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INTEGRAL HODGE CLASSES ON FOURFOLDS FIBERED BY QUADRIC BUNDLES

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Abstract. We discuss the space of sections and certain bisections on a quadric surfaces bundle X over a smooth curve. The Abel-Jacobi from these spaces to the intermediate Jacobian will be shown to be dominant with rationally connected fibers. As an application, we prove that the integral Hodge conjecture holds for degree 4 integral Hodge classes (IHC) of fourfolds fibered by quadric bundles over a smooth curve. This gives an alternative proof of a result of Colliot-Thélène and Voisin.

1. Introduction

Let X be a smooth complex projective variety of dimension n. We denote by $Hdg^{2k}_{\mathbb{Z}}(X) = H^{k,k}(X) \cap H^{2k}(X,\mathbb{Z})$ the group of integral Hodge classes of degree $2k$, and set

$$Z^{2k}(X) := Hdg^{2k}_{\mathbb{Z}}(X) / \langle Z \subset X, Z \text{ has codimension k} \rangle .$$

When $k = 2$, the group $Z^4(X)$ is a very interesting birational invariant studied recently by Voisin and Colliot-Thélène [5, 14], which is known to be nontrivial in general since 1962 (cf. [1]). Note that the famous Hodge conjecture holds for degree 4 Hodge classes if and only if the group $Z^4(X)$ is torsion. It has been proved by Conte and Murre [6] that the Hodge conjecture holds for degree 4 Hodge classes when X is a uniruled fourfold, and later Bloch and Srinivas [2] generalized this result to varieties whose 0-th Chow group is supported on subvarieties of dimension ≤ 3 (e.g. rationally connected varieties) using the diagonal decomposition. But our knowledge of $Z^4(X)$ on these varieties is still limited. For instance, it remains open when $Z^4(X) = 0$, i.e., the Hodge conjecture holds for degree 4 integral Hodge classes (IHC) on X. We should remark that the known examples where X is rationally connected and $Z^4(X)$ is nonzero are of dimension at least 6 (cf. [4], [5]).

Recently, several methods have been introduced to attack this problem. In [5], Colliot-Thélène and Voisin have shown that $Z^4(X) = 0$ when X admits a quadric surfaces fibration over some surface via a K-theoretic approach. In this paper, we give an alternative proof of the Hodge conjecture for IHC on such fourfolds via a geometric approach developed by Voisin. The main results are:

Theorem 1.1. *Let $f : X \to B$ be a family of quadric surfaces over a smooth projective curve B with at worst nondegenerate quadric cones as singular fibers. Let* $\mathrm{Sec}(X/B, h)$ *(resp.* $\mathrm{Bse}(X/B, h)$*) be the space of sections (resp. special bisections)*

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(see §4) of f. Then the morphism from $\mathrm{Bse}(X/B,h)$ *or* $\mathrm{Sec}(X/B,h)$ *to the torsor* $J(X)_h$ *defined by a Deligne cycle class map is dominant and each fiber is rational.*

Theorem 1.2. *Let $\mathbb{P}(V) \to B$ be a $\mathbb{P}^3$-bundle over a smooth projective curve B. Let $\pi : \mathfrak{X} \to T$ be a projective good family of quadric surfaces bundles in $\mathbb{P}(V)$ over a smooth projective curve T. Suppose that the general fiber $\mathfrak{X}_t$ of π satisfies the condition above. Then $Z^4(\mathfrak{X}) = 0$.*

Remark 1.3. The notion of a family being *good* is given in §1. Roughly speaking, it means that the local system $R^4\pi_*\mathbb{Z}$ is trivial and all the fibers of the family have at worst ordinary double points as singularities.

Remark 1.4. One can see that the fourfold $\mathfrak{X}$ is birational to a quadric surface bundle over a surface and hence Theorem 1.2 can be implied by Colliot-Thélène and Voisin's result. But we hope that the geometric method can be applied to fourfolds admitting a toric surfaces fibration.

1.5. **Organization of the paper.** In §2, we explain Voisin's method for proving the Hodge conjecture for integral Hodge classes on families of threefolds. The principle of her method is to find sufficiently many families of curves on the fibers $\mathfrak{X}_t$, which have, via the Abel-Jacobi map of $\mathfrak{X}_t$, the structure of a rationally connected fibration over the intermediate Jacobian $J(\mathfrak{X}_t)$.

We review the classical geometry of quadric bundles in §3 and give some natural geometric modification of the quadric bundle. In §4, we show that the morphism induced by the Abel-Jacobi map agrees with the map defined by Hassett and Tschinkel [10] (see also [16]). Theorem 1.1 is proved in Theorem 4.4 and Theorem 4.11. The main theorem is proved in the last section.

2. Intermediate Jacobian and Abel-Jacobi map

2.1. **Intermediate Jacobian.** Let X be a smooth projective threefold. The intermediate Jacobian

$$J(X) := H^3(X,\mathbb{C})/(F^2H^3(X) + H^3(X,\mathbb{Z}))$$

is a compact torus. Furthermore, when $H^{3,0}(X) = 0$ (e.g. when X is rationally connected), the intermediate Jacobian $J(X)$ is an abelian variety. It fits into the exact sequence

$$0 \to J(X) \to H^4_D(X,\mathbb{Z}(2)) \to Hdg^4_{\mathbb{Z}}(X) \to 0,$$

where $H^4_D(X,\mathbb{Z}(2)) = \mathbb{H}^4(0 \to \mathbb{Z} \to \mathcal{O}_X \to \Omega_X \to 0)$ is the Deligne cohomology group.

For any integral Hodge class $\alpha \in Hdg^4_{\mathbb{Z}}(X)$, the torsor $J(X)_\alpha$ is defined to be the preimage of α in $H^4_D(X,\mathbb{Z}(2))$. Let $CH^2(X)_\alpha \subset CH^2(X)$ be the set of codimension two cycles whose cycle class is α. Then there is a Deligne cycles class map

$$c_X : CH^2(X)_\alpha \to J(X)_\alpha \subseteq H^4_D(X,\mathbb{Z}(2)).$$

If $\alpha = 0$, it can be identified with the Abel-Jacobi map $AJ_X : CH^2(X)_0 \to J(X)$ introduced by Griffiths [7].

