Motivic hyper-Kähler resolution conjecture
I: Generalized Kummer varieties

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Given a smooth projective variety M endowed with a faithful action of a finite group G, following Jarvis–Kaufmann–Kimura (Invent. Math. 168 (2007) 23–81), and Fantechi–Göttsche (Duke Math. J. 117 (2003) 197–227), we define the orbifold motive (or Chen–Ruan motive) of the quotient stack $[M/G]$ as an algebra object in the category of Chow motives. Inspired by Ruan (Contemp. Math. 312 (2002) 187–233), one can formulate a motivic version of his cohomological hyper-Kähler resolution conjecture (CHRC). We prove this motivic version, as well as its K–theoretic analogue conjectured by Jarvis–Kaufmann–Kimura in loc. cit., in two situations related to an abelian surface A and a positive integer n. Case (A) concerns Hilbert schemes of points of A: the Chow motive of $A^{[n]}$ is isomorphic as algebra objects, up to a suitable sign change, to the orbifold motive of the quotient stack $[A^n/\mathfrak{S}_n]$. Case (B) concerns generalized Kummer varieties: the Chow motive of the generalized Kummer variety $K_n(A)$ is isomorphic as algebra objects, up to a suitable sign change, to the orbifold motive of the quotient stack $[A_0^{n+1}/\mathfrak{S}_{n+1}]$, where A_0^{n+1} is the kernel abelian variety of the summation map $A^{n+1} \to A$. As a by-product, we prove the original cohomological hyper-Kähler resolution conjecture for generalized Kummer varieties. As an application, we provide multiplicative Chow–Künneth decompositions for Hilbert schemes of abelian surfaces and for generalized Kummer varieties. In particular, we have a multiplicative direct sum decomposition of their Chow rings with rational coefficients, which is expected to be the splitting of the conjectural Bloch–Beilinson–Murre filtration. The existence of such a splitting for holomorphic symplectic varieties is conjectured by Beauville (London Math. Soc. Lecture Note Ser. 344 (2007) 38–53). Finally, as another application, we prove that over a nonempty Zariski open subset of the base, there exists a decomposition isomorphism $R\pi_*\mathbb{Q} \simeq \bigoplus R^i \pi_*\mathbb{Q}[-i]$ in $D_c^b(B)$ which is compatible with the cup products on both sides, where $\pi \colon \mathcal{K}_n(\mathcal{A}) \to B$ is the relative generalized Kummer variety associated to a (smooth) family of abelian surfaces $\mathcal{A} \to B$.

1 Introduction

1.1 Motivation 1: Ruan's hyper-Kähler resolution conjectures

In [19], Weimin Chen and Yongbin Ruan construct the orbifold cohomology ring $H^*_{\mathrm{orb}}(\mathcal{X})$ for any complex orbifold $\mathcal{X}$. As a $\mathbb{Q}$–vector space, it is defined to be the cohomology of its inertia variety $H^*(I\mathcal{X})$ (with degrees shifted by some rational numbers called *age*), but is endowed with a highly nontrivial ring structure coming from moduli spaces of curves mapping to X. An algebrogeometric treatment is contained in Abramovich–Graber–Vistoli [1], based on the construction of the moduli stack of twisted stable maps in Abramovich–Vistoli [2]. In the global quotient case,[1] some equivalent definitions are available: see for example Fantechi–Göttsche [26], Jarvis–Kaufmann–Kimura [36], Kimura [38] and Section 2.

Originating from the topological string theory of orbifolds in Dixon–Harvey–Vafa–Witten [23; 24], one observes that the stringy topological invariants of an orbifold, e.g. the orbifold Euler number and the orbifold Hodge numbers, should be related to the corresponding invariants of a crepant resolution (Batyrev [4], Batyrev–Dais [5], Yasuda [63] and Lupercio–Poddar [42]). A much deeper relation was brought forward by Ruan, who made, among others, the following cohomological hyper-Kähler resolution conjecture (CHRC) in [51]. For more general and sophisticated versions of this conjecture, see Ruan [52], Bryan–Graber [15] and Coates–Ruan [20].

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[1] In this paper, by "global quotient", we always mean the quotient of a smooth projective variety by a *finite* group.

Conjecture 1.1 (Ruan's CHRC) *Let $\mathcal{X}$ be a compact complex orbifold with underlying variety X being Gorenstein. If there is a crepant resolution $Y \to X$ with Y being hyper-Kähler, then we have an isomorphism of graded commutative $\mathbb{C}$–algebras, $H^*(Y, \mathbb{C}) \simeq H^*_{\mathrm{orb}}(\mathcal{X}, \mathbb{C})$.*

As the construction of the orbifold product can be expressed using algebraic correspondences (see Abramovich–Graber–Vistoli [1] and Section 2), one has the analogous definition of the orbifold Chow ring $\mathrm{CH}_{\mathrm{orb}}(\mathcal{X})$ (see Definition 2.7 for the global quotient case) of a smooth proper Deligne–Mumford stack $\mathcal{X}$. Motivated by the study of algebraic cycles on hyper-Kähler varieties, we propose to investigate the Chow-theoretic analogue of Conjecture 1.1. For reasons which will become clear shortly, it is more powerful and fundamental to consider the following *motivic* version of Conjecture 1.1. Let $\mathrm{CHM}_{\mathbb{C}}$ be the category of Chow motives with complex coefficients and $\mathfrak{h}$ be the (contravariant) functor that associates to a smooth projective variety its Chow motive.

Metaconjecture 1.2 (MHRC) *Let $\mathcal{X}$ be a smooth proper complex Deligne–Mumford stack with underlying coarse moduli space X being a (singular) symplectic variety. If there is a symplectic resolution $Y \to X$, then we have an isomorphism $\mathfrak{h}(Y) \simeq \mathfrak{h}_{\mathrm{orb}}(\mathcal{X})$ of commutative algebra objects in $\mathrm{CHM}_{\mathbb{C}}$, hence in particular an isomorphism of graded $\mathbb{C}$–algebras, $\mathrm{CH}^*(Y)_{\mathbb{C}} \simeq \mathrm{CH}^*_{\mathrm{orb}}(\mathcal{X})_{\mathbb{C}}$.*

See Definition 3.1 for generalities on symplectic singularities and symplectic resolutions. The reason it is only a *meta*conjecture is that the definition of orbifold Chow motive for a smooth proper Deligne–Mumford stack in general is not available in the literature and we will not develop the theory in this generality in this paper (see however Remark 2.10). From now on, let us restrict ourselves to the case where the Deligne–Mumford stack in question is of the form of a global quotient $\mathcal{X} = [M/G]$, where M is a smooth projective variety with a faithful action of a finite group G, in which case we will define the *orbifold Chow motive* $\mathfrak{h}_{\mathrm{orb}}(\mathcal{X})$ in a very explicit way in Definition 2.5.

The *motivic hyper-Kähler resolution conjecture* that we are interested in is the following more precise statement, which would contain all situations considered in this paper and its sequel.

Conjecture 1.3 (MHRC: global quotient case) *Let M be a smooth projective holomorphic symplectic variety equipped with a faithful action of a finite group G by symplectic automorphisms of M. If Y is a symplectic resolution of the quotient*

variety M/G, then we have an isomorphism of (commutative) algebra objects in the category of Chow motives with **complex** coefficients,

$$\mathfrak{h}(Y) \simeq \mathfrak{h}_{\mathrm{orb}}([M/G]) \quad \text{in} \ \mathrm{CHM}_{\mathbb{C}} \,.$$

In particular, we have an isomorphism of graded $\mathbb{C}$–algebras,

$$\mathrm{CH}^*(Y)_{\mathbb{C}} \simeq \mathrm{CH}^*_{\mathrm{orb}}([M/G])_{\mathbb{C}}.$$

The definition of the *orbifold motive* of $[M/G]$ as a *(commutative) algebra object* in the category of Chow motives with rational coefficients[2] is particularly down-to-earth; it is the G–invariant subalgebra object of some explicit algebra object:

$$\mathfrak{h}_{\mathrm{orb}}([M/G]) := \left(\bigoplus_{g \in G} \mathfrak{h}(M^g)(-\operatorname{age}(g)), \star_{\mathrm{orb}} \right)^{G},$$

where M^g is the subvariety of fixed points of g, for each $g \in G$, and the orbifold product $\star_{\mathrm{orb}}$ is defined by using natural inclusions and Chern classes of normal bundles of various fixed loci; see Definition 2.5 (or (2)) for the precise formula of $\star_{\mathrm{orb}}$ as well as the Tate twists by age (Definition 2.3) and the G–action. The orbifold Chow ring[3] is then defined as the commutative algebra

$$\mathrm{CH}^*_{\mathrm{orb}}([M/G]) := \bigoplus_i \operatorname{Hom}_{\mathrm{CHM}}(\mathbb{1}(-i), \mathfrak{h}_{\mathrm{orb}}([M/G])),$$

or, equivalently and more explicitly,

$$(1) \qquad \mathrm{CH}^*_{\mathrm{orb}}([M/G]) := \left(\bigoplus_{g \in G} \mathrm{CH}^{*-\operatorname{age}(g)}(M^g), \star_{\mathrm{orb}} \right)^{G},$$

where $\star_{\mathrm{orb}}$ is defined as follows: for two elements $g, h \in G$, the orbifold product of $\alpha \in \mathrm{CH}^{i-\operatorname{age}(g)}(M^g)$ and $\beta \in \mathrm{CH}^{j-\operatorname{age}(h)}(M^h)$ is the following element in $\mathrm{CH}^{i+j-\operatorname{age}(gh)}(M^{gh})$:

$$(2) \qquad \alpha \star_{\mathrm{orb}} \beta := \iota_*(\alpha|_{M^{<g,h>}} \cdot \beta|_{M^{<g,h>}} \cdot c_{\mathrm{top}}(F_{g,h})),$$

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[2]Strictly speaking, the orbifold Chow motive of $[M/G]$ in general lives in the larger category of Chow motives with *fractional* Tate twists. However, in our cases of interest, namely when there exists a crepant resolution, for the word "crepant resolution" to make sense we understand that the underlying singular variety M/G is at least Gorenstein, in which case all age shiftings are integers and we stay in the usual category of Chow motives. See Definitions 2.1 and 2.5 for the general notions.

[3]The definition of the orbifold Chow ring already appeared on page 211 of Fantechi–Göttsche [26] and was proved to be equivalent to the construction in Abramovich–Graber–Vistoli [1] by Jarvis–Kaufmann–Kimura [36] .

where $M^{<g,h>} := M^g \cap M^h$, the map $\iota\colon M^{<g,h>} \hookrightarrow M^{gh}$ is the natural inclusion and $F_{g,h}$ is the obstruction bundle. This construction is completely parallel to the construction of orbifold cohomology due to Fantechi–Göttsche [26], which is further simplified in Jarvis–Kaufmann–Kimura [36].

With the orbifold Chow theory briefly reviewed above, we see that in Conjecture 1.3, the fancy side of $[M/G]$ is actually the easier side, which can be used to study the motive and cycles of the hyper-Kähler variety Y. Let us turn this idea into the following working principle, which will be illustrated repeatedly in examples in the rest of the introduction.

Slogan *The cohomology theories[4] of a holomorphic symplectic variety can be understood via the hidden stack structure of its singular symplectic models.*

Interesting examples of symplectic resolutions appear when considering the Hilbert–Chow morphism of a smooth projective surface. More precisely, in his fundamental paper [7], Beauville provides such examples:

Example 1 (Beauville) Let S be a complex projective K3 surface or an abelian surface. Its Hilbert scheme of length-n subschemes, denoted by $S^{[n]}$, is a symplectic crepant resolution of the symmetric product $S^{(n)}$ via the Hilbert–Chow morphism.

The corresponding cohomological hyper-Kähler resolution conjecture was proved independently by Fantechi–Göttsche [26] and Uribe [54], making use of the work of Lehn–Sorger [41] computing the ring structure of $H^*(S^{[n]})$. The motivic hyper-Kähler resolution conjecture (Conjecture 1.3) in the case of K3 surfaces will be proved in Fu–Tian [30] and the case of abelian surfaces is our first main result:

Theorem 1.4 (MHRC for $A^{[n]}$) *Let A be an abelian surface and $A^{[n]}$ be its Hilbert scheme as before. Then we have an isomorphism of commutative algebra objects in the category* CHM *of Chow motives with rational coefficients,*

$$\mathfrak{h}(A^{[n]}) \simeq \mathfrak{h}_{\mathrm{orb,dt}}([A^n/\mathfrak{S}_n]),$$

where on the left-hand side, the product structure is given by the small diagonal of $A^{[n]} \times A^{[n]} \times A^{[n]}$ while on the right-hand side, the product structure is given by the orbifold product $\star_{\mathrm{orb}}$ with a suitable sign change, called **discrete torsion** *in Definition 3.5.*

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[4]In the large sense: Weil cohomology, Chow rings, K–theory, motivic cohomology, etc. and finally, motives.

In particular, we have an isomorphism of commutative graded $\mathbb{Q}$–algebras,

$$(3) \qquad\qquad \mathrm{CH}^*(A^{[n]})_{\mathbb{Q}} \simeq \mathrm{CH}^*_{\mathrm{orb,dt}}([A^n/\mathfrak{S}_n]).$$

Example 2 (Beauville) Let A be a complex abelian surface. The composition of the Hilbert–Chow morphism followed by the summation map $A^{[n+1]} \to A^{(n+1)} \to A$ is an isotrivial fibration. The *generalized Kummer variety* $K_n(A)$ is by definition the fiber of this morphism over the origin of A. It is a hyper-Kähler resolution of the quotient $A_0^{n+1}/\mathfrak{S}_{n+1}$, where A_0^{n+1} is the kernel abelian variety of the summation map $A^{n+1} \to A$.

The second main result of the paper is the following theorem confirming the motivic hyper-Kähler resolution conjecture (Conjecture 1.3) in this situation:

Theorem 1.5 (MHRC for $K_n(A)$) *Let $K_n(A)$ be the $2n$–dimensional generalized Kummer variety associated to an abelian surface A. Let $A_0^{n+1} := \mathrm{Ker}(+\colon A^{n+1} \to A)$ endowed with the natural $\mathfrak{S}_{n+1}$–action. Then we have an isomorphism of commutative algebra objects in the category* CHM *of Chow motives with rational coefficients,*

$$\mathfrak{h}(K_n(A)) \simeq \mathfrak{h}_{\mathrm{orb,dt}}([A_0^{n+1}/\mathfrak{S}_{n+1}]),$$

where on the left-hand side, the product structure is given by the small diagonal while on the right-hand side, the product structure is given by the orbifold product $\star_{\mathrm{orb}}$ with the sign change given by discrete torsion in Definition 3.5. In particular, we have an isomorphism of commutative graded $\mathbb{Q}$–algebras,

$$(4) \qquad\qquad \mathrm{CH}^*(K_n(A))_{\mathbb{Q}} \simeq \mathrm{CH}^*_{\mathrm{orb,dt}}([A_0^{n+1}/\mathfrak{S}_{n+1}]).$$

1.2 Consequences

We get some by-products of our main results.

Taking the Betti cohomological realization, we confirm Ruan's original cohomological hyper-Kähler resolution conjecture (Conjecture 1.1) in the case of generalized Kummer varieties:

Theorem 1.6 (CHRC for $K_n(A)$) *Let the notation be as in Theorem 1.5. We have an isomorphism of graded commutative $\mathbb{Q}$–algebras,*

$$H^*(K_n(A))_{\mathbb{Q}} \simeq H^*_{\mathrm{orb,dt}}([A_0^{n+1}/\mathfrak{S}_{n+1}]).$$

The CHRC has never been proved in the case of generalized Kummer varieties in the literature. Related work on the CHRC in this case are Nieper-Wißkirchen's description of the cohomology ring $H^*(K_n(A), \mathbb{C})$ in [47], which plays an important role in our proof; and Britze's thesis [14] comparing $H^*(A \times K_n(A), \mathbb{C})$ and the computation of the orbifold cohomology ring of $[A \times A_0^{n+1}/\mathfrak{S}_{n+1}]$ in Fantechi–Göttsche [26]. See however Remark 6.16.

From the K–theoretic point of view, we also have the following closely related conjecture (KHRC) in Jarvis–Kaufmann–Kimura [36, Conjecture 1.2], where the *orbifold K–theory* is defined in a similar way, with the top Chern class in (2) replaced by the K–theoretic Euler class; see Definition 2.8 for details.

Conjecture 1.7 (K–theoretic hyper-Kähler resolution conjecture [36]) *In the same situation as in Metaconjecture 1.2, we have isomorphisms of $\mathbb{C}$–algebras,*

$$K_0(Y)_{\mathbb{C}} \simeq K_{\mathrm{orb}}(\mathcal{X})_{\mathbb{C}} \quad and \quad K^{\mathrm{top}}(Y)_{\mathbb{C}} \simeq K_{\mathrm{orb}}^{\mathrm{top}}(\mathcal{X})_{\mathbb{C}}.$$

Using Theorems 1.4 and 1.5, we can confirm Conjecture 1.7 in the two cases considered here:

Theorem 1.8 (KHRC for $A^{[n]}$ and $K_n(A)$) *Let A be an abelian surface and n be a natural number. There are isomorphisms of commutative $\mathbb{C}$–algebras,*

$$K_0(A^{[n]})_{\mathbb{C}} \simeq K_{\mathrm{orb}}([A^n/\mathfrak{S}_n])_{\mathbb{C}};$$
$$K^{\mathrm{top}}(A^{[n]})_{\mathbb{C}} \simeq K_{\mathrm{orb}}^{\mathrm{top}}([A^n/\mathfrak{S}_n])_{\mathbb{C}};$$
$$K_0(K_n(A))_{\mathbb{C}} \simeq K_{\mathrm{orb}}([A_0^{n+1}/\mathfrak{S}_{n+1}])_{\mathbb{C}};$$
$$K^{\mathrm{top}}(K_n(A))_{\mathbb{C}} \simeq K_{\mathrm{orb}}^{\mathrm{top}}([A_0^{n+1}/\mathfrak{S}_{n+1}])_{\mathbb{C}}.$$

1.3 On explicit descriptions of the Chow rings

Let us make some remarks on the way we understand Theorem 1.4 and Theorem 1.5. For each of them, the seemingly fancy right-hand side of (3) and (4) given by the orbifold Chow ring is actually very concrete (see (1)): as groups, since all fixed loci are just various diagonals, they are direct sums of Chow groups of products of the abelian surface A, which can be handled by Beauville's decomposition of Chow rings of abelian varieties [8]; while the ring structures are given by the orbifold product, which is extremely simplified in our cases (see (2)): all obstruction bundles $F_{g,h}$ are trivial and hence the orbifold products are either the intersection product pushed forward by inclusions or simply zero.

