![t]\!]$, then the central fiber $\mathcal{X}_0$ is geometrically integral and the singular locus has codimension at least* 2.

Proof

(1) This follows directly from the semistability with respect to a weight system whose weights are $(1, \dots, 1, 0)$. To be more precise, let the $\mathbb{F}_q$-point be $[0, \dots, 0, 1]$. Being semistable with respect to this weight system means that the whole family has multiplicity at most $\frac{2(1+\cdots+1+0)}{n+1} < \frac{2(n+1)}{n+1} = 2$ at this point (see Remark 2.3). So the multiplicity has to be 1, which precisely says that the family is regular at this point.

A quadric hypersurface is geometrically integral if and only if it is neither a hyperplane with multiplicity 2 nor a union of two hyperplanes. If there is a formal section $\widehat{s}$, then $\widehat{s}(0)$ is a rational point in the central fiber and the total space is regular at this point by (1). So the central fiber is also regular (hence smooth) at this point, due to the general fact that if there is a formal section passing through a regular point of the total space, then the fiber is also regular at this point. In particular, the central fiber is geometrically reduced. If the central fiber is (geometrically) a union of two hyperplanes, then $\widehat{s}(0)$ lies in one and only one of the two irreducible components. Thus there is a linear space defined over $\mathbb{F}_q$ of the central fiber, given by the irreducible component containing the $\mathbb{F}_q$-rational point $\widehat{s}(0)$. This is impossible, since it makes the family not semistable with respect to a weight system whose weights are $(1, 0, \dots, 0)$. To see this, assume that the linear space is defined by $X_0 = 0$ so that the central fiber is given by $X_0 L(X_0, \dots, X_n) = 0$ for some linear polynomial L. Then the multiplicity of the family at the weight system is at least 1, while the semistability requires that the multiplicity is at most $\frac{2(1+0+\cdots+0)}{n+1} < 1$. Thus the central fiber is at worst a cone over a smooth geometrically integral quadric. Such a quadric has dimension at least 1, and it follows that the singular locus of the central fiber, which is the vertex of the cone, has codimension at least 2. □

As an immediate corollary, we have the following.

COROLLARY 2.12

(1) *Let $\mathcal{X} \to B$ be a semistable model of a smooth conic in $\mathbb{P}^2_{\mathbb{F}_q(B)}$. If the set of adelic points of the conic is nonempty, then all the fibers over closed points of B are smooth.*

(2) *Let $\mathcal{X} \to B$ be a semistable model of a smooth quadric hypersurface in $\mathbb{P}^n_{\mathbb{F}_q(B)}$, $n \geq 3$. If the set of adelic points of the quadric hypersurface is nonempty, then every fiber over a closed point is a geometrically integral quadric hypersurface.*

2.3. Semistable models for cubics

In this section, we study semistable models of cubic hypersurfaces.

LEMMA 2.13

Let X be a cubic hypersurface of positive dimension defined over a finite field $\mathbb{F}_q$. Then either X has an $\mathbb{F}_q$-rational point in the smooth locus or X is defined by $F(X_0, X_1, X_2) = 0$, where $F = 0$ is either a union of three Galois conjugate hyperplanes or a hyperplane with multiplicity 3.

Proof

We prove this by induction on the dimension.

If X is a geometrically integral curve, then it is either a smooth genus 1 curve or a rational curve with at most two points identified. In the smooth genus 1 curve case, there is an $\mathbb{F}_q$-point by the Lang–Weil theorem ($|\#X(\mathbb{F}_q) - 1 - q| < 2q^{\frac{1}{2}}$ in this case), or by Lang's theorem that there are no nontrivial torsors under Abelian varieties over finite fields. In the singular case, it is easy to check that there is an $\mathbb{F}_q$-point in the smooth locus. If X is reducible over $\mathbb{F}_q$, then one of the irreducible components is a line and the statement is clear. A plane cubic is geometrically reducible but irreducible over $\mathbb{F}_q$ if and only if it is the union of three Galois conjugate lines. The case of a triple line is clear.

In higher dimensions, if X is a cone, then the statement follows from the induction hypothesis.

In the following, assume that $n \geq 3$ and that X is not a cone. Any hypersurface of degree d in $\mathbb{P}^n$ over $\mathbb{F}_q$ has a rational point as long as $d \leq n$ by the Chevalley–

Warning theorem. So X has a rational point. Assume that this is $[1,0,\ldots,0]$ and that it is a singular point of X of multiplicity 2 (recall that X has a singular point of multiplicity 3 if and only if X is a cone). We may write the equation of X as

$$X_0 Q(X_1,\ldots,X_n)+C(X_1,\ldots,X_n)=0.$$

It suffices to show that there is a point $[x_1,\ldots,x_n]$ such that $Q(X_1,\ldots,X_n)\neq 0$. If this is the case, then the point

$$\big[-C(X_1,\ldots,X_n), X_1 Q(X_1,\ldots,X_n),\ldots,X_n Q(X_1,\ldots,X_n)\big]$$

is a rational point in the smooth locus. So it suffices to show that given any quadric hypersurface Q in $\mathbb{P}^n$, there is a rational point in $\mathbb{P}^n$ which is not in the quadric hypersurface. The graded ideal of all the $\mathbb{F}_q$-rational points of $\mathbb{P}^n$ is generated by degree $q+1$ homogeneous ideals $\{X_i^q X_j - X_j^q X_i, 0\leq i,j\leq n\}$. Since $q+1>2$, there is at least one rational point of $\mathbb{P}^n$ not contained in Q. □

LEMMA 2.14

Let $\mathcal{X}\to$ Spec $\mathbb{F}_q [\![t]\!]$ be a semistable family of cubic hypersurfaces in $\mathbb{P}^n$, $n\geq 3$ such that the generic fiber is smooth. The closed fiber is either geometrically integral or a union of three hyperplanes which are Galois conjugate to each other. If n is at least 6, then in the second case above, the union of the three hyperplanes is a cone over three Galois conjugate lines in $\mathbb{P}^2_{\mathbb{F}_q}$ without common intersection points.

Proof

A cubic hypersurface which is not geometrically integral always contains a hyperplane defined over an algebraic closure of the field of definition. By the definition of semistability, there is no hyperplane in the central fiber which is defined over $\mathbb{F}_q$. Thus the central fiber is either the union of three conjugate hyperplanes or geometrically integral.

If n is at least 6, the central fiber of a semistable family does not contain a linear subspace of dimension $n-2$. Thus if the central fiber is a union of three hyperplanes, the common intersection of these hyperplanes must be of dimension $n-3$, and we have the last statement. □

The following two lemmas are mostly computational. The general idea is that semistability gives bounds on the multiplicity of the total family along a linear space defined over $\mathbb{F}_q$. If the central fiber is nonnormal or a cone over a plane cubic, we can make a suitable base change so that the total family has larger multiplicity along the linear space. Then we will be able to find a new model, whose central fiber is less singular, using a change of coordinates as in the proof of Theorem 2.4.

We introduce a common hypothesis.

HYPOTHESIS 2.15

Let $\mathcal{X} \to \operatorname{Spec} \mathbb{F}_q[[t]]$ be a semistable family of cubic hypersurfaces in $\mathbb{P}^n$, $n \geq 5$ such that the generic fiber is smooth. Also assume that the characteristic is not 2 or 3.

LEMMA 2.16

Use the same assumptions as in Hypothesis 2.15. If the central fiber is a cone over a plane cubic irreducible over $\mathbb{F}_q$, and there is a formal section of the family, then there is a tower of quadratic field extensions of $\mathbb{F}_q((s))/\mathbb{F}_q((t))$ such that the base change of the generic fiber to $\mathbb{F}_q((s))$ can be extended to a family over $\operatorname{Spec} \mathbb{F}_q[[s]]$ whose central fiber is normal and not a cone over a smooth plane cubic.

Proof

Since the central fiber is a cone over an irreducible plane cubic over $\mathbb{F}_q$, we can write the equation of the family as

$$F(X_0, X_1, X_2) + tG(X_3, \ldots, X_n) + tX_0Q_0 + tX_1Q_1 + tX_2Q_2 + t^2(\ldots) = 0$$

or

$$F(X_0, X_1) + tG(X_2, \ldots, X_n) + tX_0Q_0 + tX_1Q_1 + t^2(\ldots) = 0.$$

Note that by semistability, the second case can happen only if $n = 5$ (consider a weight system whose weights are $(1, 1, 0, \ldots, 0)$).

Case I: The cubic $G(X_3, \ldots, X_n) = 0$ (or $G(X_2, \ldots, X_n) = 0$) has a smooth rational point. Without loss of generality, assume that the point is $[X_3, \ldots, X_n] = [1, 0, \ldots, 0]$ (or $[X_2, \ldots, X_n] = [1, 0, \ldots, 0]$) and that the tangent hyperplane of this point in the hypersurface $G = 0$ is given by $X_4 = 0$ (or $X_3 = 0$). Then make the following base change and change of variables:

$$t = s^2, \qquad X_0 = sY_0, \qquad X_1 = sY_1, \qquad X_2 = sY_2, \qquad X_3 = Y_3,$$
$$X_4 = sY_4, \ldots, X_n = sY_n$$

or

$$t = s^2, \qquad X_0 = sY_0, \qquad X_1 = sY_1, \qquad X_2 = Y_2, \qquad X_3 = sY_3, \ldots, X_n = sY_n.$$

The new family is

$$F(Y_0, Y_1, Y_2) + Y_3^2Y_4 + L(Y_0, Y_1, Y_2)Y_3^2 + s(\ldots) = 0$$

or

$$F(Y_0, Y_1) + Y_2^2Y_3 + L(Y_0, Y_1)Y_2^2 + s(\ldots) = 0.$$

The central fiber $F(Y_0, Y_1, Y_2) + Y_3^2(Y_4 + L(Y_0, Y_1, Y_2)) = 0$ or $F(Y_0, Y_1) + Y_2^2(Y_3 + L(Y_0, Y_1)) = 0$ defines a normal cubic hypersurface which is not a cone over a plane cubic.

Case II: The cubic curve $F(X_0, X_1, X_2) = 0$ has a smooth rational point, and $G(X_3, \dots, X_n) = 0$ does not have a rational point in the smooth locus. By Lemma 2.13, this can happen only when $G = 0$ is a hyperplane with multiplicity 3 or a union of three Galois conjugate hyperplanes. Note that $F(X_0, X_1)$ cannot have a smooth rational point; otherwise the family is not semistable. By the semistability condition, the equation $G = 0$ can define a hyperplane of multiplicity 3 only if $n = 5$ (consider a weight system whose weights are $(1, 1, 1, 1, 0, \dots, 0)$).

Case II.1: We can write the equation as

$$F(X_0, X_1, X_2) + tX_3^3 + tM(X_0, X_1, X_2; X_3, X_4, X_5) + t^2 H(X_4, X_5) + t^2(\dots) = 0,$$

where each monomial in $M(X_0, X_1, X_2; X_3, X_4, X_5)$ contains a factor of X_0, X_1, or X_2 and a factor of X_3, X_4, or X_5. By the semistability condition and the assumption that $F = 0$ has a smooth rational point, the curve defined by $F(X_0, X_1, X_2) = 0$ is a geometrically irreducible plane cubic. If there are monomials of the form $X_i X_j X_l$ in M, for some $0 \le i \le 2, 4 \le j, l \le 5$, then make the following change of variables:

$$t = s^2, \qquad X_0 = sY_0, \dots, X_3 = sY_3, \qquad X_4 = Y_4, \qquad X_5 = Y_5.$$

The new family becomes

$$\begin{aligned} &F(Y_0, Y_1, Y_2) + L_4(Y_0, Y_1, Y_2)Y_4^2 + L_5(Y_0, Y_1, Y_2)Y_5^2 \\ &\quad + L_{45}(Y_0, Y_1, Y_2)Y_4Y_5 + s(\dots) = 0, \end{aligned}$$

where L_4, L_5, and L_{45} are linear polynomials and at least one of them is nonzero. The central fiber is geometrically reduced, geometrically irreducible, normal, and not a cone over a plane cubic.

Assume that there are no monomials of the form $X_i X_j X_l$ in M, where $0 \le i \le 2, 4 \le j, l \le 5$. Make the following change of variables:

$$X_0 = tY_0, \dots, X_3 = tY_3, \qquad X_4 = Y_4, \qquad X_5 = Y_5.$$

The new family is still semistable by Lemma 2.5 and can be written as

$$H(Y_4, Y_5) + tF(Y_0, Y_1, Y_2) + t^2(\dots) = 0.$$

By semistability, $H = 0$ defines a cone over a union of three Galois conjugate points. This reduces to Case I, where $F = 0$ has a rational point in the smooth locus.

Case II.2: The equation $G = 0$ defines a cone over three Galois conjugate points or three Galois conjugate lines. Assume that $F = 0$ has a smooth rational point

$[1,0,0]$ and that the tangent line at this point is $X_1 = 0$. Make the following base change and change of variables:

$$t = s^4, \quad X_0 = s^2 Y_0, \quad X_1 = s^3 Y_1, \quad X_2 = s^3 Y_2, \quad X_3 = sY_3,$$
$$X_4 = sY_4, \dots, X_n = sY_n.$$

Then the new family is

$$G(Y_3, \dots, Y_n) + Y_0^2 Y_1 + s(\dots) = 0.$$

The equation $G(Y_3, \dots, Y_n) + Y_0^2 Y_1 = 0$ defines a normal geometrically integral cubic hypersurface which is not a cone over a plane cubic. Note that the base change can be factorized as two degree 2 base changes.

Case III: Neither F nor G has a rational point in the smooth locus. Assume that the formal section intersects the central fiber at $[0, \dots, 0, 1]$. By Lemma 2.13, $F = 0$ and $G = 0$ define two cones, both of which have multiplicity 3 along the rational point. The semistability condition requires that the total family has multiplicity strictly less than 3 along any rational point in the central fiber. Thus at least one of the two monomials $t^2 X_5^3$ and $tL(X_0, X_1) X_5^2$ has to appear with a nonzero coefficient in the defining equation. Assume that the formal section is of the form

$$[tf_0, \dots, tf_{n-1}, 1 + tf_n],$$

where f_i is a formal power series in t for each i. Thus we have

$$\begin{aligned} &F(tf_0, tf_1, tf_2) + tG(tf_0, \dots, tf_{n-1}) + at^2(1 + tf_n)^3 \\ &\quad + btL(tf_0, tf_1, tf_2)(1 + tf_n)^2 \\ &\quad + t^2(\text{a formal power series with zero constant term}) = 0, \quad a, b \in \mathbb{F}_q. \end{aligned}$$

Consider the coefficients of t^2 in the formal power series. The only possible nontrivial contribution to the coefficient comes from the terms $at^2(1 + tf_n)^3$ and $btL(tf_0, tf_1, tf_2)(1 + tf_n)^2$. The coefficient of t^2 in $btL(tf_0, tf_1, tf_2)(1 + tf_n)^2$ may be zero, even if b is nonzero. The coefficient of t^2 in $at^2(1 + tf_n)^3$ is nonzero as long as a is nonzero. If the term $tL(X_0, X_1, X_2) X_5^2$ does not appear in the defining equation (i.e., $b = 0$), then a has to be nonzero, and the coefficient of t^2 is nonzero. Therefore $tL(X_0, X_1, X_2) X_5^2$ has to appear with nonzero coefficient. Without loss of generality, assume that $L(X_0, X_1, X_2)$ equals X_0. Then make the following change of variables:

$$t = s^2, \quad X_0 = sY_0, \quad X_1 = sY_1, \quad X_2 = sY_2, \dots, X_{n-1} = sY_{n-1},$$
$$X_n = Y_n.$$

The new family is defined by

$$F(Y_0, Y_1, Y_2) + Y_0 Y_n^2 + s(\dots) = 0.$$

This defines a normal cubic hypersurface which is not a cone over a plane cubic as long as $F = 0$ is not a cone over three Galois conjugate points in a line.

Note that $F = 0$ is the central fiber of the original semistable family. Thus, by the semistability condition, it can be a cone over three Galois conjugate points in a line only if $n = 5$ (and it is never a hyperplane with multiplicity 3).

Recall that by Lemma 2.13, a cubic hypersurface over $\mathbb{F}_q$ has no rational point in the smooth locus only if it is the union of three Galois conjugate hyperplanes. Thus when $n = 5$, we may write the original family as

$$F(X_0, X_1) + tG(X_2, X_3) + tM(X_0, X_1; X_2, X_3, X_4, X_5) + t^2 H(X_4, X_5) + \cdots = 0$$

or

$$F(X_0, X_1) + tG(X_2, X_3, X_4) + tM(X_0, X_1; X_2, X_3, X_4, X_5) + at^2 X_5^3 + \cdots = 0, \quad a \neq 0,$$

where each monomial in $M(X_0, X_1; X_2, X_3, X_4, X_5)$ contains a factor of X_0 or X_1 and a factor of X_2, X_3, X_4, or X_5.

Recall that the formal section intersects the central fiber at $[0, \dots, 0, 1]$. By semistability, the multiplicity of the total family is less than 3 at this point in the central fiber. Thus at least one of the two monomials $t^2 X_5^3$ and $tL(X_0, X_1)X_5^2$ has to appear with a nonzero coefficient in the defining equation. We assume that $L = X_0$. By a similar argument as above, the term $tLX_5^2 = tX_0X_5^2$ has to appear with nonzero coefficient. Make the following change of variables:

$$X_0 = tY_0, \qquad X_1 = tY_1, \qquad X_2 = Y_2, \dots, X_5 = Y_5,$$

and the new family becomes

$$G(Y_2, Y_3) + t(\mu Y_0 Y_5^2 + \lambda Y_5^3 + \cdots) + t^2(\dots) = 0$$

or

$$G(Y_2, Y_3, Y_4) + t(\mu Y_0 Y_5^2 + \lambda Y_5^3 + \cdots) + t^2(\dots) = 0.$$

All the terms not written explicitly in $\mu X_0 X_5^2 + \lambda X_5^3 + \cdots$ have a factor of $X_i, i = 1, \dots, 4$. Then $Y_0 = -\lambda, Y_5 = \mu, Y_i = 0, i \neq 0, 5$, is a smooth point of $\mu X_0 X_5^2 + \lambda X_5^3 + \cdots = 0$. Thus we reduce this case to Case I. $\square$

LEMMA 2.17

Use the same assumptions as in Hypothesis 2.15. Assume that the central fiber is geometrically integral, nonnormal, but not a cone over a singular plane cubic. Then necessarily n is 5. Moreover, after possibly performing a ramified quadratic base change if needed, one can find a semistable model whose central fiber is normal and not a cone over an irreducible plane cubic.

Before we state the proof, first note the following.

LEMMA 2.18

A geometrically integral cubic hypersurface in $\mathbb{P}^n$ defined over $\mathbb{F}_q$ is nonnormal if and only if its singular locus has a unique irreducible component which is a linear space of dimension $n-2$ defined over $\mathbb{F}_q$.

Proof

Only the "only if" part needs a proof. To see this, we pass to an algebraic closure of $\mathbb{F}_q$. Taking $(n-2)$ general hyperplane sections, we get an irreducible singular plane cubic curve. If the cubic curve has two singular points, then the line joining the two points cannot intersect the cubic curve properly, otherwise the intersection multiplicity is at least 4. Therefore the line is necessarily an irreducible component of the cubic, which contradicts the irreducibility of the cubic. So the singular cubic has only one singular point. As a result, there is only one codimension 1 irreducible component of the singular locus of the cubic hypersurface, which is an $(n-2)$-dimensional linear subspace in $\mathbb{P}^n$. This $(n-2)$-dimensional linear space is defined over $\mathbb{F}_q$ since it is Galois invariant. $\square$

Proof of Lemma 2.17

By Lemma 2.18, the singular locus has an irreducible component consisting of an $(n-2)$-dimensional linear subspace defined over $\mathbb{F}_q$. Assume that it is defined by $X_0 = X_1 = 0$.

Note that when $n \geq 6$, by semistability with respect to a weight system whose weight vector is $(1, 1, 0, \ldots, 0)$, a semistable model does not contain an $(n-2)$-dimensional linear space defined over $\mathbb{F}_q$ in the central fiber. Thus the central fiber of the semistable model can be nonnormal only if n is 5.

We may write the family as

$$\begin{aligned}&X_0^2 L_0(X_2, \ldots, X_5) + X_1^2 L_1(X_2, \ldots, X_5) + X_0 X_1 L(X_2, \ldots, X_5) + C(X_0, X_1)\\&\quad + tF(X_2, \ldots, X_5) + t(\text{cubic forms containing } X_0, X_1) + t^2(\ldots) = 0,\end{aligned}$$

where L_0, L_1, L (resp., C, F) are homogeneous polynomials of degree 1 (resp., 3) in $X_2, \ldots, X_5$.

We have the following.

CLAIM 2.19

Assume that a semistable model with nonnormal central fiber is given by

$$X_0^2 L_0(X_2, \ldots, X_5) + X_1^2 L_1(X_2, \ldots, X_5) + X_0 X_1 L(X_2, \ldots, X_5) + C(X_0, X_1) \\ + tF(X_2, \ldots, X_5) + t(\textit{cubic forms containing } X_0, X_1) + t^2(\ldots) = 0,$$

where L_0, L_1, L (resp., C, F) are homogeneous polynomials of degree 1 (resp., 3) in $X_2, \ldots, X_5$. Then F does not vanish identically. If, moreover, the cubic hypersurface defined by $F(X_2, \ldots, X_5) = 0$ is a geometrically integral normal cubic hypersurface or a cone over a plane cubic, then Lemma 2.17 is true.