Let $Z \subset M \times X$ be a family of 1-cycles of class α, i.e., $[Z_b] = \alpha$ in $H^4(X,\mathbb{Z})$ for all $b \in M$. Then the Deligne cycle class map induces a morphism (cf. [13])

$$\phi_Z : M \to J(X)_\alpha, \tag{2.1}$$

defined by $\phi_Z(b) = c_\alpha(Z_b)$.

Remark 2.2. Assume that M is connected. Fixing $b_0 \in M$, we can define a map

$$\begin{aligned} \phi_Z^{b_0} : M &\to J(X) \\ b &\mapsto AJ_X([Z_b] - [Z_{b_0}]), \end{aligned} \tag{2.2}$$

which has the same fiber as ϕ_Z. For simplicity, we continue to use ϕ_Z to denote this morphism.

These constructions naturally extend to the relative situation. Namely, let $f : \mathfrak{X} \to T$ be a family of rationally connected threefolds over a smooth curve T, and assume that f is smooth over $T_0 \subset T$. For any section $\tilde{\alpha}$ of the local system $R^4 f_* \mathbb{Z}$, we get a family of torsors

$$\mathcal{J}_{\tilde{\alpha}} \to T_0,$$

whose fiber over $t \in T_0$ is $J(\mathfrak{X}_t)_{\tilde{\alpha}(t)}$. Given a variety $\mathcal{M}$ over T and a family of relative 1-cycles $\mathcal{Z} \subset \mathcal{M} \times_T \mathfrak{X}$ of class $\tilde{\alpha}$, i.e., $[\mathcal{Z}_t] = \tilde{\alpha}(t)$, the restriction to T_0 gives a morphism

$$\begin{array}{ccc} \Phi_{\mathcal{Z}} :\ \mathcal{M} \times_T T_0 & \longrightarrow & \mathcal{J}_{\tilde{\alpha}} \\ \downarrow & \swarrow & \\ T_0 & & \end{array} \tag{2.3}$$

In this paper, we are interested in the geometry of the map (2.3).

2.3. **Voisin's criterion.** As discussed in [15], the rational connectedness of the general fiber of (2.3) is closely related to the integral Hodge conjectures on $\mathfrak{X}$.

Before we proceed, let us first make some assumptions on our family $\mathfrak{X}$. We say that $\mathfrak{X} \to T$ is a *good* family if it satisfies the following conditions:

(1) $R^4 f_* \mathbb{Z}$ is trivial;
(2) $H^2(\mathfrak{X}_t, \mathcal{O}_{\mathfrak{X}_t}) = H^3(\mathfrak{X}_t, \mathcal{O}_{\mathfrak{X}_t}) = 0$; and $H^3(\mathfrak{X}_t, \mathbb{Z})$ is torsion free for any smooth fiber $\mathfrak{X}_t$;
(3) every fiber has at worst ordinary double points as singularities.

Remark 2.4. In the case of Theorem 1.2, we know that $H^3(\mathfrak{X}_t, \mathcal{O}_{\mathfrak{X}_t}) = 0$ because of the uniruledness. Moreover, since the integral cohomology group of $\mathbb{P}(V)$ is torsion free and $\mathfrak{X}_t$ is ample on $\mathbb{P}(V)$, $H^3(\mathfrak{X}_t, \mathbb{Z})$ is automatically torsion free by the Lefschetz hyperplane section theorem and the universal coefficients theorem.

When $\mathfrak{X} \to T$ is a good family, the following criterion is proved in [15]:

Theorem 2.5. *The group $Z^4(\mathfrak{X})$ is trivial if for any section $\tilde{\alpha}$ of $R^4 f_* \mathbb{Z}$, the following condition holds:*

$(*)$ *There exist a variety $g_{\tilde{\alpha}} : \mathcal{M}_{\tilde{\alpha}} \to T_0$ and a family of relative* 1*-cycle $\mathcal{Z}_{\tilde{\alpha}} \subset \mathcal{M}_{\alpha} \times_T \mathfrak{X}$ of class $\tilde{\alpha}$, such that the morphism $\Phi_{\mathcal{Z}_{\tilde{\alpha}}} : \mathcal{M}_{\tilde{\alpha}} \to \mathcal{J}_{\tilde{\alpha}}$ is surjective with rationally connected general fibers.*

Moreover, every algebraic cycle $W \in CH^2(\mathfrak{X})$ will induce a section $[W]$ of $R^4 f_* \mathbb{Z}$. Suppose that condition $(*)$ holds for some section $\tilde{\alpha}$; then it holds for the section $\tilde{\alpha}' := \pm\tilde{\alpha} + n[W]$ for all $n \in \mathbb{Z}$. This is because one can just take

$$\mathcal{M}_{\tilde{\alpha}'} = \mathcal{M}_{\tilde{\alpha}} \quad \text{and} \quad \mathcal{Z}_{\tilde{\alpha}'} = \pm\mathcal{Z}_{\tilde{\alpha}} + n(\mathcal{M}_{\tilde{\alpha}'} \times_T W),$$

which will naturally satisfy the condition $(*)$. Therefore, we can obtain the following result.

Corollary 2.6. *Let $A^4(\mathfrak{X}) \subseteq Hdg^4_{\mathbb{Z}}(\mathfrak{X})$ be the image of cup product*

$$\Xi : \mathrm{Pic}(\mathfrak{X}) \times \mathrm{Pic}(\mathfrak{X}) \to Hdg^4_{\mathbb{Z}}(\mathfrak{X}).$$

Then $Z^4(\mathfrak{X}) = 0$ if the condition $(*)$ holds for a set of sections representing all sections modulo $A^4(\mathfrak{X})$.

3. Classical geometry on quadric surfaces bundles

Assume that $\pi : X \to B$ is a quadric bundle over a smooth projective curve B whose singular fibers are at worst nondegenerate quadratic cones. For simplicity, we assume that $\pi : X \to B$ has nontrivial monodromy. Then the cohomology group $H^4(X, \mathbb{Z}) \cong \mathbb{Z}^2$ has rank two. Denote by $\Delta \subset B$ the set of points where the fiber is singular and set $m = |\Delta|$ to be the number of singular fibers. Then m is an even integer by [9].