In short, given an abelian surface A, Theorem 1.4 and Theorem 1.5 provide an explicit description of the Chow rings of $A^{[n]}$ and of $K_n(A)$ in terms of Chow rings of products of A (together with some combinatoric rules specified by the orbifold product). To illustrate how explicit it is, we work out two simple examples in Section 3.2: the Chow ring of the Hilbert square of a K3 surface or an abelian surface and the Chow ring of the Kummer K3 surface associated to an abelian surface.

1.4 Motivation 2: Beauville's splitting property

The original motivation for the authors to study the motivic hyper-Kähler resolution conjecture (Metaconjecture 1.2) was to understand the (rational) Chow rings, or more generally the Chow motives, of smooth projective holomorphic symplectic varieties, that is, of even-dimensional projective manifolds carrying a holomorphic 2–form which is symplectic (i.e. nondegenerate at each point). As an attempt to unify his work on algebraic cycles on abelian varieties [8] and his result with Voisin [11] on Chow rings of K3 surfaces, Beauville conjectured in [10], under the name of *the splitting property*, that for a smooth projective holomorphic symplectic variety X, there exists a canonical multiplicative splitting of the conjectural Bloch–Beilinson–Murre filtration of the rational Chow ring (see Conjecture 7.1 for the precise statement). In this paper, we will understand the splitting property as in the following motivic version (see Definition 7.2 and Conjecture 7.4):

Conjecture 1.9 (Beauville's splitting property: motives) *Let X be a smooth projective holomorphic symplectic variety of dimension $2n$. Then we have a canonical multiplicative Chow–Künneth decomposition of $\mathfrak{h}(X)$ of Bloch–Beilinson type, that is, a direct sum decomposition in the category of rational Chow motives,*

$$\text{(5)} \qquad\qquad \mathfrak{h}(X) = \bigoplus_{i=0}^{4n} \mathfrak{h}^i(X),$$

satisfying the following properties:

(i) **Chow–Künneth** *The cohomology realization of the decomposition gives the Künneth decomposition: $H^*(\mathfrak{h}^i(X)) = H^i(X)$, for each $0 \le i \le 4n$.*

(ii) **Multiplicativity** *The product $\mu\colon \mathfrak{h}(X) \otimes \mathfrak{h}(X) \to \mathfrak{h}(X)$ given by the small diagonal $\delta_X \subset X \times X \times X$ respects the decomposition: the restriction of μ on the summand $\mathfrak{h}^i(X) \otimes \mathfrak{h}^j(X)$ factorizes through $\mathfrak{h}^{i+j}(X)$.*

(iii) **Bloch–Beilinson–Murre**   *For any $i, j \in \mathbb{N}$,*

-   $\mathrm{CH}^i(\mathfrak{h}^j(X)) = 0$ *if $j < i$;*
-   $\mathrm{CH}^i(\mathfrak{h}^j(X)) = 0$ *if $j > 2i$;*
-   *the realization induces an injective map*

$$\mathrm{Hom}_{\mathrm{CHM}}(\mathbb{1}(-i), \mathfrak{h}^{2i}(X)) \to \mathrm{Hom}_{\mathbb{Q}-HS}(\mathbb{Q}(-i), H^{2i}(X)).$$

Such a decomposition naturally induces a (multiplicative) bigrading on the Chow ring $\mathrm{CH}^*(X) = \bigoplus_{i,s} \mathrm{CH}^i(X)_s$ by setting

(6) $$\mathrm{CH}^i(X)_s := \mathrm{Hom}_{\mathrm{CHM}}(\mathbb{1}(-i), \mathfrak{h}^{2i-s}(X)),$$

which is the original splitting that Beauville envisaged.

Our main results Theorem 1.4 and Theorem 1.5 allow us, for X being a Hilbert scheme of an abelian surface or a generalized Kummer variety, to achieve in Theorem 1.10 partially the goal Conjecture 1.9: we construct the candidate direct sum decomposition (5) satisfying conditions (i) and (ii) in Conjecture 1.9, namely a *self-dual multiplicative Chow–Künneth decomposition* in the sense of Shen–Vial [53] (see Definition 7.2). The remaining condition (iii) on Bloch–Beilinson–Murre properties is very much related to Beauville's *weak splitting property*, which has already been proved in Fu [29] for the case of generalized Kummer varieties; see Beauville [10], Voisin [58], Yin [64] and Rieß [50] for the complete story and more details.

Theorem 1.10 (Theorem 7.9 and Proposition 7.13) *Let A be an abelian surface and n be a positive integer. Let X be the corresponding $2n$–dimensional Hilbert scheme $A^{[n]}$ or generalized Kummer variety $K_n(A)$. Then X has a canonical self-dual multiplicative Chow–Künneth decomposition induced by the isomorphisms of Theorems 1.4 and 1.5, respectively. Moreover, via the induced canonical multiplicative bigrading on the (rational) Chow ring given in (6), the ith Chern class of X lies in $\mathrm{CH}^i(X)_0$ for any i.*

The associated filtration $F^j \mathrm{CH}^i(X) := \bigoplus_{s \geq j} \mathrm{CH}^i(X)_s$ is supposed to satisfy the Bloch–Beilinson–Murre conjecture (see Conjecture 7.11). We point out in Remark 7.12 that Conjecture 7.5 (Beauville's conjecture on abelian varieties) implies for X in our two cases some Bloch–Beilinson–Murre properties: $\mathrm{CH}^*(X)_s = 0$ for $s < 0$ and the cycle class map restricted to $\mathrm{CH}^*(X)_0$ is injective.

See Remark 7.10 for previous related results.

1.5   Cup products versus decomposition theorem

For a smooth projective morphism $\pi\colon \mathcal{X} \to B$ Deligne shows in [21] that one has an isomorphism

$$R\pi_*\mathbb{Q} \cong \bigoplus_i R^i\pi_*\mathbb{Q}[-i],$$

in the derived category of sheaves of $\mathbb{Q}$–vector spaces on B. Voisin [59] shows that, although this isomorphism cannot in general be made compatible with the product structures on both sides, not even after shrinking B to a Zariski open subset, it can be made so if π is a smooth family of projective K3 surfaces. Her result is extended in Vial [55] to relative Hilbert schemes of finite lengths of a smooth family of projective K3 surfaces or abelian surfaces. As a by-product of our main result in this paper, we can similarly prove the case of generalized Kummer varieties.

Theorem 1.11 (Corollary 8.4) *Let* $\mathcal{A} \to B$ *be an abelian surface over* B. *Consider* $\pi\colon K_n(\mathcal{A}) \to B$, *the relative generalized Kummer variety. Then there exist a decomposition isomorphism*

(7) $$R\pi_*\mathbb{Q} \cong \bigoplus_i R^i\pi_*\mathbb{Q}[-i]$$

and a nonempty Zariski open subset U *of* B *such that this decomposition becomes multiplicative for the restricted family over* U.

Conventions and notation   Throughout the paper, all varieties are defined over the field of complex numbers.

- The notation CH (resp. $\mathrm{CH}_{\mathbb{C}}$) means Chow groups with *rational* (resp. *complex*) coefficients. CHM is the category of Chow motives over the complex numbers with *rational* coefficients.

- For a variety X, its small diagonal, always denoted by δ_X, is $\{(x, x, x) \mid x \in X\} \subset X \times X \times X$.

- For a smooth surface X, its Hilbert scheme of length-n subschemes is always denoted by $X^{[n]}$. It is smooth of dimension $2n$ by Fogarty [27].

- An (even) dimensional smooth projective variety is *holomorphic symplectic* if it has a holomorphic symplectic (i.e. nondegenerate at each point) 2–form. When talking about resolutions, we tend to use the word *hyper-Kähler* as its synonym, which usually (but not in this paper) requires also "irreducibility", that is, the simple connectedness of

the variety and the uniqueness up to scalars of the holomorphic symplectic 2–form. In particular, punctual Hilbert schemes of abelian surfaces are examples of holomorphic symplectic varieties.

• An abelian variety is always supposed to be connected. Its nonconnected generalization causes extra difficulty and is dealt with in Section 6.2.

• When working with 0–cycles on an abelian variety A, to avoid confusion, for a collection of points $x_1, \ldots, x_m \in A$, we will write $[x_1] + \cdots + [x_m]$ for the 0–cycle of degree m (or equivalently, a point in $A^{(m)}$, the mth symmetric product of A) and $x_1 + \cdots + x_m$ will stand for the usual sum using the group law of A, which is therefore a point in A.

2 Orbifold motives and orbifold Chow rings

To fix the notation, we start by a brief reminder of the construction of pure motives (see [3]). In order to work with Tate twists by age functions (Definition 2.3), we have to extend slightly the usual notion of pure motives by allowing twists by a rational number.

Definition 2.1 (Chow motives with fractional Tate twists) The category of *Chow motives with fractional Tate twists* with rational coefficients, denoted by $\widetilde{\mathrm{CHM}}$, has as objects finite direct sums of triples of the form (X, p, n), with X a connected smooth projective variety, $p \in \mathrm{CH}^{\dim X}(X \times X)$ a projector and $n \in \mathbb{Q}$ a *rational* number. Given two objects (X, p, n) and (Y, q, m), the morphism space between them consists of correspondences:

$$\mathrm{Hom}_{\widetilde{\mathrm{CHM}}}((X, p, n), (Y, q, m)) := q \circ \mathrm{CH}^{\dim X + m - n}(X \times Y) \circ p,$$

where we simply impose that all Chow groups of a variety with noninteger codimension are zero. The composition law of correspondences is the usual one. Identifying $(X, p, n) \oplus (Y, q, n)$ with $(X \amalg Y, p \amalg q, n)$ makes $\widetilde{\mathrm{CHM}}$ a $\mathbb{Q}$–linear category. Moreover, $\widetilde{\mathrm{CHM}}$ is a rigid symmetric monoidal pseudoabelian category with unit $\mathbb{1} := (\mathrm{Spec}\,\mathbb{C}, \mathrm{Spec}\,\mathbb{C}, 0)$, tensor product defined by $(X, p, n) \otimes (Y, q, m) := (X \times Y, p \times q, n + m)$ and duality given by $(X, p, n)^{\vee} := (X, {}^{t}p, \dim X - n)$. There is a natural contravariant functor $\mathfrak{h}\colon \mathrm{SmProj}^{\mathrm{op}} \to \widetilde{\mathrm{CHM}}$ sending a smooth projective variety X to its Chow motive $\mathfrak{h}(X) = (X, \Delta_X, 0)$ and a morphism $f\colon X \to Y$ to its transposed graph ${}^{t}\Gamma_f \in \mathrm{CH}^{\dim Y}(Y \times X) = \mathrm{Hom}_{\widetilde{\mathrm{CHM}}}(\mathfrak{h}(Y), \mathfrak{h}(X))$.

Remarks 2.2 Some general remarks are in order:

(i) The category $\widetilde{\mathrm{CHM}}_{\mathbb{C}}$ of *Chow motives with fractional Tate twists* with complex coefficients is defined similarly by replacing all Chow groups with rational coefficients CH by Chow groups with complex coefficients $\mathrm{CH}_{\mathbb{C}}$ in the above definition.

(ii) The usual category of Chow motives with rational (resp. complex) coefficients CHM (resp. $\mathrm{CHM}_{\mathbb{C}}$; see [3]) is identified with the full subcategory of $\widetilde{\mathrm{CHM}}$ (resp. $\widetilde{\mathrm{CHM}}_{\mathbb{C}}$) consisting of objects (X, p, n) with $n \in \mathbb{Z}$.

(iii) Thanks to the extension of the intersection theory (with rational coefficients) of Fulton [32] to the so-called $\mathbb{Q}$–*varieties* by Mumford [43], the motive functor $\mathfrak{h}$ defined above can actually be extended to the larger category of finite group quotients of smooth projective varieties, or more generally to $\mathbb{Q}$–varieties with global Cohen–Macaulay cover; see for example [17, Sections 2.2–2.3]. Indeed, for global quotients one defines $\mathfrak{h}(M/G) := \big(M, (1/|G|) \sum_{g \in G} {}^{t}\Gamma_g, 0\big) := \mathfrak{h}(M)^{G}$. (Note that it is essential to work with rational coefficients.) Denoting by $\pi\colon M \to M/G$ the quotient morphism and letting X be an auxiliary variety, a morphism from $\mathfrak{h}(X)$ to $\mathfrak{h}(M/G)$ is a correspondence in $\mathrm{CH}^{\dim X}(X \times M/G)$, which, under the above identification $\mathfrak{h}(M/G) = \mathfrak{h}(M)^{G}$, is regarded as a G–invariant element of $\mathrm{CH}^{\dim X}(X \times M)$ via

the pullback $\mathrm{id}_X \times \pi^*$, where π^* is defined in [32, Example 1.7.6]. The latter has the property that $\pi_* \pi^* = |G| \cdot \mathrm{id}$ while $\pi^* \pi_* = \sum_{g \in G} {}^t \Gamma_g$. It is useful to observe that if we replace G by $G \times H$, where H acts trivially on M, the pullback π^* changes by the factor $|H|$. We will avoid this kind of confusion by only considering *faithful* quotients when dealing with Chow groups of quotient varieties.

Let M be an m–dimensional smooth projective complex variety equipped with a faithful action of a finite group G. We adapt the constructions in [26] and [36] to define the orbifold motive of the smooth proper Deligne–Mumford stack $[M/G]$. For any $g \in G$, $M^g := \{x \in M \mid gx = x\}$ is the fixed locus of the automorphism g, which is a smooth subvariety of M. The following notion is due to Reid (see [49]).

Definition 2.3 (age) Given an element $g \in G$, let $r \in \mathbb{N}$ be its order. The *age* of g, denoted by $\mathrm{age}(g)$, is the locally constant $\mathbb{Q}_{\geq 0}$–valued function on M^g defined as follows. Let Z be a connected component of M^g. Choosing any point $x \in Z$, we have the induced automorphism $g_* \in \mathrm{GL}(T_x M)$, whose eigenvalues, repeated according to multiplicities, are

$$\{e^{2\pi\sqrt{-1}(\alpha_1/r)}, \ldots, e^{2\pi\sqrt{-1}(\alpha_m/r)}\},$$

with $0 \leq \alpha_i \leq r - 1$. One defines

$$\mathrm{age}(g)|_Z := \frac{1}{r} \sum_{i=1}^{m} \alpha_i.$$

It is obvious that the value of $\mathrm{age}(g)$ on Z is independent of the choice of $x \in Z$ and it takes values in $\mathbb{N}$ if $g_* \in \mathrm{SL}(T_x M)$. Also immediate from the definition, we have $\mathrm{age}(g) + \mathrm{age}(g^{-1}) = \mathrm{codim}(M^g \subset M)$ as locally constant functions. Thanks to the natural isomorphism $h \colon M^g \to M^{hgh^{-1}}$ sending x to $h.x$, for any $g, h \in G$, the age function is invariant under conjugation.

Example 2.4 Let S be a smooth projective variety of dimension d and n a positive integer. The symmetric group $\mathfrak{S}_n$ acts by permutation on $M = S^n$. For each $g \in \mathfrak{S}_n$, a straightforward computation (see Paragraph 5.1.3) shows that $\mathrm{age}(g)$ is the constant function $\frac{1}{2}d(n - |O(g)|)$, where $O(g)$ is the set of orbits of g as a permutation of $\{1, \ldots, n\}$. For example, when S is a surface (i.e. $d = 2$), the age is always a nonnegative integer and we have $\mathrm{age}(\mathrm{id}) = 0$, $\mathrm{age}(12 \ldots r) = r - 1$, $\mathrm{age}(12)(345) = 3$, etc.

Recall that an *algebra object* in a symmetric monoidal category $(\mathcal{M}, \otimes, \mathbb{1})$ (for example, CHM, $\widetilde{\mathrm{CHM}}$, etc.) is an object $A \in \mathrm{Obj}\,\mathcal{M}$ together with a morphism $\mu \colon A \otimes A \to A$

in $\mathcal{M}$, called the *multiplication* or *product structure*, satisfying the associativity axiom $\mu \circ (\mu \otimes \mathrm{id}) = \mu \circ (\mathrm{id} \otimes \mu)$. An algebra object A in $\mathcal{M}$ is called *commutative* if $\mu \circ \iota = \mu$, where $\iota \colon A \otimes A \to A \otimes A$ is the structural symmetry isomorphism of $\mathcal{M}$. For each smooth projective variety X, its Chow motive $\mathfrak{h}(X)$ is naturally a commutative algebra object in CHM (hence in $\widetilde{\mathrm{CHM}}$, $\widetilde{\mathrm{CHM}}_{\mathbb{C}}$, etc.) whose multiplication is given by the small diagonal $\delta_X \in \mathrm{CH}^{2 \dim X}(X \times X \times X) = \mathrm{Hom}_{\mathrm{CHM}}(\mathfrak{h}(X) \otimes \mathfrak{h}(X), \mathfrak{h}(X))$.