Proof of Claim 2.19

The multiplicity of the family with respect to the weights $(1, 1, 0, \ldots, 0)$ is at least 1. It is 1 if and only if F does not vanish identically. On the other hand, semistability requires that the multiplicity with respect to this weight system is at most 1. Thus F is not identically zero.

Make the following change of variables:

$$X_0 = tY_0, \qquad X_1 = tY_1, \qquad X_2 = Y_2, \ldots, X_5 = Y_5.$$

Then the new family is still semistable by Lemma 2.5 and is given by

$$F(Y_2, \ldots, Y_5) + t(Y_0^2 L_0 + Y_1^2 L_1 + Y_0 Y_1 L + \cdots) + t^2(\ldots) = 0.$$

If $F(X_2, \ldots, X_5) = 0$ defines a geometrically integral normal cubic hypersurface or a cone over a plane cubic, then we are either done or have reduced to Lemma 2.16. $\square$

Therefore, in the following, we may assume that $F(X_2, \ldots, X_5) = 0$ defines a nonnormal cubic surface in $\mathbb{P}^3$ that is not a cone. Assume that the $(n-2)$-dimensional linear space of the singular locus of the cubic surface defined by $F = 0$ is defined by $X_2 = X_3 = 0$ (see Lemma 2.18). Write

$$F(X_2, X_3, X_4, X_5) = X_2^2 L_2(X_0, X_1, X_4, X_5) + X_3^2 L_3(X_0, X_1, X_4, X_5) \\ + X_2 X_3 L(X_0, X_1, X_4, X_5) + C_F(X_2, X_3),$$

where L_2, L_3, L are linear forms and C_F is a cubic form.

To prove the lemma, we first find another model whose central fiber is easier to understand. Make the following change of variables:

$$X_0 = tY_0, \qquad X_1 = tY_1, \qquad X_2 = Y_2, \dots, X_5 = Y_5.$$

Then the new family is still semistable by Lemma 2.5 and is given by

$$F(Y_2, \dots, Y_5) + tG(Y_0, Y_1, Y_4, Y_5) + t(\text{terms containing } Y_2, Y_3) + t^2(\dots) = 0.$$

By Claim 2.19, we know that G does not vanish identically, and we may assume that G defines a nonnormal cubic surface in $\mathbb{P}^3$ that is not a cone.

We remark here that we have not really improved the singularities of the central fiber after this step. Rather, we just put the equation in a more tractable way so that the calculation that follows becomes easier.

Make the following change of variables:

$$t = s^2, \qquad Y_2 = sZ_2, \qquad Y_3 = sZ_3, \qquad Y_0 = Z_0, \qquad Y_1 = Z_1,$$
$$Y_4 = Z_4, \qquad Y_5 = Z_5.$$

The new family becomes

$$\begin{aligned} &Z_2^2 L_2(Z_4, Z_5) + Z_3^2 L_3(Z_4, Z_5) + Z_2 Z_3 L(Z_4, Z_5) \\ &\quad + G(Z_0, Z_1, Z_4, Z_5) + s(\dots) = 0. \end{aligned}$$

The only thing left to prove is that the central fiber is a normal cubic hypersurface, but is not a cone over a smooth plane cubic. To prove this, it suffices to show that the singular locus has dimension at most 2, and that the singular locus is not a 2-dimensional linear space, along which the hypersurface

$$Z_2^2 L_2(Z_4, Z_5) + Z_3^2 L_3(Z_4, Z_5) + Z_2 Z_3 L(Z_4, Z_5) + G(Z_0, Z_1, Z_4, Z_5) = 0$$

has multiplicity 3.

We can compute the singular locus after making a base change to an algebraic closure of $\mathbb{F}_q$. By the assumption that $F = 0$ defines a nonnormal cubic surface that is not a cone over a plane cubic, the linear forms L_0, L_3, L are not all proportional to some fixed linear form. Up to making linear combinations of Z_2, Z_3 and Z_4, Z_5 over $\bar{\mathbb{F}}_q$, we may assume that either $L_2 = Z_4, L_3 = Z_5, L = 0$ or $L_2 = Z_4, L_3 = 0, L = Z_5$. So it suffices to show that the singular locus of

$$Z_2^2 Z_4 + Z_3^2 Z_5 + G(Z_0, Z_1, Z_4, Z_5) = 0$$

and

$$Z_2^2 Z_4 + Z_2 Z_3 Z_5 + G(Z_0, Z_1, Z_4, Z_5) = 0$$

has dimension at most 2, and that either the singular locus is not a 2-dimensional linear subspace, or if the singular locus is a 2-dimensional linear subspace, then the hypersurface has multiplicity 2 along this subspace.

In the first case, the singular locus is defined by

$$Z_2 Z_4 = Z_3 Z_5 = \frac{\partial G}{\partial Z_0} = \frac{\partial G}{\partial Z_1} = \frac{\partial G}{\partial Z_4} + Z_2^2 = \frac{\partial G}{\partial Z_5} + Z_3^2 = 0.$$

If $Z_2 = Z_3 = 0$, then

$$\frac{\partial G}{\partial Z_0} = \frac{\partial G}{\partial Z_1} = \frac{\partial G}{\partial Z_4} = \frac{\partial G}{\partial Z_5} = 0.$$

Thus the singular locus is a 1-dimensional linear space.

If $Z_2 = 0, Z_3 \neq 0$, then $Z_5 = 0$. We also have $\frac{\partial G}{\partial Z_5} + Z_3^2 = 0$. Thus the singular locus has dimension at most 2. Similarly, if $Z_2 \neq 0, Z_3 = 0$, then the singular locus has dimension at most 2. In both cases, the singular locus cannot be a 2-dimensional linear space.

If $Z_2 \neq 0, Z_3 \neq 0$, then $Z_4 = Z_5 = 0$. Furthermore, $\frac{\partial G}{\partial Z_4} + Z_2^2 = \frac{\partial G}{\partial Z_5} + Z_3^2 = 0$. Thus the singular locus has dimension at most 1.

In the second case, the singular locus is defined by

$$2Z_2 Z_4 + Z_3 Z_5 = Z_2 Z_5 = \frac{\partial G}{\partial Z_0} = \frac{\partial G}{\partial Z_1} = \frac{\partial G}{\partial Z_4} + Z_2^2 = \frac{\partial G}{\partial Z_5} + Z_2 Z_3 = 0.$$

If $Z_2 = 0$, then $Z_3 Z_5 = 0$ and $\frac{\partial G}{\partial Z_0} = \frac{\partial G}{\partial Z_1} = \frac{\partial G}{\partial Z_4} = \frac{\partial G}{\partial Z_5} = 0$. The last conditions on G define a codimension 2 linear space $M_1(Z_0, Z_1, Z_4, Z_5) = M_2(Z_0, Z_1, Z_4, Z_5) = 0$. Thus the singular locus has dimension at most 2. If there is a 2-dimensional irreducible component of the singular locus, then it has to be of the form $Z_2 = M_1(Z_0, Z_1, Z_4, Z_5) = M_2(Z_0, Z_1, Z_4, Z_5) = 0$. It is straightforward to check that the central fiber has multiplicity 2 along this component.

If $Z_2 \neq 0$, then $Z_5 = Z_4 = 0$ and

$$\frac{\partial G}{\partial Z_0} = \frac{\partial G}{\partial Z_1} = \frac{\partial G}{\partial Z_4} + Z_2^2 = \frac{\partial G}{\partial Z_5} + Z_2 Z_3 = 0.$$

We can use proof by contradiction to show that the singular locus is not a 2-dimensional linear subspace. Assume that the singular locus is a 2-dimensional linear subspace. Then the linear space is necessarily contained in $Z_4 = Z_5 = 0$. Note that the partial derivatives $\frac{\partial G}{\partial Z_i}, i = 0, 1, 4, 5$, are nonzero polynomials in Z_0, Z_1, Z_4, Z_5, otherwise $G = 0$ defines a cone over a plane cubic. Since we have assumed that the singular locus is a 2-dimensional linear space, the quadratic form $\frac{\partial G}{\partial Z_5} + Z_2 Z_3$ is a product of linear forms modulo Z_4 and Z_5. This forces the quadratic form $\frac{\partial G}{\partial Z_5}$ to lie in the homogeneous ideal generated by Z_4, Z_5. It follows that this 2-dimensional linear subspace in the singular locus must be of the form $Z_4 = Z_5 = Z_3 = 0$ (since we have assumed that $Z_2 \neq 0$ in the beginning). However, the polynomial $\frac{\partial G}{\partial Z_4} + Z_2^2$ does not vanish identically on this linear subspace since $Z_2 \neq 0$ by assumption. One gets a contradiction. □

We can globalize the base change and birational modifications to obtain the following.

COROLLARY 2.20
Let $\mathcal{X} \to B$ be a model of smooth cubic n-folds ($n \geq 4$) defined over the function field $\mathbb{F}_q(B)$ of a smooth projective curve B defined over $\mathbb{F}_q$ of characteristic at least 5. Assume that the set of adelic points of the generic fiber of $\mathcal{X} \to B$ is nonempty. Then there is a tower of degree 2 branched covers $C = C_1 \to C_2 \to \cdots \to B$ such that there is a model $\mathcal{X}_C \to C$ of $\mathbb{F}_q(C) \times_B \mathcal{X}$, whose fibers over closed points are geometrically reduced, geometrically irreducible, and normal. Moreover, none of the fibers is a cone over a smooth plane cubic.

2.4. Semistable models for complete intersections of two quadrics

This section is a straightforward computation using the theory of semistable models. First note the following.

LEMMA 2.21
Let X be a geometrically integral, nonnormal complete intersection of two quadrics in $\mathbb{P}^n$ defined over a perfect field k of characteristic at least 3. Then the singular locus of X has a unique $(n-3)$-dimensional component which is a linear space defined over k.

Proof
We base change to an algebraic closure $\bar{k}$ of k and take general hyperplane sections repeatedly until the complete intersection is a reduced and irreducible curve in $\mathbb{P}^3$. Then it is a singular curve of arithmetic genus 1 contained in a pencil of quadric surfaces. The curve has only one singular point. Indeed, if there are two singular points, take the plane spanned by these two points together with any third point. Then the intersection of the plane with the curve cannot be a proper intersection, otherwise the intersection multiplicity is at least 5 but the curve has degree 4. Therefore, the whole curve is contained in the plane since the curve is irreducible. But a $(2,2)$-complete intersection curve cannot not be contained in any hyperplane. Thus there is

a unique irreducible component of the singular locus which is an $(n-3)$-dimensional linear space. This linear space is defined over k since it is Galois invariant. $\square$

LEMMA 2.22
Let $\mathcal{X} \to \operatorname{Spec} \mathbb{F}_q[\![t]\!]$ be a semistable family of complete intersections of two quadrics defined by $Q = Q' = 0$ in $\mathbb{P}^n, n \geq 3$. Assume that the characteristic is not 2 and that the generic fiber of $\mathcal{X} \to \operatorname{Spec} \mathbb{F}_q[\![t]\!]$ is smooth. Let Q_0 and Q'_0 be the reductions of Q and Q' modulo t.

(1) *If $n \geq 4$, none of the quadrics defined by $\lambda Q_0 + \mu Q'_0 = 0, [\lambda, \mu] \in \mathbb{P}^1(\mathbb{F}_q)$ has a linear factor defined over $\mathbb{F}_q$.*
(2) *If $n \geq 4$, the closed fiber defined by $Q_0 = Q'_0 = 0$ does not contain a linear subspace of dimension $n-2$ defined over $\mathbb{F}_q$.*
(3) *For any formal section $\widehat{s}$, at most one of the two quadrics defined by $Q_0 = 0$ and $Q'_0 = 0$ is singular at the $\mathbb{F}_q$-rational point $\widehat{s}(0)$.*
(4) *The closed fiber is geometrically reduced.*

Proof
The first two observations follow directly from the definition of semistable families.

For (3), use the fact that if a formal section of a fibration intersects the closed fiber at a singular point, then the total space has to be singular at this point. Thus if both Q_0 and Q'_0 are singular at $\widehat{s}(0)$, the total space of the two families of quadric hypersurfaces defined by $Q = 0$ and $Q' = 0$ over $\operatorname{Spec} \mathbb{F}_q[\![t]\!]$ is both singular at $\widehat{s}(0)$. Assume that the point is $[1, 0, \ldots, 0]$. Then the multiplicity of Q and Q' at the weight system W whose weights are $(0, 1, \ldots, 1)$ is at least 2. It follows that

$$\operatorname{mult}_W(Q, Q') \geq \operatorname{mult}_W Q + \operatorname{mult}_W Q' = 4 > \frac{4n}{n+1} = \frac{4(0+1+\cdots+1)}{n+1},$$

which violates the semistability hypothesis.

For (4), if the closed fiber is not geometrically reduced, there is a whole irreducible component which is not geometrically reduced. Since the closed fiber has degree 4 and the closed fiber does not contain a linear subspace of dimension $n-2$ defined over $\mathbb{F}_q$, the only possibilities for the closed fiber are:

- a union of two Galois conjugate $(n-2)$-dimensional linear subspaces, each having multiplicity 2;
- a quadric of dimension $n-2$ with multiplicity 2.

In the first case, the two linear spaces intersect at an $(n-3)$-dimensional linear space, as one can see by taking $n-3$ general hyperplane sections.

In all cases, there is a unique linear form H and a quadratic form q such that the reduced closed fiber is defined by $H = q = 0$. The quadratic polynomials Q_0 and Q'_0 are contained in the ideal generated by H and q. So $Q_0 = H \cdot L_0 + a_0 q, Q'_0 =$

$H \cdot L_0' + a_0' q$. Then one of the quadrics $\lambda Q_0 + \mu Q_0' = 0, [\lambda, \mu] \in \mathbb{P}^1(\mathbb{F}_q)$ has a linear factor defined over $\mathbb{F}_q$. This is impossible by (1). □

For the next few results, we introduce a common hypothesis.

HYPOTHESIS 2.23
Let $\mathcal{X} \to$ Spec $\mathbb{F}_q[\![t]\!]$ be a semistable family of complete intersections of two quadrics defined by $Q = Q' = 0$ in $\mathbb{P}^n, n \geq 5$. Assume that the characteristic is not 2 and that the generic fiber of $\mathcal{X} \to$ Spec $\mathbb{F}_q[\![t]\!]$ is smooth.

LEMMA 2.24
Use the same notation and hypotheses as in 2.23. If there is a formal section of the family, then the closed fiber is either geometrically irreducible or there is a member of the pencil spanned by Q_0, Q_0' of the form $X_0^2 - aX_1^2 = 0$, where a is not a square in $\mathbb{F}_q$, and X_0, X_1 are linear forms.

Proof
If the closed fiber is geometrically reduced and geometrically reducible, then, by Lemma 2.22(2), the only possibilities of the closed fiber are:

- two quadrics of dimension $n-2$;
- a union of an $(n-2)$-dimensional quadric and two $(n-2)$-dimensional linear subspaces, and none of the linear subspaces is defined over $\mathbb{F}_q$;
- a union of four linear subspaces of dimension $n-2$, none of which is defined over $\mathbb{F}_q$.

For the first case, each of the quadrics is contained in a unique hyperplane. So if one of the quadrics is defined over $\mathbb{F}_q$, then the corresponding hyperplane is defined over $\mathbb{F}_q$, and one of the quadrics $\lambda Q_0 + \mu Q_0' = 0, [\lambda, \mu] \in \mathbb{P}^1(\mathbb{F}_q)$ has a linear factor defined over $\mathbb{F}_q$, which is impossible by Lemma 2.22(1). Otherwise the two quadrics are conjugate to each other. So the product of the two linear forms of the corresponding hyperplanes is defined over $\mathbb{F}_q$ and has the form $X_0^2 - aX_1^2 = 0$, where a is not a square in $\mathbb{F}_q$. Clearly, this is one of quadrics in the pencil.

For the second case, the quadric is defined over $\mathbb{F}_q$ by $H = q = 0$. This is impossible by the same consideration as in the proof of Lemma 2.22(4).

For the last case, we first pass to an algebraic closure and take $(n-3)$ general linear sections. Then we get a reducible complete intersection in $\mathbb{P}^3$, which is a union of four lines. Furthermore, two lines intersect if and only if they come from two linear subspaces intersecting at an $(n-3)$-dimensional subspace. Two lines are disjoint if and only if they come from two linear subspaces intersecting at an $(n-4)$-dimensional subspace.

In $\mathbb{P}^3$, the four lines are defined by a pencil of quadrics. First consider the case where one of the members in the pencil is a smooth quadric surface. In this case, the lines are the rulings and the closed fiber $\mathcal{X}_0$ is a cone over the four lines. Two of the lines belong to one family of the ruling and the other two belong to the other family of the ruling. The formal section cannot intersect the closed fiber at the vertex by Lemma 2.22(3). It cannot intersect the closed fiber at the smooth locus since none of the linear spaces is defined over $\mathbb{F}_q$. Thus it has to intersect the closed fiber at the intersection of two of the linear subspaces and not the vertex. Furthermore, the triple intersection of the linear subspaces is empty. So this formal section determines two linear subspaces which intersect along a linear subspace of dimension $n-3$. The union of these two linear subspaces is Galois invariant and is defined by $H = q = 0$ over $\mathbb{F}_q$. By the same argument as in the proof of Lemma 2.22(4), this is impossible.

If none of the members of the pencil of quadrics is smooth, then they are all cones over (possibly singular) plane conics with the same vertex. So the four lines intersect at the vertex, and the four $(n-2)$-dimensional linear subspaces intersect at a subspace of dimension $n-3$. Since none of the linear subspaces is defined over $\mathbb{F}_q$, the formal section must intersect the central fiber at the vertex. But this is impossible by Lemma 2.22(3). □

It will become clear in the proof of the Hasse principle that we need the singular fibers to be geometrically integral and not a cone over a curve in $\mathbb{P}^3$. Luckily, we can always find a semistable model which satisfies this requirement.

LEMMA 2.25

Use the same notation and hypotheses as in Hypothesis 2.23. Assume that the family admits a formal section. If the central fiber is geometrically integral and nonnormal, or if it is a cone over an irreducible complete intersection curve in $\mathbb{P}^3$ of degree $(2,2)$, then necessarily $n = 5$. Moreover, in the case where the central fiber is a cone over a curve in $\mathbb{P}^3$, one can find another semistable model whose closed fiber is geometrically integral, nonnormal, but no longer a cone over a curve in $\mathbb{P}^3$.

Proof

First recall that the singular locus of a nonnormal geometrically integral complete intersection of two quadrics contains a unique $(n-3)$-dimensional linear space, necessarily defined over the field of definition of the complete intersection (see Lemma 2.21). If the central fiber is a cone over a geometrically irreducible curve of genus 1, then it also contains an $(n-3)$-dimensional linear space, which is a cone over an $\mathbb{F}_q$-rational point of the curve (the curve is either a smooth curve of genus 1 or a rational curve and hence has an $\mathbb{F}_q$-rational point in both cases). But by the semistability assumption, there is no $(n-3)$-dimensional linear space defined over $\mathbb{F}_q$ in the closed fiber when $n \geq 6$. So these cases can occur only when n is 5.

Assume that the closed fiber is a cone over an irreducible curve in $\mathbb{P}^3$. There is an $\mathbb{F}_q$-point in the smooth locus of the curve. Up to a change of coordinates over $\mathbb{F}_q[\![t]\!]$, we may assume the point $[1,0,0,0,0,0]$ is a rational point of the generic fiber over $\mathbb{F}_q(\!(t)\!)$, as well as a smooth point in the central fiber. Assume that the tangent hyperplanes to the two quadrics defining the central fiber are $X_1 = 0$ and $X_2 = 0$. Then we may write the equations as

$$\begin{cases} X_0X_1 + q_0(X_1, X_2, X_3) + tq_1(X_4, X_5) + tX_1(\dots) + tX_2(\dots) \\ \quad + tX_3(\dots) + t^2(\dots) = 0, \\ X_0X_2 + q_0'(X_1, X_2, X_3) + tq_1'(X_4, X_5) + tX_1(\dots) + tX_2(\dots) \\ \quad + tX_3(\dots) + t^2(\dots) = 0. \end{cases}$$

We use the following change of variables:

$$\begin{aligned} &X_0 = Y_0, \qquad X_1 = tY_1, \qquad X_2 = tY_2, \qquad X_3 = tY_3, \\ &X_4 = Y_4, \qquad X_5 = Y_5. \end{aligned}$$

Note that both of the defining equations have multiplicity 1 along a weight system whose weights are $(0,1,1,1,0,0)$. Thus the new family is semistable by Lemma 2.10. The new defining equations are

$$\begin{cases} Y_0Y_1 + q_1(Y_4, Y_5) + t(\dots) = 0, \\ Y_0Y_2 + q_1'(Y_4, Y_5) + t(\dots) = 0. \end{cases}$$

Note that q_1 and q_1' are not proportional. Otherwise the pencil of quadrics would contain a member which is the union of two hyperplanes both defined over $\mathbb{F}_q$, and the family cannot be semistable by Lemma 2.22(1). Also q_1 and q_1' cannot have a common linear factor, otherwise the complete intersection contains a 3-dimensional linear space defined over $\mathbb{F}_q$, contradicting the semistability by Lemma 2.22(2). Thus $q_1(Y_4, Y_5) = q_1'(Y_4, Y_5) = 0$ has no solution over $\bar{\mathbb{F}}_q$. Therefore, the new central fiber is geometrically integral, not a cone over a curve in $\mathbb{P}^3$, and nonnormal with singular locus $Y_0 = Y_4 = Y_5 = 0$. □

LEMMA 2.26

Use the same notation and hypotheses as in Hypothesis 2.23. Assume that there is a formal section. If there is a member $q(X_0, X_1)$ defined over the ground field of the pencil spanned by Q_0, Q_0' which defines two Galois conjugate hyperplanes, then there is another semistable model whose closed fiber is geometrically integral and not a cone over a curve in $\mathbb{P}^3$.