3.1. **Fano scheme of lines.** Let $\mathcal{F}$ be the space of lines in the fibers of π and $\mathcal{U} \subset \mathcal{F} \times X$ the universal family. The Stein factorization

$$\mathcal{F} \to C \xrightarrow{g} B$$

is the composition of a smooth $\mathbb{P}^1$-bundle and a double cover branched along the discriminant divisor Δ. Following the notion of [10], we may call C the discriminant curve of $X \to B$ and it endows with a natural involution $\iota : C \to C$.

Remark 3.2. If the morphism $\pi : X \to B$ is smooth and hence $\Delta = \emptyset$, then $g : C \to B$ is an étale covering of degree two. Moreover, in the case $B = \mathbb{P}^1$, π is smooth if and only if the fibration $X \to \mathbb{P}^1$ has trivial monodromy.

Given a section $\delta : C \to \mathcal{F}$, the restriction $F_\delta \subseteq \mathcal{U}$ of the universal family to the image $\delta(C) \cong C$ gives a one dimensional family of lines on X, that is, a diagram:

$$\begin{array}{ccc} F_\delta & \xrightarrow{p_\delta} & X \\ \Big\downarrow{\scriptstyle q_\delta} & & \\ C & & \end{array} \tag{3.1}$$

This yields a cylinder homomorphism

$$\begin{aligned} \Psi_\delta : H_1(C, \mathbb{Z}) &\to H_3(X, \mathbb{Z}) \\ \gamma &\mapsto p_\delta(q_\delta^{-1}(\gamma)), \end{aligned} \tag{3.2}$$

which is an isomorphism by [12] Lecture 5. Then we get an isomorphism

$$J(X) \xrightarrow{\sim} J(C)$$

induced by this cylinder homomorphism, which is a morphism of Hodge structures. Moreover, using the identification of intermediate Jacobians or Jacobians via the cycle class map, we can rewrite the above isomorphism in terms of cycles. This is the functoriality of the Abel-Jacobi map (e.g. [13]).

Lemma 3.3. *The map*

$$\psi_\delta : J(C) \to J(X) \tag{3.3}$$

defined by $\psi_\delta(\sum_i n_i[c_i]) = c_X(\sum_i n_i[p_\delta(q_\delta^*(c_i))])$ is an isomorphism, where c_i are points in C and $\sum_i n_i = 0$.

We may omit the notation c_X in the latter sections for simplicity.

3.4. **Geometric modifications.** The ruled surface F_δ obtained in (3.1) is birational to the Fano scheme $\mathcal{F}$ of lines in the fibers of $X \to B$. To describe this birational map, let us recall a useful modification of the family $X \to B$ in [10].

Let $\widetilde{Y}$ be the blow-up of $X \times_B C$ along the nodes of the singular fibers and let $\mu : \widetilde{Y} \to Y$ be the blowing-down of the strict transform of the singular fibers of $X \times_B C$. Then $Y \to C$ is a smooth family of quadric surfaces with trivial monodromy.

Moreover, the universal family $\mathcal{U}$ is the small resolution of $X \times_B C$. Indeed, $X \times_B C$ has a singular point for each singular fiber. Using the family $\mathcal{F}$ of lines in the fibers, one can construct a smooth surface S in $X \times_B C$ passing through a given singular point: the blow-up of S in $X \times_B C$ is then isomorphic to $\mathcal{U}$. Hence we have a commutative diagram:

$$
\begin{array}{ccccccc}
 & & \mathcal{U} & \xleftarrow{\ \mu'\ } & \widetilde{Y} & \xrightarrow{\ \mu\ } & Y \\
 & \swarrow^{q} & \downarrow^{p} & \swarrow^{\eta} & \downarrow & & \\
\mathcal{F} & & X & \longleftarrow & X \times_B C & & \\
\downarrow^{r} & & \downarrow^{\pi} & & \downarrow & & \\
C & \xrightarrow{\ g\ } & B & \longleftarrow & C & &
\end{array}
\tag{3.4}
$$

where the arrow $\mu' : \tilde{Y} \to \mathcal{U}$ blows down one of the rulings of the exceptional divisors of $\mathcal{U} \to X$ over the nodes.

Lemma 3.5 ([11]). *The Fano scheme of lines of the fibers of $Y \to C$ is a disjoint union of two ruled surfaces $\mathcal{F} \cup \iota^*\mathcal{F}$. Each of them parametrizes one of the two rulings on the fibers of $Y \to C$ and the corresponding universal family $\mathcal{W}$ (or $\iota^*\mathcal{W}$) is isomorphic to Y.*

With the same notation as in §3.1, let $\delta : C \to \mathcal{F}$ be a section. We let $R_\delta := \mathcal{W}|_{\delta(C)}$ (resp. $\iota^* R_\delta$) be the restriction of $\mathcal{W}$ (resp. $\iota^*\mathcal{W}$) to the curve $\delta(C) \subseteq \mathcal{F}$. Then $j : R_\delta \to C$ and $j' : \iota^* R_\delta \to C$ are families of lines in one of the two rulings on the fibers of Y. Note that the ruled surface R_δ is actually isomorphic to $\mathcal{F}$ in this case. We thus obtain birational maps from (3.4) as follows:

$$
\begin{array}{ccccc}
\mathcal{W} \cong Y & \dashrightarrow & \mathcal{U} & \longrightarrow & X \\
{\scriptstyle i}\uparrow & & \uparrow & & \\
\mathcal{F} \cong R_\delta & \overset{\Lambda_\delta}{\dashrightarrow} & \iota^* F_\delta & &
\end{array}
\tag{3.5}
$$

There is a similar commutative diagram for $\iota^* R_\delta$ and we denote by $i' : \iota^* R_\delta \hookrightarrow Y$ the embedding of $\iota^* R_\delta$ in Y.

Geometrically, for $z \in \mathcal{F}$, if we denote by $\ell(z)$ the corresponding lines on X, the birational map $\Lambda_\delta : \mathcal{F} \dashrightarrow \mathcal{F}_\delta$ is defined away from the fibers over points in Δ and is given by

$$
\Lambda_\delta(z) = \ell(\iota(z)) \cap \ell(\delta \circ r(z)) \in \ell(\delta \circ r(z)),
\tag{3.6}
$$

for all $z \in \mathcal{F} \backslash r^{-1}(\Delta)$.