Definition 2.5 (orbifold Chow motive) We define first of all an auxiliary (in general noncommutative) algebra object $\mathfrak{h}(M, G)$ of $\widetilde{\mathrm{CHM}}$ in several steps:

(i) As a Chow motive with fractional twists, $\mathfrak{h}(M, G)$ is defined to be the direct sum over G of the motives of fixed loci twisted à la Tate by $-\mathrm{age}$:

$$\mathfrak{h}(M, G) := \bigoplus_{g \in G} \mathfrak{h}(M^g)(-\mathrm{age}(g)).$$

(ii) $\mathfrak{h}(M, G)$ is equipped with a natural G–action: each element $h \in G$ induces for each $g \in G$ an isomorphism $h \colon M^g \to M^{hgh^{-1}}$ by sending x to $h.x$, hence an isomorphism between the direct summands $\mathfrak{h}(M^g)(-\mathrm{age}(g))$ and $\mathfrak{h}(M^{hgh^{-1}})(-\mathrm{age}(hgh^{-1}))$ by the conjugation invariance of the age function.

(iii) For any $g \in G$, let r be its order. We have a natural automorphism g_* of the vector bundle $TM|_{M^g}$. Consider its eigen-subbundle decomposition

$$TM|_{M^g} = \bigoplus_{j=0}^{r-1} W_{g,j},$$

where $W_{g,j}$ is the subbundle associated to the eigenvalue $e^{2\pi\sqrt{-1}(j/r)}$. Define

$$S_g := \sum_{j=0}^{r-1} \frac{j}{r}[W_{g,j}] \in K_0(M^g)_{\mathbb{Q}}.$$

Note that the virtual rank of S_g is nothing but $\mathrm{age}(g)$ by Definition 2.3.

(iv) For any $g_1, g_2 \in G$, let $M^{<g_1, g_2>} = M^{g_1} \cap M^{g_2}$ and $g_3 = g_2^{-1} g_1^{-1}$. Define the following element in $K_0(M^{<g_1, g_2>})_{\mathbb{Q}}$:

F_{g_1, g_2}
$$:= S_{g_1}|_{M^{<g_1,g_2>}} + S_{g_2}|_{M^{<g_1,g_2>}} + S_{g_3}|_{M^{<g_1,g_2>}} + TM^{<g_1,g_2>} - TM|_{M^{<g_1,g_2>}}.$$

Note that its virtual rank is

(8) $\mathrm{rk}\, F_{g_1, g_2} = \mathrm{age}(g_1) + \mathrm{age}(g_2) - \mathrm{age}(g_1 g_2) - \mathrm{codim}(M^{<g_1,g_2>} \subset M^{g_1 g_2}).$

In fact, this class in the Grothendieck group is represented by a genuine obstruction vector bundle that is constructed in [26] (see [36]). In particular, the quantity $\mathrm{age}(g_1) + \mathrm{age}(g_2) - \mathrm{age}(g_1g_2)$ is always an integer.

(v) The product structure $\star_{\mathrm{orb}}$ on $\mathfrak{h}(M, G)$ is defined to be multiplicative with respect to the G–grading and for each $g_1, g_2 \in G$, the *orbifold product*

$$\star_{\mathrm{orb}} \colon \mathfrak{h}(M^{g_1})(-\mathrm{age}(g_1)) \otimes \mathfrak{h}(M^{g_2})(-\mathrm{age}(g_2)) \to \mathfrak{h}(M^{g_1g_2})(-\mathrm{age}(g_1g_2))$$

is the correspondence determined by the algebraic cycle

$$\delta_*(c_{\mathrm{top}}(F_{g_1,g_2}))$$
$$\in \mathrm{CH}^{\dim M^{g_1} + \dim M^{g_2} + \mathrm{age}(g_1) + \mathrm{age}(g_2) - \mathrm{age}(g_1g_2)}(M^{g_1} \times M^{g_2} \times M^{g_1g_2}),$$

where $\delta \colon M^{<g_1,g_2>} \to M^{g_1} \times M^{g_2} \times M^{g_1g_2}$ is the natural morphism sending x to (x, x, x) and c_{top} means the top Chern class of F_{g_1,g_2}. One can check easily that the product structure $\star_{\mathrm{orb}}$ is invariant under the action of G.

(vi) The associativity of $\star_{\mathrm{orb}}$ is nontrivial. The proof in [36, Lemma 5.4] is completely algebraic hence also works in our motivic case.

(vii) Finally, the *orbifold Chow motive* of $[M/G]$, denoted by $\mathfrak{h}_{\mathrm{orb}}([M/G])$, is the G–invariant subalgebra object[5] of $\mathfrak{h}(M, G)$, which turns out to be a commutative algebra object in $\widetilde{\mathrm{CHM}}$:

$$(9) \qquad \mathfrak{h}_{\mathrm{orb}}([M/G]) := \mathfrak{h}(M, G)^G = \left(\bigoplus_{g \in G} \mathfrak{h}(M^g)(-\mathrm{age}(g)), \star_{\mathrm{orb}} \right)^G.$$

We still use $\star_{\mathrm{orb}}$ to denote the orbifold product on this subalgebra object $\mathfrak{h}_{\mathrm{orb}}([M/G])$.

Remark 2.6 With Definition 2.5(ii) in mind, the correspondence

$$p := \frac{1}{|G|} \sum_{h \in G} \Gamma_h \in \bigoplus_{h \in G} \bigoplus_{g \in G} \mathrm{CH}^{\dim M^g}(M^g \times M^{hgh^{-1}})$$

defines an idempotent endomorphism of the Chow motive $\mathfrak{h}(M, G)$, which is equal to $\bigoplus_{g \in G} \mathfrak{h}(M^g)(-\mathrm{age}(g))$. Under this identification, and ignoring the algebra structure, the Chow motive $\mathfrak{h}_{\mathrm{orb}}([M/G])$ is defined explicitly as the image of p (which exists, since the category of Chow motives is pseudoabelian). Composing a correspondence in $\mathrm{Hom}_{\mathrm{CHM}}((Y, q, m), \mathfrak{h}(M, G))$ with p amounts to symmetrizing. The orbifold product

---

[5]Here we use the fact that the category $\widetilde{\mathrm{CHM}}$ is $\mathbb{Q}$–linear and pseudoabelian to define the G–invariant part A^G of a G–object A as the image of the projector $(1/|G|)\sum_{g \in G} g \in \mathrm{End}(A)$.

on $\mathfrak{h}_{\mathrm{orb}}([M/G])$ is then given by the symmetrization of the orbifold product of $\mathfrak{h}(M,G)$ (Definition 2.5(v)), that is, by $p \circ \star_{\mathrm{orb}} \circ (p \otimes p)\colon \mathfrak{h}(M,G) \otimes \mathfrak{h}(M,G) \to \mathfrak{h}(M,G)$. We note that p is self-dual, so that, by [56, Lemma 3.3], $p \circ \star_{\mathrm{orb}} \circ (p \otimes p) = (p \otimes p \otimes p)_* \star_{\mathrm{orb}}$ if $\star_{\mathrm{orb}}$ is viewed as a cycle on $\mathfrak{h}(M,G) \otimes \mathfrak{h}(M,G) \otimes \mathfrak{h}(M,G)$.

By replacing the rational equivalence relation by another adequate equivalence relation (see [3]), the same construction gives the *orbifold homological motives*, *orbifold numerical motives*, etc. associated to a global quotient smooth proper Deligne–Mumford stack as algebra objects in the corresponding categories of pure motives (with fractional Tate twists).

The definition of the orbifold Chow ring then follows in the standard way and agrees with the one in [26; 36; 1].

Definition 2.7 (orbifold Chow ring) The *orbifold Chow ring* of $[M/G]$ is the commutative $\mathbb{Q}_{\geq 0}$–graded $\mathbb{Q}$–algebra $\mathrm{CH}^*_{\mathrm{orb}}([M/G]) := \bigoplus_{i \in \mathbb{Q}_{\geq 0}} \mathrm{CH}^i_{\mathrm{orb}}([M/G])$ with

(10) $$\mathrm{CH}^i_{\mathrm{orb}}([M/G]) := \mathrm{Hom}_{\widetilde{\mathrm{CHM}}}(\mathbb{1}(-i), \mathfrak{h}_{\mathrm{orb}}([M/G])).$$

The ring structure on $\mathrm{CH}^*_{\mathrm{orb}}([M/G])$, called *orbifold product*, denoted again by $\star_{\mathrm{orb}}$, is determined by the product structure $\star_{\mathrm{orb}}\colon \mathfrak{h}_{\mathrm{orb}}([M/G]) \otimes \mathfrak{h}_{\mathrm{orb}}([M/G]) \to \mathfrak{h}_{\mathrm{orb}}([M/G])$ in Definition 2.5.

More concretely, $\mathrm{CH}^*_{\mathrm{orb}}([M/G])$ is the G–invariant $\mathbb{Q}$–subalgebra of an auxiliary (noncommutative) finitely $\mathbb{Q}_{\geq 0}$–graded $\mathbb{Q}$–algebra $\mathrm{CH}^*(M,G)$, which is defined by

$$\mathrm{CH}^*(M,G) := \left(\bigoplus_{g \in G} \mathrm{CH}^{*-\mathrm{age}(g)}(M^g), \star_{\mathrm{orb}} \right),$$

where for two elements $g, h \in G$, the orbifold product of $\alpha \in \mathrm{CH}^{i-\mathrm{age}(g)}(M^g)$ and $\beta \in \mathrm{CH}^{j-\mathrm{age}(h)}(M^h)$ is the following element in $\mathrm{CH}^{i+j-\mathrm{age}(gh)}(M^{gh})$:

(11) $$\alpha \star_{\mathrm{orb}} \beta := \iota_*(\alpha|_{M^{<g,h>}} \cdot \beta|_{M^{<g,h>}} \cdot c_{\mathrm{top}}(F_{g,h})),$$

where $\iota\colon M^{<g,h>} \hookrightarrow M^{gh}$ is the natural inclusion.

Similarly, the orbifold K–theory is defined as follows. Recall that for a smooth variety X and for $F \in K_0(X)$, we have the *lambda operation* $\lambda_t\colon K_0(X) \to K_0(X)[\![t]\!]$, where $\lambda_t(F)$ is a formal power series $\sum_{i=0}^{\infty} t^i \lambda^i(F)$ subject to the multiplicativity relation $\lambda_t(F \oplus F') = \lambda_t(F) \cdot \lambda_t(F')$ for all objects $F, F' \in K(X)$, and such

that, for any rank-r vector bundle E over X, we have $\lambda_t([E]) = \sum_{i=0}^{r} t^i [\bigwedge^i E]$. See [61, Chapter II, Section 4]. Finally $\lambda_{-1}(F)$ is defined by evaluating at $t = -1$ in $\lambda_t(F)$ and is called the *K–theoretic Euler class* of $F^\vee$; see also [36, page 34].

Definition 2.8 (orbifold K–theory) The *orbifold K–theory* of $[M/G]$, denoted by $K_{\mathrm{orb}}([M/G])$, is the subalgebra of G–invariant elements of the $\mathbb{Q}$–algebra $K(M, G)$, which is defined by

$$K(M, G) := \left(\bigoplus_{g \in G} K_0(M^g), \star_{\mathrm{orb}} \right),$$

where for two elements $g, h \in G$, the orbifold product of $\alpha \in K_0(M^g)$ and $\beta \in K_0(M^h)$ is the following element in $K_0(M^{gh})$:

$$\alpha \star_{\mathrm{orb}} \beta := \iota_* (\alpha|_{M^{<g,h>}} \cdot \beta|_{M^{<g,h>}} \cdot \lambda_{-1}(F_{g,h}^\vee)),$$

where $\iota \colon M^{<g,h>} \hookrightarrow M^{gh}$ is the natural inclusion and $\lambda_{-1}(F_{g,h}^\vee)$ is the *K–theoretic Euler class* of $F_{g,h}$ as defined above.

Remark 2.9 The main interest of the paper lies in the situation when the underlying singular variety of the orbifold has at worst Gorenstein singularities. Recall that an algebraic variety X is *Gorenstein* if it is Cohen–Macaulay and the dualizing sheaf is a line bundle, denoted by ω_X. In the case of a global quotient M/G, being Gorenstein is implied by the local G–triviality of the canonical bundle ω_M, which means that the stabilizer of each point $x \in M$ is contained in $\mathrm{SL}(T_x M)$. In this case, it is straightforward to check that the age function actually takes values in the integers $\mathbb{Z}$ and therefore the orbifold motive lies in the usual category of pure motives (without fractional twists) CHM. In particular, the orbifold Chow ring and orbifold cohomology ring are $\mathbb{Z}$–graded. Example 2.4 exhibits a typical situation that we would like to study; see also Remark 3.2.

Remark 2.10 (nonglobal quotients) In the broader setting of smooth proper Deligne–Mumford stacks which are not necessarily finite group global quotients, the orbifold Chow ring is still well defined in [1] but the down-to-earth construction as above, which is essential for the applications (see our slogan in Section 1), is lost (see however the equivariant treatment [25]). Another problem is that the definition of the orbifold Chow motive in this general setting is neither available in the literature nor covered in this paper. In the case where the coarse moduli space is projective with Gorenstein singularities, the orbifold Chow motive is constructed in [31, Section 2.3] in the spirit of [1].

3 Motivic hyper-Kähler resolution conjecture

3.1 A motivic version of the cohomological hyper-Kähler resolution conjecture

In [51], as part of the broader picture of *stringy geometry and topology of orbifolds*, Ruan proposed the cohomological hyper-Kähler resolution conjecture (CHRC), which says that the orbifold cohomology ring of a compact Gorenstein orbifold is isomorphic to the Betti cohomology ring of a hyper-Kähler crepant resolution of the underlying singular variety if one takes $\mathbb{C}$ as coefficients; see Conjecture 1.1 in the introduction for the statement. As explained in Ruan [52], the plausibility of the CHRC is justified by some considerations from theoretical physics as follows. Topological string theory predicts that the quantum cohomology theory of an orbifold should be equivalent to the quantum cohomology theory of a/any crepant resolution of (possibly some deformation of) the underlying singular variety. On the one hand, the orbifold cohomology ring constructed by Chen–Ruan [19] is the classical part (genus zero with three marked points) of the quantum cohomology ring of the orbifold (see [18]); on the other hand, the classical limit of the quantum cohomology of the resolution is the so-called *quantum corrected* cohomology ring [52]. However, if the crepant resolution has a hyper-Kähler structure, then all its Gromov–Witten invariants as well as the quantum corrections vanish and one expects therefore an equivalence, i.e. an isomorphism of $\mathbb{C}$–algebras, between the orbifold cohomology of the orbifold and the usual Betti cohomology of the hyper-Kähler crepant resolution.

Before moving on to a more algebrogeometric study, we have to recall some standard definitions and facts on (possibly singular) symplectic varieties (see [9; 46]):

Definition 3.1 • A *symplectic form* on a smooth complex algebraic variety is a closed holomorphic 2–form that is nondegenerate at each point. A smooth variety is called *holomorphic symplectic* or just *symplectic* if it admits a symplectic form. Projective examples include deformations of Hilbert schemes of K3 surfaces and of abelian surfaces, generalized Kummer varieties, etc. A typical nonprojective example is provided by the cotangent bundle of a smooth variety.

• A (possibly singular) *symplectic variety* is a normal complex algebraic variety such that its smooth part admits a symplectic form whose pullback to a/any resolution extends to a holomorphic 2–form. A germ of such a variety is called a *symplectic singularity*. Such singularities are necessarily rational Gorenstein [9] and conversely,

by a result of Namikawa [46], a normal variety is symplectic if and only if it has rational Gorenstein singularities and its smooth part admits a symplectic form. The main examples that we are dealing with are of the form of a quotient by a finite group of symplectic automorphisms of a smooth symplectic variety, e.g. the symmetric products $S^{(n)} = S^n / \mathfrak{S}_n$ of smooth algebraic surfaces S with trivial canonical bundle.

• Given a singular symplectic variety X, a *symplectic resolution* or *hyper-Kähler resolution* is a resolution $f : Y \to X$ such that the pullback of a symplectic form on the smooth part X_{reg} extends to a symplectic form on Y. Note that a resolution is symplectic if and only if it is *crepant*: $f^* \omega_X = \omega_Y$. The definition is independent of the choice of a symplectic form on X_{reg}. A symplectic resolution is always semismall. The existence of symplectic resolutions and the relations between them form a highly attractive topic in holomorphic symplectic geometry. An interesting situation, which will not be touched upon in this paper however, is the normalization of the closure of a nilpotent orbit in a complex semisimple Lie algebra, whose symplectic resolutions are extensively studied in the literature (see [28; 13]). For examples relevant to this paper, see Examples 3.4.

Returning to the story of the hyper-Kähler resolution conjecture, in order to study algebraic cycles and motives of holomorphic symplectic varieties, especially with a view toward the splitting property conjecture of Beauville [10] (see Section 7), we would like to propose the motivic version of the CHRC; see Metaconjecture 1.2 in the introduction for the general statement. As we are dealing exclusively with the global quotient case in this paper and its sequel, we will concentrate on this more restricted case and on the more precise formulation Conjecture 1.3 in the introduction.

Remark 3.2 (integral grading) We use the same notation as in Conjecture 1.3. Then, since G preserves a symplectic form (hence a canonical form) of M, the quotient variety M/G has at worst Gorenstein singularities. As is pointed out in Remark 2.9, this implies that the age functions take values in $\mathbb{Z}$, the orbifold motive $\mathfrak{h}_{\mathrm{orb}}([M/G])$ is in CHM, the usual category of (rational) Chow motives, and the orbifold Chow ring $\mathrm{CH}^*_{\mathrm{orb}}([M/G])$ is integrally graded.