Proof
Assume the family is given by

$$\begin{cases} q_0(X_0, X_1) + tq_1(X_2, \dots, X_n) + tX_0(\dots) + tX_1(\dots) + t^2(\dots) = 0, \\ X_0L_0 + X_1L_1 + q_0'(X_2, \dots, X_n) + t(\dots) = 0, \end{cases}$$

where q_0 is irreducible over $\mathbb{F}_q$.

Up to a change of coordinates by taking $\mathbb{F}_q[\![t]\!]$-linear combinations of $X_2, \dots, X_5$, we may assume that the point $[0, 0, 1, 0, \dots, 0]$ is a rational point over $\mathbb{F}_q(\!(t)\!)$. It follows that there is no monomial of the form $t^e X_2^2, e \geq 1$ in the equations above. By Lemma 2.22(3), the closed fiber of the second family of quadric hypersurfaces is smooth along $[0, 0, 1, 0, \dots, 0]$. The generic fibers of both families are smooth at the point $[0, 0, 1, 0, \dots, 0]$. Assume that the tangent hyperplanes of the two quadric hypersurfaces along the point $[0, 0, 1, 0, \dots, 0]$ are $X_i = 0$ and $X_j = 0$ for some i and j, respectively. Then the only monomials in the defining equations of the two quadric hypersurfaces which have a factor of X_2 are of the form $t^{k_1}u(t)X_2X_i$ with $u(t)$ a unit and $X_2X_j, i \neq j$, respectively. Note that the second equation already has a monomial X_2X_j for some j in the zeroth order term since the central fiber of the family defined by the second equation is smooth along $[0, 0, 1, 0, \dots, 0]$.

First, we claim that for a semistable family we cannot have $\{i, j\} = \{0, 1\}$. In the following, we explain why it is impossible to have $(i, j) = (0, 1)$. The case $(i, j) = (1, 0)$ is similar. We make the following change of variables:

$$X_0 = tY_0, \qquad X_1 = t^2Y_1, \qquad X_2 = Y_2, \qquad X_k = tY_k, \quad k \geq 3.$$

The multiplicities of the two equations at the weight system whose weights are $(1, 2, 0, 1, \dots, 1)$, are both equal to 2, the sum of which is exactly $\frac{4(1+2+0+1+\dots+1)}{6} = 4$. Since the original family is semistable, the new family is also semistable by Lemma 2.10. The new family is defined by

$$\begin{cases} aY_0^2 + t^{k_1-1}Y_2Y_0 + t(\dots) = 0, \quad a \neq 0, \\ Y_0(\dots) + Y_1(\text{linear forms that do not contain } Y_2) + Y_2Y_1 \\ \quad + (\text{quadratic forms that do not contain } Y_0, Y_1, Y_2) + t(\dots) = 0. \end{cases}$$

But this is not semistable since the second equation has a linear factor defined over the ground field in the zeroth order term, which contradicts Lemma 2.22(1).

So without loss of generality, we may assume that $(i, j) = (3, 0), (0, 3)$, or $(3, 4)$. We make the following change of variables:

$$X_j = t^2Y_j, \qquad X_2 = Y_2, \qquad X_k = tY_k, \quad k \neq 2, j.$$

As before, the new family is semistable by Lemma 2.10 and is defined by

$$\begin{cases} aY_1^2 + t^{k_1-1}Y_2Y_3 + t(\dots) = 0, \\ Y_0(\text{linear forms that do not contain} Y_2) + Y_1(\dots) + Y_2Y_0 \\ \quad + (\text{quadratic forms that do not contain } Y_0, Y_1, Y_2) + t(\dots) = 0; \end{cases}$$

if $(i, j) = (3, 0)$,

$$\begin{cases} q_0(Y_0, Y_1) + t^{k_1-1}Y_2Y_0 + t(\dots) = 0, \\ Y_0(\dots) + Y_1(\dots) + Y_2Y_3 \\ \quad + (\text{quadratic forms that do not contain } Y_0, Y_1, Y_2) + t(\dots) = 0; \end{cases}$$

if $(i, j) = (0, 3)$,

$$\begin{cases} q_0(Y_0, Y_1) + t^{k_1-1}Y_2Y_3 + t(\dots) = 0, \\ Y_0(\dots) + Y_1(\dots) + Y_2Y_4 \\ + (\text{quadratic forms that do not contain } Y_0, Y_1, Y_2) + t(\dots) = 0. \end{cases}$$

if $(i, j) = (3, 4)$.

So if $(i, j) = (3, 0)$, then $k_1 = 1$ and $a \neq 0$. Otherwise the first equation has a linear factor defined over the ground field in the zeroth order term, which contradicts Lemma 2.22(1). In the other cases, if $k_1 \neq 1$, then the zeroth order term of the first defining equation is still $q(Y_0, Y_1)$ and we can use the same type of change of variables and produce a third semistable family. We may continue this process until we get Y_2Y_3 in the zeroth order term for the case $(i, j) = (3, 4)$ and Y_2Y_0 in the zeroth order term for the case $(i, j) = (0, 3)$.

In the end, we have a new semistable family whose central fiber is defined by

$$\begin{cases} aY_1^2 + Y_2Y_3 = 0, \\ Y_0(Y_2 + \text{linear forms that do not contain } Y_2) + Y_1(\dots) \\ \quad + (\text{quadratic forms that do not contain } Y_0, Y_1, Y_2) = 0; \end{cases} \qquad \text{if } (i, j) = (3, 0),$$

$$\begin{cases} q_0(Y_0, Y_1) + Y_2Y_0 = 0, \\ Y_0(\dots) + Y_1(\dots) + Y_2Y_3 \\ \quad + (\text{quadratic forms that do not contain } Y_0, Y_1, Y_2) = 0; \end{cases} \qquad \text{if } (i, j) = (0, 3),$$

$$\begin{cases} q_0(Y_0, Y_1) + Y_2Y_3 = 0, \\ Y_0(\dots) + Y_1(\dots) + Y_2Y_4 \\ \quad + (\text{quadratic forms that do not contain } Y_0, Y_1, Y_2) = 0, \end{cases} \qquad \text{if } (i, j) = (3, 4).$$

Then the central fiber is geometrically integral. Indeed, the new family is still semistable and has a formal section, so it is geometrically reduced by Lemma 2.22(4).

Note that the only monomials in the defining equations above that have a factor of Y_2 are Y_2Y_0, Y_2Y_3, or Y_2Y_4. So the point $[0,0,1,0,\dots,0]$ is a smooth point of the central fiber; in particular, a smooth point of every quadric in the pencil defining the central fiber. Then none of the quadrics defining the central fiber is a union of two Galois conjugate hyperplanes. By Lemma 2.24, the central fiber is geometrically irreducible.

If the central fiber is a cone over a $(2,2)$-complete intersection curve in $\mathbb{P}^3$, we can use Lemma 2.25 to produce a new family whose fiber is geometrically reduced, geometrically irreducible, and not a cone over a curve. □

We may glue local semistable families together to obtain the following.

COROLLARY 2.27
Assume that the characteristic is not 2*. Consider a smooth complete intersection of two quadrics in* $\mathbb{P}^n, n \geq 5$ *defined over* $\mathbb{F}_q(B)$*. Assuming that the set of adelic points is nonempty, then there is a semistable model over* B *whose closed fibers are geometrically integral, and are not cones over a* $(2,2)$*-complete intersection curve in* $\mathbb{P}^3$*. The closed fibers can be nonnormal only if* $n = 5$*.*

3. Asymptotically canonical sequence of spaces of sections

3.1. The main construction
In this section, we discuss the main construction used in the proof of the main theorem, which is essentially due to [19].

Definition 3.1
Let $\mathcal{X} \to B$ be a family of algebraic varieties defined over a field. A section $s : B \to \mathcal{X}$ is *m-free* if $s(B)$ is contained in the smooth locus of $\mathcal{X}$ and $H^1(B, \mathcal{N}_s(-D_m)) = \{0\}$ for any effective divisor D_m of degree m on B, where $\mathcal{N}_s$ is the normal sheaf of s in $\mathcal{X}$. A 1-free section is also called a *free section*. A morphism $f : B \to X$ is *m-free* if the induced section of the trivial family $X \times B \to B$ is m-free.

Remark 3.2
In case of a morphism from $\mathbb{P}^1$, 1-freeness is the same as free and 2-freeness is the same as very free. If the generic fiber is smooth projective and separably rationally connected, then the existence of a section in the smooth locus implies the existence of an m-free section for any $m > 0$.

We now introduce some basic hypotheses on the family $\mathcal{X} \to B$.

HYPOTHESIS 3.3
Given a family $\mathcal{X} \to B$ of Fano complete intersections defined over a perfect field k, assume that the following are satisfied.

(1) *The Fano scheme of lines of a general fiber $\mathcal{X}_b$ is smooth.*
(2) *Choose an algebraic closure $\bar{k}$ of k. A general line (defined over $\bar{k}$) in a general fiber (defined over $\bar{k}$) is a free line.*
(3) *The relative dimension of $\mathcal{X} \to B$ is at least* 3.
(4) *There is a free section.*

We need to introduce one more notion. Let $\mathcal{X} \to B$ be as above, and let $F(B) \to B$ be the relative Fano scheme of lines of the family. The Fano scheme of lines is connected on a smooth Fano complete intersection of dimension at least 3 (see, e.g., [18, Théorème 2.1]). Thus by the smoothness assumption, it is geometrically irreducible for a general fiber, and there is an open subset U of the base B such that the relative Fano scheme of lines $F(U)$ over U is irreducible. Let $\bar{F} \to B$ be the closure of $F(U)$ in $F(B)$. Denote by $\mathcal{L} \to \bar{F}$ the universal family of lines for the fibration $\mathcal{X} \to B$ restricted to the irreducible component $\bar{F}$, and let $\mathrm{ev}_L : \mathcal{L} \to \mathcal{X}$ be the natural evaluation morphism. The morphism ev_L is proper and surjective.

The morphism $\mathcal{L} \to \mathcal{X}$ factors through a variety $\mathcal{Z}$ via $\mathcal{L} \to \mathcal{Z} \to \mathcal{X}$ such that a general fiber of $\mathcal{L} \to \mathcal{Z}$ is geometrically irreducible and $\mathcal{Z} \to \mathcal{X}$ is finite and generically étale (see [25, (9)]). Let $\mathcal{X}^0$ be the open locus of $\mathcal{X}$, and let $\mathcal{Z}^0$ be the inverse image of $\mathcal{X}^0$ in $\mathcal{Z}$ such that $\mathcal{Z}^0 \to \mathcal{X}^0$ is étale.

Finally, let $\mathcal{X}^1 \subset \mathcal{X}$ be the open subvariety of $\mathcal{X}$ over which the evaluation morphism ev_L has constant fiber dimension. The complement $\mathcal{X} - \mathcal{X}^1$ has codimension at least 2 in $\mathcal{X}$.

Definition 3.4
Use the same notation as above. A section $s : B \to \mathcal{X}$ is a *nice section* if it is 2-free, intersects the locus $\mathcal{X}^0 \cap \mathcal{X}^1$ and the fiber product $B \times_{\mathcal{X}} \mathcal{L}$ is geometrically irreducible.

It follows from the definition that a nice section is always contained in $\mathcal{X}^1$.

When the Fano pseudoindex (i.e., the minimal intersection number of $-K_X$ with an effective curve class) of a general fiber of $\mathcal{X} \to B$ is at least 3, the existence of a nice section is easy. Since the complement of $\mathcal{X}^1$ in $\mathcal{X}$ has codimension at least 2, a general deformation of a 2-free section lies in $\mathcal{X}^1$. Then checking the irreducibility of the base change amounts to checking the irreducibility of the family of lines through a general point of the section. We first prove the following lemma.

LEMMA 3.5
Let $\mathcal{X} \to B$ be a family of Fano complete intersections defined over an algebraically closed field $\bar{k}$ which satisfies Hypothesis 3.3. Assume that the Fano pseudoindex of a general fiber is at least 3*. Then a* 2*-free section defined over $\bar{k}$ is nice if it contains a general $\bar{k}$-point of $\mathcal{X}$ and lies in $\mathcal{X}^1$.*

As indicated above, the condition of a point being "general" can be taken to be that the family of lines through this point is geometrically irreducible. The only nonobvious thing to check is that these points form a *nonempty* open subset of $\mathcal{X}$.

Proof
By [28, Corollary 9], a general fiber of $\mathcal{X} \to B$ is separably rationally connected. So there are 2-free sections over $\bar{k}$.

Note that for any curve $T \to \mathcal{X}^1$, every irreducible component of the base change $T \times_{\mathcal{X}} \mathcal{Z}$ dominates T (see [25, Proof of (7)]). Thus it suffices to show that the family of lines through a general point of a general fiber of $\mathcal{X}$ is geometrically irreducible.

We look at the evaluation map $f : F_b \to \mathcal{X}_b$ of the Fano scheme of lines on a general fiber over $b \in B$. Let K_b be the function field of $\mathcal{X}_b$. The Fano scheme F_b is smooth by assumption. So the generic fiber F_{K_b} of the morphism f is normal. Moreover, the family of lines through a general point in $\mathcal{X}_b$ is a complete intersection of positive dimension since the Fano pseudoindex is assumed to be at least 3. So $H^0(F_{K_b}, \mathcal{O}_{F_{K_b}}) = K_b$. Then it is geometrically irreducible by [25, Lemma 10] and a general fiber is geometrically irreducible (see [21, Théorème 9.7.7]). Note that this is easy in characteristic 0 since we know the normality of F_{K_b} after base change to an algebraic closure. Only in characteristic p does one need to be careful, since normality is not preserved by passing to an algebraic closure. ☐

In general, one can still show the existence of a nice section over $\bar{k}$ for every family satisfying Hypothesis 3.3. The proof is a simple application of the results of Kollár [25] on the fundamental groups of separably rationally connected varieties. For the reader's convenience, we summarize some results concerning the property of a section being "nice" in the following. These are proved in Appendix B.

LEMMA 3.6 (Lemma B.6)
Let $\mathcal{X} \to B$ be a family defined over an algebraically closed field k satisfying Hypothesis 3.3.

(1) *There is a nice section.*
(2) *Let W be a geometrically irreducible component of the space of sections, and let $\mathcal{S} \to W$ be the universal family such that there is a geometric point $w \in W$*

that parameterizes a nice section $\mathcal{S}_w$. Then a general geometric point of W parameterizes a nice section.

(3) *Let $\mathcal{S} \to W$ be a geometrically irreducible component of the space of sections such that a general geometric point parameterizes a nice section. Then $\mathcal{S} \times_{\mathcal{X}} \mathcal{L}$ is geometrically irreducible and generically smooth. Furthermore, it is contained in a unique geometrically irreducible component of the Kontsevich moduli space of stable sections which contains an open substack parameterizing nice sections.*

Now we introduce the following hypotheses on the family $\mathcal{X} \to B$.

HYPOTHESIS 3.7

Given a family $\mathcal{X} \to B$ of Fano complete intersections defined over a perfect field k, assume that the following are satisfied.

(1) *The Fano scheme of lines of a general fiber $\mathcal{X}_b$ is smooth.*

(2) *Choose an algebraic closure $\bar{k}$ of k. A general line (defined over $\bar{k}$) in a general fiber (defined over $\bar{k}$) is a free line.*

(3) *The relative dimension of $\mathcal{X} \to B$ is at least* 3.

(4) *There is a family of nice sections $\mathcal{S} \to W$ parameterized by a geometrically irreducible variety W defined over k.*

When the field k is algebraically closed, this is equivalent to Hypothesis 3.3.

Given a family $\mathcal{X} \to B$ defined over a perfect field k and satisfying Hypothesis 3.7, we have the following construction due to de Jong, He, and Starr [19, Section 5].

CONSTRUCTION 3.8

Start with the family $\mathcal{X} \to B$ and the family of nice sections $\mathcal{S} \to W$. We will define a sequence of irreducible components $M_i(W), i = 0, 1, \dots$ of the moduli space of sections and their compactifications $\overline{M}_i(W), i = 0, 1, \dots$ as follows.

Define $\overline{M}_0(W)$ to be the unique irreducible component obtained as the Zariski closure of W in the Kontsevich moduli space of stable sections, and define $M_0(W)$ to be the Zariski-dense open substack of $\overline{M}_0(W)$ parameterizing nice sections. Denote by $\mathcal{S}_0 \to M_0(W)$ the universal family of sections.

Then we define $\overline{M}_1(W)$ to be the unique geometrically irreducible component containing the family of stable sections $\mathcal{S}_0 \times_{\mathcal{X}} \mathcal{L}$. A general point in $\overline{M}_1(W)$ parameterizes a nice section by Lemma 3.6(3). Take $M_1(W)$ to be the geometrically irreducible open substack parameterizing nice sections, and let $\mathcal{S}_1 \to M_1(W)$ be the universal family.

The fibration $\mathcal{X} \to B$ together with the family of nice sections parameterized by $M_1(W)$ also satisfies Hypothesis 3.7 by Lemma 3.6. Then we can continue with the above construction, replacing W by $M_1(W)$ and so on. This process produces a sequence of geometrically irreducible components $M_i(W) \subset \overline{M}_i(W), i = 0, 1, \ldots$.

Definition 3.9
Given a family of Fano complete intersections $\mathcal{X} \to B$ defined over a perfect field k satisfying Hypothesis 3.7, we say that $\mathcal{X} \to B$ has an *asymptotically canonical sequence* if for any two geometrically irreducible components of the spaces of sections $\mathcal{S}_1 \to W_1$ and $\mathcal{S}_2 \to W_2$ whose general points parameterize nice sections, there are numbers N_1, N_2 such that $M_{N_1+i}(W_1) = M_{N_2+i}(W_2)$ for all $i \geq 0$.

Remark 3.10
One can be more specific about the numbers N_1 and N_2. There is an ample divisor H on $\mathcal{X}$ whose restriction to each fiber of $\mathcal{X} \to B$ is isomorphic to $\mathcal{O}(1)$. Then $N_2 - N_1 = H \cdot (\beta_2 - \beta_1)$, where β_i is the curve class of the family of sections parameterized by W_i for $i = 1, 2$. In particular, if W_1 and W_2 parameterize sections of the same curve class, which is the case if they are Galois conjugate families, then $N_1 = N_2$.

The importance of this property is that it allows us to get geometrically irreducible components defined over a perfect field even though we do not know the existence of a free section over this field.

LEMMA 3.11
Let $\mathcal{X} \to B$ be a family of Fano complete intersections defined over a perfect field k such that the base change to an algebraic closure $\bar{k}$ satisfies Hypothesis 3.3 or, equivalently, Hypothesis 3.7. Let $\mathcal{S} \to W$ be a nice section of the family defined over $\bar{k}$. Furthermore, assume that the sequence $M_i(W), i \geq 0$ over $\bar{k}$ is an asymptotically canonical sequence. Then for i large enough, every component $M_i(W)$ is Galois invariant, that is, defined over k.

Proof
The family of nice sections $\mathcal{S} \to W$ is defined over a finite Galois field extension k'/k. Thus the sequence $M_i(W), i = 0, 1, \ldots$ is defined over k'. It suffices to show that the Galois group $\mathrm{Gal}(k'/k)$ fixes $M_i(W)$ for i large. There are only finitely many geometrically irreducible components of nice sections $\mathcal{S}_1 \to W_1, \ldots, \mathcal{S}_n \to W_n$ defined over k' which are Galois conjugate to the family $\mathcal{S} \to W$. So for each $i > 0$ the spaces of nice sections $M_i(W_j), j = 1, \ldots, n$ are defined over k' and are Galois

conjugate to $M_i(W)$. Furthermore, these are all the Galois orbits of $M_i(W)$. By assumption, there is a finite number N such that for all $i > N$ and for any $j = 1, \ldots, n$, $M_i(W_j) = M_i(W)$ (see Remark 3.10). Thus the components $M_i(W)$ are Galois invariant, that is, defined over k. □

There are several natural questions related to this construction.

(1) Given a family of Fano complete intersections $\mathcal{X} \to B$ defined over a perfect field k satisfying Hypothesis 3.7, when can we find an asymptotically canonical sequence?

(2) Given a family of Fano complete intersections $\mathcal{X} \to B$ defined over a perfect field k such that the base change to an algebraic closure $\bar{k}$ satisfies Hypothesis 3.7, is the sequence $M_i(W), i = 0, 1, \ldots$, constructed over $\bar{k}$, Galois invariant for i large enough?

(3) In view of Manin's conjecture on asymptotic behavior of rational points, one could ask, in the case when the sequence is asymptotically canonical, is there a limit of $[M_i] \cdot \mathbb{L}^{-K_{\mathcal{X}} \cdot s - iK_{\mathcal{X}} \cdot L}$ in the Grothendieck ring of varieties? If the family is defined over a finite field $\mathbb{F}_q$ and has an asymptotically canonical sequence, is there a limit of the number $\frac{\# M_i(\mathbb{F}_q)}{q^{K_{\mathcal{X}} \cdot s + iK_{\mathcal{X}} \cdot L}}$?