Remark 3.6. We can see that the discriminant curve of $Y \to C$ is the disjoint union of two curves C and $\iota^* C \cong C$. Similarly as in (3.2) and Lemma 3.3, choosing a family of two rulings $R_\delta \cup \iota^* R_\delta$ in the fibers of $Y \to C$, the cylinder homomorphism

$$H_1(C, \mathbb{Z}) \oplus H_1(C, \mathbb{Z}) \to H_3(Y, \mathbb{Z})$$

induces an isomorphism $\Psi'_\delta : J(C) \times J(C) \xrightarrow{\sim} J(Y)$ given by $\Psi'_\delta([\gamma] + [\gamma']) = i_*(j^*[\gamma]) + i'_*(j'^*[\gamma'])$ for $[\gamma], [\gamma'] \in J(C)$.

3.7. **Weil restriction.** A beautiful geometry fact is that the quadric fibration $X \to B$ can be reconstructed from the ruled surface $\mathcal{F} \to C$ by Weil restriction. More precisely, one can define a contravariant functor

$$\begin{aligned} \mathfrak{R}_{C/B}\mathcal{F} : (\mathcal{S}ch_B)^\circ &\longrightarrow \text{(Sets)} \\ T &\longrightarrow \mathrm{Hom}_C(T \times_B C, \mathcal{F}). \end{aligned} \tag{3.7}$$

It is known (cf. [3]) that the functor $\mathfrak{R}_{C/B}\mathcal{F}$ is representable by a scheme $\mathrm{Res_{C/B}}\mathcal{F}$ over B and there is a functorial isomorphism

$$\mathrm{Hom_B}(\mathrm{V}, \mathrm{Res_{C/B}}\mathcal{F}) \xrightarrow{\sim} \mathrm{Hom_C}(\mathrm{V} \times_{\mathrm{B}} \mathrm{C}, \mathcal{F}) \tag{3.8}$$

of functors in V, where V varies over all B-schemes.

As shown in [10], Hassett and Tschinkel indicate the following diagram:

$$\begin{array}{ccccc} & & \widehat{X} & & \\ & \stackrel{\beta}{\swarrow} & & \stackrel{\gamma}{\searrow} & \\ \mathrm{Res_{C/B}}\mathcal{F} & & & & X \\ & \stackrel{\varpi}{\searrow} & & \stackrel{\pi}{\swarrow} & \\ & & B & & \end{array} \tag{3.9}$$

where the arrows are described as follows:

(1) $\varpi^{-1}(b) = \mathrm{Sym}^2(\mathrm{r}^{-1}(\mathrm{b}))$ is isomorphic to $\mathbb{P}^2$ over $b \in \Sigma$;
(2) β is the blowing up of $\mathrm{Res_{C/B}}\mathcal{F}$ along the diagonal in $\varpi^{-1}(b)$ over each point $b \in \Sigma$;
(3) γ is the blowing down of $\widehat{X}$ along the proper transform of $\varpi^{-1}(b)$ in $\widehat{X}$ over each point $b \in \Sigma$.

In particular, if $X \to B$ is a smooth quadric bundle, the Weil restriction $\mathrm{Res_{C/B}}\mathcal{F}$ is isomorphic to X.

4. Sections on quadric fibrations

With the same assumption as in §3, we are going to find families of curves on X satisfying the conditions in Theorem 2.5. The natural candidates are families of sections and bisections of $X \to B$. In this section, we prove Theorem 1.1 and show that there exist infinitely many families of sections and bisections satisfying $(*)$. Throughout this section, we assume that the family $X \to B$ has nontrivial monodromy.

4.1. **Notation and conventions.** We denote by $\mathcal{S}ch_S$ or $\mathcal{S}ch_{\mathbb{C}}$ the category of schemes over a scheme S or complex numbers $\mathbb{C}$.

Let $\pi : X \to B$ be a projective family of varieties over a curve B. The moduli functor

$$\mathfrak{S}(X/B) : (\mathcal{S}ch_{\mathbb{C}})^{\circ} \to (\mathrm{Set})$$

sends any $T \in \mathrm{Sch}_{\mathbb{C}}$ to the set of families of sections of $X \to B$ over T. By [8] Part IV4.c, $\mathfrak{S}(X/B)$ is representable by a scheme $\mathrm{Sec}(X/B)$, which is a union of countably many quasi-projective varieties.

In this section, for a cycle $z \in CH_1(X)$, we will use $[z]$ to denote the cycle class $c_X(z) \in H^4(X, \mathbb{Z})$. Moreover, if $\sigma : B \to X$ is a section of π, we continue to use σ to represent the cycle class of $\sigma(B)$ on X.

4.2. **Families of sections.** Let us consider the space of sections on X. For any nonnegative integer $h \in \mathbb{Z}^{\geq 0}$, we define

$$\mathrm{Sec}(X/B, h) := \{\sigma : B \to X | \ \sigma_*[B] = (1, h) \in H^4(X, \mathbb{Z})\}$$

to be the space of smooth sections of $X \to B$ of class $(1, h)$. Here, the cohomology class $(1, h)$ means that its degree over B is 1 while its degree computed via $\mathcal{O}_X(1)$ is h. Then $\mathrm{Sec}(X/B)$ is the union of all $\mathrm{Sec}(X/B, h)$ for $h \in \mathbb{Z}^{\geq 0}$.

Remark 4.3. The space $\mathrm{Sec}(X/B, h)$ is equivalent to the space defined in [10] using the *height* of sections. Here, the height of a section $\sigma \in \mathrm{Sec}(X/B, h)$ is defined as $\deg(\sigma^* \omega_\pi^{-1})$, which only depends on h.

The space $\mathrm{Sec}(X/B, h)$ is quasi-projective with natural compactifications. In this paper, we regard $\mathrm{Sec}(X/B, h)$ as an open subset of the Hilbert scheme of X and denote by $\overline{\mathrm{Sec}}(X/B, h)$ the closure of $\mathrm{Sec}(X/B, h)$ in the Hilbert scheme parameterizing sections in $\mathcal{F}$.