Remark 3.3 (K–theoretic analogue) As noted in the introduction (Conjecture 1.7), we are also interested in the K–theoretic version of the hyper-Kähler resolution conjecture (KHRC) proposed in [36, Conjecture 1.2]. We want to point out that in Conjecture 1.3, the statement for Chow rings is more or less equivalent to the KHRC; however, the full formulation for Chow motivic algebras is, on the other hand, strictly richer. In fact, in

all cases that we are able to prove the KHRC, in this paper as well as in the upcoming one [30], we have to first solve the MHRC on the motive level and deduce the KHRC as a consequence. See Section 4 for the proof of Theorem 1.8.

Examples 3.4 All examples studied in this paper are in the following situation: let M and G be as in Conjecture 1.3 and Y be (the principal component of) the G–Hilbert scheme G–Hilb(M) of G–*clusters* of M, that is, a 0–dimensional G–invariant subscheme of M whose global functions form the regular G–representation (see [34; 45]). In some interesting cases, Y gives a symplectic resolution of M/G:

- Let S be a smooth algebraic surface, and let $G = \mathfrak{S}_n$ act on $M = S^n$ by permutation. By the result of Haiman [33, Theorem 5.1], $Y = \mathfrak{S}_n$–Hilb(S^n) is isomorphic to the nth punctual Hilbert scheme $S^{[n]}$, which is a crepant resolution, hence a symplectic resolution if S has trivial canonical bundle, of $M/G = S^{(n)}$, the nth symmetric product.

- Let A be an abelian surface, and let M be the kernel of the summation map $s \colon A^{n+1} \to A$ and $G = \mathfrak{S}_{n+1}$ act on M by permutations. Then $Y = G$–Hilb(M) is isomorphic to the generalized Kummer variety $K_n(A)$ and is a symplectic resolution of M/G.

Although both sides of the isomorphism in Conjecture 1.3 are in the category CHM of motives with *rational* coefficients, it is in general necessary to make use of roots of unity to realize such an isomorphism of algebra objects. However, in some situations, it is possible to stay in CHM by making a suitable sign change, which is related to the notion of *discrete torsion* in theoretical physics:

Definition 3.5 (discrete torsion) For any $g, h \in G$, let

$$(12) \qquad\qquad \epsilon(g, h) := \tfrac{1}{2}(\text{age}(g) + \text{age}(h) - \text{age}(gh)).$$

It is easy to check that

$$(13) \qquad\qquad \epsilon(g_1, g_2) + \epsilon(g_1 g_2, g_3) = \epsilon(g_1, g_2 g_3) + \epsilon(g_2, g_3).$$

In the case when $\epsilon(g, h)$ is an integer for all $g, h \in G$, we can define the *orbifold Chow motive with discrete torsion* of a global quotient stack $[M/G]$, denoted by $\mathfrak{h}_{\text{orb,dt}}([M/G])$, by the following simple change of sign in step (v) of Definition 2.5: the orbifold product with discrete torsion

$$\star_{\text{orb,dt}} \colon \mathfrak{h}(M^{g_1})(-\text{age}(g_1)) \otimes \mathfrak{h}(M^{g_2})(-\text{age}(g_2)) \to \mathfrak{h}(M^{g_1 g_2})(-\text{age}(g_1 g_2))$$

is the correspondence determined by the algebraic cycle

$$(-1)^{\epsilon(g_1,g_2)} \cdot \delta_*(c_{\text{top}}(F_{g_1,g_2}))$$
$$\in \text{CH}^{\dim M^{g_1} + \dim M^{g_2} + \text{age}(g_1) + \text{age}(g_2) - \text{age}(g_1g_2)}(M^{g_1} \times M^{g_2} \times M^{g_1g_2}).$$

Thanks to (13), $\star_{\text{orb,dt}}$ is still associative. Similarly, the *orbifold Chow ring with discrete torsion* of $[M/G]$ is obtained by replacing (11) in Definition 2.7 by

$$(14) \qquad \alpha \star_{\text{orb,dt}} \beta := (-1)^{\epsilon(g,h)} \cdot i_*(\alpha|_{M^{<g,h>}} \cdot \beta|_{M^{<g,h>}} \cdot c_{\text{top}}(F_{g,h})),$$

which is again associative by (13).

Thanks to the notion of discrete torsion, we can have the following version of motivic hyper-Kähler resolution conjecture, which takes place in the category of rational Chow motives and involves only rational Chow groups.

Conjecture 3.6 (MHRC: global quotient case with discrete torsion) *In the same situation as Conjecture 1.3, suppose that $\epsilon(g,h)$ of Definition 3.5 is an integer for all $g, h \in G$. Then we have an isomorphism of (commutative) algebra objects in the category of Chow motives with **rational** coefficients,*

$$\mathfrak{h}(Y) \simeq \mathfrak{h}_{\text{orb,dt}}([M/G]) \quad \text{in CHM}.$$

In particular, we have an isomorphism of graded $\mathbb{Q}$–algebras,

$$\text{CH}^*(Y) \simeq \text{CH}^*_{\text{orb,dt}}([M/G]).$$

Remark 3.7 It is easy to see that Conjecture 3.6 implies Conjecture 1.3: to get rid of the discrete torsion sign change $(-1)^{\epsilon(g,h)}$, it suffices to multiply the isomorphism on each summand $\mathfrak{h}(M^g)(-\text{age}(g))$, or $\text{CH}(M^g)$, by $\sqrt{-1}^{\text{age}(g)}$, which involves of course the complex numbers (roots of unity at least).

3.2 Toy examples

To better illustrate the conjecture as well as the proof in Section 4, we present in this section some explicit computations for two of the simplest nontrivial cases of the MHRC.

3.2.1 Hilbert squares of K3 surfaces Let S be a K3 surface or an abelian surface. Consider the involution f on $S \times S$ flipping the two factors. The relevant Deligne–Mumford stack is $[S^2/f]$; its underlying singular symplectic variety is the second

symmetric product $S^{(2)}$, and $S^{[2]}$ is its symplectic resolution. Let $\widetilde{S^2}$ be the blowup of S^2 along its diagonal Δ_S:

$$
\begin{array}{ccc}
E & \xrightarrow{\ j\ } & \widetilde{S^2} \\
\pi \downarrow & \square & \downarrow \epsilon \\
\Delta_S & \xrightarrow[\ \Delta\ ]{} & S \times S
\end{array}
$$

Then f lifts to a natural involution on $\widetilde{S^2}$ and the quotient is

$$
q\colon \widetilde{S^2} \twoheadrightarrow S^{[2]}.
$$

On the one hand, $\mathrm{CH}^*(S^{[2]})$ is identified, via q^*, with the invariant part of $\mathrm{CH}^*(\widetilde{S^2})$; on the other hand, by Definition 2.7, $\mathrm{CH}^*_{\mathrm{orb}}([S^2/\mathfrak{S}_2]) = \mathrm{CH}^*(S^2, \mathfrak{S}_2)^{\mathrm{inv}}$. Therefore to check either MHRC Conjecture 1.3 or Conjecture 3.6 (at the level of Chow rings *only*) in this case, we only have to show the following:

Proposition 3.8 *We have an isomorphism of $\mathbb{C}$–algebras,*

$$
\mathrm{CH}^*(\widetilde{S^2})_{\mathbb{C}} \simeq \mathrm{CH}^*(S^2, \mathfrak{S}_2)_{\mathbb{C}}.
$$

In fact, taking into account the discrete torsion, there is an isomorphism of $\mathbb{Q}$–algebras,

$$
\mathrm{CH}^*(S^{[2]}) \simeq \mathrm{CH}^*_{\mathrm{orb,dt}}([S^2/\mathfrak{S}_2]).
$$

Proof A straightforward computation using (iii) and (iv) of Definition 2.5 shows that all obstruction bundles are trivial (at least in the Grothendieck group). Hence, by Definition 2.7,

$$
\mathrm{CH}^*(S^2, \mathfrak{S}_2) = \mathrm{CH}^*(S^2) \oplus \mathrm{CH}^{*-1}(\Delta_S),
$$

whose ring structure is explicitly given by

$$
\begin{aligned}
\alpha \star_{\mathrm{orb}} \beta &= \alpha \cdot \beta & \in \mathrm{CH}^{i+j}(S^2) & \quad \text{for any } \alpha \in \mathrm{CH}^i(S^2) \text{ and } \beta \in \mathrm{CH}^j(S^2), \\
\alpha \star_{\mathrm{orb}} \beta &= \alpha|_\Delta \cdot \beta & \in \mathrm{CH}^{i+j}(\Delta_S) & \quad \text{for any } \alpha \in \mathrm{CH}^i(S^2) \text{ and } \beta \in \mathrm{CH}^j(\Delta_S), \\
\alpha \star_{\mathrm{orb}} \beta &= \Delta_*(\alpha \cdot \beta) & \in \mathrm{CH}^{i+j+2}(S^2) & \quad \text{for any } \alpha \in \mathrm{CH}^i(\Delta_S) \text{ and } \beta \in \mathrm{CH}^j(\Delta_S).
\end{aligned}
$$

The blowup formula (see for example, [57, Theorem 9.27]) provides an a priori only additive isomorphism

$$
(\epsilon^*, j_* \pi^*)\colon \mathrm{CH}^*(S^2) \oplus \mathrm{CH}^{*-1}(\Delta_S) \xrightarrow{\simeq} \mathrm{CH}^*(\widetilde{S^2}),
$$

whose inverse is given by $(\epsilon_*, -\pi_* j^*)$.

With everything given explicitly as above, it is straightforward to check that this isomorphism respects also the multiplication up to a sign change:

- For any $\alpha \in \mathrm{CH}^i(S^2)$ and $\beta \in \mathrm{CH}^j(S^2)$, one has

$$\epsilon^*(\alpha \star_{\mathrm{orb}} \beta) = \epsilon^*(\alpha \cdot \beta) = \epsilon^*(\alpha) \cdot \epsilon^*(\beta).$$

- For any $\alpha \in \mathrm{CH}^i(S^2)$ and $\beta \in \mathrm{CH}^j(\Delta_S)$, the projection formula yields

$$j_* \pi^*(\alpha \star_{\mathrm{orb}} \beta) = j_* \pi^*(\alpha|_\Delta \cdot \beta) = j_*(j^* \epsilon^*(\alpha) \cdot \pi^* \beta) = \epsilon^*(\alpha) \cdot j_* \pi^*(\beta).$$

- For any $\alpha \in \mathrm{CH}^i(\Delta_S)$ and $\beta \in \mathrm{CH}^j(\Delta_S)$, we make the sign change $\alpha \star_{\mathrm{orb,dt}} \beta = -\Delta_*(\alpha \cdot \beta)$ and we get

$$\begin{aligned}
j_* \pi^*(\alpha) \cdot j_* \pi^*(\beta) &= j_*(j^* j_* \pi^* \alpha \cdot \pi^* \beta) \\
&= j_*(c_1(N_{E/\widetilde{S^2}}) \cdot \pi^* \alpha \cdot \pi^* \beta) \\
&= -\epsilon^* \Delta_*(\alpha \cdot \beta) \\
&= \epsilon^*(\alpha \star_{\mathrm{orb,dt}} \beta),
\end{aligned}$$

where in the next-to-last equality one uses the excess intersection formula for the blowup diagram together with the fact that $N_{E/\widetilde{S^2}} = \mathcal{O}_\pi(-1)$ while the excess normal bundle is

$$\pi^* T_S / \mathcal{O}_\pi(-1) \simeq T_\pi \otimes \mathcal{O}_\pi(-1) \simeq \mathcal{O}_\pi(1),$$

where one uses the assumption that $K_S = 0$ to deduce that $T_\pi \simeq \mathcal{O}_\pi(2)$.

As the sign change is exactly the one given by discrete torsion (Definition 3.5), we have an isomorphism of $\mathbb{Q}$–algebras,

$$\mathrm{CH}^*(S^{[2]}) \simeq \mathrm{CH}^*_{\mathrm{orb,dt}}([S^2/\mathfrak{S}_2]).$$

By Remark 3.7, this yields, without making any sign change, an isomorphism of $\mathbb{C}$–algebras,

$$\mathrm{CH}^*(S^{[2]})_\mathbb{C} \simeq \mathrm{CH}^*_{\mathrm{orb}}([S^2/\mathfrak{S}_2])_\mathbb{C},$$

which concludes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

3.2.2 Kummer K3 surfaces   Let A be an abelian surface. We always identify $A_0^2 := \mathrm{Ker}(A \times A \xrightarrow{+} A)$ with A by $(x, -x) \mapsto x$. Under this identification, the associated Kummer K3 surface $S := K_1(A)$ is a hyper-Kähler crepant resolution of the symplectic

quotient A/f, where f is the involution of multiplication by -1 on A. Consider the blowup of A along the fixed locus F which is the set of 2–torsion points of A:

$$
\begin{array}{ccc}
E & \xrightarrow{\ j\ } & \widetilde{A} \\
\pi \downarrow & \square & \downarrow \epsilon \\
F & \xrightarrow{\ i\ } & A
\end{array}
$$

Then S is the quotient of $\widetilde{A}$ by $\widetilde{f}$, the lifting of the involution f. As in the previous toy example, the MHRC at the level of Chow rings *only* in the present situation is reduced to the following:

Proposition 3.9 *We have an isomorphism of $\mathbb{C}$–algebras, $\mathrm{CH}^*(\widetilde{A})_{\mathbb{C}} \simeq \mathrm{CH}^*(A, \mathfrak{S}_2)_{\mathbb{C}}$. In fact, taking into account the discrete torsion, there is an isomorphism of $\mathbb{Q}$–algebras, $\mathrm{CH}^*(K_1(A)) \simeq \mathrm{CH}^*_{\mathrm{orb,dt}}([A/\mathfrak{S}_2])$.*

Proof Because the computation is quite similar to that of Proposition 3.8, we only give a sketch of the proof. By Definition 2.7, $\mathrm{age}(\mathrm{id}) = 0$, $\mathrm{age}(\widetilde{f}) = 1$ and $\mathrm{CH}^*(A, \mathfrak{S}_2) = \mathrm{CH}^*(A) \oplus \mathrm{CH}^{*-1}(F)$, whose ring structure is given by

$$
\begin{aligned}
\alpha \star_{\mathrm{orb}} \beta &= \ \ \alpha \bullet \beta \ \ \in \mathrm{CH}^{i+j}(A) && \text{for any } \alpha \in \mathrm{CH}^i(A) \text{ and } \beta \in \mathrm{CH}^j(A), \\
\alpha \star_{\mathrm{orb}} \beta &= \alpha|_F \bullet \beta \ \in \mathrm{CH}^i(F) && \text{for any } \alpha \in \mathrm{CH}^i(A) \text{ and } \beta \in \mathrm{CH}^0(F), \\
\alpha \star_{\mathrm{orb}} \beta &= i_*(\alpha \bullet \beta) \in \mathrm{CH}^2(A) && \text{for any } \alpha \in \mathrm{CH}^0(F) \text{ and } \beta \in \mathrm{CH}^0(F).
\end{aligned}
$$

Again by the blowup formula, we have an isomorphism

$$
(\epsilon^*, j_*\pi^*)\colon \mathrm{CH}^*(A) \oplus \mathrm{CH}^{*-1}(F) \xrightarrow{\ \cong\ } \mathrm{CH}^*(\widetilde{A}),
$$

whose inverse is given by $(\epsilon_*, -\pi_* j^*)$. It is now straightforward to check that they are moreover ring isomorphisms with the left-hand side equipped with the orbifold product. The sign change comes from the negativity of the self-intersection of (the components of) the exceptional divisor. $\qquad\square$

4 Main results and steps of the proofs

The main results of the paper are the verification of Conjecture 3.6, hence Conjecture 1.3 by Remark 3.7, in the following two cases, Case (A) and Case (B). See Theorem 1.4 and Theorem 1.5 in the introduction for the precise statements of the results. These two theorems are proved in Section 5 and Section 6 respectively.

Let A be an abelian surface and n be a positive integer.

Case (A) Hilbert schemes of abelian surfaces M is equal to A^n endowed with the natural action of $G = \mathfrak{S}_n$. The symmetric product $A^{(n)} = M/G$ is a singular symplectic variety and the Hilbert–Chow morphism

$$\rho\colon Y = A^{[n]} \to A^{(n)}$$

gives a symplectic resolution.

Case (B) Generalized Kummer varieties M is equal to $A_0^{n+1} := \operatorname{Ker}(A^{n+1} \xrightarrow{s} A)$ endowed with the natural action of $G = \mathfrak{S}_{n+1}$. The quotient $A_0^{n+1}/\mathfrak{S}_{n+1} = M/G$ is a singular symplectic variety. Recall that the *generalized Kummer variety* $K_n(A)$ is the fiber over O_A of the isotrivial fibration $A^{[n+1]} \to A^{(n+1)} \xrightarrow{s} A$. The restriction of the Hilbert–Chow morphism

$$Y = K_n(A) \to A_0^{n+1}/\mathfrak{S}_{n+1}$$

gives a symplectic resolution.

Let us deduce the KHRC (Conjecture 1.7) in these two cases from our main results:[6]

Proof of Theorem 1.8   Let M and G be as in either Case (A) or Case (B). Without using discrete torsion, we have an isomorphism of $\mathbb{C}$–algebras $\operatorname{CH}^*(M)_{\mathbb{C}} \simeq \operatorname{CH}^*_{\mathrm{orb}}([M/G])_{\mathbb{C}}$ by Theorems 1.4 and 1.5. An *orbifold Chern character* is constructed in [36], which by [36, Main result 3] provides an isomorphism of $\mathbb{Q}$–algebras,

$$\operatorname{ch}_{\mathrm{orb}}\colon K_{\mathrm{orb}}([M/G])_{\mathbb{Q}} \xrightarrow{\cong} \operatorname{CH}^*_{\mathrm{orb}}([M/G])_{\mathbb{Q}}.$$

The desired isomorphism of algebras is then obtained by the composition of $\operatorname{ch}_{\mathrm{orb}}$ (tensored with $\mathbb{C}$), the Chern character isomorphism $\operatorname{ch}\colon K(Y)_{\mathbb{Q}} \xrightarrow{\cong} \operatorname{CH}^*(Y)_{\mathbb{Q}}$ tensored with $\mathbb{C}$, and the isomorphism $\operatorname{CH}^*(M)_{\mathbb{C}} \simeq \operatorname{CH}^*_{\mathrm{orb}}([M/G])_{\mathbb{C}}$ from our main results.