3.2. Application to the Hasse principle

For the purpose of this article, the importance of questions (1) and (2) is because of the following (clearly, an affirmative answer to (1) and (2) implies the existence of the component Σ).

LEMMA 3.12

Let $\mathcal{X} \to B$ be a family of varieties defined over $\mathbb{F}_q$ such that the generic fiber X has dimension at least 1 *and is either a smooth quadric hypersurface, a smooth cubic hypersurface, or a complete intersection of two quadrics. If there is an irreducible component Σ of the space of sections which is geometrically irreducible, then there is a section defined over $\mathbb{F}_q$.*

Proof

By the Lang–Weil estimate, the variety Σ has an $\mathbb{F}_{q^n}$-point for every n large enough. Equivalently, the generic fiber X of the family $\pi : \mathcal{X} \to B$ has an $\mathbb{F}_{q^n}(B)$-rational point for every n large enough.

A smooth quadric hypersurface Q in arbitrary characteristic or a complete intersection of two quadrics defined over a field of odd characteristic has a rational point if and only if there is a rational point in some odd degree field extension (see [27] for the case of quadrics and [6] for the case of complete intersection of two quadrics).

Thus in these two cases we are done. These are all we need for the proof of the main theorem.

We sketch an argument which proves the lemma in all characteristics. First note that if there is an $\mathbb{F}_{q^{3n}}(B)$-rational point, then there is an $\mathbb{F}_{q^n}(B)$-rational point. To see this, denote the $\mathbb{F}_{q^{3n}}$-point and its Galois conjugate points by x, y, z and consider the linear space spanned by these points. If this is a line, then it has to lie in X since every quadric which intersects a line at three points has to contain the line. Thus there is a rational point of X coming from a rational point in the line. If this is a plane contained in the hypersurface X, then there are rational points contained in the plane. If this is a plane which is not contained in X, then the intersection of the plane with X is either a possibly singular conic or a zero-cycle of degree 4. In any case, there is an effective zero-cycle of degree 3 which is contained in the intersection and defined over $\mathbb{F}_{q^n}(B)$. It then follows that there is a rational point over $\mathbb{F}_{q^n}(B)$. As discussed above, the variety X has a rational point in $\mathbb{F}_{q^{3^k}}(B)$ for some k large enough. Then there is an $\mathbb{F}_q(B)$-point by iterating this argument.

For smooth cubic hypersurfaces, we claim that if there is a rational point of X defined over $\mathbb{F}_{q^{2n}}(B)$, then there is a rational point defined over $\mathbb{F}_{q^n}(B)$. As discussed at the beginning of the proof, the cubic hypersurface X has a rational point in $\mathbb{F}_{q^{2^k}}(B)$ for some k large enough. Then by the claim there is an $\mathbb{F}_q(B)$-point. To see this claim, note that given any $\mathbb{F}_{q^{2n}}(B)$-point and its conjugate, they span a unique line L defined over $\mathbb{F}_{q^n}(B)$. Then either the line is contained in the hypersurface X or it intersects X at a third intersection point. In any case, we have a rational point over $\mathbb{F}_{q^n}(B)$. $\square$

Remark 3.13

The existence of a geometrically irreducible component is almost a necessary condition. More precisely, if there is a section of $\mathcal{X} \to B$ which lies in the smooth locus of $\mathcal{X}$, then after adding sufficiently many very free curves in general fibers (over $\bar{\mathbb{F}}_q$) and their Galois conjugates, we have a smooth point of an irreducible component of the Kontsevich moduli space (see [20, Section 2.1]). Furthermore, a general point of this component parameterizes a section of $\mathcal{X}$.

However, there is no guarantee that if there is a section, then it lies in the smooth locus of $\mathcal{X}$ in general. For a semistable family of quadric hypersurfaces, this is automatic by the definition of semistability. For a complete intersection of two quadrics of dimension at least 3, the existence of a section implies that all the fibers are geometrically integral and thus have a rational point in the smooth locus. Furthermore, we also know weak approximation holds once there is a rational point (see Theorem 6.3). So we can find a section which intersects every singular fiber at a smooth point, in particular, that lies in the smooth locus. For semistable families of cubic hypersurfaces, closed fibers may not have a rational point in the smooth locus. But after a tower of degree 2 base changes, we can always achieve this. However, the weak approximation problem is still open.

On the other hand, if we can resolve the singularities of $\mathcal{X} \to B$ (which seems possible even though resolution of singularities is unknown in general, since the singularities are fairly explicit), then we can apply the argument at the beginning of this remark to the resolution. The irreducible component of the space of sections of the resolution gives a geometrically irreducible subvariety of the space of sections of $\mathcal{X} \to B$.

COROLLARY 3.14
Let $\mathcal{X} \to B$ be a family of varieties defined over $\mathbb{F}_q$ such that the generic fiber X has dimension at least 3 and is either a smooth quadric hypersurface, a smooth cubic hypersurface, or a complete intersection of two quadrics. If the family $\mathcal{X} \to B$ satisfies Hypothesis 3.7 and the Construction 3.8 gives an asymptotically canonical sequence of spaces of sections, then there is a section.

Later in the proof we have to work with unions of sections and high degree curves in fibers. Thus it is necessary to know when such a curve lies in the components $\bar{M}_i(W)$ constructed above.

We first define the notion of a comb (with broken teeth).

Definition 3.15
A comb (with broken teeth) defined over an algebraically closed field k is a projective connected at worst nodal curve $C = C_0 \cup R_1 \cup \cdots \cup R_l$ together with a morphism $f : C \to X$ to a variety X, where C_0 is a smooth projective curve, and $R_i, 1 \leq i \leq l$, are disjoint chains of rational curves attached to C_0 at distinct points. The curve C_0 is called the *handle* and each $R_i, 1 \leq i \leq l$, is a *(broken) tooth.*

We need the following simple observation.

LEMMA 3.16
Let $\mathcal{X} \to B$ be a family of Fano complete intersections over an algebraically closed field satisfying Hypothesis 3.7, and let $s : B \to \mathcal{X}$ be a nice section parameterized by a general point of W. Assume that $C \subset \mathcal{X}$ is a comb with handle $s(B)$ and teeth $C_i, i = 1, \ldots, n$, which are free curves in smooth fibers $\mathcal{X}_{b_i}, i = 1, \ldots, n$. Furthermore, assume that every C_i deforms to a chain of free lines in the fiber $\mathcal{X}_{b_i}$ and that the deformation can be parameterized by an irreducible curve. Then the comb C lies in $\overline{M}_k(W)$ for some k and corresponds to a smooth point in $\overline{M}_k(W)$. In particular,

$\overline{M}_k(W)$ is the unique irreducible component containing the point corresponding to the comb C.

Proof

For each point $b_i \in B$, define U_i to be the open subset of $\mathcal{X}_{b_i}$ containing all the geometric points x such that there is a chain of free lines $\mathcal{L} = L_1 \cup \cdots \cup L_k$ containing x in the smooth locus of $\mathcal{L}$ and such that the chain $\mathcal{L}$ lies in the same irreducible component of the Kontsevich moduli space of stable maps to $\mathcal{X}_{b_i}$ as C_i.

By the assumptions, the complement of U_i in $\mathcal{X}_{b_i}$ is a proper closed subset. Thus a general deformation of the section $s(B)$ meets the fibers $\mathcal{X}_{b_i}$ in U_i.

Therefore, there are families of curves $\mathcal{S} \to T, \mathcal{C}_i \to T, i = 1, \ldots, n$ over an irreducible curve T with sections $s_0 : T \to \mathcal{S}, s_i : T \to \mathcal{C}_i$ and evaluation morphisms $\mathrm{ev}_0 : \mathcal{S} \to \mathcal{X}, \mathrm{ev}_i : \mathcal{C}_i \to \mathcal{X}, i = 1, \ldots, n$ such that

(1) the family $\mathcal{S} \to T$ is a family of sections of $\mathcal{X} \to B$ and the families $\mathcal{C}_i \to T$ are families of curves in the fiber $\mathcal{X}_{b_i}$;

(2) the families $\mathcal{S} \to T, \mathcal{C}_i \to T$ glue together along the sections $s_0, s_1, \ldots, s_n$ and form a family of combs $\mathcal{C} \to T$;

(3) there are two points t_1, t_2 in T such that $\mathcal{C}_{t_1}$ is the comb C and $\mathcal{C}_{t_2}$ is a comb whose handle is a nice section $s'(B)$ and whose teeth are chains of free lines.

Clearly, the comb C and the comb $\mathcal{C}_{t_2}$ are smooth points of the *same* irreducible component of the Kontsevich space of stable sections. So it suffices to show that the comb $\mathcal{C}_{t_2}$ lies in one of the $M_i(W)$'s. This follows from a simple induction on the number of lines in the teeth of the comb. □

4. Hasse principle for quadrics

In this section, we prove the Hasse principle for smooth quadrics defined over a global function field as a warm-up for the later, more technical proofs.

First, consider the case of a semistable family of quadrics $\mathcal{X} \to B$ over $\mathbb{F}_q$ of relative dimension at least 3 with smooth generic fiber. Still denote the base change of the family to an algebraic closure $\bar{\mathbb{F}}_q$ by $\pi : \mathcal{X} \to B$. Then this family over $\bar{\mathbb{F}}_q$ satisfies Hypothesis 3.3. The only nontrivial condition to check is the existence of a free section. To see this, first note that over $\bar{\mathbb{F}}_q$ all the singular fibers are integral, and thus there is a point in the smooth locus. Over each singular fiber, choose a smooth point of the fiber. We know that the family $\mathcal{X} \to B$ satisfies weak approximation (see, e.g., [8, p. 128, paragraph after the proof of Théorème 4.4]). So we can find a section s_0 which passes through the chosen smooth points of the singular fibers. In particular, the section s_0 lies in the smooth locus of $\mathcal{X}$. We can add very free curves in general fibers and take a general deformation of the comb to produce a free section. In this way, we get nice sections by Lemma 3.6.

By Corollary 3.14, the existence of a section over $\mathbb{F}_q$ follows from the subsequent lemma.

LEMMA 4.1
Let $\pi : \mathcal{X} \to B$ be a semistable family of quadric hypersurfaces of dimension at least 3 defined over $\bar{\mathbb{F}}_q$. Then there is an asymptotically canonical sequence of sections.

Proof
Given two geometrically irreducible components of nice sections, we need to show that they produce the same irreducible components when the degree of the section becomes large enough. To show this, let s_1 and s_2 be two nice sections belonging to the two families. Then it suffices to show that, after adding enough lines in general fibers, there is a deformation parameterized by an irreducible curve of the union of s_1 with lines to the union of s_2 with lines. The general idea to find such a deformation is to construct a ruled surface in $\mathcal{X}$ over B such that s_1 and s_2 appear as sections of the ruled surface. Below is the construction.

Let $b_1, \dots, b_n$ be the points in B whose fiber $\mathcal{X}_{b_j}$ is singular. By taking a general deformation of s_2, we may assume that the line spanned by the points $s_1(b_j)$ and $s_2(b_j)$ is not contained in the fiber $\mathcal{X}_{b_j}$.

Over each point b_j, choose a third general point x_j such that the line spanned by x_j and $s_1(b_j)$ (resp., $s_2(b_j)$) does not lie in $\mathcal{X}_{b_j}$. Since weak approximation holds for this family, there is a section s_3 such that

(1) $s_3(b_j) = x_j$;

(2) if b is a point such that the line spanned by the two points $s_1(b)$ and $s_2(b)$ lie in the fiber $\mathcal{X}_b$ (the fiber $\mathcal{X}_b$ is necessarily smooth and there are only finitely many such fibers), then the line spanned by $s_3(b)$ and $s_1(b)$ (resp., $s_2(b)$) is not contained in $\mathcal{X}_b$.

Then take the family of planes $\Pi \to B$ spanned by $s_1(b), s_2(b), s_3(b)$. The intersection of Π with $\mathcal{X}$ is a ruled surface S fibered over B whose fibers are conics $\{R_b\}_{b \in B}$.

By the choice of s_3, the conics in the singular fibers, R_{b_i}, are smooth conics, and R_b is reduced for all b. Furthermore, s_1 and s_2 only intersect fibers of $S \to B$ in the smooth locus.

The surface S might have singularities when the conic R_b is a union of two lines and the singularity is locally given by $xy = t^n$. Let $\tilde{S}$ be the minimal resolution. Note that the fibers of $\tilde{S} \to B$ over each point are still reduced by a local computation. On the surface $\tilde{S}$, the strict transform of the two sections s_1 and s_2 are linearly equivalent modulo some vertical fibers. In fact, we have

$$s_1 + \sum \mathrm{ch}_j + \sum_{i=1}^{k} R_{g_i} \sim s_2 + \sum_{j=k+1}^{n} R_{g_j}, \tag{1}$$

where ch_j is a chain of irreducible components (each with multiplicity 1) in the fibers, s_1 and s_2 are not in the same irreducible component, and R_{g_i}, R_{g_j} are general fibers of $\tilde{S} \to B$. This defines a pencil in the linear system whose general member is a smooth curve and is a section of $\tilde{S} \to B$.

Thus there is a pencil $\tilde{\mathcal{C}} \to \mathbb{P}^1$ spanned by the above two divisors, and a general member is a smooth curve. The map $\tilde{\mathcal{C}} \to \mathcal{X}$ factors through a new family $\mathcal{C} \to \mathcal{X}$ which contracts all the exceptional divisors of $\tilde{S} \to S$.

Using the relation (1), one can assemble two combs C_1, C_2 with handles s_1 and s_2 and teeth consisting of conics and lines in general fibers such that there is a family of stable maps $\mathcal{C} \to \mathbb{P}^1$ with the following properties.

(1) A general member of the family $\mathcal{C}_t \to \mathcal{X}$ is a section of $\mathcal{X} \to B$.

(2) $\mathcal{C}_0 = C_1$ and $\mathcal{C}_\infty = C_2$.

Note that for any smooth conic in a quadric hypersurface, there is a one-parameter degeneration to a union of two lines. So by Lemma 3.16, the proof is finished. □

Remark 4.2

The above argument is a prototype of what we will do in the next two sections to prove the Hasse principle for cubics and complete intersection of two quadrics. One first checks that Hypothesis 3.7 is satisfied. This requires producing a 2-free section by using weak approximation results over $\bar{\mathbb{F}}_q$. Then one constructs a ruled surface containing two general nice sections. In this step, one has to make sure of the following.

- The ruled surface has reduced fibers.
- The two sections lie in the smooth locus of the ruled surface.
- Over every fiber $\mathcal{X}_b$ that is singular, the two sections lie in the same irreducible component. In the case of cubics, one can make sure that the fiber of the ruled surface has only one irreducible component for every b such that $\mathcal{X}_b$ is singular. However, in the case of $(2,2)$-complete intersection, there could be two irreducible components. But luckily one can show that the two sections intersect the same irreducible component.

Once we can show these, we can use the constructions in Section 3 to conclude the proof of Hasse principle.

The case of conics is even easier.

LEMMA 4.3

Let $\mathcal{X} \to B$ be a semistable model of a smooth conic in $\mathbb{P}^2_{\mathbb{F}_q(B)}$ which has local *sections everywhere. Then there is a geometrically irreducible component of the space of sections.*

Proof

By Corollary 2.12, the fibration $\mathcal{X} \to B$ is smooth. Over $\bar{\mathbb{F}}_q$, this family is isomorphic to $\mathbb{P}(E) \to B$ for some locally free sheaf E of rank 2. Thus a section whose $\mathcal{O}_{\mathbb{P}(E)}(1)$ degree is d is the same as a surjection $E \to L \to 0$ to some invertible sheaf L of degree d. When d is large enough, the moduli space of sections is fibered over the Jacobian of C with fibers an open subset of the projective space $\mathbb{P}(H^0(C, E^* \otimes L))$, thus irreducible. Obviously this component is Galois invariant and defined over $\mathbb{F}_q$. $\square$

This finishes the proof of Hasse principle for conics by Lemma 3.12.

For quadric surfaces, we can reduce to the case of conics by considering the relative Fano scheme of lines. This is a family of conics fibered over a curve C which has a generically étale map to B of degree 2. Given a rational point of a quadric surface, there are exactly two lines containing this point. Conversely, given two lines of different families of the rulings, we get a rational point by taking the intersection of the two lines. So the original family has a local section everywhere if and only if the family of conics over C has a section everywhere locally. And there is section of $\mathcal{X} \to B$ if and only if there is a section for the family of conics over C.

5. Hasse principle for cubics

5.1. R-connectedness of cubic hypersurfaces

We first review the construction of Madore [26].

Let X be a smooth cubic hypersurface of dimension at least 4 defined over a C_1-field k, and let x, y be two k-rational points in X. Then there is a chain of rational curves connecting x and y by a result of Madore [26, Lemma 1.3].

In the following, we give a description of the chain of rational curves that connects two general points under some extra assumptions. This is all we need. The more general case can be treated similarly.

LEMMA 5.1

Let k be a C_1-field. Let Y be a geometrically integral singular cubic hypersurface in $\mathbb{P}^n$ defined over k, and let z be a k-rational point in the singular locus of Y with multiplicity 2. Assume that the set of k-rational points of the projective tangent cone of Y at z is Zariski-dense. Then for a general point u in Y, there is a map $f : \mathbb{P}^1 \to Y$ defined over k such that $f(0) = z$, $f(\infty) = u$.

Proof
Projection from z to a hyperplane gives a birational map $Y \dashrightarrow \mathbb{P}^{n-1}$. Equivalently, one can blow up the projective space at the point z and take the strict transform $\tilde{Y}$ of Y. There is a birational morphism $\tilde{Y} \to \mathbb{P}^{n-1}$. The projective tangent cone of Y at the point z is a quadric in the exceptional divisor E of $Bl_z\mathbb{P}^n \to \mathbb{P}^n$. Denote by v the image of u in $\mathbb{P}^{n-1}$ under this morphism.

Assume that the coordinate of z is $[1,0,\dots,0]$ and that the hyperplane $\mathbb{P}^{n-1}$ is defined by $X_0 = 0$. Write the equation of Y as

$$X_0 Q(X_1,\dots,X_n) + C(X_1,\dots,X_n) = 0.$$

Then the projective tangent cone in E is defined by $Q(X_1,\dots,X_n) = 0$. The inverse birational map is the following:

$$\begin{aligned} &\mathbb{P}^{n-1} \dashrightarrow Y,\\ &[X_1,\dots,X_n] \dashrightarrow \big[-C(X_1,\dots,X_n), X_1 Q(X_1,\dots,X_n),\dots,X_n Q(X_2,\dots,X_n)\big]. \end{aligned}$$

Thus the map is not defined when $Q = C = 0$, which is precisely the locus parameterizing the family of lines in Y containing z. This is also clear from the geometric description of the birational map as a projection. A general point in $Q = 0$ is mapped to z. Choose one such general point such that the line spanned by this point and v avoids the locus $Q = C = 0$. This is possible by our assumption. Then the rational curve we want is the restriction of the birational map to the line. □

Remark 5.2
The assumption is satisfied if k is algebraically closed or if k is the function field of a curve over an algebraically closed field and the projective tangent cone is not a union of two Galois conjugate hyperplanes.

Given two general points x, y in a smooth cubic hypersurface X of dimension at least 4, denote by H_x and H_y the tangent hyperplane of X at x and y. Then $H_x \cap H_y \cap X$ is a smooth cubic hypersurface of dimension at least 2. Thus it has a rational point z over any C_1-field. If z is general, we can apply Lemma 5.1 to x, z and y, z to get a chain of two rational curves.

5.2. *Geometry of cubics*
In this section, we collect some useful results about the geometry of cubic hypersurfaces. All of these results are well known in characteristic 0, and some of them fail in small characteristic. We include a proof here since we could not find a precise reference for the results needed.

LEMMA 5.3
Let X be a normal cubic surface defined over an algebraically closed field k of characteristic not equal to $2, 3, 5$. Assume furthermore that X is not a cone over a plane cubic curve.

(1) *There are only finitely many lines on X.*
(2) *Let $x \in X$ be a general point, and let H_x be the tangent hyperplane at x. Then $H_x \cap X$ is a nodal cubic plane curve.*
(3) *For any point $x \in X^{\mathrm{sm}}$, there is a very free curve.*
(4) *For any two general points $x, y \in X$, the intersection of their tangent hyperplanes with X, $H_x \cap H_y \cap X$, is smooth (i.e., three distinct points).*

Proof
The first statement is well known.

For the second one, we consider the Gauss map defined on the smooth locus:

$$g : X \dashrightarrow \mathbb{P}^3,$$
$$x \in X^{\mathrm{sm}} \dashrightarrow H_x.$$

The second fundamental form of the cubic surface at a smooth point is the same as the differential of the Gauss map at the point. Thus the tangent hyperplane intersection at a point x has a node at x if and only if the Gauss map is smooth (or equivalently, étale) at the point x.

Denote by Y the closure of the image. If we write the defining equation of X as $F(X_0, \ldots, X_3) = 0$, then the above map is the restriction of the map

$$\tilde{g} : \mathbb{P}^3 \dashrightarrow \mathbb{P}^{3*},$$
$$[X_0, \ldots, X_3] \dashrightarrow \Big[\frac{\partial F}{\partial X_0}, \ldots, \frac{\partial F}{\partial X_3}\Big].$$

If the characteristic is not 3 and the surface X is smooth, then the map is a well-defined morphism on $\mathbb{P}^3$. Clearly, $\tilde{g}^*\mathcal{O}(1) = \mathcal{O}(2)$. Thus $\deg g \cdot \deg Y = 12$.