Recall that we have a morphism induced by the Deligne cycle class map

$$\phi_h : \mathrm{Sec}(X/B, h) \to J(X)_{(1,h)} \tag{4.1}$$

with $\phi_h(\sigma) = [\sigma]$, which naturally extends to $\overline{\mathrm{Sec}}(X/B, h)$. Our first main result is:

Theorem 4.4. *For $h \gg 0$, the morphism $\phi_h : \mathrm{Sec}(X/B, h) \to J(X)_{(1,h)}$ is the composition of an open inclusion and a projective bundle morphism. In particular, the extended map $\bar{\phi}_h : \overline{\mathrm{Sec}}(X/B, h) \to J(X)_{(1,h)}$ is surjective with rationally connected general fibers.*

The proof relies on the standard argument of the "reduction to the discriminant argument" (cf. [10], §3). We now review this reduction and divide the proof into two steps:

Step 1. Denoting by $\mathrm{Sec}(\mathcal{F}/C)$ the space of sections on the ruled surface $\mathcal{F} \to C$, then there is a natural one-to-one map

$$\theta : \mathrm{Sec}(X/B) \to \mathrm{Sec}(\mathcal{F}/C), \tag{4.2}$$

since there are exactly two lines passing through a given point on a quadric surface.

Lemma 4.5. *The map* (4.2) *is an isomorphism.*

Proof. Given a section $\sigma : B \to X$, since the smooth fibers of $X \to B$ and $\mathrm{Res}_{\mathrm{C/B}}\mathcal{F} \to \mathrm{B}$ are isomorphic and the ambient spaces are smooth, the pullback γ^* and composition β_* of σ remains a section $\beta_*\gamma^*(\sigma) : B \to \mathrm{Res}_{\mathrm{C/B}}\mathcal{F}$ of ϖ. By representability of moduli functors, this actually gives a morphism

$$\begin{aligned} \rho : \mathrm{Sec}(X/B) &\longrightarrow \mathrm{Sec}(\mathrm{Res}_{\mathrm{C/B}}\mathcal{F}/\mathrm{C}) \\ \sigma &\longmapsto \beta_*\gamma^*(\sigma) \end{aligned} \tag{4.3}$$

induced from the natural transformation $\mathfrak{S}(X/B) \Longrightarrow \mathfrak{S}(\mathrm{Res}_{\mathrm{C/B}}\mathcal{F}/\mathrm{B})$ using the base change γ^* and composition β_*.

Similarly, we construct a natural transformation between two moduli functors $\mathfrak{S}(\mathrm{Res}_{\mathrm{C/B}}\mathcal{F}/\mathrm{B})$ and $\mathfrak{S}(\mathcal{F}/C)$ using the universal property of Weil restriction. Namely, for a family of sections

$$g : \Sigma \to S \times \mathrm{Res}_{\mathrm{C/B}}\mathcal{F},$$

over a scheme S, we get a unique map

$$\mathfrak{R}(g) : \Sigma \times_B C \to S \times \mathcal{F} \tag{4.4}$$

by the canonical isomorphism (3.8). Moreover, it is easy to see that (4.4) is a family of sections $\mathcal{F} \to C$ over S. This construction defines a natural transformation

$$\vartheta : \mathfrak{S}(\mathrm{Res}_{\mathrm{C/B}}\mathcal{F}/\mathrm{B}) \longrightarrow \mathfrak{S}(\mathcal{F}/\mathrm{C}). \tag{4.5}$$

Once again, we obtain a morphism

$$\theta' : \mathrm{Sec}(\mathrm{Res}_{\mathrm{C/B}}\mathcal{F}/\mathrm{B}) \to \mathrm{Sec}(\mathcal{F}/\mathrm{C})$$

from (4.5) because of the representability of two functors $\mathfrak{S}(\mathrm{Res}_{\mathrm{C/B}}\mathcal{F}/\mathrm{B})$ and $\mathfrak{S}(\mathcal{F}/C)$.

Then the map $\theta = \theta' \circ \rho$ is the composition of θ' and ρ, and hence is a morphism. Moreover, it is separated and bijective. As we work over the field $\mathbb{C}$ and $\mathrm{Sec}(\mathcal{F}/C)$ is smooth, then it is an isomorphism by Zariski's main theorem for quasi-finite morphisms. □

Considering a section $\delta : C \to \mathcal{F}$ as a curve on $\mathcal{F}$, we define

$$\mathrm{Sec}(\mathcal{F}/C, d) = \{\delta \in \mathrm{Sec}(\mathcal{F}/C) |\ \delta^2 = d\},$$

which is an irreducible component of $\mathrm{Sec}(\mathcal{F}/C)$. We have

$$\mathrm{Sec}(X/B, h) \xrightarrow{\sim} \mathrm{Sec}(\mathcal{F}/C, d)$$

via the isomorphism (3.3) for d satisfying

$$d = (1, h) \cdot c_1(X) - 2 + \frac{m}{2} - 2g(B). \tag{4.6}$$

Step 2. Before we proceed, for simplicity of notation, we use Remark 2.2 to modify our map ϕ_h as follows:

$$\begin{aligned} \phi_h : \mathrm{Sec}(X/B, h) &\to J(X) \\ \sigma &\mapsto [\sigma] - [\sigma_0] \end{aligned} \tag{4.7}$$

where we fix a section $\sigma_0 \in \mathrm{Sec}(X/B, h)$. Then it is equivalent to show that our first assertion holds for this refined map.

Furthermore, for any $\delta_0 \in \mathrm{Sec}(\mathcal{F}/C)$, Hassett and Tschinkel [10] have defined a map

$$(4.8) \qquad \begin{aligned} \widehat{\phi}_d : \mathrm{Sec}(\mathcal{F}/C,\, d) &\to J(C) \\ \delta \quad &\longmapsto \delta^*([\delta_0]) - D, \end{aligned}$$

where $D \in \mathrm{Pic}(C)$ is a given divisor with $\deg(D) = \delta^*([\delta_0])$. They have shown that this map $\widehat{\phi}_d$ is the composition of an open immersion and a projective bundle map for h sufficiently large.