Similarly, for topological K–theory one uses the orbifold topological Chern character, which is also constructed in [36],

$$\operatorname{ch}_{\mathrm{orb}}\colon K^{\mathrm{top}}_{\mathrm{orb}}([M/G])_{\mathbb{Q}} \xrightarrow{\cong} H^*_{\mathrm{orb}}([M/G], \mathbb{C}),$$

together with $\operatorname{ch}\colon K^{\mathrm{top}}(Y)_{\mathbb{Q}} \xrightarrow{\cong} H^*(Y, \mathbb{Q})$ and the cohomological hyper-Kähler resolution conjecture,

$$H^*_{\mathrm{orb}}([M/G], \mathbb{C}) \simeq H^*(Y, \mathbb{C}),$$

---

[6]Our proof for the KHRC passes through Chow rings, thus a direct geometric (sheaf-theoretic) description of the isomorphism between $K(Y)_{\mathbb{C}}$ and $K_{\mathrm{orb}}([M/G])_{\mathbb{C}}$ is still missing.

which is proved in Case (A) in [26] and [54] based on [41] and in Case (B) in Theorem 1.6. □

In the rest of this section, we explain the main steps of the proofs of Theorem 1.4 and Theorem 1.5. For both cases, the proof proceeds in three steps. For each step, Case (A) is quite straightforward and Case (B) requires more subtle and technical arguments.

Step (i) Recall the notation $\mathfrak{h}(M,G) := \bigoplus_{g \in G} \mathfrak{h}(M^g)(-\operatorname{age}(g))$. Denote by

$$\iota \colon \mathfrak{h}(M,G)^G \hookrightarrow \mathfrak{h}(M,G) \quad \text{and} \quad p \colon \mathfrak{h}(M,G) \twoheadrightarrow \mathfrak{h}(M,G)^G$$

the inclusion of and the projection onto the G–invariant part $\mathfrak{h}(M,G)^G$, which is a direct factor of $\mathfrak{h}(M,G)$ inside CHM. We will first construct an a priori just additive G–equivariant morphism of Chow motives $\mathfrak{h}(Y) \to \mathfrak{h}(M,G)$, given by some correspondences $\{(-1)^{\operatorname{age}(g)} U^g \in \operatorname{CH}(Y \times M^g)\}_{g \in G}$ inducing an (additive) isomorphism

$$\phi = p \circ \sum_g (-1)^{\operatorname{age}(g)} U^g \colon \mathfrak{h}(Y) \xrightarrow{\simeq} \mathfrak{h}_{\operatorname{orb}}([M/G]) = \mathfrak{h}(M,G)^G.$$

The isomorphism ϕ will have the property that its inverse is $\psi := \left((1/|G|) \sum_g {}^t U^g\right) \circ \iota$ (see Proposition 5.2 and Proposition 6.4 for Case (A) and Case (B) respectively). Note that since $\sum_g (-1)^{\operatorname{age}(g)} U^g$ is G–equivariant, we have $\iota \circ \phi = \sum_g (-1)^{\operatorname{age}(g)} U^g$ and likewise $\psi \circ p = (1/|G|) \sum_g {}^t U^g$. Our goal is then to prove that these morphisms are moreover multiplicative (after the sign change by discrete torsion), i.e. that the diagram

(15)
$$\begin{array}{ccc} \mathfrak{h}(Y)^{\otimes 2} & \xrightarrow{\delta_Y} & \mathfrak{h}(Y) \\ {\scriptstyle \phi^{\otimes 2}}\downarrow & & \downarrow{\scriptstyle \phi} \\ \mathfrak{h}_{\operatorname{orb}}([M/G])^{\otimes 2} & \xrightarrow[\star_{\operatorname{orb,dt}}]{} & \mathfrak{h}_{\operatorname{orb}}([M/G]) \end{array}$$

is commutative, where the algebra structure $\star_{\operatorname{orb,dt}}$ on the Chow motive $\mathfrak{h}_{\operatorname{orb}}([M/G])$ is the symmetrization of the algebra structure $\star_{\operatorname{orb,dt}}$ on $\mathfrak{h}(M,G)$ defined in Definition 3.5 (in the same way that the algebra structure $\star_{\operatorname{orb}}$ on the Chow motive $\mathfrak{h}_{\operatorname{orb}}([M/G])$ is the symmetrization of the algebra structure $\star_{\operatorname{orb}}$ on $\mathfrak{h}(M,G)$; see Remark 2.6).

The main theorem will then be deduced from the following:

Proposition 4.1 *With the notation as before, the following two algebraic cycles have the same symmetrization in* $\operatorname{CH}\left(\left(\coprod_{g \in G} M^g\right)^3\right)$:

- $W := \left((1/|G|) \sum_g U^g \times (1/|G|) \sum_g U^g \times \sum_g (-1)^{\mathrm{age}(g)} U^g \right)_* (\delta_Y)$;

- *the algebraic cycle Z determining the orbifold product (Definition 2.5(v)) with the sign change by discrete torsion (Definition 3.5):*

$$Z|_{M^{g_1} \times M^{g_2} \times M^{g_3}} = \begin{cases} 0 & \text{if } g_3 \neq g_1 g_2, \\ (-1)^{\epsilon(g_1, g_2)} \cdot \delta_* c_{\mathrm{top}}(F_{g_1, g_2}) & \text{if } g_3 = g_1 g_2. \end{cases}$$

*Here the **symmetrization** of a cycle in $\left(\coprod_{g \in G} M^g \right)^3$ is the operation*

$$\gamma \mapsto (p \otimes p \otimes p)_* \gamma = \frac{1}{|G|^3} \sum_{g_1, g_2, g_3 \in G} (g_1, g_2, g_3).\gamma.$$

Proposition 4.1 implies Theorems 1.4 and 1.5 The only thing to show is the commutativity of (15), which is of course equivalent to the commutativity of the diagram

$$
\begin{array}{ccc}
\mathfrak{h}(Y)^{\otimes 2} & \xrightarrow{\ \delta_Y\ } & \mathfrak{h}(Y) \\
{\scriptstyle \psi^{\otimes 2}} \big\uparrow & & \big\downarrow {\scriptstyle \phi} \\
\mathfrak{h}_{\mathrm{orb}}([M/G])^{\otimes 2} & \xrightarrow[\ \star_{\mathrm{orb,dt}}\ ]{} & \mathfrak{h}_{\mathrm{orb}}([M/G])
\end{array}
$$

By the definition of ϕ and ψ, we need to show the following diagram is commutative:

(16)
$$
\begin{array}{ccc}
\mathfrak{h}(Y)^{\otimes 2} & \xrightarrow{\ \delta_Y\ } & \mathfrak{h}(Y) \\
{\scriptstyle \left((1/|G|) \sum_g {}^t U^g \right)^{\otimes 2}} \big\uparrow & & \big\downarrow {\scriptstyle \sum_g (-1)^{\mathrm{age}(g)} U^g} \\
\mathfrak{h}(M, G)^{\otimes 2} & & \mathfrak{h}(M, G) \\
{\scriptstyle \iota^{\otimes 2}} \big\uparrow & & \big\downarrow {\scriptstyle p} \\
\mathfrak{h}_{\mathrm{orb}}([M/G])^{\otimes 2} & \xrightarrow[\ \star_{\mathrm{orb,dt}}\ ]{} & \mathfrak{h}_{\mathrm{orb}}([M/G])
\end{array}
$$

It is easy to see that the composition $\sum_g (-1)^{\mathrm{age}(g)} U^g \circ \delta_Y \circ \left((1/|G|) \sum_g {}^t U^g \right)^{\otimes 2}$ is the morphism (or correspondence) induced by the cycle W in Proposition 4.1; see e.g. [56, Lemma 3.3]. On the other hand, $\star_{\mathrm{orb,dt}}$ for $\mathfrak{h}_{\mathrm{orb}}([M/G])$ is by definition $p \circ Z \circ \iota^{\otimes 2}$. Therefore, the desired commutativity, hence also the main results, amounts to the equality $p \circ W \circ \iota^{\otimes 2} = p \circ Z \circ \iota^{\otimes 2}$, which says exactly that the symmetrizations of W and of Z are equal in $\mathrm{CH}\left(\left(\coprod_{g \in G} M^g \right)^3 \right)$. $\square$

One is therefore reduced to show Proposition 4.1 in both Case (A) and Case (B).

Step (ii)   We prove that W on the one hand and Z on the other hand, as well as their symmetrizations, are both symmetrically distinguished in the sense of O'Sullivan [48] (see Definition 5.4). To avoid confusion, let us point out that the cycle W is already symmetrized. In Case (B) concerning the generalized Kummer varieties, we have to generalize the category of abelian varieties and the corresponding notion of symmetrically distinguished cycles, in order to deal with algebraic cycles on "nonconnected abelian varieties" in a canonical way. By the result of O'Sullivan [48] (see Theorem 5.5 and Theorem 5.6), it suffices for us to check that the symmetrizations of W and Z are numerically equivalent.

Step (iii)   Finally, in Case (A), explicit computations of the cohomological realization of ϕ show that the induced (iso)morphism $\phi \colon H^*(Y) \to H^*_{\mathrm{orb}}([M/G])$ is the same as the one constructed in [41]. While in Case (B), based on the result of [47], one can prove that the cohomological realization of ϕ satisfies Ruan's original cohomological hyper-Kähler resolution conjecture. Therefore the symmetrizations of W and Z are homologically equivalent, which finishes the proof by Step (ii).

5   Case (A): Hilbert schemes of abelian surfaces

We prove Theorem 1.4 in this section. Our notation is as before: $M := A^n$ with the action of $G := \mathfrak{S}_n$ and the quotient $A^{(n)} := M/G$. Then the Hilbert–Chow morphism

$$\rho \colon A^{[n]} =: Y \to A^{(n)}$$

gives a symplectic resolution.

5.1   A recap of $\mathfrak{S}_n$–equivariant geometry

To fix the conventions and terminology, let us collect here a few basic facts concerning $\mathfrak{S}_n$–equivariant geometry:

5.1.1   The conjugacy classes of the group $\mathfrak{S}_n$ consist of permutations of the same cycle type; hence the conjugacy classes are in bijection to partitions of n. The number of disjoint cycles whose composition is $g \in \mathfrak{S}_n$ is exactly the number $|O(g)|$ of orbits in $\{1, \ldots, n\}$ under the permutation action of $g \in \mathfrak{S}_n$. We will say that $g \in \mathfrak{S}_n$ is *of partition type* λ, denoted by $g \in \lambda$, if the partition determined by g is λ.

5.1.2 Let X be a variety of pure dimension d. Given a permutation $g \in \mathfrak{S}_n$, the fixed locus $(X^n)^g := \operatorname{Fix}_g(X^n)$ can be described explicitly as the partial diagonal

$$\{(x_1, \ldots, x_n) \in X^n \mid x_i = x_j \text{ if } i \text{ and } j \text{ are in the same orbit under the action of } g\}.$$

As in [26], we therefore have the natural identification

$$(X^n)^g = X^{O(g)}.$$

In particular, the codimension of $(X^n)^g$ in X^n is $d(n - |O(g)|)$.

5.1.3 Since g and g^{-1} belong to the same conjugacy class, it follows from the equality $\operatorname{age}(g) + \operatorname{age}(g^{-1}) = \operatorname{codim}((X^n)^g \subset X^n)$ that

$$\operatorname{age}(g) = \frac{d}{2}(n - |O(g)|),$$

as was stated in Example 2.4.

5.1.4 Let $\mathscr{P}(n)$ be the set of partitions of n. Given such a partition

$$\lambda = (\lambda_1 \geq \cdots \geq \lambda_l) = (1^{a_1} \ldots n^{a_n}),$$

where $l := |\lambda|$ is the *length* of λ and $a_i = |\{j \mid 1 \leq j \leq n \text{ and } \lambda_j = i\}|$, we define

$$\mathfrak{S}_\lambda := \mathfrak{S}_{a_1} \times \cdots \times \mathfrak{S}_{a_n}.$$

For $g \in \mathfrak{S}_n$ a permutation of partition type λ, its centralizer $C(g)$, i.e. the stabilizer under the action of $\mathfrak{S}_n$ on itself by conjugation, is isomorphic to the semidirect product:

$$C(g) \simeq (\mathbb{Z}/\lambda_1 \times \cdots \times \mathbb{Z}/\lambda_l) \rtimes \mathfrak{S}_\lambda.$$

Note that the action of $C(g)$ on X^n restricts to an action on $(X^n)^g = X^{O(g)} \simeq X^l$ and the action of the normal subgroup $\mathbb{Z}/\lambda_1 \times \cdots \times \mathbb{Z}/\lambda_l \subseteq C(g)$ is trivial. We denote the quotient by $X^{(\lambda)} := (X^n)^g/C(g) = (X^n)^g/\mathfrak{S}_\lambda$, and we regard the motive $\mathfrak{h}(X^{(\lambda)})$ as the direct summand $\mathfrak{h}((X^n)^g)^{\mathfrak{S}_\lambda}$ inside $\mathfrak{h}((X^n)^g)$ via the pullback along the projection $(X^n)^g \to (X^n)^g/\mathfrak{S}_\lambda$; see Remarks 2.2(iii).

5.2 Step (i): Additive isomorphisms

In this section, we establish an isomorphism between $\mathfrak{h}(Y)$ and $\mathfrak{h}_{\operatorname{orb}}([M/G])$ by using results of [16], and more specifically by constructing correspondences similar to the ones used therein.

Let

(17) $U^g := (A^{[n]} \times_{A^{(n)}} (A^n)^g)_{\mathrm{red}}$

$= \{(z, x_1, \ldots, x_n) \in A^{[n]} \times (A^n)^g \mid \rho(z) = [x_1] + \cdots + [x_n]\}$

be the incidence variety, where $\rho \colon A^{[n]} \to A^{(n)}$ is the Hilbert–Chow morphism. As the notation suggests, U^g is the fixed locus of the induced automorphism g on the *isospectral Hilbert scheme*

$U := U^{\mathrm{id}} = A^{[n]} \times_{A^{(n)}} A^n = \{(z, x_1, \ldots, x_n) \in A^{[n]} \times A^n \mid \rho(z) = [x_1] + \cdots + [x_n]\}.$

Note that $\dim U^g = n + |O(g)| = 2n - \mathrm{age}(g)$ [12] and $\dim(A^{[n]} \times (A^n)^g) = 2 \dim U^g$. For each $g \in G$, we consider the correspondence

(18) $\Gamma_g := (-1)^{\mathrm{age}(g)} U^g \in \mathrm{CH}^{2n - \mathrm{age}(g)}(A^{[n]} \times (A^n)^g),$

which defines a morphism of Chow motives

(19) $\Gamma := \displaystyle\sum_{g \in G} \Gamma_g \colon \mathfrak{h}(A^{[n]}) \to \bigoplus_{g \in G} \mathfrak{h}((A^n)^g)(-\mathrm{age}(g)) =: \mathfrak{h}(A^n, \mathfrak{S}_n),$

where we used the notation from Definition 2.5.

Lemma 5.1 *The algebraic cycle Γ in (19) defines an $\mathfrak{S}_n$–equivariant morphism with respect to the trivial action on $A^{[n]}$ and the action on $\mathfrak{h}(A^n, \mathfrak{S}_n)$ of Definition 2.5.*

Proof For each $g, h \in G$, as the age function is invariant under conjugation, it suffices to show that the following composition is equal to $\Gamma_{hgh^{-1}}$:

$$\mathfrak{h}(A^{[n]}) \xrightarrow{\Gamma_g} \mathfrak{h}((A^n)^g)(-\mathrm{age}(g)) \xrightarrow{h} \mathfrak{h}((A^n)^{hgh^{-1}})(-\mathrm{age}(g)).$$

This follows from the fact that the diagram

$$
\begin{array}{ccccc}
A^{[n]} & \longleftarrow & U^g & \xrightarrow[\simeq]{h} & U^{hgh^{-1}} \\
 & & \downarrow & & \downarrow \\
 & & (A^n)^g & \xrightarrow[\simeq]{h} & (A^n)^{hgh^{-1}}
\end{array}
$$

is commutative. $\square$

As before, $\iota \colon \mathfrak{h}(A^n, G)^G \hookrightarrow \mathfrak{h}(A^n, G)$ and $p \colon \mathfrak{h}(A^n, G) \twoheadrightarrow \mathfrak{h}(A^n, G)^G$ are the inclusion of and the projection onto the G–invariant part. Thanks to Lemma 5.1, we obtain

the desired morphism

(20) $$\phi := p \circ \Gamma \colon \mathfrak{h}(A^{[n]}) \to \mathfrak{h}_{\mathrm{orb}}([A^n/G]) = \mathfrak{h}(A^n, G)^G,$$

which satisfies $\Gamma = \iota \circ \phi$.

Now one can reformulate the result of de Cataldo–Migliorini [16], which actually works for all surfaces, as follows:

Proposition 5.2 *The morphism ϕ is an isomorphism, whose inverse is given by* $\psi := (1/n!)\left(\sum_{g \in G} {}^t U^g\right) \circ \iota$, *where*

$$ {}^t U^g \colon \mathfrak{h}((A^n)^g)(-\operatorname{age}(g)) \to \mathfrak{h}(A^{[n]}) $$

is the transposed correspondence of U^g.