The map is generically one-to-one as long as X is normal and not a cone. To see this, take two points $x, y \in X^{\mathrm{sm}}$. If they are mapped to the same points under the Gauss map, then they share the same tangent hyperplane. In particular, the line spanned by x and y intersects X at a 0-dimensional scheme of length 4, and thus is contained in X. So under the assumption that X is normal and not a cone, the Gauss map is generically one-to-one. By smoothing and degeneration, we see that $\deg g \leq 12$ for all normal cubic surfaces which are not cones.

We claim that if the characteristic of the base field k is not $2, 3, 5$, then the map g is separable when considered as a morphism onto its image. Admitting this, the map

g is étale at a general point since it is generically one-to-one. Take a general point x which does not lie in any line on X, and let H_x be the tangent hyperplane at x. Then $H_x \cap X$ has a nodal singularity at x. If $y \in H_x \cap X$ is another singular point of $H_x \cap X$, then the intersection $H_x \cap X$ is reducible, and there is a line through x contradicting the choice of X. Thus $H_x \cap X$ is a nodal plane cubic and, in particular, lies in the smooth locus.

To see the claim, note that a map is nonseparable only if the degree of g is divisible by the characteristic. So the map is separable except possibly in the case $\deg g = \operatorname{char} k = 7$ or 11. In these two cases, if the map is nonseparable, then the image Y is a hyperplane. Let $[1,0,0,0]$ be a smooth point not contained in any line in X, and write $F = X_0^2 X_1 + X_0 Q(X_2, X_3) + C(X_1, X_2, X_3)$. If the image of X is a hyperplane, then there are constants $\lambda_i, i = 0, \dots, 3$ such that

$$\lambda_0 \frac{\partial F}{\partial X_0} + \cdots + \lambda_3 \frac{\partial F}{\partial X_3} = 0 \tag{2}$$

over X. We may write $Q(X_2, X_3) = aX_2^2 + bX_3^2$. If $ab \neq 0$, then the projective tangent cone is nondegenerate and we are done. At least one of a, b is nonzero, otherwise the point x is contained in one of the lines defined by $X_1 = C(X_1, X_2, X_3) = 0$. Assume that $a = 1, b = 0$ in the following. The equation (2) becomes

$$\lambda_0 X_0 X_1 + \lambda_0 X_2^2 + \lambda_1 X_0^2 + \lambda_1 \frac{\partial C}{\partial X_1} + 2\lambda_2 X_0 X_1 + \lambda_2 \frac{\partial C}{\partial X_2} + \lambda_3 \frac{\partial C}{\partial X_3} = 0.$$

This equation means that the left-hand side vanishes identically on X and forces $\lambda_0 = \lambda_1 = \lambda_2 = 0$. Also the derivative $\frac{\partial F}{\partial X_3} = \frac{\partial C}{\partial X_3}$ cannot be identically zero, otherwise X is a cone with vertex $[0,0,0,1]$. So λ_3 has to be zero too, and the image of X is never a hyperplane. This finishes the proof of the claim.

For (3), take a general point x which does not lie in any line on X, and let H_x be the tangent hyperplane at the point x. Then $H_x \cap X$ is a nodal curve. Let $f : \mathbb{P}^1 \to H_x \cap X \subset X$ be the composition of the normalization and the inclusion map. Then f is an immersion. It follows that the normal sheaf defined as the quotient of $T_{\mathbb{P}^1} \to f^* T_X$ is an invertible sheaf and isomorphic to $\mathcal{O}(1)$ by a simple Chern class computation. Thus f is very free.

To show the last statement, it suffices to show that there is a rational curve contained in the smooth locus through every point $x \in X^{\mathrm{sm}}$. Once we have this curve, we can add to it many very free curves which are general tangent hyperplane sections in the smooth locus. Then a general smoothing with the point x fixed produces a very free curve at x.

Choose y to be a general point such that $C_y = H_y \cap X$ is a nodal plane cubic contained in X^{sm} and such that the line spanned by x, y intersects the cubic surface at a point z which does not lie on any line contained in X.

There is a birational involution:

$$i_z : X \dashrightarrow X,$$
$$p \mapsto q,$$

where p, q, z are collinear. Then $i_z(C_y)$ is a rational curve in X^{sm} and contains x. □

The next two lemmas study the Fano scheme of lines on a cubic hypersurface.

LEMMA 5.4
Let X be a normal cubic hypersurface in $\mathbb{P}^n$ which is not a cone over a smooth plane cubic. Then the family of lines through a general point has the expected dimension $n-4$.

Proof
We first count the dimension of the Fano variety of lines of X at each of its geometric points. There are three cases.

Case I: The line contains a point $x \in X$ that is a vertex of a cone. The family of lines through x has dimension $n-2$. Thus the family of lines through a vertex has dimension at most

$$\dim\{x \in X | x \text{ is a vertex}\} + n - 2 = 2n - 6.$$

Case II: The line contains a point $x = [1, 0, \dots, 0] \in X$ that is a singular point with multiplicity 2. We may write the defining equation of X as

$$X_0 Q(X_1, \dots, X_n) + C(X_1, \dots, X_n) = 0.$$

The family of lines through x is defined by $Q(X_1, \dots, X_n) = C(X_1, \dots, X_n) = 0$ in $\mathbb{P}^{n-1}$ (with coordinates $[X_1, \dots, X_n]$). The polynomials Q and C have no common factor; otherwise X is reducible. Thus, the family of lines through x has dimension $n-3$. Therefore, the family of lines through a singular point with multiplicity 2 has dimension at most $\dim X^{\text{sing}} + n - 3 \leq 2n - 6$.

Case III: The line lies in the smooth locus of X. The Fano scheme of lines is smooth at this point and has dimension $2n-6$.

So under the assumptions on X, the evaluation map of the Fano scheme of lines has fiber dimension at most $2n - 6 + 1 - (n-1) = n - 4$ at a general point. □

LEMMA 5.5
(1) *Let X be a smooth cubic hypersurface in $\mathbb{P}^n, n \geq 4$ over an algebraically closed field of characteristic at least 5. Then the locus of points in X through which there does not exist a free line has codimension at least 2.*

(2) *With the assumptions as above, the locus of points in X through which the family of lines has dimension more than $n-4$ has codimension at least* 2.

Proof

First, assume that $\dim X = 3$. Then the evaluation map of the universal family of lines is finite surjective of degree 6, thus separable and generically étale when the characteristic of the field is at least 5. So every line through a general point of a cubic 3-fold is free. For a smooth cubic hypersurface of dimension higher than 3, we can take general hyperplane sections to cut it down to a 3-fold. A free line in the hyperplane section is also a free line in the hypersurface X. So there is a free line through a general point of X, and the locus has codimension at least 1. But if the locus of points in X through which there does not exist a free line has a codimension 1 component, then it is an ample divisor which will intersect a free line. Thus the locus has to have codimension at least 2.

For the second part, note that the evaluation map of the universal family of lines has generic fiber dimension $n-4$ and is flat in codimension 1. ☐

The following two results are also needed later in the proof. They guarantee that some auxiliary fibration used in the construction of the ruled surface containing two sections has reduced fiber everywhere.

LEMMA 5.6

Let $X \subset \mathbb{P}^n, n \geq 5$ be a normal cubic hypersurface which is not a cone over a plane cubic. Given two general points x, y, denote by H_x and H_y the tangent hyperplanes at x and y. Then the intersection $H_x \cap H_y \cap X$ is a reduced cubic hypersurface. If X is smooth, then this is true for any two points. In fact, the intersection of X with any two hyperplanes is reduced.

Proof

If we take $(n-3)$ general hyperplane sections of X, then we get a normal cubic surface which is not a cone. The normality is clear. Since the family of lines of X has the expected dimension, the cubic surface constructed has only finitely many lines. So it is not a cone. Without loss of generality, assume that the hyperplanes are $X_0 = X_1 = \cdots = X_{n-4} = 0$. Denote by $F(X_0, \ldots, X_n)$ the defining equation for X. Then take the family $F(tX_0, \ldots, tX_{n-4}, X_{n-3}, \ldots, X_n) = 0$ in $\mathbb{P}^n \times \mathbb{A}^1$. This is an isotrivial family $\mathcal{V} \to T \cong \mathbb{A}^1$ such that the fiber over zero is a cone over a normal cubic surface which is not a cone over a plane cubic, and a general fiber is isomorphic to X.

Since the condition that $H_x \cap H_y \cap X$ is a reduced cubic hypersurface is an open condition, it suffices to show this for two general points in the central fiber, which is Lemma 5.3(4).

Next we discuss the case where X is a smooth cubic hypersurface. Let H_1 and H_2 be two hyperplanes. Without loss of generality, assume that $H_1 = \{X_0 = 0\}$ and $H_2 = \{X_1 = 0\}$. If $X \cap H_1 \cap H_2$ is nonreduced, then the defining equation of X has the form

$$X_2^2 X_3 + X_0 Q_0 + X_1 Q_1 = 0$$

or

$$X_2^3 + X_0 Q_0 + X_1 Q_1 = 0.$$

Clearly, X is singular along $X_0 = X_1 = X_2 = Q_1 = Q_2 = 0$ in $\mathbb{P}^n, n \geq 5$, which is nonempty. □

LEMMA 5.7
Let X be a normal cubic hypersurface of dimension at least 2 defined over an algebraically closed field of characteristic at least 7. Assume that X is not a cone over a plane cubic. Then for a general point x, the projective tangent cone of X at x is reduced. If, furthermore, X is smooth, then the projective tangent cone is a smooth quadric hypersurface.

Proof
The tangent hyperplane at a general point of a normal cubic surface which is not a cone intersects the cubic surface at a nodal plane cubic. Thus the projective tangent cone at this point is reduced. We can use the same degeneration as in Lemma 5.6 to show this for the cubic hypersurface X.

If the hypersurface X is smooth, it suffices to show that the Gauss map is generically étale, or equivalently, separable. This is true since the degree of the Gauss map is 12 (see proof of Lemma 5.3(2)) and the characteristic is not 2 or 3. □

Finally, we show that we can degenerate a higher degree rational curve to a union of free lines. Note that for a smooth cubic hypersurface of dimension at least 4 (the case we actually need), Coskun and Starr [14, Theorem 1.1] proved that the Kontsevich moduli space of stable maps is geometrically irreducible. The proof is given in characteristic 0, but the same argument works in general with minor modifications. (Compare also the proof of irreducibility by using analytic number theory; see [5]. Some arguments in the next few paragraphs also appear in [22].)

LEMMA 5.8
Let X be a smooth cubic hypersurface of dimension at least 3, and let C be a conic or the intersection of X with a tangent plane which has only one node as the singularity. Then C is a free curve and there is a deformation of C, parameterized by an irreducible curve, to a chain of free lines.

Proof
First of all, a conic and a nodal plane cubic in a smooth cubic surface is always free. Then the freeness in the higher dimensional case can be proved by taking general hyperplane sections containing the curve C and using the normal sheaf exact sequence. In fact, a nodal plane cubic is a very free curve in a smooth cubic surface and thus is also very free in X.

If C is a nodal plane cubic, then one can take a general deformation of C, which yields an embedded rational curve of degree 3, that is, a twisted cubic. The twisted cubic determines a unique $\mathbb{P}^3$. If the twisted cubic is general, then the $\mathbb{P}^3$ is general.

If C is a conic, then one can also find a general $\mathbb{P}^3$ containing (a general deformation of) C.

For a general $\mathbb{P}^3 \subset \mathbb{P}^n$, the intersection of X with the $\mathbb{P}^3$ is a smooth cubic surface all of whose 27 lines are free lines in X. One can find a deformation of a conic (resp., twisted cubic) degenerate to a chain of lines in a cubic surface. Given a conic, one can take the residual line and project from it. It is well known that this gives a conic bundle structure of the cubic surface, and there are five degenerate fibers consisting of chains of two lines. This gives a degeneration. For twisted cubics, the easiest way to write down the degeneration is to note that the linear system corresponding to a twisted cubic is basepoint-free and gives a birational morphism to $\mathbb{P}^2$, which contracts six lines. The linear system is the pullback of $|\mathcal{O}_{\mathbb{P}^2}(1)|$. Using this one can easily write down the degeneration. □

5.3. Asymptotically canonical sequence
By Corollary 2.20, there is a tower of degree 2 base changes $C_0 \to C_1 \to \cdots \to C_n = B$ such that for the cubic hypersurface X defined over $\mathbb{F}_q(B)$, the base change $X \times_{\mathbb{F}_q(B)} \mathbb{F}_q(C_0)$ has an integral model over C_0 each of whose closed fibers is normal and is not a cone over a plane cubic.

The cubic hypersurface X has an $\mathbb{F}_q(C_i)$-rational point if and only if X has a rational point over $\mathbb{F}_q(C_{i-1})$ (see proof of the case of cubic hypersurfaces in Lemma 3.12). Thus the Hasse principle for X will follow from the subsequent lemma.

LEMMA 5.9
Let $\mathcal{X} \to B$ be a family of cubic hypersurfaces in $\mathbb{P}^n, n \geq 5$ defined over an alge-

braically closed field of characteristic greater than 5. Assume that every fiber is reduced and irreducible, normal, and not a cone over a plane cubic. Then there is an asymptotically canonical sequence of sections.

Proof

First note that there exist free sections of the family. The proof is similar to the case of quadric hypersurface fibrations. We choose a point in the smooth locus of every singular fiber of the family, and there is a section through these points since weak approximation is true for the family (see Theorem A.1). This section lies in the smooth locus of $\mathcal{X}$. After adding enough very free rational curves in general fibers and taking a general smoothing, we get a free section. So we can find families of nice sections and apply Construction 3.8.

Given two geometrically irreducible components of nice sections, choose two nice sections s_1 and s_2 in each family. Let $\mathcal{H}_1 \to B$ (resp., $\mathcal{H}_2 \to B$) be the family of tangent hyperplanes of $\mathcal{X}$ along s_1 (resp., s_2). Let $\mathcal{Y} \to B$ be the intersection of $\mathcal{H}_1, \mathcal{H}_2$ and $\mathcal{X}$. Then $\mathcal{Y} \to B$ is a family of cubic hypersurfaces in $\mathbb{P}^{n-2}, n-2 \geq 3$. Up to replacing the two sections with general deformations, we may assume the following.

(1) For every point $b \in B$ such that $\mathcal{X}_b$ is singular, $s_1(b)$ and $s_2(b)$ are general points in $\mathcal{X}_b$. That is, the conclusions of Lemmas 5.4, 5.6, and 5.7 hold.

(2) For every point $b \in B$ such that $\mathcal{X}_b$ is smooth, there is at least one free line through $s_1(b)$ (resp., $s_2(b)$) and the family of lines through $s_1(b)$ (resp., $s_2(b)$) has dimension $n-4$. This is possible by Lemma 5.5.

(3) For a general point $b \in B$, the projective tangent cone of $\mathcal{H}_{s_1(b)} \cap \mathcal{X}_b$ at $s_1(b)$ (resp., $s_2(b)$) is a smooth quadric hypersurface. In addition, the fiber $\mathcal{Y}_b$ is smooth for a general point b (see Lemma 5.7).

(4) For every point $b \in B$, the projective tangent cone of $\mathcal{X}_b$ at $s_1(b)$ (resp., $s_2(b)$) is a reduced quadric hypersurface (see Lemma 5.7).

(5) The intersection $\mathcal{Y} \cap \mathcal{X}^0 \cap \mathcal{X}^1$ is a nonempty open subset of $\mathcal{Y}$. For the definition of $\mathcal{X}^0$ and $\mathcal{X}^1$, see the paragraph before Definition 3.4.

Note that (4) follows from (2) when the fiber $\mathcal{X}_b$ is smooth since if the projective tangent cone at a point is nonreduced, then the family of lines through this point has no smooth point.

By Lemma 5.6 and our choices of s_1, s_2, every fiber of $\mathcal{Y}$ is reduced.

By weak approximation for the family $\mathcal{Y} \to B$ (see Theorem A.1), given finitely many points in the smooth locus of different fibers, one can find a 2-free section s of $\mathcal{Y} \to B$ passing through these points, which can be considered as a section of $\mathcal{X}$ by composing with the inclusion $\mathcal{Y} \to \mathcal{X}$. This section s, considered as a section of $\mathcal{X}$, is 2-free provided that its $\mathcal{O}_{\mathcal{X}}(1)$ degree is sufficiently large compared with the genus of B. This condition can be achieved by adding very free curves in general fibers of $\mathcal{Y} \to B$ to s and taking a general deformation of the comb in $\mathcal{Y}$. In other words, given finitely many points in the smooth locus of different fibers of $\mathcal{Y} \to B$, there is a 2-free section s of $\mathcal{Y} \to B$ passing through these points. When considered as a section of $\mathcal{X} \to B$, this section s is also 2-free. Moreover, by Lemma 3.5 and property (5) of our construction, a general deformation of s is a nice section.

There are two families of projective spaces $\mathcal{P}_1 = \mathrm{Proj}(E) \to B$ and $\mathcal{P}_2 = \mathrm{Proj}(E) \to B$ together with birational maps $\mathcal{P}_1 \dashrightarrow \mathcal{X} \cap \mathcal{H}_1$ and $\mathcal{P}_2 \dashrightarrow \mathcal{X} \cap \mathcal{H}_2$, whose restriction to every fiber is the birational map $\mathbb{P}^{n-2} \dashrightarrow \mathcal{X}_b \cap \mathcal{H}_{1b}$ and $\mathbb{P}^{n-2} \dashrightarrow \mathcal{X}_b \cap \mathcal{H}_{2b}$ discussed in Lemma 5.1. The space of lines through $s_1(b)$ (resp., $s_2(b)$) has dimension $n-4$ by (2). Since the indeterminacy locus in the fiber $\mathcal{P}_{1b}$ (resp., $\mathcal{P}_{2b}$) corresponds to the family of lines through $s_1(b)$ (resp., $s_2(b)$), it has codimension 2 in each fiber. Inside the family of projective spaces, there is a family of quadric hypersurfaces $\mathcal{Q}_1 \subset \mathcal{P}_1$ (resp., $\mathcal{Q}_2 \subset \mathcal{P}_2$) which corresponds the family of projective tangent cones of $\mathcal{X} \to B$ along the section $s_1 : B \to \mathcal{X}$ (resp., s_2). By Lemma 5.7, every fiber of $\mathcal{Q}_1$ and $\mathcal{Q}_2$ is reduced.

Choose a general 2-free section s_3 of $\mathcal{Y} \to B$ such that

- for every point $b \in B$ such that $\mathcal{X}_b$ is singular, the point $s_3(b)$ is in the locus where the birational maps $\mathcal{P}_{1b} \dashrightarrow \mathcal{X}_b \cap \mathcal{H}_{1b}$ and $\mathcal{P}_{2b} \dashrightarrow \mathcal{X}_b \cap \mathcal{H}_{1b}$ are isomorphisms;
- after composing with the inclusion $\mathcal{Y} \subset \mathcal{X}$, the section is a nice section of the family $\mathcal{X} \to B$.

It is easy to find 2-free sections of $\mathcal{Y} \to B$ whose composition with the inclusion $\mathcal{Y} \subset \mathcal{X}$ is a 2-free section of $\mathcal{X} \to B$. Then it is a nice section by (5) and Lemma 3.5.

In the following, we show how to find a ruled surface which contains the two sections s_1 and s_3.

There is a section σ_3 of the family $\mathcal{P}_1 \to B$ whose image under the birational map $\mathcal{P}_1 \dashrightarrow \mathcal{X} \cap \mathcal{H}_1$ is the section s_3. Take a section σ_1 of the family of quadrics $\mathcal{Q}_1 \subset \mathcal{P}_1 \to B$ such that for each point b where $\mathcal{X}_b$ is singular, the line spanned by $\sigma_1(b)$ and $\sigma_3(b)$ inside $\mathcal{P}_{1b}$ does not meet the indeterminacy locus of $\mathcal{P}_{1b} \dashrightarrow \mathcal{X} \cap \mathcal{H}_1$. This is possible since the indeterminacy locus, which is the same as the locus parameterizing the family of lines through $s_1(b)$, has codimension at least 2 along every fiber over b by (2). Then we use the fact that weak approximation is true for a smooth quadric hypersurface of positive dimension over the function field of a curve to find such a section.

Let $\mathcal{L}_1 \to B$ be the family of lines in $\mathcal{P}_1 \to B$ spanned by the two sections σ_1 and σ_3. There is a rational map $\mathcal{L}_1 \dashrightarrow \mathcal{X}$. Denote by $\mathcal{S}_1 \to B$ the normalization of the closure of the image of $\mathcal{L}_1 \dashrightarrow \mathcal{X}$. There is a morphism $\mathrm{ev} : \mathcal{S}_1 \to \mathcal{X}$. By construction:

- Each fiber of $\mathcal{S}_1 \to B$ is either smooth or is a union of two $\mathbb{P}^1$'s, both with multiplicity 1.
- Each fiber over b is mapped to a plane cubic which is the intersection of a tangent plane at $s_1(b)$ and $\mathcal{X}_b$.
- The two sections σ_1 and σ_3 are also sections of the family $\mathcal{S}_1 \to B$, which lie in the smooth locus of $\mathcal{S}_1$, and such that $\mathrm{ev} \circ \sigma_1 = s_1, \mathrm{ev} \circ \sigma_3 = s_3$.
- For any point $b \in B$ such that $\mathcal{X}_b$ is singular, the fiber $\mathcal{S}_{1b}$ is mapped to an irreducible nodal plane cubic.