Remark 4.6. We also recommend [16] to the readers for a more general construction for homogeneous fibrations.

The map $\widehat{\phi}_d$ is not canonical and depends on the choice of δ_0 and D. However, if we take $\delta_0 \in \mathrm{Sec}(\mathcal{F}/C, d)$ and $D = \delta_0^*([\delta_0])$, we claim that (4.8) is the same as the morphism ϕ_h up to an isomorphism $J(X) \cong J(C)$, that is,

Lemma 4.7. *The diagram*

$$(4.9) \qquad \begin{array}{ccc} \mathrm{Sec}(\mathcal{F}/C, d) & \xrightarrow{\widehat{\phi}_d} & J(C) \\ \downarrow{\scriptstyle \theta} & & \downarrow{\scriptstyle \psi_{\delta_0}} \\ \mathrm{Sec}(X/B, h) & \xrightarrow{\phi_h} & J(X) \end{array}$$

is commutative, where the isomorphisms θ and ψ_{δ_0} are given in Lemma 3.3 *and Lemma* 4.5*.*

Proof. To prove the assertion, we first show that the diagram (4.9) is commutative up to the translation by a two torsion element, i.e., there exists a two torsion element $x \in J(X)$ such that

$$\psi_{\delta_0} \circ \widehat{\phi}_d(\delta) = \phi_h \circ \theta(\delta) + x,$$

for all $\delta \in \mathrm{Sec}(\mathcal{F}/C, h)$.

Let us first describe $\psi_{\delta_0} \circ \widehat{\phi}_d(\delta)$ and $\phi_h \circ \theta(\delta)$ as cycle classes in $CH^2(X)$. Recall that there is a diagram

$$(4.10) \qquad \begin{array}{ccccc} F_\delta & \hookrightarrow & \mathcal{U} & \xrightarrow{p} & X \\ \downarrow & & \downarrow & & \\ C & \overset{\delta}{\hookrightarrow} & \mathcal{F} & & \end{array}$$

(similarly for F_{δ_0}). From the definition of ψ_{δ_0}, one can easily get

$$(4.11) \qquad \psi_{\delta_0}(\phi_d(\delta)) = [p_*(F_\delta \cdot F_{\delta_0} - F_{\delta_0}^2)] \in CH^2(X)$$

as a union of lines on X.

Next, for every $\delta \in \mathrm{Sec}(\mathcal{F}/C)$, one can view $F_\delta \subseteq \mathcal{U}$ as the incident variety on $C \times X$ given by

$$F_\delta = \{(c, z) | z \in \delta(c)\} \subseteq C \times X.$$

Observe that for another $\delta' \in \mathrm{Sec}(\mathcal{F}/C)$, there exists a natural section $D_\delta^{\delta'}$ of $F_\delta \to C$ given by

$$D_\delta^{\delta'} = \{(c, z) | z \in \delta(c) \cap \delta'(\iota(c))\} \subseteq F_\delta.$$

This is because on a smooth quadric surface, two lines passing though a point will meet the two lines passing through another $\delta_0(b)$ at exactly two points.

Let us write $D_\delta = D_\delta^\delta$. Then we have

$$[p_*(D_\delta^{\delta'})] = [p_*(D_{\delta'}^{\delta})] \quad \text{and} \quad [p_*(D_\delta)] = 2[\theta(\delta)] \tag{4.12}$$

from our construction. Coming back to the diagram (4.9), we can get

$$2(\phi_h \circ \theta(\delta)) = [p_*(D_\delta - D_{\delta_0})]. \tag{4.13}$$

Furthermore, let $f \in CH^2(\mathcal{U})$ be the fiber class of $\mathcal{U} \to \mathcal{F}$; then we have the following relations among these cycles in $CH^2(\mathcal{U})$:

(1) $D_{\delta_0} \equiv_{rat} D_{\delta_0}^{\delta} + (D_{\delta_0} \cdot D_{\delta_0}^{\delta} - D_{\delta_0}^2)f$ and $D_\delta \equiv_{rat} D_\delta^{\delta_0} + (D_\delta \cdot D_\delta^{\delta_0} - D_\delta^2)f$, where $D_\delta \cdot D_\delta^{\delta_0}$ (resp. $D_{\delta_0} \cdot D_{\delta_0}^{\delta}$) denotes the intersection number of D_δ (resp. D_{δ_0}) and $D_\delta^{\delta_0}$ (resp. $D_{\delta_0}^{\delta}$) in the ruled surface $\mathcal{F}_\delta$ (resp. $\mathcal{F}_{\delta_0}$).

(2) $(D_{\delta_0}^{\delta} \cdot D_{\delta_0})f \equiv_{rat} F_\delta \cdot F_{\delta_0} \equiv_{rat} (D_\delta^{\delta_0} \cdot D_\delta)f$.

(3) $F_\delta^2 \equiv_{rat} (D_\delta^2)f$ and $F_{\delta_0}^2 \equiv_{rat} (D_{\delta_0}^2)f$.

Here (1) follows easily from the geometry of the ruled surfaces, while (2) and (3) come from the definition of $D_\delta^{\delta'}$. Putting these together, we get

$$\begin{aligned} 2\phi_h \circ \theta(\delta) &= [p_*(D_\delta - D_{\delta_0})] \\ &= [p_*(D_\delta - D_\delta^{\delta_0})] + [p_*(D_{\delta_0}^{\delta} - D_{\delta_0})] \\ &= [p_*((D_\delta^2)f - (D_{\delta_0}^2)f_0)] \\ &= [p_*(F_\delta^2 - F_{\delta_0}^2)] \\ &= 2[p_*(F_\delta \cdot F_{\delta_0} - F_{\delta_0}^2)] = 2\psi_{\delta_0} \circ \widehat{\phi}_d(\delta). \end{aligned} \tag{4.14}$$

Therefore, $2(\phi_h \circ \theta - \psi_{\delta_0} \circ \widehat{\phi}_d) = 0$ which implies that the diagram is commutative up to some two-torsion element x_δ. Note that the two-torsion elements in $J(X)$ are discrete. This means the two-torsion element $x_\delta = x$ is constant because $\mathrm{Sec}(\mathcal{F}/C, d)$ is connected. Then one can conclude that the two-torsion x is 0 by specializing at the point $\delta = \delta_0$. Indeed, since the pullback and pushforward preserves the rational equivalence relation, we have

$$\phi_h \circ \theta(\delta_0) = 0 = \psi_{\delta_0} \circ \widehat{\phi}_d(\delta_0).$$

□

4.8. **Families with trivial monodromy.** If $X \to B$ is a smooth quadric bundle with trivial monodromy, then $H^4(X, \mathbb{Z}) \cong \mathbb{Z}^3$ and the intermediate Jacobian $J(X)$ is isomorphic to the direct sum of two copies of the Jacobian $J(B)$. As discussed in Remark 3.6, the discriminant curve C becomes a disjoint union of two curves B and $\iota^* B \cong B$.