Proof Let $\lambda = (\lambda_1 \geq \cdots \geq \lambda_l) \in \mathscr{P}(n)$ be a partition of n of length l and let A^λ be A^l, equipped with the natural action of $\mathfrak{S}_\lambda$ and with the natural morphism to $A^{(n)}$ that sends $(x_1, \ldots, x_l)$ to $\sum_{j=1}^{l} \lambda_j[x_j]$. Define the incidence subvariety $U^\lambda := (A^{[n]} \times_{A^{(n)}} A^\lambda)_{\mathrm{red}}$. Denote the quotients by $A^{(\lambda)} := A^\lambda/\mathfrak{S}_\lambda$ and $U^{(\lambda)} := U^\lambda/\mathfrak{S}_\lambda$, where the latter is also regarded as a correspondence between $A^{[n]}$ and $A^{(\lambda)}$. See Remarks 2.2(iii) for the use of Chow motives of global quotients, and Paragraph 5.1.4 for our case at hand (i.e. $A^{(\lambda)}$).

The main theorem in [16] asserts that the following correspondence is an isomorphism:

$$\phi' := \sum_{\lambda \in \mathscr{P}(n)} U^{(\lambda)} \colon \mathfrak{h}(A^{[n]}) \xrightarrow{\simeq} \bigoplus_{\lambda \in \mathscr{P}(n)} \mathfrak{h}(A^{(\lambda)})(|\lambda| - n);$$

moreover, the inverse of ϕ' is given by

$$\psi' := \sum_{\lambda \in \mathscr{P}(n)} \frac{1}{m_\lambda} \cdot {}^t U^{(\lambda)} \colon \bigoplus_{\lambda \in \mathscr{P}(n)} \mathfrak{h}(A^{(\lambda)})(|\lambda| - n) \xrightarrow{\simeq} \mathfrak{h}(A^{[n]}),$$

where $m_\lambda = (-1)^{n-|\lambda|} \prod_{j=1}^{|\lambda|} \lambda_j$ is a nonzero constant. To relate our morphism ϕ to the isomorphism ϕ' and also to ψ and ψ' one uses the following elementary lemma:

Lemma 5.3 *One has a natural isomorphism,*

(21) $$\left(\bigoplus_{g \in \mathfrak{S}_n} \mathfrak{h}((A^n)^g)(-\operatorname{age}(g)) \right)^{\mathfrak{S}_n} \xrightarrow{\simeq} \bigoplus_{\lambda \in \mathscr{P}(n)} \mathfrak{h}(A^{(\lambda)})(|\lambda| - n).$$

Proof By regrouping permutations by their partition types, we clearly have

$$\left(\bigoplus_{g \in \mathfrak{S}_n} \mathfrak{h}((A^n)^g)(-\operatorname{age}(g)) \right)^{\mathfrak{S}_n} \cong \bigoplus_{\lambda \in \mathscr{P}(n)} \left(\bigoplus_{g \in \lambda} \mathfrak{h}((A^n)^g) \right)^{\mathfrak{S}_n} (|\lambda| - n).$$

So it suffices to give a natural isomorphism, for any fixed partition $\lambda \in \mathscr{P}(n)$, between $\left(\bigoplus_{g \in \lambda} \mathfrak{h}((A^n)^g) \right)^{\mathfrak{S}_n}$ and $\mathfrak{h}(A^{(\lambda)})$. However, such an isomorphism of motives follows from the isomorphism of quotient varieties

(22)
$$\left(\coprod_{g \in \lambda} (A^n)^g \right) / \mathfrak{S}_n \cong A^\lambda / \mathfrak{S}_\lambda = A^{(\lambda)},$$

where the first isomorphism can be obtained by choosing a permutation $g_0 \in \lambda$ and observing that the centralizer of g_0 is isomorphic to the semidirect product

$$(\mathbb{Z}/\lambda_1 \times \cdots \times \mathbb{Z}/\lambda_l) \rtimes \mathfrak{S}_\lambda,$$

where the normal subgroup $\mathbb{Z}/\lambda_1 \times \cdots \times \mathbb{Z}/\lambda_l$ acts trivially. We remark that there are some other natural choices for the isomorphism in (21), due to different points of view and conventions; but they only differ from ours by a nonzero constant. $\qquad\square$

Now it is easy to conclude the proof of Proposition 5.2. The idea is to relate the isomorphisms ϕ' and ψ' of de Cataldo and Migliorini recalled above to our morphisms ϕ and ψ. Given a partition $\lambda \in \mathscr{P}(n)$, for any $g \in \lambda$, the isomorphism between $(A^n)^g$ and A^λ will identify U^g to U^λ. We have the commutative diagram

$$\begin{array}{ccc}
\coprod_{g \in \lambda} U^g & \longrightarrow & \coprod_{g \in \lambda} (A^n)^g \\
\downarrow{\scriptstyle q} & & \downarrow{\scriptstyle q} \\
U^{(\lambda)} & \longrightarrow & A^{(\lambda)} \\
\downarrow & & \\
A^{[n]} & &
\end{array}$$

where the degree of each of the two quotient-by-$\mathfrak{S}_n$ morphisms q is easily computed: $\deg(q) = n!/\left(\prod_{j=1}^{|\lambda|} \lambda_j \right)$. The natural isomorphism (21) of Lemma 5.3 is simply given by

$$\frac{1}{\deg q} q_* \circ \iota : \mathfrak{h}\left(\coprod_{g \in \lambda} (A^n)^g \right)^{\mathfrak{S}_n} \xrightarrow{\simeq} \mathfrak{h}(A^{(\lambda)}),$$

with inverse given by $p \circ q^*$ (in fact the image of q^* is already $\mathfrak{S}_n$–invariant). Thus the composition of ϕ with the natural isomorphism (21) is equal to

$$\sum_{\lambda \in \mathscr{P}(n)} \left(\frac{1}{\deg q} q_* \circ \sum_{g \in \lambda} (-1)^{\mathrm{age}(g)} U^g \right) = \sum_{\lambda \in \mathscr{P}(n)} \frac{1}{\deg q} q_* \circ q^* \circ (-1)^{|\lambda|-n} U^{(\lambda)}$$

$$= \sum_{\lambda \in \mathscr{P}(n)} (-1)^{|\lambda|-n} U^{(\lambda)},$$

where we used the commutative diagram above for the first equality. As a consequence, ϕ is an isomorphism as $\phi' = \sum_{\lambda \in \mathscr{P}(n)} U^{(\lambda)}$ is one.

Similarly, the composition of the inverse of (21) with ψ is equal to

$$\sum_{\lambda \in \mathscr{P}(n)} \left(\frac{1}{n!} \sum_{g \in \lambda} {}^t U^g \circ q^* \right) = \sum_{\lambda \in \mathscr{P}(n)} \frac{1}{n!} \cdot {}^t U^{(\lambda)} \circ q_* \circ q^*$$

$$= \sum_{\lambda \in \mathscr{P}(n)} \frac{\deg q}{n!} \cdot {}^t U^{(\lambda)} = \sum_{\lambda \in \mathscr{P}(n)} \frac{(-1)^{|\lambda|-n}}{m_\lambda} \cdot {}^t U^{(\lambda)}.$$

Since $\psi' = \sum_{\lambda \in \mathscr{P}(n)} (1/m_\lambda) \cdot {}^t U^{(\lambda)}$ is the inverse of ϕ' by [16], as noted above, ψ is the inverse of ϕ. □

Then to show Theorem 1.4, it suffices to prove Proposition 4.1 in this situation, which will be done in the next two steps.

5.3 Step (ii): Symmetrically distinguished cycles on abelian varieties

The following definition is due to O'Sullivan [48]. Recall that all Chow groups are with rational coefficients. As in [48], we denote in this section by $\overline{\mathrm{CH}}$ the $\mathbb{Q}$–vector space of algebraic cycles modulo the numerical equivalence relation.

Definition 5.4 (symmetrically distinguished cycles [48]) Let A be an abelian variety and $\alpha \in \mathrm{CH}^i(A)$. For each integer $m \geq 0$, denote by $V_m(\alpha)$ the $\mathbb{Q}$–vector subspace of $\mathrm{CH}(A^m)$ generated by elements of the form

$$p_*(\alpha^{r_1} \times \alpha^{r_2} \times \cdots \times \alpha^{r_n}),$$

where $n \leq m$ and $r_j \geq 0$ are integers, and $p \colon A^n \to A^m$ is a closed immersion with each component $A^n \to A$ being either a projection or the composite of a projection with $[-1] \colon A \to A$. Then α is called *symmetrically distinguished* if for every m the restriction of the projection $\mathrm{CH}(A^m) \to \overline{\mathrm{CH}}(A^m)$ to $V_m(\alpha)$ is injective.

Despite their seemingly complicated definition, symmetrically distinguished cycles behave very well. More precisely, we have:

Theorem 5.5 (O'Sullivan [48]) *Let A be an abelian variety. Then:*

(i) *The symmetric distinguished cycles in $\mathrm{CH}^i(A)$ form a $\mathbb{Q}$–vector subspace.*

(ii) *The fundamental class of A is symmetrically distinguished and the intersection product of two symmetrically distinguished cycles is symmetrically distinguished. They form therefore a graded $\mathbb{Q}$–subalgebra of $\mathrm{CH}^*(A)$.*

(iii) *Let $f\colon A \to B$ be a morphism of abelian varieties. Then $f_*\colon \mathrm{CH}(A) \to \mathrm{CH}(B)$ and $f^*\colon \mathrm{CH}(B) \to \mathrm{CH}(A)$ preserve symmetrically distinguished cycles.*

The reason why this notion is very useful in practice is that it allows us to conclude an equality of algebraic cycles modulo rational equivalence from an equality modulo numerical equivalence (or, a fortiori, modulo homological equivalence):

Theorem 5.6 (O'Sullivan [48]) *The composition $\mathrm{CH}(A)_{\mathrm{sd}} \hookrightarrow \mathrm{CH}(A) \twoheadrightarrow \overline{\mathrm{CH}}(A)$ is an isomorphism of $\mathbb{Q}$–algebras, where $\mathrm{CH}(A)_{\mathrm{sd}}$ is the subalgebra of symmetrically distinguished cycles. In other words, in each numerical class of algebraic cycle on A, there exists a unique symmetrically distinguished algebraic cycle modulo rational equivalence. In particular, a (polynomial of) symmetrically distinguished cycles is trivial in $\mathrm{CH}(A)$ if and only if it is numerically trivial.*

Returning to the proof of Theorem 1.4, it remains to prove Proposition 4.1. Keeping the same notation as in Step (i), we first prove that in our situation the two cycles in Proposition 4.1 are symmetrically distinguished.

Proposition 5.7 *The following two algebraic cycles, as well as their symmetrizations, are symmetrically distinguished in $\mathrm{CH}\big(\big(\coprod_{g\in G}(A^n)^g\big)^3\big)$:*

• $W := \big((1/|G|)\sum_g U^g \times (1/|G|)\sum_g U^g \times \sum_g (-1)^{\mathrm{age}(g)} U^g\big)_*(\delta_{A^{[n]}})$;

• *the algebraic cycle Z determining the orbifold product (Definition 2.5(v)) with the sign change by discrete torsion (Definition 3.5):*

$$Z|_{M^{g_1}\times M^{g_2}\times M^{g_3}} = \begin{cases} 0 & \text{if } g_3 \neq g_1 g_2, \\ (-1)^{\epsilon(g_1,g_2)}\cdot \delta_* c_{\mathrm{top}}(F_{g_1,g_2}) & \text{if } g_3 = g_1 g_2. \end{cases}$$

Proof For W, it amounts to showing that $(U^{g_1} \times U^{g_2} \times U^{g_3})_*(\delta_{A^{[n]}})$ is symmetrically distinguished in $\mathrm{CH}((A^n)^{g_1} \times (A^n)^{g_2} \times (A^n)^{g_3})$, for any $g_1, g_2, g_3 \in G$. Indeed, by [60, Proposition 5.6], $(U^{g_1} \times U^{g_2} \times U^{g_3})_*(\delta_{A^{[n]}})$ is a polynomial of big diagonals of $(A^n)^{g_1} \times (A^n)^{g_2} \times (A^n)^{g_3} =: A^N$. However, all big diagonals of A^N are clearly symmetrically distinguished since $\Delta_A \in \mathrm{CH}(A \times A)$ is. By Theorem 5.5, W is symmetrically distinguished.

As for Z, for any fixed $g_1, g_2 \in G$, it is easy to see that F_{g_1, g_2} is always a trivial vector bundle, at least virtually, hence its top Chern class is either 0 or 1 (the fundamental class), which is of course symmetrically distinguished. Also recall that (Definition 2.5)

$$\delta \colon (A^n)^{<g_1, g_2>} \hookrightarrow (A^n)^{g_1} \times (A^n)^{g_2} \times (A^n)^{g_1 g_2},$$

which is a (partial) diagonal inclusion, in particular a morphism of abelian varieties. Thus $\delta_*(c_{\mathrm{top}}(F_{g_1, g_2}))$ is symmetrically distinguished by Theorem 5.5, hence so is Z.

Finally, since any automorphism in $G \times G \times G$ preserves symmetrically distinguished cycles, symmetrizations of Z and W remain symmetrically distinguished. $\qquad \square$

By Theorem 5.6, in order to show Proposition 4.1, it suffices to show on the one hand that the symmetrizations of Z and W are both symmetrically distinguished, and on the other hand that they are numerically equivalent. The first part is exactly the previous Proposition 5.7 and we now turn to an a priori stronger version of the second part in the following final step.

5.4 Step (iii): Cohomological realizations

We will show in this section that the symmetrizations of the algebraic cycles W and Z have the same (rational) cohomology class. To this end, it is enough to show the following:

Proposition 5.8 *The cohomology realization of the (additive) isomorphism*

$$\phi \colon \mathfrak{h}(A^{[n]}) \xrightarrow{\simeq} \left(\bigoplus_{g \in G} \mathfrak{h}((A^n)^g)(-\mathrm{age}(g)) \right)^{\mathfrak{S}_n}$$

is an isomorphism of $\mathbb{Q}$–algebras

$$\bar{\phi} \colon H^*(A^{[n]}) \xrightarrow{\simeq} H^*_{\mathrm{orb,dt}}([A^n/\mathfrak{S}_n]) = \left(\bigoplus_{g \in G} H^{*-2\,\mathrm{age}(g)}((A^n)^g), \star_{\mathrm{orb,dt}} \right)^{\mathfrak{S}_n}.$$

In other words, $\mathrm{Sym}(W)$ *and* $\mathrm{Sym}(Z)$ *are homologically equivalent.*

Before we proceed to the proof of Proposition 5.8, we need to do some preparation on the Nakajima operators (see [44]). Let S be a smooth projective surface. Recall that given a cohomology class $\alpha \in H^*(S)$, the Nakajima operator

$$\mathfrak{p}_k(\alpha) \colon H^*(S^{[r]}) \to H^*(S^{[r+k]}),$$

for any $r \in \mathbb{N}$, is by definition $\beta \mapsto I_{r;k}{}^*(\alpha \times \beta) := q_*(p^*(\alpha \times \beta) \cdot [I_{r;k}])$, where $p \colon S^{[r+k]} \times S \times S^{[r]} \to S \times S^{[r]}$ and $q \colon S^{[r+k]} \times S \times S^{[r]} \to S^{[r+k]}$ are the natural projections and the cohomological correspondence $I_{r;k}$ is defined as the unique irreducible component of maximal dimension of the incidence subscheme

$$\{(\xi', x, \xi) \in S^{[r+k]} \times S \times S^{[r]} \mid \xi \subset \xi' \text{ and } \rho(\xi') = \rho(\xi) + k[x]\}.$$

Here and in the sequel, ρ is always the Hilbert–Chow morphism. To the best of our knowledge, it is still not known whether the above incidence subscheme is irreducible but we do know that there is only one irreducible component with maximal dimension ($= 2r + k + 1$); see [44, Section 8.3; 40, Lemma 1.1].

For our purpose, we need to consider the following generalized version of such correspondences in a fashion similar to [40]. Following that work, the shorthand $S^{[n_1],\dots,[n_r]}$ means the product $S^{[n_1]} \times \cdots \times S^{[n_r]}$. A sequence of $[1]$'s of length n is denoted by $[1]^n$. For any $r, n, k_1, \dots, k_n \in \mathbb{N}$, we consider the closed subscheme of $S^{[r+\sum k_i],[1]^n,[r]}$ whose closed points are given by (see [40] for the natural scheme structure)

$$J_{r;k_1,\dots,k_n} := \left\{ (\xi', x_1, \dots, x_n, \xi) \,\middle|\, \xi \subset \xi' \text{ and } \rho(\xi') = \rho(\xi) + \sum_{i=1}^n k_i [x_i] \right\}.$$

As far as we know, the irreducibility of $J_{r;k_1,\dots,k_n}$ is unknown in general, but we will only need its component of maximal dimension. To this end, we consider the following locally closed subscheme of $S^{[r+\sum k_i],[1]^n,[r]}$ by adding an open condition:

$$I^0_{r;k_1,\dots,k_n}$$
$$:= \left\{ (\xi', x_1, \dots, x_n, \xi) \,\middle|\, \xi \subset \xi', x_i \neq x_j, x_i \cap \xi = \varnothing \text{ and } \rho(\xi') = \rho(\xi) + \sum_i k_i [x_i] \right\}.$$

Let $I_{r;k_1,\dots,k_n}$ be its Zariski closure. By Briançon [12] (see also [40, Lemma 1.1]), $I_{r;k_1,\dots,k_n}$ is irreducible of dimension $2r + n + \sum k_i$ and it is the unique irreducible component of maximal dimension of $J_{r;k_1,\dots,k_n}$. In particular, the correspondence $I_{r;k}$ used by Nakajima mentioned above is the special case when $n = 1$. Let us also mention

that when $r = 0$, we actually have that $J_{0;k_1,\ldots,k_n}$ is irreducible [16, Remark 2.0.1], and hence is equal to $I_{0;k_1,\ldots,k_n}$.