Using the ruled surface $\mathcal{S}_1 \to B$, we can assemble two combs C_1, C_3. The handle of C_1 (resp., C_3) is s_1 (resp., s_3) and the teeth are free lines, conics, and nodal plane cubics. We may construct the two combs in such a way that none of the teeth are contained in the singular fibers of $\mathcal{X}$. This is because that $\mathcal{S}_1$ has irreducible fiber over any point $b \in B$ over which the fiber $\mathcal{X}_b$ is singular. Furthermore, there is a deformation of the two combs given by a pencil in the ruled surface $\mathcal{S}_1$.

In the following, we use $M_i(s_0)$ and so on to denote the irreducible component constructed from the family of nice sections containing the section s_0. Then by Lemmas 5.8 and 3.16, there exist numbers N_1, N_3 such that $M_{N_1+i}(s_1) = M_{N_3+i}(s_3)$ for all $i \geq 0$.

Similarly, we have $M_{N_2+i}(s_2) = M_{N_3'+i}(s_3)$ for some N_2, N_3' and all $i \geq 0$. So there is an asymptotically canonical sequence. □

6. Hasse principle and weak approximation for complete intersection of two quadrics

6.1. R-equivalence and weak approximation

In this section, we briefly review a construction in [10].

Let X be a smooth complete intersection of two quadrics in $\mathbb{P}^n, n \geq 5$, defined over the field k, and let x, y be two k-rational points of X. Assume that the point x has coordinate $[1, 0, \dots, 0]$, and write the equation of X as

$$\begin{cases} X_0 X_1 + q(X_1, \dots, X_n) = 0, \\ X_0 X_2 + q'(X_1, \dots, X_n) = 0. \end{cases}$$

Consider the pencil of tangent hyperplanes $\lambda X_1 + \mu X_2$ at x. There is exactly one member of the pencil $\lambda X_1 + \mu X_2$ containing the point y. Denote it by H. The intersection of this hyperplane H and X is a singular $(2,2)$-complete intersection in $\mathbb{P}^{n-1}$. Projection from x gives one a quadric hypersurface in $\mathbb{P}^{n-2}$ defined by the equation

$$\lambda q(X_1, \dots, X_n) + \mu q'(X_1, \dots, X_n)|_{\lambda X_1 + \mu X_2 = 0} = 0.$$

Definition 6.1

In the situation as above, we say that x, y are *general points* if the quadric hypersurface

$$\lambda q(X_1, \dots, X_n) + \mu q'(X_1, \dots, X_n)|_{\lambda X_1 + \mu X_2 = 0} = 0$$

is smooth and its hyperplane section

$$\lambda q(0, 0, X_3, \dots, X_n) + \mu q'(0, 0, X_3, \dots, X_n) = 0$$

is geometrically integral.

LEMMA 6.2

Let k be a field of odd characteristic such that the set of rational points on every smooth quadric hypersurface of positive dimension is Zariski-dense once nonempty. Let X be a smooth complete intersection of two quadrics in $\mathbb{P}^n, n \geq 5$, defined over the field k, and let x, y be two general k-rational points of X. Then there is a map $f : \mathbb{P}^1 \to X$ defined over k such that $f(0) = f(\infty) = x, f(1) = y$.

Proof

We use the same notation as in the beginning of this section. For any infinite field k and a general $[\lambda, \mu] \in \mathbb{P}^1$, the hypersurface

$$\lambda q(X_1, \dots, X_n) + \mu q'(X_1, \dots, X_n)|_{\lambda X_1 + \mu X_2 = 0} = 0$$

is smooth.

If $n = 5$ and there is a free line through x, a general choice of $[\lambda, \mu] \in \mathbb{P}^1$ gives a smooth conic

$$\lambda q(0, 0, X_3, \dots, X_5) + \mu q'(0, 0, X_3, \dots, X_5) = 0.$$

To see this, just note that if there is a free line, then q and q' cannot be simultaneously singular at the same point; otherwise they are cones with the same vertex, and there is only one line through x with multiplicity 4.

If $n \geq 6$, then we can take $(n-5)$-general hyperplane sections to cut down to the case $n = 5$. Clearly, for general points x, y, the quadric hypersurface

$$\lambda q(X_1, \dots, X_n) + \mu q'(X_1, \dots, X_n)|_{\lambda X_1 + \mu X_2 = 0} = 0$$

is smooth. The hyperplane section

$$\lambda q(0, 0, X_3, \dots, X_n) + \mu q'(0, 0, X_3, \dots, X_n) = 0$$

is geometrically integral if the $(n-3)$-general hyperplane sections have the same properties.

For simplicity of the discussion, we assume in the following that the tangent hyperplane at x containing y is given by $X_1 = 0$. Thus the assumption that the point y is general means that $q(0, X_2, \ldots, X_n) = 0$ (resp., $q(0, 0, X_3, \ldots, X_n) = 0$) defines a smooth quadric hypersurface Q in $\mathbb{P}^{n-2}$ (resp., a geometrically integral hyperplane section of Q). The hypersurface Q, by construction, is birational to the singular $(2, 2)$-complete intersection. The birational map is explicitly given as

$$\begin{aligned} Q &\dashrightarrow X \cap H, \\ [X_2, \ldots, X_n] &\dashrightarrow \left[-q'(0, X_2, \ldots, X_n), X_2^2, X_2X_3, \ldots, X_2X_n\right]. \end{aligned}$$

The generic point of the hyperplane section $X_2 = 0$ of Q is mapped to $x = [1, 0, \ldots, 0]$. The map is not defined on the locus in Q satisfying $q' = X_2 = 0$, which is the locus parameterizing lines through x. This is also clear from the geometric description of the birational map $X \cap H \dashrightarrow Q$ as a projection.

The point y is mapped to a k-rational point in Q, denoted by u, which does not lie in the hyperplane section $X_2 = 0$. Then it is straightforward to check that there is a smooth conic through the point u and two general k-rational points in the hyperplane section $X_2 = 0$ which also satisfies $q' \neq 0$. Here we use the assumption on the field k. This conic with the rational points u, v, w is the rational curve we are looking for. $\square$

Examples of fields satisfying conditions in the lemma are $\mathbb{F}_q(\!(t)\!)$, $\bar{\mathbb{F}}_q(\!(t)\!)$, finite extensions of $\mathbb{Q}_p$, $\mathbb{F}(B)$, $\bar{\mathbb{F}}(B)$ (where B is a smooth curve), and number fields. Using this construction and the fibration method, Colliot-Thélène, Sansuc, and Swinnerton-Dyer [10] proved the following theorem.

THEOREM 6.3 ([10, Theorems 3.10, 3.11])
Let X be a smooth complete intersection of two quadrics in $\mathbb{P}^n, n \geq 5$, defined over a global field of odd characteristic or characteristic 0. *Assume that X has a rational point. Then X satisfies weak approximation.*

This is proved in [10] for the case of number fields using the above construction and the fibration method (see [10, Theorems 3.10, 3.11]). But the proof works in this setup as well. We refer the interested reader to [10] for details.

For later reference, note that during the proof of Lemma 6.2 we proved the following lemma.

LEMMA 6.4
Let X be a smooth complete intersection of two quadrics in $\mathbb{P}^5$ defined over an alge-

braically closed field k of odd characteristic, and let x be a point in X. Assume that there is a free line through the point x. Also assume that x has coordinates $[1, 0, \dots, 0]$, and write the equation of X as

$$\begin{cases} X_0X_1 + q(X_1, \dots, X_5) = 0, \\ X_0X_2 + q'(X_1, \dots, X_5) = 0. \end{cases}$$

Then for a general choice of λ, μ, the quadric hypersurface

$$\lambda q(X_1, \dots, X_5) + \mu q'(X_1, \dots, X_5)|_{\lambda X_1 + \mu X_2 = 0} = 0$$

and its hyperplane section

$$\lambda q(0, 0, X_3, \dots, X_5) + \mu q'(0, 0, X_3, \dots, X_5) = 0$$

are smooth.

We also have the following observation.

LEMMA 6.5
Use the same hypotheses as those of Lemma 6.4. Assume furthermore that all the lines through the point x are free lines. Then for any $[\lambda, \mu] \in \mathbb{P}^1$, the quadric hypersurface defined by $\lambda q(X_3, \dots, X_5) + \mu q'(X_3, \dots, X_5) = 0$ is reduced.

Proof
The assumption implies that $q(X_3, X_4, X_5) = q'(X_3, X_4, X_5) = 0$ defines a smooth 0-dimensional subscheme of $\mathbb{P}^2$. If any member of the pencil is nonreduced, the base locus of the pencil cannot be a smooth subscheme. $\square$

6.2. Geometry of complete intersection of two quadrics
In this section, we collect some useful facts about complete intersections of two quadrics.

LEMMA 6.6
Let X be a smooth complete intersection of two quadrics in $\mathbb{P}^n, n \geq 5$, defined over an algebraically closed field of odd characteristic.

(1) *If $n = 5$, then there are four lines through a general point, all of which are free.*
(2) *Given any point x in X, the family of lines through x has dimension $n - 5$.*
(3) *The locus $\{x \in X \mid \text{there is no free line through } x\}$ has codimension at least 2 in X.*

Proof
Every $(2,2)$-complete intersection of dimension at least 2 contains at least a line by a simple dimension count. Given a line L in a smooth complete intersection of two quadrics X, we have the short exact sequence of normal bundles for the embedding:

$$0 \to N_{L/X} \cong \bigoplus_{i=1}^{n-3} \mathcal{O}(e_i) \to N_{L/\mathbb{P}^n} \cong \oplus^{n-1}\mathcal{O}(1) \to N_{X/\mathbb{P}^n}|_L \cong \mathcal{O}(2) \oplus \mathcal{O}(2) \to 0.$$

Thus all the e_i's are less than or equal to 1. Then by considering the first Chern class, all of them are greater than -2.

When $n = 5$, the normal bundle of every line is either $\mathcal{O}(1) \oplus \mathcal{O}(-1)$ or $\mathcal{O} \oplus \mathcal{O}$. In any case, the normal bundle has no H^1. So the Fano scheme U is smooth of dimension 2. It is well known that U is connected (see [23, Theorem V.4.3.3], [18, Théorème 2.1]), thus also irreducible. Moreover, the evaluation map of the universal family is dominant.

Assume that the defining equations can be written as

$$\begin{cases} X_0X_1 + q(X_1, \dots, X_5) = 0, \\ X_0X_2 + q'(X_1, \dots, X_5) = 0. \end{cases}$$

Thus the family of lines through $[1, 0, \dots, 0]$ is defined by

$$X_1 = X_2 = q(X_1, \dots, X_5) = q'(X_1, \dots, X_5)$$

in $\mathbb{P}^4$ (with coordinates $X_1, \dots, X_5$). If this point is general, there are only finitely many lines through this point by a dimension argument. Thus this family is a 0-dimensional scheme of length 4. In other words, the evaluation map of the universal family has degree 4. Since the field has odd characteristic, the evaluation map is separable. So it is generically smooth. Then the first statement follows from a standard argument of deformation theory.

For (2), we still assume that the point x is $[1, 0, \dots, 0]$ and write the defining equations as

$$\begin{cases} X_0X_1 + q(X_1, \dots, X_n) = 0, \\ X_0X_2 + q'(X_1, \dots, X_n) = 0. \end{cases}$$

Then there is an $(n-4)$-dimensional family of lines through x if and only if $q(0, 0, X_3, \dots, X_n)$ and $q'(0, 0, X_3, \dots, X_n)$ have a common linear factor $L(X_3, \dots, X_5)$. If such a factor exists, then X contains the $(n-3)$-dimensional linear space $X_1 = X_2 = L(X_3, \dots, X_5)$, which is impossible by the Lefschetz hyperplane theorem for Picard groups. Or, as pointed out by the referee, this implies that the complete intersection X is singular along $X_0 = X_1 = X_2 = L = 0$, contradicting the smoothness assumption.

For (3), by taking hyperplane sections and using the first statement, we know that given any complete intersection of two quadrics in $\mathbb{P}^n, n \geq 5$, there is a free line through a general point. Thus the locus

$$\{x \in X \mid \text{There is no free line through } x.\}$$

has codimension at least 1. But X has Picard number 1. So this locus will intersect a free line if it has a codimension 1 component, which contradicts the definition of the locus. Thus it has codimension at least 2. $\square$

LEMMA 6.7

Let X be a complete intersection of two quadrics in $\mathbb{P}^n$ defined over an algebraically closed field of odd characteristic. Assume that X is reduced and irreducible. Then the dimension of the family of lines through a general point $x \in X$ is $n-5$, unless X is nonnormal or is a cone over a $(2,2)$-complete intersection curve in $\mathbb{P}^3$.

Proof

If X is smooth, the statement holds for any point by Lemma 6.6. So in the following we assume that X is singular.

Let y be a singular point of X which is not a vertex if X is a cone. Assume the coordinates of the point y are $[1, 0, \ldots, 0]$. Thus we can write the equation of X as

$$\begin{cases} q(X_1, \ldots, X_n) = 0, \\ X_0 X_2 + q'(X_1, \ldots, X_n) = 0. \end{cases}$$

Note that $q(X_1, 0, X_3, \ldots, X_n)$ and $q'(X_1, 0, X_3, \ldots, X_n)$ have no common linear factor. Otherwise the variety X contains a linear space of dimension $n-2$ and thus is reducible. So the family of lines through a nonvertex singular point has dimension $n-4$. The family of lines through a vertex has dimension $n-3$.

If a line L lies in the smooth locus of X, then the same computation as in Lemma 6.6(1) implies that the normal bundle of the line has no H^1. So the Fano scheme is smooth at this point and has dimension $2n-8$.

If X is normal and not a cone over a $(2,2)$-complete intersection in $\mathbb{P}^3$, then by the above computation, the family of lines containing a singular nonvertex point has dimension at most $\dim X^{\text{sing}} + n - 4 \leq n - 4 + n - 4 = 2n - 8$, and the family of lines containing a vertex has dimension at most $n - 5 + n - 3 = 2n - 8$. Thus every irreducible component of the Fano scheme has dimension $2n-8$, and the fiber of the evaluation map over a general point on X has dimension $n-5$. $\square$

Finally, we need the following result about degenerations of low degree rational curves.

LEMMA 6.8
Let X be a smooth complete intersection of two quadrics in $\mathbb{P}^n, n \geq 5$, defined over an algebraically closed field of odd characteristic, and let x be a general point in X. Furthermore, let C be either a smooth conic, a twisted cubic, or a degree 4 *nodal rational curve which is the intersection of X with a $\mathbb{P}^3$ tangent to X at the point x. Then there is a deformation of C, parameterized by an irreducible curve, to a chain of free lines.*

Proof
It is easy to check that the curves are free.

Taking a general deformation of C, we get an embedded rational curve of degree 2, 3, or 4, which is contained in a linear projective space of dimension 2, 3, or 4.

By taking $(n-4)$-general hyperplanes containing a general deformation of the curve C, we may assume that the curve C is contained in a smooth complete intersection of two quadrics Y in $\mathbb{P}^4$. Furthermore, we may assume that every line in Y is a free line in X. So it suffices to prove that C degenerates to a chain of lines in Y. The linear system $|\mathcal{O}_Y(C)|$ is basepoint-free in all three cases. In the conic case, the linear system gives a conic bundle structure with four degenerate fibers, which consists of two lines. In the twisted cubic case, the linear system defines a birational morphism to $\mathbb{P}^2$ that contracts five exceptional divisors, and the linear system is the pullback of the linear system $|\mathcal{O}_{\mathbb{P}^2}(1)|$. It is easy to write down a degeneration explicitly in these two cases. In the degree 4 case, the linear system defines a birational morphism to a smooth quadric surface in $\mathbb{P}^3$, contracting four exceptional divisors. In this case, the linear system is the pullback of $|\mathcal{O}_{\mathbb{P}^3}(1)|$. Take three exceptional divisors and a conic in the quadric surface containing the image of these three divisors. The union of the strict transform of the conic and these three exceptional divisors gives the desired degeneration. □

6.3. Asymptotically canonical sequence
In the following, we prove the Hasse principle. First of all, we study nonnormal complete intersections of two quadrics.

LEMMA 6.9
Let X be a geometrically integral, nonnormal complete intersection of two quadrics in $\mathbb{P}^5$ defined over a field k of characteristic at least 3. *Assume that the unique $(n-3)$-dimensional component of the singular locus is defined by $X_0 = X_1 = X_2 = 0$ (see*

Lemma 2.21). Then over $\bar{k}$ (up to projective isomorphism), the variety X (in $\mathbb{P}^5$) is defined by one of the following equations:

$$\begin{cases} X_0X_3 + X_1^2 = 0, \\ X_0X_4 + X_2^2 = 0, \end{cases}$$

$$\begin{cases} X_0X_3 + X_1X_4 + X_2X_5 = 0, \\ X_0L(X_1, X_2) + Q(X_1, X_2) = 0, \end{cases}$$

$$\begin{cases} X_0X_3 + X_1X_4 + X_2^2 = 0, \\ X_0L(X_1, X_2) + Q(X_1, X_2) = 0, \end{cases}$$

$$\begin{cases} X_0X_3 + X_1^2 + X_2^2 = 0, \\ X_0L(X_1, X_2) + Q(X_1, X_2) = 0. \end{cases}$$

Proof

The defining equation of X can be written in the form

$$\begin{cases} X_0L_0(X_3, X_4, X_5) + X_1L_1(X_3, X_4, X_5) \\ \quad + X_2L_2(X_3, X_4, X_5) + Q(X_0, X_1, X_2) = 0, \\ X_0L'_0(X_3, X_4, X_5) + X_1L'_1(X_3, X_4, X_5) \\ \quad + X_2L'_2(X_3, X_4, X_5) + Q'(X_0, X_1, X_2) = 0. \end{cases}$$

Since it is singular along $X_0 = X_1 = X_2 = 0$, the Jacobian matrix at every point in $X_0 = X_1 = X_2 = 0$ is of the form

$$\begin{pmatrix} L_0(X_3, X_4, X_5) & L_1(X_3, X_4, X_5) & L_2(X_3, X_4, X_5) & 0 & 0 & 0 \\ L'_0(X_3, X_4, X_5) & L'_1(X_3, X_4, X_5) & L'_2(X_3, X_4, X_5) & 0 & 0 & 0 \end{pmatrix},$$

and has rank at most 1.

Up to a change of variables, we may assume that $L_0(X_3, X_4, X_5) = X_3$.

If L'_0 is not a multiple of X_3, then we may assume that it is X_4. It then follows that L_1, L_2 are multiples of X_3 and L'_1, L'_2 are multiples of X_4. So after a change of coordinates, the equations can be written as

$$\begin{cases} X_0X_3 + Q(X_1, X_2) = 0, \\ X_0X_4 + Q'(X_1, X_2) = 0. \end{cases}$$

Note that Q and Q' have no common factors; otherwise X is reducible. Over an algebraically closed field of odd characteristic, we may modify the equations by taking linear combinations of the two equations and a new combination of coordinates X_3, X_4. Then the new equation becomes

$$\begin{cases} X_0X_3 + X_1^2 = 0, \\ X_0X_4 + X_2^2 = 0. \end{cases}$$

If L_0' is a multiple of X_3, write it as λX_3. It follows that $L_i' = \lambda L_i, i = 1, 2$. So we may assume that the second equation is of the form $Q'(X_0, X_1, X_2)$. Depending on the dimension of the $\bar{k}$-span of L_0, L_1, L_2, and up to a linear change of coordinates, we may assume that the first equation is one of the following (e.g., we can eliminate monomials containing X_0 by replacing X_3 with a linear combination of X_3 and other linear coordinates):

$$X_0X_3 + X_1X_4 + X_2X_5 = 0,$$
$$X_0X_3 + X_1X_4 + X_2^2 = 0,$$
$$X_0X_3 + X_1^2 + X_2^2 = 0.$$

Up to a change of coordinates, we may assume that there is a smooth point of X of the form $[1, 0, \dots, 0]$, and we may write the second equation as $X_0L(X_1, X_2) + Q(X_1, X_2) = 0$ while the first equation remains the same. $\square$

With this classification, we can prove the following crucial fact.

LEMMA 6.10
Let X be a reduced and irreducible complete intersection of two quadrics defined over an algebraically closed field k of odd characteristic. Assume that X is not a cone over a complete intersection curve in $\mathbb{P}^3$. Let x be a general smooth point of X. Assume that the point x is $[1, 0, \dots, 0]$, and write the defining equation of X as

$$\begin{cases} X_0L_1(X_1, \dots, X_n) + Q(X_1, \dots, X_n) = 0, \\ X_0L_2(X_1, \dots, X_n) + Q'(X_1, \dots, X_n) = 0. \end{cases}$$

Then for a general $[\lambda, \mu] \in \mathbb{P}^1(k)$, the two quadrics

$$\lambda Q(X_1, \dots, X_n) + \mu Q'(X_1, \dots, X_n)|_{L_1 = L_2 = 0} = 0$$

and

$$\lambda Q(X_1, \dots, X_n) + \mu Q'(X_1, \dots, X_n)|_{\lambda L_1 + \mu L_2 = 0} = 0$$

are reduced.