For a class $\epsilon = (1, a, b) \in H^4(X, \mathbb{Z})$, we consider the natural (Abel-Jacobi) map

$$\mathrm{Sec}(X/B, \epsilon) \to J(X). \tag{4.15}$$

Note that $J(X)$ is isomorphic to $J(B) \times J(B)$. We can easily apply the same proof as Theorem 4.4 to show that the image of (4.15) dominates the diagonal $J(B) \hookrightarrow J(B) \times J(B)$ and the general fibers of $\mathrm{Sec}(X/B, \epsilon) \to J(B)$ are rational when a or b is large enough.

4.9. **Bisections on quadric fibrations.** Next, we will consider families of *special* bisections on $\pi : X \to B$. With the same notation as before, let $\Sigma = Sing(\pi^{-1}(\Delta))$ be the set of points where the morphism π is not smooth. We denote by m the number of points in Σ.

Let us denote by $\mathrm{Bse}(X/B, h)$ the space of bisections of class $(2, h)$ on X which are ramified over all the points in Σ. Note that such a bisection σ corresponds to a genus $g = \frac{m}{2} + 2g(B) - 1$ and bidegree $(2, h)$ curve passing through all points in Σ. One can view $\mathrm{Bse}(X/B, h)$ as an open subset of the Hilbert scheme which parameterizes curves on X of genus g, bidegree $(2, h)$ and passes through all the points in Σ. We denote by $\overline{\mathrm{Bse}}(X/B, h)$ the Zariski closure in this Hilbert scheme.

Remark 4.10. Note that if $B \cong \mathbb{P}^1$, then m is always greater than zero under the assumption of nontrivial monodromy.

Recall that we have a smooth family of quadric surfaces $Y \to C$ by (3.4). According to Remark 3.6, we have

$$\psi'_\delta : J(C) \xrightarrow{\sim} J(Y)$$

via the cylinder morphism Ψ'_δ. Similarly as in the nonsmooth case, we also have $\mathrm{Sec}(Y/C) \cong \mathrm{Sec}(\mathcal{F}/C)$.

Let $\zeta \in H^4(Y, \mathbb{Z})$ be an integral class and denote by $\mathrm{Sec}(Y/C, \zeta)$ the space of sections of class ζ, which is a component of $\mathrm{Sec}(\mathcal{F}/C)$. By the same argument as in the proof of Theorem 4.4, we get that the morphism

$$\phi_\zeta : \mathrm{Sec}(Y/C, \zeta) \to J(Y)_\zeta \tag{4.16}$$

induced by the cycle class map is the composition of an open immersion and a projective bundle map when $\zeta \cdot ([-K_Y]) \gg 0$.

Since $\mathrm{Sec}(Y/C, \zeta) \cong \mathrm{Bse}(X/B, h)$ via the pushforward η_* and pullback μ^*, where h is uniquely (up to the involution) determined by ζ, then we get

Theorem 4.11. *The morphism $\phi_{(2,h)} : \mathrm{Bse}(X/B, h) \to J(X)_{(2,h)}$ defined by the Deligne cycle class map is dominant and each fiber is rational. The extended map*

$$\overline{\phi}_{(2,h)} : \overline{\mathrm{Bse}}(X/B, h) \to J(X)_{(2,h)}$$

is surjective with rationally connected general fibers.

Proof. Still, we only need to show the first assertion. This comes from the commutativity of the following diagram:

$$\begin{array}{ccc} \mathrm{Sec}(Y/C, \zeta) & \xrightarrow{\phi_\zeta} & J(Y)_\zeta \\ \downarrow \simeq & & \downarrow \Lambda \\ \mathrm{Bse}(X/B, h) & \xrightarrow{\phi_{(2,h)}} & J(X)_{(2,h)} \end{array}$$

where the vertical arrows are induced from the pushforwad η_* and the pullback μ^*. As we explained in §4.8, the image of ϕ_ζ dominates the diagonal $J(C) \hookrightarrow J(C) \times J(C) \cong J(Y)$ (after twist), and it is easy to see that the diagonal maps isomorphically to $J(X)_{(2,h)}$ via the map Λ. According to §4.8, we know that the composition $\Lambda \circ \phi_\zeta$ and hence $\phi_{(2,h)}$ is dominant with rationally connected general fibers. This proves the claim. □

Then Theorem 1.1 follows from Theorem 4.4 and Theorem 4.11.

5. Proof of Theorem 1.2

Proof of Theorem 1.2. We assume that the general fiber of π admits a quadric fibration with nontrivial monodromy. Suppose $\pi : \mathfrak{X} \to T$ is smooth over the open subset $T_0 \subseteq T$. Then the section $\tilde{\alpha}$ of $R^4\pi_*\mathbb{Z}$ is trivial over T_0, and is determined by the cohomology class of bidegree (d_1, d_2) on the smooth fiber $\mathfrak{X}_t, t \in T_0$.