For any $r, n, m, k_1, \ldots, k_n, l_1, \ldots, l_m \in \mathbb{N}$, consider the following diagram, analogous to the one found in [40, page 181]:

$$S^{[r+\sum k_i + \sum l_j],[1]^m,[r+\sum k_i]} \xleftarrow{\;p_{123}\;} S^{[r+\sum k_i + \sum l_j],[1]^m,[r+\sum k_i],[1]^n,[r]} \xrightarrow{\;p_{345}\;} S^{[r+\sum k_i],[1]^n,[r]}$$

$$\left\downarrow{p_{1245}}\right.$$

$$S^{[r+\sum k_i + \sum l_j],[1]^{m+n},[r]}$$

By a similar argument as in [40, page 181] (actually easier since we only need weaker dimension estimates), we see that

- p_{1245} induces an isomorphism from $p_{123}^{-1}(I^0_{r+\sum k_i;l_1,\ldots,l_m}) \cap p_{345}^{-1}(I^0_{r;k_1,\ldots,k_n})$ to $I^0_{r;k_1,\ldots,k_n,l_1,\ldots,l_m}$;
- the complement of $I^0_{r;k_1,\ldots,k_n,l_1,\ldots,l_m}$ in

$$p_{1245}(p_{123}^{-1}(I^0_{r+\sum k_i;l_1,\ldots,l_m}) \cap p_{345}^{-1}(I^0_{r;k_1,\ldots,k_n}))$$

  is of dimension $< 2r + n + m + \sum k_i + \sum l_j = \dim I^0_{r;k_1,\ldots,k_n,l_1,\ldots,l_m}$;
- the intersection of $p_{123}^{-1}(I^0_{r+\sum k_i;l_1,\ldots,l_m})$ and $p_{345}^{-1}(I^0_{r;k_1,\ldots,k_n})$ is transversal at the generic point of $p_{123}^{-1}(I_{r+\sum k_i;l_1,\ldots,l_m}) \cap p_{345}^{-1}(I_{r;k_1,\ldots,k_n})$.

Combining these, we have in particular that

$$(23) \qquad p_{1245,*}(p_{123}^*[I_{r+\sum k_i;l_1,\ldots,l_m}] \cdot p_{345}^*[I_{r;k_1,\ldots,k_n}]) = [I_{r;k_1,\ldots,k_n,l_1,\ldots,l_m}].$$

We will only need the case $r = 0$ and $m = 1$ in the proof of Proposition 5.8.

Proof of Proposition 5.8 The existence of an isomorphism of $\mathbb{Q}$–algebras between the two cohomology rings $H^*(A^{[n]})$ and $H^*_{\mathrm{orb,dt}}([A^n/\mathfrak{S}_n])$ is established by Fantechi and Göttsche [26, Theorem 3.10] based on the work of Lehn and Sorger [41]. Therefore by the definition of ϕ in Step (i), it suffices to show that the cohomological correspondence

$$\Gamma_* := \sum_{g \in \mathfrak{S}_n} (-1)^{\mathrm{age}(g)} U^g_{\;*} \colon H^*(A^{[n]}) \to \bigoplus_{g \in \mathfrak{S}_n} H^{*-2\,\mathrm{age}(g)}((A^n)^g)$$

coincides with the following inverse of the isomorphism Ψ used in Fantechi–Göttsche [26, Theorem 3.10]:

$$\Phi \colon H^*(A^{[n]}) \to \bigoplus_{g \in \mathfrak{S}_n} H^{*-2\,\mathrm{age}(g)}((A^n)^g),$$

$$\mathfrak{p}_{\lambda_1}(\alpha_1) \cdots \mathfrak{p}_{\lambda_l}(\alpha_l) \, \mathbb{1} \mapsto n! \cdot \mathrm{Sym}(\alpha_1 \times \cdots \times \alpha_l).$$

Let us explain the notation from [26] in the above formula: $\alpha_1, \ldots, \alpha_l \in H^*(A)$; the symbol $\times$ stands for the exterior product $\prod \mathrm{pr}_i^*(-)$; $\mathfrak{p}$ is the Nakajima operator; $\mathbb{1} \in H^0(A^{[0]}) \simeq \mathbb{Q}$ is the fundamental class of a point; $\lambda = (\lambda_1, \ldots, \lambda_l)$ is a partition of n; $g \in \mathfrak{S}_n$ is a permutation of type λ with a numbering $\{1, \ldots, l\} \xrightarrow{\cong} O(g)$ of orbits of g (as a permutation) chosen so that λ_j is the length of the jth orbit; the class $\alpha_1 \times \cdots \times \alpha_l$ is placed in the direct summand indexed by g; and Sym means the symmetrization operation $(1/n!) \sum_{h \in \mathfrak{S}_n} h$. Note that $\mathrm{Sym}(\alpha_1 \times \cdots \times \alpha_l)$ is independent of the choice of g, numbering, etc.

A repeated use of (23) with $r = 0$ and $m = 1$, along with the projection formula, yields

$$\mathfrak{p}_{\lambda_1}(\alpha_1) \cdots \mathfrak{p}_{\lambda_l}(\alpha_l) \mathbb{1} = I^*_{0;\lambda_1,\ldots,\lambda_l}(\alpha_1 \times \cdots \times \alpha_l) = U^{\lambda^*}(\alpha_1 \times \cdots \times \alpha_l),$$

where the second equality comes from the definition and the irreducibility of U^λ (see [16, Remark 2.0.1]). As a result, one only has to show that

(24) $$\sum_{g \in \mathfrak{S}_n} (-1)^{\mathrm{age}(g)} U^g_* \circ U^{\lambda^*}(\alpha_1 \times \cdots \times \alpha_l) = n! \cdot \mathrm{Sym}(\alpha_1 \times \cdots \times \alpha_l).$$

Indeed, for a given $g \in G$, if g in not of type λ, then by [16, Proposition 5.1.3], we know that $U^g_* \circ U^{\lambda^*} = 0$. For any $g \in G$ of type λ, fix a numbering $\varphi \colon \{1, \ldots, l\} \xrightarrow{\cong} O(g)$ such that $|\varphi(j)| = \lambda_j$ and let $\widetilde{\varphi} \colon A^\lambda = A^l \to A^{O(g)}$ be the induced isomorphism. Then denoting by $q \colon A^\lambda \twoheadrightarrow A^{(\lambda)}$ the quotient map by $\mathfrak{S}_\lambda$, the computation [16, Proposition 5.1.4] implies that for such $g \in \lambda$,

$$
\begin{aligned}
U^g_* \circ U^{\lambda^*}(\alpha_1 \times \cdots \times \alpha_l) &= \widetilde{\varphi}_* \circ U^\lambda_* \circ U^{\lambda^*}(\alpha_1 \times \cdots \times \alpha_l) \\
&= m_\lambda \cdot \widetilde{\varphi}_* \circ q^* \circ q_*(\alpha_1 \times \cdots \times \alpha_l) \\
&= m_\lambda \cdot \deg(q) \cdot \mathrm{Sym}(\alpha_1 \times \cdots \times \alpha_l) \\
&= m_\lambda \cdot |\mathfrak{S}_\lambda| \cdot \mathrm{Sym}(\alpha_1 \times \cdots \times \alpha_l),
\end{aligned}
$$

where $m_\lambda = (-1)^{n-|\lambda|} \prod_{i=1}^{|\lambda|} \lambda_i$ as before. Putting those together, we have

$$
\begin{aligned}
\sum_{g \in \mathfrak{S}_n} (-1)^{\mathrm{age}(g)} U^g_* U^{\lambda^*}(\alpha_1 \times \cdots \times \alpha_l) &= \sum_{g \in \lambda} (-1)^{n-|\lambda|} U^g_* U^{\lambda^*}(\alpha_1 \times \cdots \times \alpha_l) \\
&= \sum_{g \in \lambda} \left(\prod_{i=1}^{|\lambda|} \lambda_i \right) \cdot |\mathfrak{S}_\lambda| \cdot \mathrm{Sym}(\alpha_1 \times \cdots \times \alpha_l) \\
&= n! \cdot \mathrm{Sym}(\alpha_1 \times \cdots \times \alpha_l),
\end{aligned}
$$

where the last equality is the orbit-stabilizer formula for the action of $\mathfrak{S}_n$ on itself by conjugation. The desired equality (24), hence also the proposition, is proved. □

As explained in Section 4, the proof of Theorem 1.4 is now complete: Proposition 5.7 and Proposition 5.8 together imply that $\mathrm{Sym}(W)$ and $\mathrm{Sym}(Z)$ are rationally equivalent using Theorem 5.6. Therefore Proposition 4.1 holds in our Case (A), which means exactly that the isomorphism ϕ in Proposition 5.2 (defined in (20)) is also multiplicative with respect to the product structure on $\mathfrak{h}(A^{[n]})$ given by the small diagonal and the orbifold product with sign change by discrete torsion on $\mathfrak{h}(A^n, \mathfrak{S}_n)^{\mathfrak{S}_n}$.

6 Case (B): Generalized Kummer varieties

We prove Theorem 1.5 in this section. Our notation is as in the beginning of Section 4:

$$M = A_0^{n+1} := \mathrm{Ker}(A^{n+1} \xrightarrow{+} A),$$

which is noncanonically isomorphic to A^n, with the action of $G = \mathfrak{S}_{n+1}$ and the quotient $X := A_0^{(n+1)} := M/G$. Then the restriction of the Hilbert–Chow morphism to the *generalized Kummer variety*

$$K_n(A) =: Y \xrightarrow{f} A_0^{(n+1)}$$

is a symplectic resolution.

6.1 Step (i): Additive isomorphisms

We use the result in [17] to establish an additive isomorphism $\mathfrak{h}(Y) \xrightarrow{\simeq} \mathfrak{h}_{\mathrm{orb}}([M/G])$.

Recall that a morphism $f \colon Y \to X$ is called *semismall* if for all integers $k \geq 0$, the codimension of the locus $\{x \in X \mid \dim f^{-1}(x) \geq k\}$ is at least $2k$. In particular, f is generically finite. Consider a (finite) Whitney stratification $X = \coprod_a X_a$ by connected strata such that for any a, the restriction $f|_{f^{-1}(X_a)} \colon f^{-1}(X_a) \to X_a$ is a topological fiber bundle of fiber dimension d_a. Then the semismallness condition says that $\mathrm{codim}\, X_a \geq 2d_a$ for any a. In that case, a stratum X_a is said to be *relevant* if equality holds: $\mathrm{codim}\, X_a = 2d_a$.

The result we need for Step (i) is de Cataldo–Migliorini [17, Theorem 1.0.1]. Let us only state their theorem in the special case where all fibers over relevant strata are irreducible, which is enough for our purpose:

Theorem 6.1 [17] *Let $f \colon Y \to X$ be a semismall morphism of complex projective varieties with Y being smooth. Suppose that all fibers over relevant strata are irreducible and that for each connected relevant stratum X_a of codimension $2d_a$ (and fiber dimension d_a), the normalization $\overline{Z}_a$ of the closure $\overline{X}_a$ is projective and admits*

a stratification with strata being finite group quotients of smooth varieties. Then (the closure of) the incidence subvarieties between X_a and Y induce an isomorphism of Chow motives,

$$\bigoplus_a \mathfrak{h}(\overline{Z}_a)(-d_a) \simeq \mathfrak{h}(Y).$$

Moreover, the inverse isomorphism is again given by the incidence subvarieties but with different nonzero coefficients.

Remarks 6.2 • The normalizations $\overline{Z}_a$ are singular, but they are $\mathbb{Q}$–varieties, for which the usual intersection theory works with rational coefficients (see Remarks 2.2).

• The statement about the correspondence inducing isomorphisms as well as the (nonzero) coefficients of the inverse correspondence is contained in [17, Section 2.5].

• Since any symplectic resolution of a (singular) symplectic variety is semismall, the previous theorem applies to the situation of Conjectures 1.3 and 3.6.

• The correspondence in [16] which is used in Section 5 for Case (A) is a special case of Theorem 6.1.

• Theorem 6.1 is used in [62] to deduce a motivic decomposition of generalized Kummer varieties equivalent to Corollary 6.3.

Let us start by making precise a Whitney stratification for the (semismall) symplectic resolution $Y = K_n(A) \to X = A_0^{(n+1)}$. Our notation is as in the proof of Proposition 5.2. Let $\mathscr{P}(n+1)$ be the set of partitions of $n+1$. Then

$$X = \coprod_{\lambda \in \mathscr{P}(n+1)} X_\lambda,$$

where the locally closed strata are defined by

$$X_\lambda := \left\{ \sum_{i=1}^{|\lambda|} \lambda_i [x_i] \in A^{(n+1)} \mid \sum_{i=1}^{|\lambda|} \lambda_i x_i = 0 \text{ with } x_i \text{ distinct} \right\},$$

with normalization of closure

$$\overline{Z}_\lambda = \overline{X}_\lambda^{\text{norm}} = A_0^{(\lambda)} := A_0^\lambda / \mathfrak{S}_\lambda,$$

where

(25) $$A_0^\lambda = \left\{ (x_1, \ldots, x_{|\lambda|}) \in A^\lambda \mid \sum_{i=1}^{|\lambda|} \lambda_i x_i = 0 \right\}.$$

It is easy to see that $\dim X_\lambda = \dim A_0^\lambda = 2(|\lambda| - 1)$ while the fibers over X_λ are isomorphic to a product of Briançon varieties [12] $\prod_{i=1}^{|\lambda|} \mathfrak{B}_{\lambda_i}$, which is irreducible of dimension $\sum_{i=1}^{|\lambda|} (\lambda_i - 1) = n + 1 - |\lambda| = \frac{1}{2} \operatorname{codim} X_\lambda$.

In conclusion, $f\colon K_n(A) \to A_0^{(n+1)}$ is a semismall morphism with all strata being relevant and all fibers over strata being irreducible. One can therefore apply Theorem 6.1 to get the following:

Corollary 6.3 *For each $\lambda \in \mathscr{P}(n+1)$, let*

$$V^\lambda := \left\{ (\xi, x_1, \ldots, x_{|\lambda|}) \,\Big|\, \rho(\xi) = \sum_{i=1}^{|\lambda|} \lambda_i [x_i] \text{ and } \sum_{i=1}^{|\lambda|} \lambda_i x_i = 0 \right\} \subset K_n(A) \times A_0^\lambda$$

be the incidence subvariety, whose dimension is $n - 1 + |\lambda|$. Then the quotients $V^{(\lambda)} := V^\lambda / \mathfrak{S}_\lambda \subset K_n(A) \times A_0^{(\lambda)}$ induce an isomorphism of rational Chow motives,

$$\phi'\colon \mathfrak{h}(K_n(A)) \xrightarrow{\;\simeq\;} \bigoplus_{\lambda \in \mathscr{P}(n+1)} \mathfrak{h}(A_0^{(\lambda)})(|\lambda| - n - 1).$$

Moreover, the inverse $\psi' := \phi'^{-1}$ is induced by $\sum_{\lambda \in \mathscr{P}(n+1)} (1/m_\lambda) V^{(\lambda)}$, where $m_\lambda = (-1)^{n+1-|\lambda|} \prod_{i=1}^{|\lambda|} \lambda_i$ is a nonzero constant.

Similarly to Proposition 5.2 for Case (A), the previous Corollary 6.3 allows us to establish an additive isomorphism between $\mathfrak{h}(K_n(A))$ and $\mathfrak{h}_{\mathrm{orb}}([A_0^{n+1}/\mathfrak{S}_{n+1}])$:

Proposition 6.4 *Let $M = A_0^{n+1}$ with the action of $G = \mathfrak{S}_{n+1}$. Let p and ι denote the projection onto and the inclusion of the G–invariant part of $\mathfrak{h}(M, G)$. For each $g \in G$, let*

(26) $$V^g := (K_n(A) \times_{A_0^{(n+1)}} M^g)_{\mathrm{red}} \subset K_n(A) \times M^g$$

be the incidence subvariety. Then they induce an isomorphism of rational Chow motives,

$$\phi := p \circ \sum_{g \in G} (-1)^{\mathrm{age}(g)} V^g \colon \mathfrak{h}(K_n(A)) \xrightarrow{\;\simeq\;} \left(\bigoplus_{g \in G} \mathfrak{h}(M^g)(-\operatorname{age}(g)) \right)^G.$$

Moreover, its inverse ψ is given by $(1/(n+1)!) \sum_{g \in G} {}^t V^g \circ \iota$.

Proof The proof goes exactly as for Proposition 5.2, with Lemma 5.3 replaced by the following canonical isomorphism:

$$(27) \quad \left(\bigoplus_{g \in \mathfrak{S}_{n+1}} \mathfrak{h}\left((A_0^{n+1})^g \right)(-\operatorname{age}(g)) \right)^{\mathfrak{S}_{n+1}} \xrightarrow{\simeq} \bigoplus_{\lambda \in \mathscr{P}(n+1)} \mathfrak{h}(A_0^{(\lambda)})(|\lambda| - n - 1).$$

Indeed, if λ is the partition determined by g, then it is easy to compute that $\operatorname{age}(g) = n + 1 - |O(g)| = n + 1 - |\lambda|$ and moreover that the quotient of $(A_0^{n+1})^g$ by the centralizer of g, which is $\prod_{i=1}^{|\lambda|} \mathbb{Z}/\lambda_i \mathbb{Z} \rtimes \mathfrak{S}_\lambda$ with $\prod_{i=1}^{|\lambda|} \mathbb{Z}/\lambda_i \mathbb{Z}$ acting trivially, is exactly $A_0^{(\lambda)}$. $\qquad\square$

To show Theorem 1.5, it remains to show Proposition 4.1 in this situation (where all cycles U are actually V of Proposition 6.4).

6.2 Step (ii): Symmetrically distinguished cycles on abelian torsors with torsion structures

Observe that we have the extra technical difficulty that $(A_0^{n+1})^g$ is in general an extension of a finite abelian group by an abelian variety, thus nonconnected. To deal with algebraic cycles on not necessarily connected "abelian varieties" in a canonical way as well as the property of being symmetrically distinguished, we introduce the following category. Roughly speaking, this is the category of *abelian varieties with origin fixed only up to torsion*. It lies between the category of abelian varieties (with origin fixed) and the category of abelian torsors (i.e. varieties isomorphic to an abelian variety, thus without a chosen origin).