Proof
First consider the case that X is normal and not a cone over a curve in $\mathbb{P}^3$. Then

for a general point x, the family of lines through x has dimension $n-5$. Thus $L_1 = L_2 = Q = Q' = 0$ cut out a complete intersection scheme in $\mathbb{P}^{n-1}$. In particular, $Q(X_1, \dots, X_n)|_{L_1=L_2=0}$ and $Q'(X_1, \dots, X_n)|_{L_1=L_2=0}$ have no common factor. Then a general linear combination of them defines a reduced quadric hypersurface.

Next consider that case that X is nonnormal, but not a cone over a curve. We can cut down the variety X by general hyperplane sections until X is a 3-fold in $\mathbb{P}^5$. If we can prove the statement in this case, the general case also follows.

So now assume that X is a nonnormal 3-fold in $\mathbb{P}^5$ and is not a cone. Lemma 6.9 classifies the defining equations. The last one is a cone over a curve in $\mathbb{P}^3$. So we only consider the first three cases. This property is an open condition for points. So it suffices to find one point satisfying the condition. For the first two cases, one checks easily that the point $[1, 0, \dots, 0]$ satisfies the condition.

Finally, consider the case that X is defined by

$$\begin{cases} X_0X_3 + X_1X_4 + X_2^2 = 0, \\ X_0L(X_1, X_2) + Q(X_1, X_2) = 0. \end{cases}$$

If $L(X_1, X_2)$ is of the form $aX_1 + bX_2, b \neq 0$, one directly checks that such (λ, μ) exists for the point $[1, 0, \dots, 0]$. So in the following, assume that $L(X_1, X_2)$ equals X_1. The quadratic polynomial Q contains the monomial X_2^2; otherwise X is reducible. Then by a linear change of coordinates, we may assume that X is defined as

$$\begin{cases} X_0X_3 + X_1X_4 = 0, \\ X_0X_1 + Q(X_1, X_2) = 0. \end{cases}$$

We may further write it as

$$\begin{cases} X_0X_3 + X_1X_4 = 0, \\ (X_0 + cX_1 + dX_2)X_1 + X_2^2 = 0. \end{cases}$$

Then after a change of coordinates as

$$\begin{cases} X_1X_4 + X_0X_3 - dX_2X_3 = 0, \\ X_1X_0 + X_2^2 = 0. \end{cases}$$

Thus if d is nonzero, the point $[0, 1, 0, \dots, 0, 0]$ is what we want. In the following, assume that $d = 0$. Then $[-e^2, 1, e, 0, 0, 0], e \neq 0$, is a smooth point. Make the change of coordinates

$$Y_0 = X_0 + e^2X_1, \qquad Y_1 = X_1, \qquad Y_2 = X_2 - eX_1, \qquad Y_3 = X_3,$$
$$Y_4 = X_4 - e^2X_3.$$

The new equation becomes

$$\begin{cases} Y_1Y_4 + Y_0Y_3 = 0, \\ Y_1(Y_0 + 2eY_2) + Y_2^2 = 0. \end{cases}$$

Clearly, $[0, 1, 0, 0, 0, 0]$ is a point we want. □

Lemma 6.11 below shows the existence of a canonical sequence for a family of smooth $(2, 2)$-complete intersections in $\mathbb{P}^5$. Therefore, the Hasse principle holds for such a variety.

In principle, this proof should also prove the general case. But there is a subtle technical point. The author does not know whether the generic fiber of the family $\mathcal{E}$ appearing in the proof (i.e., the hyperplane section in Lemma 6.2) is a smooth quadric hypersurface when n is at least 6. One can show that this is the case if every line through a general point of a general fiber is free. However, the author does not know how to show this in general. What can be proved is that this is a quadric hypersurface with at worst one singular point. It might also be possible to deal with this case by some careful analysis. But the author prefers to work with smooth varieties to avoid such complexity.

There are two ways to prove the Hasse principle in the general case. We may either use the standard fibration method, or note that to prove the existence of an asymptotically canonical sequence, we can take a general family of hyperplane sections to reduce to the case $n = 5$ and construct the ruled surface.

LEMMA 6.11
Let $\pi : \mathcal{X} \to B$ be a family of complete intersections of two quadric hypersurfaces in $\mathbb{P}^5$ defined over an algebraically closed field k of odd characteristic. Assume that every fiber is integral and that none of the fibers is a cone over a $(2, 2)$-complete intersection curve in $\mathbb{P}^3$. Then there is an asymptotically canonical sequence.

Proof
First of all, we show the existence of a nice section. This basically follows the same line of argument as before. We use the fact that there are smooth points in every closed fiber and that weak approximation holds for smooth complete intersection of two quadrics (e.g., by [8]) to find a section in the smooth locus of $\mathcal{X}$. Then the standard argument of attaching very free curves and smoothing produces a free section and then a nice section by Lemma 3.6.

Given two different irreducible families of sections, we will show that Construction 3.8 eventually gives the same irreducible components of spaces of sections.

Choose two general nice sections s_1 and s_2 in each of these two families. We may assume the following.

(1) Over every point $b \in B$ where the fiber is singular, the points s_1 and s_2 are general in the sense that the conditions of Lemma 6.10 hold.

(2) There are only finitely many points $b \in B$ such that there is a nonfree line in $\mathcal{X}_b$ passing through $s_1(b)$.

(3) For any point $b \in B$ such that the fiber $\mathcal{X}_b$ is smooth, there is a free line through $s_1(b)$.

Now we construct a ruled surface $\mathcal{S} \to B$ and a B-morphism $f : \mathcal{S} \to \mathcal{X}$ together with two sections σ_1 and σ_2 such that

- $f \circ \sigma_i = s_i, i = 1, 2$;
- for any point $b \in B$, the points $\sigma_1(b)$ and $\sigma_2(b)$ lie in the smooth locus of the fiber;
- over any point $b \in B$, the fiber of $\mathcal{S}$ over B is reduced and has at most two irreducible components;
- over each b such that $\mathcal{X}_b$ is singular, the fiber of $\mathcal{S}$ over b is either smooth or a union of two $\mathbb{P}^1$'s such that σ_1 and σ_2 lie in the smooth locus of the same irreducible component;
- a general fiber of $\mathcal{S}$ over $b \in B$ is mapped to the intersection of $\mathcal{X}_b$ with a tangent $\mathbb{P}^3$ at $s_1(b) = f \circ \sigma_1(b)$; each irreducible component of reducible fibers of S is mapped to embedded rational curves of degree 1, 2, or 3.

The construction of such a surface uses Lemma 6.2. First of all, one takes the family of hyperplanes tangent at s_1, and that also contain the section s_2 as in the proof of Lemma 6.2. Following that proof, project each hyperplane section to a quadric surface. The generic fiber is a smooth quadric surface Q defined over $k(B)$ by Lemma 6.4. Over the generic fiber, there is a hyperplane section E of this quadric whose image under the birational map is the rational point corresponding to s_1. This hyperplane section E is smooth over $k(B)$ by Lemma 6.4.

Denote by $\mathcal{Q} \to B$ and $\mathcal{E} \to B$ the family of quadric hypersurfaces and its hyperplane sections. There is a rational map $\mathcal{Q} \dashrightarrow \mathcal{X}$. There is a multisection of degree 4 of $\mathcal{E} \to B$ over which the rational map is not defined. This multisection is the closure of a zero-cycle of degree 4 in the generic fiber which parameterizes the lines through the $k(B)$-rational point corresponding to the section s_1. By Lemma 6.10, Lemma 6.5, and our choice of s_1 and s_2, both the family of quadric surfaces and the family of hyperplane sections have reduced fibers over every point $b \in B$.

By condition (1) above, the indeterminacy locus of the map $\mathcal{Q} \dashrightarrow \mathcal{X}$ is precisely the multisection of degree 4 in a neighborhood of every point $b \in B$ such that the fiber of $\mathcal{X} \to B$ over the point b is normal (see the proof of Lemma 6.2).

If a fiber $\mathcal{X}_b$ of the family $\mathcal{X} \to B$ is nonnormal, then the fiber $\mathcal{E}_b$ of $\mathcal{E} \to B$ is a union of two lines. The rational map $\mathcal{Q} \dashrightarrow \mathcal{X}$ may not be defined along one of the lines. But there is always one line along which the rational map is defined. It is amusing to work out the detailed geometric picture, and we leave this task to the interested reader. Here we indicate the reason by looking at the equations. Assume that the section s_1 intersects the fiber over the point b at the smooth point $[1, 0, \ldots, 0]$. By Lemma 6.9, the defining equation along a formal neighborhood of the nonnormal fiber is one of the following:

$$\begin{cases} X_0X_3 + X_1^2 + t(\ldots) = 0, \\ X_0X_4 + X_2^2 + t(\ldots) = 0, \end{cases}$$

$$\begin{cases} X_0X_3 + X_1X_4 + X_2X_5 + t(\ldots) = 0, \\ X_0L(X_1, X_2) + Q(X_1, X_2) + t(\ldots) = 0, \end{cases}$$

$$\begin{cases} X_0X_3 + X_1X_4 + X_2^2 + t(\ldots) = 0, \\ X_0L(X_1, X_2) + Q(X_1, X_2) + t(\ldots) = 0. \end{cases}$$

If we write the family as

$$\begin{cases} X_0L_1 + Q(X_1, \ldots, X_5) + t(\ldots) = 0, \\ X_0L_2 + Q'(X_1, \ldots, X_5) + t(\ldots) = 0, \end{cases}$$

then locally around the point b, the indeterminacy locus is given by

$$\begin{cases} L_1 + t(\ldots) = 0, \\ L_2 + t(\ldots) = 0, \\ Q(X_1, \ldots, X_5) + t(\ldots) = 0, \\ Q'(X_1, \ldots, X_5) + t(\ldots) = 0. \end{cases}$$

One can check the above claim using the explicit equations.

Since smooth quadratic fibrations of positive dimension over a curve B satisfy weak approximation, and the families $\mathcal{Q} \to B$ and $\mathcal{E} \to B$ have reduced fibers over every point, we may find three sections $\tilde{\sigma}_1, \tilde{\sigma}_2, \tilde{\sigma}_3$ of the family $\mathcal{Q} \to B$ such that

- the section $\tilde{\sigma}_2$ is contained in the locus where the rational map $\mathcal{Q} \dashrightarrow \mathcal{X}$ is defined (this is possible since the rational map is undefined along some codimension 2 locus, and we can choose general sections to be disjoint from this codimension 2 locus);
- the section $\tilde{\sigma}_2$ is mapped to the section s_2 under this birational map;
- the sections $\tilde{\sigma}_1$ and $\tilde{\sigma}_3$ are also sections of the family $\mathcal{E} \to B$;

- if a point $b \in B$ is a point such that the fiber $\mathcal{X}_b$ of the family $\mathcal{X} \to B$ is nonnormal, then the sections $\tilde{\sigma}_1$ and $\tilde{\sigma}_3$ intersect the fiber $\mathcal{Q}_b$ of $\mathcal{Q} \to B$ in the locus where the rational map $\mathcal{Q} \to \mathcal{X}$ is defined;
- if the fiber of $\mathcal{Q}$ is reducible over a point b, then $\tilde{\sigma}_1(b)$ and $\tilde{\sigma}_2(b)$ lie in the same irreducible component of $\mathcal{Q}_b$;
- if the fiber of $\mathcal{Q}$ is singular but irreducible over a point b, then the plane spanned by $\tilde{\sigma}_1(b)$ and $\tilde{\sigma}_2(b)$ and $\tilde{\sigma}_3(b)$ intersects $\mathcal{Q}_b$ at a smooth conic;
- the sections $\tilde{\sigma}_1$ and $\tilde{\sigma}_3$ intersect the indeterminacy locus of $\mathcal{Q} \dashrightarrow \mathcal{X}$ transversely over the points b where the fibers of $\mathcal{X}$, $\mathcal{Q}$, and $\mathcal{E}$ are smooth.

Let $\Pi \to B$ be the family of planes spanned by the three sections $\tilde{\sigma}_1, \tilde{\sigma}_2, \tilde{\sigma}_3$, and let $\mathcal{S}_0$ be the family of conics of the intersection of the family of planes Π and the family $\mathcal{Q}$. There is a rational map $S_0 \dashrightarrow \mathcal{X}$. The surface $\mathcal{S}$ is constructed as the normalization of the closure of the image of $\mathcal{S}_0$. The sections σ_1 and σ_2 are strict transforms of $\tilde{\sigma}_1$ and $\tilde{\sigma}_2$.

As before, we can construct a pencil inside the minimal resolution of $\mathcal{S}$ giving a deformation of a comb with handle σ_1 and a comb with handle σ_2. The pencil induces a deformation of a comb with handle s_1 and a comb with handle s_2 in $\mathcal{X}$. By construction, we may take the teeth to be free lines, conics, twisted cubics, and a nodal rational curve tangent to $\mathcal{X}_b$ along $s_1(b)$ for some b. Furthermore, these curves all lie in the smooth fibers of $\mathcal{X} \to B$. This is because, by construction over a point $b \in B$, where $\mathcal{X}_b$ is singular, the sections σ_1 and σ_2 intersect the fiber of $\mathcal{S}$ over b at the same irreducible component even if the fiber of $\mathcal{S}$ over b could be reducible. So there is no need to add curves in singular fibers of $\mathcal{X}$ to construct the comb. Finally, the existence of the canonical sequence follows from Lemmas 6.8 and 3.16. ☐

Appendices

A. Weak approximation for cubic hypersurfaces defined over function fields of curves

In this appendix, we indicate how to modify the argument of [29] to prove the following theorem.

THEOREM A.1
Let X be a smooth cubic hypersurface in $\mathbb{P}^n, n \geq 3$, defined over the function field $K(B)$ of a smooth curve B over an algebraically closed field K of characteristic not equal to 2, 3, 5. *Then X satisfies weak approximation.*

The first result we need is the following.

THEOREM A.2 ([13, Theorem 2.15], [24, Proposition 6.4.1])
Let X be smooth projective cubic surface in $\mathbb{P}^3$ over $K(B)$, the function field of a curve C defined over an algebraically closed field K of characteristic not equal to $2, 3$. Then there is an integral model $\mathcal{X} \to C$ such that

(1) *each closed fiber is an integral cubic surface,*

(2) *the total space $\mathcal{X}$ has terminal singularities and is Gorenstein.*

Such a model is called a *standard model* of $X/K(B)$ in [13]. Corti gives an algorithm to produce such a model in [13] and shows that if the algorithm terminates, the end product is the so-called standard model satisfying the conditions in the above theorem (see [13, Theorem 2.15]). When K is a field of characteristic 0, the existence of such a model (i.e., the termination of the algorithm) is proved by Corti. The general case follows from Kollár's result on the existence of semistable models (see [24, Proposition 6.4.1]).

Essentially the only thing in [29] that needs to be changed is the proof of the following lemma.

LEMMA A.3 ([29, Lemma 5.1])
Let $\pi : \mathcal{X} \to B$ be a standard model of families of cubic surfaces over a smooth projective curve B, and let $s : B \to \mathcal{X}$ be a section. Given finitely many points $b_1, \dots, b_k$ in B, and a positive integer N, there is a section $s' : B \to \mathcal{X}$ such that s' is congruent to s modulo $\mathfrak{m}_{B,b_i}^N$, and $s'(B - \cup b_i)$ lies in the smooth locus of $\pi : \mathcal{X} \to B$.

The author proves this lemma in characteristic 0 in [29]. The proof given there uses the fact that a dominant map in characteristic 0 is separable. In the following, we give a variant of the original proof which avoids the use of such a statement.

Proof
We can assume that the base field K is uncountable.

We first resolve the singularities of $\mathcal{X}$ along the fibers over b_i in such a way that the resolution is an isomorphism except along the singular locus in the fibers over b_i (see [15], [16], and for the case of characteristic at least 7, the much earlier result of Abhyankar [1, Chapter 0]). Then we use the iterated blowup construction (see [29, Section 2.2]) according to the jet data of s near the points b_i. After sufficiently many iterated blowups, fixing the jet data is the same as passing through fixed components. Call the new space $\mathcal{X}_1$.

Then the lemma is reduced to showing that there is a section of the new family $\pi_1 : \mathcal{X}_1 \to B$ having desired intersection numbers with irreducible components of the fibers over $b_1, \dots, b_k$ in B and which lies in the smooth locus of $\pi_1 : \mathcal{X}_1 \to B$.

Denote by $f_1 : B \to \mathcal{X}_1$ the strict transform of the given section $s : B \to \mathcal{X}$. The section f_1 has the correct intersection numbers but may intersect the singular locus of $\mathcal{X}_1$. We will show below that if the section f_1 passes through only one singular point of $\mathcal{X}_1$, then we can deform the section away from the singularity.

Assuming this, the general case can be proved by induction. Namely, one first resolves all but one singularity along the section and then applies this argument to deform the section away from this singularity (after adding enough very free curves in general fibers). In this way, one gets a section which passes through fewer singular points. In this argument, we only take general deformations. So the condition of the intersection numbers is always preserved.

In the following, we explain why a general deformation of f_1 deforms outside the singularities of $\mathcal{X}_1$. We may also assume that the singularity on the section is the only singularity of the total space $\mathcal{X}_1$. Denote by b the image of the singular point of $\mathcal{X}_1$ in B. Take a resolution of singularities $\pi_{21} : \mathcal{X}_2 \to \mathcal{X}_1$ which is an isomorphism over the smooth locus of $\mathcal{X}_1$ such that the exceptional locus in $\mathcal{X}_2$ consists of simple normal crossing divisors $E_i, i = 1, \ldots, n$. After adding very free curves in general fibers and smoothing, we may assume that the strict transform of the section f_1, denoted by $f_2 : B \to \mathcal{X}_2$, satisfies $H^1(B, \mathcal{N}_{f_2}(-p)) = 0$ for any point $p \in B$, where $\mathcal{N}_{f_2}$ is the normal sheaf of the section in $\mathcal{X}_2$. This in particular implies that $\mathcal{N}_{f_2}$ is globally generated and the deformation of the section is unobstructed.

Let V be an irreducible component of the Kontsevich moduli space of stable maps of $\mathcal{X}_1$ containing the point represented by the map $f_1 : B \to \mathcal{X}_1$. There is a forgetful map F from the Kontsevich space of stable sections of the fibration $\mathcal{X}_2 \to B$ to the Kontsevich space of stable sections of the fibration $\mathcal{X}_1 \to B$. Every section of $\mathcal{X}_1 \to B$ lifts to a section of $\mathcal{X}_2 \to B$. Hence the forgetful map F is surjective on closed points. Since the field K is uncountable and the Kontsevich space has only countably many irreducible components, there is an irreducible component U of the moduli space of stable sections which maps surjectively onto V. Furthermore, a general point of U parameterizes a section of $\mathcal{X}_2 \to B$ (i.e., the domain is irreducible). Let $f_2' : B \to \mathcal{X}_2$ be a section parameterized by a very general point in U, and let f_1' be the composition $\pi_{21} \circ f_2'$.

If $f_1' : B \to \mathcal{X}_1$ already avoids the singular locus, then we are done. In the following, assume that $f_1'(B)$ still passes through the (unique) singular point of $\mathcal{X}_1$.

Since the map $F : U \to V$ is surjective, there is a stable map $\tilde{f}_2$ from a possibly reducible domain to $\mathcal{X}_2$ whose composition with π_{21} is the section $f_1 : B \to \mathcal{X}_1$. We claim that the domain of $\tilde{f}_2$ has to be reducible. Assume the contrary. Then the stable map f_2' is a deformation of $\tilde{f}_2 = f_2$. Thus a general point of U is also unobstructed, in particular a smooth point of U, and U has the expected dimension $-f_2'^* K_{\mathcal{X}_2} \cdot B$ at this point.

The standard model $\mathcal{X}$ has 3-fold terminal and local complete intersection singularities. So does the new total space $\mathcal{X}_1$ by construction. Therefore, every irreducible component containing the point $f_1' : B \to \mathcal{X}_2 \to \mathcal{X}_1$ has dimension at least $-f_1'^* K_{\mathcal{X}_1} \cdot B$. Furthermore, by definition of terminal singularities, we have

$$-\pi_{21}^* K_{\mathcal{X}_1} = -K_{\mathcal{X}_2} + \sum_{i=1}^{n} a_i E_i, \quad a_i > 0,$$

where the sum is over all exceptional divisors of $\pi_{21} : \mathcal{X}_2 \to \mathcal{X}_1$.

Since the image of $f_1'(B)$ in $\mathcal{X}_1$ intersects the singular locus, $-f_1'^* K_{\mathcal{X}_1} \cdot B$ is strictly larger than $-f_2'^* K_{\mathcal{X}_2} \cdot B$. Hence $\dim V > \dim U$, which is impossible since U surjects onto V.

Denote by $\tilde{f}_2 : \tilde{B} = B \cup R \to \mathcal{X}_2$ the stable map from a reducible domain whose composition with π_{21} is the section f_1. Note that the resolution of singularities $\mathcal{X}_2 \to \mathcal{X}_1$ is an isomorphism away from the fiber over the point b. So the curve R is supported in the exceptional divisors of the fiber of $\mathcal{X}_2$ over the point b.