There exist two divisors $\mathcal{H}_1, \mathcal{H}_2$ on $\mathfrak{X}$ such that the restriction of $\mathcal{H}_i, i = 1, 2$ to the general fiber is respectively $i^*_{\mathfrak{X}_t}\mathcal{O}_{\mathbb{P}(V)}(1)$ and a fiber class. Then the restriction of the cycles $\mathcal{H}_1^2$ and $\mathcal{H}_1 \cdot \mathcal{H}_2$ induces two sections of $R^4\pi_*\mathbb{Z}$ of class $(k, 2)$ and $(2, 0)$ for some $k > 0$. As discussed in Corollary 2.6, one only needs to check that the condition $(*)$ holds for sections $\tilde{\alpha}$ of $R^4\pi_*\mathbb{Z}$ modulo the group spanned by $[H_1^2]$ and $[H_1H_2]$. Since $[H_1^2]$ and $[H_1H_2]$ are of class $(2, d)$ and $(2, 0)$, this allows us to assume that $\tilde{\alpha}$ is of class $(1, h)$ or $(2, h)$ for some $h \gg 0$.

Let us first consider the case where the section $\tilde{\alpha}$ is of class $(1, h)$. Let g be the genus of the curve B. Take $\mathcal{M}_{\tilde{\alpha}}$ to be a desingularization of the relative Hilbert scheme of genus g curves of bidegree $(1, h)$ in $\mathfrak{X}|_{T_0}$ and let $\mathcal{Z}_\alpha \subseteq \mathcal{M}_{\tilde{\alpha}} \times_T \mathfrak{X}$ be the pullback of the universal family. By Theorem 4.4, the map

$$\phi_{\mathcal{Z}_{\tilde{\alpha}}} : \mathcal{M}_{\tilde{\alpha}} \to \mathcal{J}_{\tilde{\alpha}}(\mathfrak{X})$$

is surjective, and general fibers are rationally connected.

Similarly, for a section $\tilde{\alpha}$ of bidegree $(2, h)$, we choose $\mathcal{M}_{\tilde{\alpha}}$ to be a desingularization of the relative Hilbert scheme of genus $\frac{m-2}{2}$ curves in $\mathfrak{X}|_{T_0}$ of bidegree $(2, h)$, passing through all the vertices of the quadric cones in $\mathfrak{X}_t$. By Theorem 4.11, if we take $\mathcal{Z}_{\tilde{\alpha}}$ to be the pullback of the universal family, the induced morphism

$$\phi_{\mathcal{Z}_{\tilde{\alpha}}} : \mathcal{M}_{\tilde{\alpha}} \to \mathcal{J}_{\tilde{\alpha}}(\mathfrak{X})$$

is surjective with rationally connected general fibers. This completes the proof. □

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References

[1] M. F. Atiyah and F. Hirzebruch, *Analytic cycles on complex manifolds*, Topology **1** (1962), 25–45. MR0145560 (26 #3091)

[2] S. Bloch and V. Srinivas, *Remarks on correspondences and algebraic cycles*, Amer. J. Math. **105** (1983), no. 5, 1235–1253, DOI 10.2307/2374341. MR714776 (85i:14002)

[3] Siegfried Bosch, Werner Lütkebohmert, and Michel Raynaud, *Néron models*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 21, Springer-Verlag, Berlin, 1990. MR1045822 (91i:14034)

[4] Jean-Louis Colliot-Thélène and Manuel Ojanguren, *Variétés unirationnelles non rationnelles: au-delà de l'exemple d'Artin et Mumford* (French), Invent. Math. **97** (1989), no. 1, 141–158, DOI 10.1007/BF01850658. MR999316 (90m:14012)

[5] Jean-Louis Colliot-Thélène and Claire Voisin, *Cohomologie non ramifiée et conjecture de Hodge entière* (French, with English and French summaries), Duke Math. J. **161** (2012), no. 5, 735–801, DOI 10.1215/00127094-1548389. MR2904092

[6] A. Conte and J. P. Murre, *The Hodge conjecture for fourfolds admitting a covering by rational curves*, Math. Ann. **238** (1978), no. 1, 79–88, DOI 10.1007/BF01351457. MR510310 (80a:14018)

[7] Phillip A. Griffiths, *On the periods of certain rational integrals. I, II*, Ann. of Math. (2) 90 (1969), 460-495; ibid. (2) **90** (1969), 496–541. MR0260733 (41 #5357)

[8] Alexander Grothendieck, *Fondements de la géométrie algébrique. [Extraits du Séminaire Bourbaki, 1957–1962.]* (French), Secrétariat mathématique, Paris, 1962. MR0146040 (26 #3566)

[9] Joe Harris and Loring W. Tu, *On symmetric and skew-symmetric determinantal varieties*, Topology **23** (1984), no. 1, 71–84, DOI 10.1016/0040-9383(84)90026-0. MR721453 (85c:14032)

[10] Brendan Hassett and Yuri Tschinkel, *Spaces of sections of quadric surface fibrations over curves*, Compact moduli spaces and vector bundles, Contemp. Math., vol. 564, Amer. Math. Soc., Providence, RI, 2012, pp. 227–249, DOI 10.1090/conm/564/11161. MR2894638

[11] Brendan Hassett, Anthony Várilly-Alvarado, and Patrick Varilly, *Transcendental obstructions to weak approximation on general K3 surfaces*, Adv. Math. **228** (2011), no. 3, 1377–1404, DOI 10.1016/j.aim.2011.06.017. MR2824558 (2012i:14025)

[12] A. N. Tjurin, *Five lectures on three-dimensional varieties* (Russian), Uspehi Mat. Nauk **27** (1972), no. 5, (167), 3–50. MR0412196 (54 #323)

[13] Claire Voisin, *Hodge theory and complex algebraic geometry. I*, Reprint of the 2002 English edition, Cambridge Studies in Advanced Mathematics, vol. 76, Cambridge University Press, Cambridge, 2007. Translated from the French by Leila Schneps. MR2451566 (2009j:32014)

[14] Claire Voisin, *Some aspects of the Hodge conjecture*, Jpn. J. Math. **2** (2007), no. 2, 261–296, DOI 10.1007/s11537-007-0639-x. MR2342587 (2008g:14012)

[15] Claire Voisin, *Abel-Jacobi map, integral Hodge classes and decomposition of the diagonal*, J. Algebraic Geom. **22** (2013), no. 1, 141–174, DOI 10.1090/S1056-3911-2012-00597-9. MR2993050

[16] Yi Zhu, *Homogeneous Fibrations over Curves*, ProQuest LLC, Ann Arbor, MI, 2012. Thesis (Ph.D.)–State University of New York at Stony Brook. MR3067976

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