Let H_1 be an ample divisor on $\mathcal{X}_1$, and let $H_2 = \pi_{21}^* H_1$. Then there are positive rational numbers $b_1, \ldots, b_n$ such that $H_2 - \sum b_i E_i$ is an ample divisor on $\mathcal{X}_2$. Up to perturbing the numbers b_i and renumbering the index, we may assume that $b_1 < b_2 < \cdots < b_n$. The section $f_2(B)$ intersects E_k for some k. Assume that $f_2'(B)$ intersects the divisor $E_{k'}$ for some k'. We have the following inequality (let us recall that the subscript of f indicates the target of the map is to $\mathcal{X}_1$ or $\mathcal{X}_2$ and a general point in the moduli is represented by f' with a suitable subscript):

$$\begin{aligned}
&f_2'^* H_2 \cdot B - b_{k'} \\
&\quad = f_2'^* \Big(H_2 - \Big(\sum b_i E_i \Big) \Big) \cdot B \\
&\quad = \tilde{f}_2^* \Big(H_2 - \sum b_i E_i \Big) \cdot (B + R) \\
&\qquad \text{(since } \tilde{f}_2 : B \cup R \to \mathcal{X}_1 \text{ and } f_2' : B \to \mathcal{X}_1 \text{ are deformation equivalent)} \\
&\quad = f_2^* \Big(H_2 - \sum b_i E_i \Big) \cdot B + \Big(H_2 - \sum b_i E_i \Big) \cdot R \\
&\quad > f_2^* \Big(H_2 - \sum b_i E_i \Big) \cdot B \\
&\quad = f_2^* H_2 \cdot B - b_k.
\end{aligned}$$

Since $f_2'^* H_2 \cdot B = f_1'^* H_1 \cdot B = f_1^* H_1 \cdot B = f_2^* H_2 \cdot B$, we have $b_{k'} < b_k$. Thus $k' < k$.

To sum up, we start with a section $f_1 : B \to \mathcal{X}_1$ whose strict transform $f_2 : B \to \mathcal{X}_2$ is an unobstructed section which intersects the exceptional divisor E_k. If the deformation of the section $f_1 : B \to \mathcal{X}_1$ in the irreducible component V still intersects the singular locus, then we produce a new section $f_2' : B \to \mathcal{X}_2$ which intersects the exceptional divisor $E_{k'}$ for some $k' < k$. Clearly, in the process we may keep all the desired intersection numbers unchanged.

Continuing this process, we will eventually find a section $s' : B \to \mathcal{X}_1$ which has the desired intersection numbers and lies in the smooth locus of the total space $\mathcal{X}_1$. Finally, note that if a section lies in the smooth locus of the total space $\mathcal{X}_1$, then the section lies in the smooth locus of the morphism $\pi_1 : \mathcal{X}_1 \to B$. □

Once this lemma is proved, the proof proceeds exactly as in [29]. For the reader's convenience we sketch the proof below and refer those interested to the original article for the full proof.

By Lemma A.3, there is a section $s : B \to \mathcal{X}$ that lies in the smooth locus of $\pi : \mathcal{X} \to B$. For simplicity, we assume that there is only one formal section $\widehat{s}_1$ in a formal neighborhood of $b \in B$ to approximate. The proof works the same in the general case, only with more complicated notation. Denote by $\widehat{s}_0$ the restriction of s to the formal neighborhood of b. The formal sections $\widehat{s}_0$ and $\widehat{s}_1$ determine a family of lines $\hat{L}$ in the formal neighborhood of b. Denote by $\widehat{s}_2$ the formal section given by the third intersection point of $\hat{L}$ with $\mathcal{X}$. Note that the space of lines through a point is isomorphic to a projective space. In particular, weak approximation holds for this space. So we can find a family of lines $\mathcal{L} \to B$ that contains the section s and approximates $\hat{L}$ to any prespecified order. The intersection of $\mathcal{L}$ with $\mathcal{X}$ consists of the section s and a multisection C of degree 2.

Note that once there is a multisection of degree 2, one gets a section given by the third intersection point of the family of lines determined by this multisection. This reduces the original weak approximation problem (finding a section approximating $\widehat{s}_1$) to a new weak approximation problem for the multisection C approximating $\widehat{s}_2$ and $\widehat{s}_0$.

By construction, the multisection C already approximates $\widehat{s}_2$ to the prespecified order. Applying Lemma A.3 once more to the new family $\mathcal{X} \times_B C \to C$, formed by the base change $C \to B$, we find a multisection of degree 2 of $\mathcal{X} \to B$ that approximates the formal section $\widehat{s}_2$ to the prespecified order and otherwise lies in the smooth locus. Recall that the section s lies in the smooth locus. So does the formal section $\widehat{s}_0$. So one only needs to approximate formal sections that lie in the smooth locus. Such a problem is handled in [29, Lemma 4.5]. The proof of this lemma depends on three things: strong rational connectedness of the smooth locus of a cubic surface with ADE singularities (which is true under the assumption on the characteristic by Lemma 5.3), a computation of base change and birational modifications of the integral model (see [29, Proposition 3.4]), and G-equivariant techniques (see [29, Lemma 3.6, Theorem 4.1]). When the characteristic is not $2, 3, 5$, the proof of these results needs no change. In [29, Proposition 3.4], the author computed the base change needed for the new central fiber to have ADE singularities. They are of degrees $2, 3, 4, 5, 6$. Thus under the assumption on the characteristic, all the base changes needed are Galois, and the Galois groups are cyclic groups of order prime to the characteristic. The G-equivariant techniques apply in these cases.

B. Fundamental group of rationally connected fibrations

In this section, we collect some easy corollaries of Kollár's results on the fundamental group of (separably) rationally connected varieties. All the fundamental groups mentioned below are to be understood as the algebraic fundamental group.

THEOREM B.1
Let $\pi : \mathcal{X} \to B$ be projective family of varieties over a smooth projective connected curve B defined over a field k. Assume that the generic fiber is smooth and separably rationally connected. Furthermore, assume that there is a free section $s_0 : B \to \mathcal{X}$ (see Definition 3.1). Choose a k-point $x \in s_0(B)$. Then there is a geometrically irreducible component of the space of sections with a marked point defined over k, that is, a family $\mathcal{S} \to W$ together with a section $p : W \to \mathcal{S}$ and an evaluation morphism $\mathrm{ev} : \mathcal{S} \to \mathcal{X}$ *such that the following hold.*

(1) *We have* $\mathrm{ev}(p(W)) = x$.
(2) *A general geometric point w in W parameterizes a 2-free section $\mathcal{S}_w$.*
(3) *Choose an algebraic closure K of k. For any open subvariety $\mathcal{X}^0 \subset \mathcal{X}$ defined over K containing the point x, the map of fundamental groups*

$$\pi_1\big(\mathrm{ev}^{-1}(\mathcal{X}^0), p(w)\big) \to \pi_1(\mathcal{X}^0, x)$$

is surjective.

Proof
Recall the following theorem of Kollár on the fundamental group of separably rationally connected varieties.

THEOREM B.2 ([25, (3)–(5)])
Let X be a smooth projective separably rationally connected variety defined over a field k, and let $x \in X$ be a k-rational point. Then there is a dominating family of rational curves through x (defined over k)

$$F : U \times \mathbb{P}^1 \to X, F\big(U \times [0, 1]\big) = x$$

with the following properties.

(1) *The variety U is geometrically irreducible, smooth, and open in*

$$\mathrm{Hom}\big(\mathbb{P}^1, X, [1,0] \mapsto x\big).$$

The morphism $F : U \times [1,0] \to X$ is smooth.

(2) *For every geometric point u of U, $F_u^* T_X$ is ample.*

(3) *Choose an algebraic closure K of k. For any K-open subvariety $X^0 \subset X$ containing the point x, the map*

$$\pi_1\big(F^{-1}(X^0) \cap (U \times \mathbb{P}^1), \big(u, [0,1]\big)\big) \to \pi_1(X^0, x)$$

is surjective.

(4) *Under the same assumptions as before, for any étale cover $Y^0 \to X^0$ defined over K, there is a nonempty open subset $U(Y_0)$ of U such that for any K-point $u \in U(Y_0)$, the fiber product $\mathbb{P}^1_u \times_X Y^0$ is geometrically irreducible.*

While the last statement is not explicitly stated in [25], it is very easy to deduce (see, e.g., the proof of Lemma B.6(1).

We apply this theorem to the generic fiber of the fibration with the rational point given by the section. Then we get a family of very free curves defined over the function field of B, which can be "spread out" to a geometrically irreducible family, still denoted by U, of rational curves in general fibers passing through the section. Over a general fiber, the family of rational curves in U still satisfies the conditions in Kollár's theorem, if we choose the basepoint to be the intersection point of the section s_0 with the fiber.

Consider the fiber product

$$B \times_{\mathcal{X}} \mathcal{S} \times_{\mathcal{X}} \cdots \times_{\mathcal{X}} \mathcal{S},$$

where the factor $\mathcal{S}$ appears sufficiently many times. A general point of the fiber product parameterizes a comb with handle the free section $s_0 : B \to \mathcal{X}$ and teeth very free curves in the family U. We can add all the Galois orbits of the teeth to the comb. This gives a smooth k-point of the fiber product

$$B \times_{\mathcal{X}} \mathcal{S} \times_{\mathcal{X}} \cdots \times_{\mathcal{X}} \mathcal{S},$$

with a certain number of the factor $\mathcal{S}$.

Choose the unique irreducible component $\mathcal{S} \to W$ containing the k-rational point corresponding to the comb. A general deformation of the comb with the point x fixed is a 2-free section. After shrinking the base, we may assume that the total space $\mathcal{S}$ is normal and W still contains all the points which parameterize a stable map consisting of the union of s_0 and general very free curves of the family U in general fibers.

Given an open subset $\mathcal{X}^0 \subset \mathcal{X}$, if the induced map on fundamental groups

$$\pi_1\big(\mathrm{ev}^{-1}(\mathcal{X}^0), p(w)\big) \to \pi_1(\mathcal{X}^0, x)$$

is not surjective, then there is a finite étale cover $\mathcal{Y}^0 \to \mathcal{X}^0 \subset \mathcal{X}$ from an irreducible variety $\mathcal{Y}^0$ defined over K, such that there is a point $y \in \mathcal{Y}^0$ which is mapped to x under the morphism, and the fiber product $\mathcal{S} \times_{\mathcal{X}} \mathcal{Y}^0$ is reducible. The total family of $\mathcal{S}$ is normal, and so is the fiber product. Thus it is disconnected.

We choose a reducible fiber $\mathcal{S}_0$ in the family consisting of the free section s_0 and general very free curves in general fibers in the following way. We first choose general points $b_1, \dots, b_k \in B$ so that the points $s_0(b_i)$ are contained in the open subset $\mathcal{X}^0$ and the family of very free curves in the generic fiber specializes to a family of very free curves in the fibers over $b_1, \dots, b_k$ which induces surjections on the fundamental group. The cover $\mathcal{Y}^0 \to \mathcal{X}^0$ induces a finite (possibly disconnected) étale cover $\mathcal{Y}^0_{b_i} \to \mathcal{X}^0_{b_i}$ for each $1 \le i \le k$. We then choose a very free curve in the fiber over b_i to be a general curve C_i such that the base change $C_i \times \mathcal{Y}^0_{b_i}$ has the same number of geometrically irreducible components as geometrically connected components of $\mathcal{Y}^0_b$. That is to say, the base change of each geometrically irreducible component of $\mathcal{Y}^0_{b_i}$ to C_i is geometrically irreducible. This is possible by Kollár's results.

We now look at the fiber product $\mathcal{S}_0 \times_{\mathcal{X}} \mathcal{Y}^0$. This is geometrically connected. To see this, simply note that each geometrically irreducible component of $C_i \times_{\mathcal{X}} \mathcal{Y}^0_{b_i}$ is connected by one (in fact, any) geometrically irreducible component of the inverse image of the section s_0 and that all the geometrically irreducible components of the inverse image of the section s_0 are connected by the base change of any curve C_i. So $\mathcal{S}_0 \times_{\mathcal{X}} \mathcal{Y}^0$ connects every geometrically irreducible component of $\mathcal{S} \times_{\mathcal{X}} \mathcal{Y}^0$. Thus we get a contradiction, and the map on fundamental groups

$$\pi_1\big(\mathrm{ev}^{-1}(\mathcal{X}^0), p(w)\big) \to \pi_1(\mathcal{X}^0, x)$$

is surjective. □

We also have the following version without specifying a basepoint.

THEOREM B.3

Let $\pi : \mathcal{X} \to B$ be projective family of varieties over a smooth projective connected curve B defined over a field k. Assume that the generic fiber is smooth and separably rationally connected. Furthermore, assume that there is a free section $s_0 : B \to \mathcal{X}$. Then there is a geometrically irreducible component of the space of sections defined over k, that is, a family of sections $\mathcal{S} \to W$ and an evaluation morphism $\mathrm{ev} : \mathcal{S} \to \mathcal{X}$, *such that the following hold.*

(1) *A general geometric point w in W parameterizes a 2-free section $\mathcal{S}_w$.*

(2) *Choose an algebraic closure K of k. For any dominant map $f : \mathcal{Z} \to \mathcal{X}$ from an irreducible variety $\mathcal{Z}$, there is an open subset W^0 of W such that for any geometric point $w \in W^0$, the fiber product $\mathcal{S}_w \times_{\mathcal{X}} \mathcal{Z}$ is geometrically irreducible.*

Proof

The free section s_0 determines a geometrically irreducible family of sections $\mathcal{S} \to W$ defined over k. Moreover, there is a dominant map $\mathcal{S} \to \mathcal{X}$. Let K be the function field of $\mathcal{S}$, and let η be the generic point of $\mathcal{S}$. Apply Theorem B.1 over the field K to the K-point η of $\mathcal{X} \times_k K$. (For details, see the proof of [25, (6)].) □

In the following, we specialize to the case of a family of Fano complete intersections satisfying Hypothesis 3.3. For the reader's convenience, we reproduce the hypothesis below.

HYPOTHESIS B.4 (Hypothesis 3.3)

Given a family $\mathcal{X} \to B$ of Fano complete intersections defined over an algebraically closed field, assume the following are satisfied.

(1) *There is free section $s : B \to \mathcal{X}$.*

(2) *The Fano scheme of lines of a general fiber $\mathcal{X}_b$ is smooth.*

(3) *A general line in a general fiber is a free line.*

(4) *The relative dimension of $\mathcal{X} \to B$ is at least* 3.

To apply the results of Kollár, we first need to check that a general fiber is separably rationally connected. This is taken care of by the following result.

THEOREM B.5 ([28, Corollary 9])

Let X be a smooth Fano complete intersection of dimension at least 3. *Then X is separably rationally connected if and only if it is separably uniruled.*

Thus all the previous results apply to the fibration $\mathcal{X} \to B$. Recall that $F \to B$ is the relative Hilbert scheme of lines and that $F(U) \to U$ is the relative Hilbert scheme of lines for the smooth fibers. Finally, $\overline{F} \to B$ is the closure of $F(U)$ in F and $\mathcal{L} \to \overline{F}$ is the universal families of lines restricted to $\overline{F}$. There is a natural B-morphism $\mathcal{L} \to \mathcal{X}$.

LEMMA B.6

Let $\mathcal{X} \to B$ be a family defined over an algebraically closed field k satisfying Hypothesis 3.3.

(1) *There is a nice section.*

(2) *Let $\mathcal{S} \to W$ be an irreducible component of the space of sections such that there is a geometric point $w \in W$ which parameterizes a nice section $\mathcal{S}_w$. Then a general point of W parameterizes a nice section.*

(3) *Let $\mathcal{S} \to W$ be a geometrically irreducible component of the space of sections such that a general geometric point parameterizes a nice section. Then $\mathcal{S} \times_{\mathcal{X}} \mathcal{L}$ is geometrically irreducible and generically smooth. Furthermore, it is contained in a unique irreducible component of the Kontsevich moduli space of stable sections which contains an open substack parameterizing nice sections.*

Proof

The morphism $\mathcal{L} \to \mathcal{X}$ factors through a variety $\mathcal{Z}$: $\mathcal{L} \to \mathcal{Z} \to \mathcal{X}$ such that a general fiber of $\mathcal{L} \to \mathcal{Z}$ is geometrically irreducible and such that $\mathcal{Z} \to \mathcal{X}$ is finite and generically étale (see [25, (9)]). Let $\mathcal{X}^0$ be the open locus of $\mathcal{X}$ such that $\mathcal{Z} \to \mathcal{X}$ is étale when restricted to the inverse image of $\mathcal{X}^0$, and let $\mathcal{X}^1$ be the open locus of $\mathcal{X}$ such that $\mathcal{Z} \to \mathcal{X}$ has constant fiber dimension. Let $\mathcal{Z}^0$ be the inverse image of $\mathcal{X}^0$ in $\mathcal{Z}$. The complement of $\mathcal{X}^1$ in $\mathcal{X}$ has codimension at least 2. Thus a general deformation of a free section lies in $\mathcal{X}^1$.

Choose a general point x in $\mathcal{X}^0$. Consider the family of sections $\mathcal{S} \to W$ containing the point x constructed in Theorem B.1. By Theorem B.1, the fiber product $\mathcal{S} \times_{\mathcal{X}} \mathcal{Z}^0$ is geometrically irreducible. By shrinking W, we may assume that W is smooth and that the morphism $\mathcal{S} \to W$ is a smooth morphism. Furthermore, there is a section from W to $\mathcal{S} \times_{\mathcal{X}} \mathcal{Z}^0$ by choosing a point in $\mathcal{Z}^0$ lying over x. The generic fiber of $\mathcal{S} \times_{\mathcal{X}} \mathcal{Z}^0 \to W$ is smooth and contains a $k(W)$-rational point, thus geometrically irreducible since $\mathcal{S} \times_{\mathcal{X}} \mathcal{Z}^0$ is geometrically irreducible. Then a general fiber of the morphism $\mathcal{S} \times_{\mathcal{X}} \mathcal{Z}^0 \to W$ is geometrically irreducible.

A 2-free section $s : B \to \mathcal{X}$ is nice if

(1) it is contained in $\mathcal{X}^1$,

(2) the fiber product $B \times_{\mathcal{X}} \mathcal{Z}^0$ is irreducible.

So a general member of the family $\mathcal{S} \to W$ is a nice section.

For the second statement, let $\mathcal{S}_w$ be a nice section, and let x be a general point in $\mathcal{S}_w \cap \mathcal{X}^0$. We first deform the nice section $\mathcal{S}_w$ with the point x fixed. This gives a covering family $\mathcal{T} \to U$, $F : \mathcal{T} \to \mathcal{X}$, with a section $p : U \to \mathcal{T}$ such that $F(p(U)) = x$. The deformation covers an open subset of $\mathcal{X}$ since $\mathcal{S}_w$ is 2-free. Furthermore, a general deformation is a 2-free section which lies in the locus $\mathcal{X}^1$.

Consider the fiber product $\mathcal{T} \times_{\mathcal{X}} \mathcal{Z}^0$ and its projection to U. A section T_u over a general geometric point $u \in U$ is 2-free. Thus by replacing U with a smaller open subset, we may assume that $\mathcal{T}, U$ are smooth and that the fibration $\mathcal{T} \to U$ is smooth.

So $\mathcal{T} \times_{\mathcal{X}} \mathcal{Z}^0$ is irreducible if and only if it is connected. There is a section of the map $\mathcal{T} \times_{\mathcal{X}} \mathcal{Z}^0 \to U$ by lifting x to one of its inverse images in $\mathcal{Z}^0$. This determines a geometrically irreducible component of $\mathcal{T} \times_{\mathcal{X}} \mathcal{Z}^0$. Since $\mathcal{S}_w \times_{\mathcal{X}} \mathcal{L}$ is irreducible, $\mathcal{S}_w \times_{\mathcal{X}} \mathcal{Z}^0$ is irreducible. Thus it lies in a unique irreducible component of $\mathcal{T} \times_{\mathcal{X}} \mathcal{Z}^0$. Then there can be only one irreducible component, independent of the chosen inverse image of x. Once we know that the total space is geometrically irreducible and that the generic fiber has a rational point in the smooth locus, a similar argument as above shows that for a general point $u \in U$, the section $\mathcal{T}_u$ is nice.

To sum up, given a nice section $\mathcal{S}_w$ and a general point $x \in \mathcal{S}_w \cap \mathcal{X}^0$, a general deformation of the section $\mathcal{S}_w$ with the point x fixed is a nice section. Since the section $\mathcal{S}_w$ is 2-free, the deformation with x fixed covers an open subset of $\mathcal{X}$. Thus there is a nice section over any general point in $\mathcal{X}$.

Consider the evaluation map of the total family $\mathcal{S} \to W$ to $\mathcal{X}$, $\mathrm{ev}_S : \mathcal{S} \to \mathcal{X}$. The previous paragraphs shows that for a general point $y \in \mathcal{X}^0$, there is one irreducible component of the fiber of ev_S over the point y, whose general points parameterize nice sections.

The locus in W such that the fiber of $\mathcal{S} \times_{\mathcal{X}} \mathcal{Z}^0 \to W$ is geometrically irreducible is a constructible set by [21, Corollaire 9.7.9]. Therefore, it contains an open subset of W.

For the last statement, first note that a standard computation using the exact sequence of normal sheaves shows that a general deformation of the union of $\mathcal{S}_w$ and a free line is a 2-free section. Furthermore, if we deform the union of a nice section and a free line with one general point of the section fixed, then a general deformation is a 2-free section. Call this family $\mathcal{C} \to V$. We need to show that a general member of this family is a nice section. We may assume that $V, \mathcal{C}$ are normal after restricting to a smaller open subset. The proof proceeds similarly to the proof of Theorem B.1. Namely, one first shows that the total space of the fiber product $\mathcal{C} \times_{\mathcal{X}} \mathcal{Z}^0$ is geometrically irreducible. One can show this by specializing to the union of a nice section and a general free line. Then a general member of the family is a nice section. So a general point of the family parameterizes a nice section by the second statement. $\square$

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