Definition 6.5 (abelian torsors with torsion structure) One defines the following category $\mathscr{A}$.

An object of $\mathscr{A}$, called an *abelian torsor with torsion structure*, or an ATTS, is a pair (X, Q_X), where X is a connected smooth projective variety and Q_X is a subset of X such that there exists an isomorphism, as complex algebraic varieties, $f: X \to A$ from X to an abelian variety A which induces a bijection between Q_X and $\operatorname{Tor}(A)$, the set of all torsion points of A. The point here is that the isomorphism f, called a *marking*, usually being noncanonical in practice, is not part of the data of an ATTS.

A morphism between two objects (X, Q_X) and (Y, Q_Y) is a morphism of complex algebraic varieties $\phi: X \to Y$ such that $\phi(Q_X) \subset Q_Y$. Compositions of morphisms are defined in the natural way. Note that by choosing markings, a morphism between

two objects in $\mathscr{A}$ is essentially the composition of a morphism between two abelian varieties followed by a torsion translation.

Denote by $\mathscr{A}\mathscr{V}$ the category of abelian varieties. Then there is a natural functor $\mathscr{A}\mathscr{V} \to \mathscr{A}$ sending an abelian variety A to $(A, \mathrm{Tor}(A))$.

The following elementary lemma provides the kind of examples we will be considering:

Lemma 6.6 (constructing ATTSs and compatibility) *Let A be an abelian variety. Let $f \colon \Lambda \to \Lambda'$ be a morphism of lattices[7] and $f_A \colon A \otimes_\mathbb{Z} \Lambda \to A \otimes_\mathbb{Z} \Lambda'$ be the induced morphism of abelian varieties. Then:*

(i) $\mathrm{Ker}(f_A)$ *is canonically a disjoint union of ATTSs such that*

$$Q_{\mathrm{Ker}(f_A)} = \mathrm{Ker}(f_A) \cap \mathrm{Tor}(A \otimes_\mathbb{Z} \Lambda).$$

(ii) *If one has another morphism of lattices $g \colon \Lambda' \to \Lambda''$ inducing a morphism of abelian varieties $g_A \colon A \otimes_\mathbb{Z} \Lambda' \to A \otimes_\mathbb{Z} \Lambda''$, then the natural inclusion $\mathrm{Ker}(f_A) \hookrightarrow \mathrm{Ker}(g_A \circ f_A)$ is a morphism of ATTSs (on each component).*

Proof For (i), we have the following two short exact sequences of abelian groups:

$$0 \to \mathrm{Ker}(f) \to \Lambda \xrightarrow{\pi} \mathrm{Im}(f) \to 0 \quad \text{and} \quad 0 \to \mathrm{Im}(f) \to \Lambda' \to \mathrm{Coker}(f) \to 0,$$

with $\mathrm{Ker}(f)$ and $\mathrm{Im}(f)$ being lattices. Tensoring them with A, one has exact sequences

$$0 \to A \otimes_\mathbb{Z} \mathrm{Ker}(f) \to A \otimes_\mathbb{Z} \Lambda \xrightarrow{\pi_A} A \otimes_\mathbb{Z} \mathrm{Im}(f) \to 0,$$

$$0 \to T \to A \otimes_\mathbb{Z} \mathrm{Im}(f) \to A \otimes_\mathbb{Z} \Lambda',$$

where $T := \mathrm{Tor}^\mathbb{Z}(A, \mathrm{Coker}(f))$ is a finite abelian group consisting of some torsion points of $A \otimes_\mathbb{Z} \mathrm{Im}(f)$. Then

$$\mathrm{Ker}(f_A) = \pi_A^{-1}(T)$$

is an extension of the finite abelian group T by the abelian variety $A \otimes_\mathbb{Z} \mathrm{Ker}(f)$. Choosing a section of π makes $A \otimes_\mathbb{Z} \Lambda$ the product of $A \otimes_\mathbb{Z} \mathrm{Ker}(f)$ and $A \otimes_\mathbb{Z} \mathrm{Im}(f)$, inside of which $\mathrm{Ker}(f_A)$ is the product of $A \otimes_\mathbb{Z} \mathrm{Ker}(f)$ and the finite subgroup T of $A \otimes_\mathbb{Z} \mathrm{Im}(f)$. This shows that $Q_{\mathrm{Ker}(f_A)} := \mathrm{Ker}(f_A) \cap \mathrm{Tor}(A \otimes_\mathbb{Z} \Lambda)$, which is independent of the choice of the section, makes each connected component of $\mathrm{Ker}(f_A)$, a fiber over T, an ATTS.

---

[7]A *lattice* is a free abelian group of finite rank.

With (i) proved, (ii) is trivial: the torsion structures on $\mathrm{Ker}(f_A)$ and on $\mathrm{Ker}(g_A \circ f_A)$ are both defined by claiming that a point is torsion if it is a torsion point in $A \otimes_{\mathbb{Z}} \Lambda$. $\square$

Before generalizing the notion of symmetrically distinguished cycles to the new category $\mathscr{A}$, we have to first prove the following well-known fact.

Lemma 6.7 *Let A be an abelian variety and $x \in \mathrm{Tor}(A)$ be a torsion point. Then the corresponding torsion translation*

$$t_x \colon A \to A, \quad y \mapsto x + y,$$

acts trivially on $\mathrm{CH}(A)$.

Proof The following proof, which we reproduce for the sake of completeness, is taken from [37, Lemma 2.1]. Let m be the order of x. Let Γ_{t_x} be the graph of t_x. Then one has $\boldsymbol{m}^*(\Gamma_{t_x}) = \boldsymbol{m}^*(\Delta_A)$ in $\mathrm{CH}(A \times A)$, where $\boldsymbol{m}$ is the multiplication by m map of $A \times A$. However, $\boldsymbol{m}^*$ is an isomorphism of $\mathrm{CH}(A \times A)$ by Beauville's decomposition [8]. We conclude that $\Gamma_{t_x} = \Delta_A$, and hence the induced correspondences, which are t_x^* and the identity, respectively, are the same. $\square$

Definition 6.8 (symmetrically distinguished cycles in $\mathscr{A}$) Given an ATTS $(X, Q_X) \in \mathscr{A}$ (see Definition 6.5), an algebraic cycle $\gamma \in \mathrm{CH}(X)$ is called *symmetrically distinguished* if, for a marking $f \colon X \to A$, the cycle $f_*(\gamma) \in \mathrm{CH}(A)$ is symmetrically distinguished in the sense of O'Sullivan (Definition 5.4). By Lemma 6.7, this definition is independent of the choice of marking. An algebraic cycle on a disjoint union of ATTSs is symmetrically distinguished if it is so on each component. We denote by $\mathrm{CH}(X)_{\mathrm{sd}}$ the subspace consisting of symmetrically distinguished cycles.

The following proposition is clear from Theorem 5.5 and Theorem 5.6.

Proposition 6.9 *Let $(X, Q_X) \in \mathrm{Obj}(\mathscr{A})$ be an ATTS. Then:*

(i) $\mathrm{CH}^*(X)_{\mathrm{sd}}$, *the space of symmetric distinguished cycles in* $\mathrm{CH}^*(X)$, *is a graded $\mathbb{Q}$–subalgebra of* $\mathrm{CH}^*(X)$.

(ii) *If $f \colon (X, Q_X) \to (Y, Q_Y)$ is a morphism in $\mathscr{A}$, then $f_* \colon \mathrm{CH}(X) \to \mathrm{CH}(Y)$ and $f^* \colon \mathrm{CH}(Y) \to \mathrm{CH}(X)$ preserve symmetrically distinguished cycles.*

(iii) *The composition $\mathrm{CH}(X)_{\mathrm{sd}} \hookrightarrow \mathrm{CH}(X) \twoheadrightarrow \overline{\mathrm{CH}}(X)$ is an isomorphism. In particular, a (polynomial of) symmetrically distinguished cycles is trivial in $\mathrm{CH}(X)$ if and only if it is numerically trivial.*

We will need the following easy fact to prove that some cycles on an ATTS are symmetrically distinguished by checking it in an ambient abelian variety.

Lemma 6.10 *Let* $i\colon B \hookrightarrow A$ *be a morphism of* ATTS*s which is a closed immersion. Let* $\gamma \in \mathrm{CH}(B)$ *be an algebraic cycle. Then* γ *is symmetrically distinguished in* B *if and only if* $i_*(\gamma)$ *is so in* A.

Proof   One implication is clear from Proposition 6.9(ii). For the other one, assume $i_*(\gamma)$ is symmetrically distinguished in A. By choosing markings, one can suppose that A is an abelian variety and B is a torsion translation by $\tau \in \mathrm{Tor}(A)$ of a subabelian variety of A. Thanks to Lemma 6.7, changing the origin of A to τ does not change the cycle class $i_*(\gamma) \in \mathrm{CH}(A)$, hence one can further assume that B is a subabelian variety of A. By Poincaré reducibility, there is a subabelian variety $C \subset A$ such that the natural morphism $\pi\colon B \times C \to A$ is an isogeny. We have the following diagram:

Because $\pi^*\colon \mathrm{CH}(A) \to \mathrm{CH}(B \times C)$ is an isomorphism with inverse $(1/\deg(\pi))\pi_*$,

$$\gamma = \mathrm{pr}_{1*} \circ j_*(\gamma) = \mathrm{pr}_{1*} \circ \pi^* \circ \frac{1}{\deg(\pi)} \pi_* \circ j_*(\gamma) = \frac{1}{\deg(\pi)} \mathrm{pr}_{1*} \circ \pi^* \circ i_*(\gamma).$$

Since π and pr_1 are morphisms of abelian varieties, the hypothesis that $i_*(\gamma)$ is symmetrically distinguished implies that γ is also symmetrically distinguished by Proposition 6.9(ii).   □

We now turn to the proof of Proposition 4.1 in Case (B), which takes the following form. As is explained in Section 4, with Step (i) being done (Proposition 6.4), this would finish the proof of Theorem 1.5.

Proposition 6.11   (Proposition 4.1 in Case (B))   *In* $\mathrm{CH}\big(\big(\coprod_{g \in G} M^g\big)^3\big)$, *the symmetrizations of the following two algebraic cycles are rationally equivalent:*

- $W := \big((1/|G|) \sum_g V^g \times (1/|G|) \sum_g V^g \times \sum_g (-1)^{\mathrm{age}(g)} V^g\big)_*(\delta_{K_n(A)})$;
- Z *is the cycle determining the orbifold product (Definition 2.5(v))* with the sign change by discrete torsion (Definition 3.5):

$$Z|_{M^{g_1} \times M^{g_2} \times M^{g_3}} = \begin{cases} 0 & \text{if } g_3 \neq g_1 g_2, \\ (-1)^{\epsilon(g_1, g_2)} \cdot \delta_* c_{\mathrm{top}}(F_{g_1, g_2}) & \text{if } g_3 = g_1 g_2. \end{cases}$$

To this end, we apply Proposition 6.9(iii) by proving in this section that they are both symmetrically distinguished (Proposition 6.12) and then verifying in Section 6.3 that they are homologically equivalent (Proposition 6.13).

Let M be the abelian variety $A_0^{n+1} = \{(x_1, \ldots, x_{n+1}) \in A^{n+1} \mid \sum_i x_i = 0\}$ as before. For any $g \in G$, the fixed locus

$$M^g = \left\{ (x_1, \ldots, x_{n+1}) \in A^{n+1} \; \middle| \; \sum_i x_i = 0 \text{ and } x_i = x_{g.i} \text{ for all } i \right\}$$

has the decomposition into connected components

$$(28) \qquad\qquad\qquad M^g = \coprod_{\tau \in A[d]} M_\tau^g,$$

where $d := \gcd(g)$ is the greatest common divisor of the lengths of orbits of the permutation g, $A[d]$ is the set of d–torsion points and the connected component M_τ^g is described as follows.

Let $\lambda \in \mathscr{P}(n+1)$ be the partition determined by g and $l := |\lambda|$ be its length. Choose a numbering $\varphi \colon \{1, \ldots, l\} \xrightarrow{\simeq} O(g)$ of orbits such that $|\varphi(i)| = \lambda_i$. Then $d = \gcd(\lambda_1, \ldots, \lambda_l)$ and φ induces an isomorphism

$$(29) \qquad\qquad\qquad \widetilde{\varphi} \colon A_0^\lambda \xrightarrow{\simeq} M^g,$$

sending $(x_1, \ldots, x_l)$ to $(y_1, \ldots, y_{n+1})$ with $y_j = x_i$ if $j \in \varphi(i)$. Here A_0^λ is defined in (25), which has obviously the decomposition into connected components

$$(30) \qquad\qquad\qquad A_0^\lambda = \coprod_{\tau \in A[d]} A_\tau^{\lambda/d},$$

where

$$A_\tau^{\lambda/d} = \left\{ (x_1, \ldots, x_l) \in A^\lambda \; \middle| \; \sum_{i=1}^l \frac{\lambda_i}{d} x_i = \tau \right\}$$

is connected (noncanonically isomorphic to A^{l-1} as varieties) and is equipped with a canonical ATTS (Definition 6.5) structure, namely, a point of $A_\tau^{\lambda/d}$ is defined to be of torsion (i.e. in $Q_{A_\tau^{\lambda/d}}$) if and only if it is a torsion point (in the usual sense) in the abelian variety A^λ. The decomposition (28) of M^g is the transportation of the decomposition (30) of A_0^λ via the isomorphism (29): $A_\tau^{\lambda/d} \xrightarrow[\simeq]{\widetilde{\varphi}} M_\tau^g$. The component M_τ^g hence acquires a canonical ATTS structure. It is clear that the decomposition (28) and the ATTS structure on components are both independent of the choice of φ. One can also define the ATTS structure on M^g by using Lemma 6.6.

Similar to Proposition 5.7, here is the main result of this section:

Proposition 6.12 *Our notation is as in Proposition 6.11. W and Z, as well as their symmetrizations, are symmetrically distinguished in $\mathrm{CH}\left(\left(\coprod_{g \in G} M^g\right)^3\right)$, where M^g is viewed as a disjoint union of ATTSs as in (28) and symmetrical distinguishedness is in the sense of Definition 6.8.*

Proof For W, it is enough to show that $q_* \circ p^* \circ \delta_*(\mathbb{1}_{K_n(A)})$ is symmetrically distinguished for any $g_1, g_2, g_3 \in G$, where the notation is explained in the following commutative diagram, whose squares are all cartesian and without excess intersections:

$$(31)$$

$$
\begin{array}{c}
(A^{[n+1]})^3 \xleftarrow{\ p''\ } U^{g_1} \times U^{g_2} \times U^{g_3} \xrightarrow{\ q''\ } (A^{n+1})^{g_1} \times (A^{n+1})^{g_2} \times (A^{n+1})^{g_3} \\
\end{array}
$$

$$
A^{[n+1]} \hookrightarrow^{\delta'} (A^{[n+1]})^3/A \xleftarrow{\ p'\ } U^{g_1} \times_A U^{g_2} \times_A U^{g_3} \xrightarrow{\ q'\ } (A^{n+1})^{g_1} \times_A (A^{n+1})^{g_2} \times_A (A^{n+1})^{g_3}
$$

$$
K_n(A) \hookrightarrow^{\delta} K_n(A)^3 \xleftarrow{\ p\ } V^{g_1} \times V^{g_2} \times V^{g_3} \xrightarrow{\ q\ } M^{g_1} \times M^{g_2} \times M^{g_3}
$$

where the incidence subvarieties U^g are defined in (17) in Section 5.2 (with n replaced by $n + 1$); all fiber products in the second row are over A; the second row is the base change by the inclusion of small diagonal $A \hookrightarrow A^3$ of the first row; the third row is the base change by $O_A \hookrightarrow A$ of the second row; finally, δ, δ' and δ'' are various (absolute or relative) small diagonals.

Observe that the two inclusions i and j are in the situation of Lemma 6.6: Let

$$\Lambda := \mathbb{Z}^{O(g_1)} \oplus \mathbb{Z}^{O(g_2)} \oplus \mathbb{Z}^{O(g_3)},$$

which admits a natural morphism u to $\Lambda' := \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$ by weighted sum on each factor (with weights being the lengths of orbits). Let $v \colon \Lambda' \to \Lambda'' := \mathbb{Z} \oplus \mathbb{Z}$ be given by $(m_1, m_2, m_3) \mapsto (m_1 - m_2, m_1 - m_3)$. Then it is clear that i and j are identified with the inclusions

$$\operatorname{Ker}(u_A) \xhookrightarrow{i} \operatorname{Ker}(v_A \circ u_A) \xhookrightarrow{j} A \otimes_{\mathbb{Z}} \Lambda.$$

By Lemma 6.6, $(A^{n+1})^{g_1} \times_A (A^{n+1})^{g_2} \times_A (A^{n+1})^{g_3}$ and $M^{g_1} \times M^{g_2} \times M^{g_3}$ are naturally disjoint unions of ATTSs and the inclusions i and j are morphisms of ATTSs on each component.

Now by functorialities and the base change formula (see [32, Theorem 6.2]), we have

$$j_* \circ q'_* \circ p'^* \circ \delta'_*(\mathbb{1}_{A^{[n+1]}}) = q''_* \circ p''^* \circ \delta''_*(\mathbb{1}_{A^{[n+1]}}),$$

which is a polynomial of big diagonals of $A^{|O(g_1)|+|O(g_2)|+|O(g_3)|}$ by Voisin's result [60, Proposition 5.6] and therefore symmetrically distinguished in particular. By Lemma 6.10, $q'_* \circ p'^* \circ \delta'_*(\mathbb{1}_{A^{[n+1]}})$ is symmetrically distinguished on each component of $(A^{n+1})^{g_1} \times_A (A^{n+1})^{g_2} \times_A (A^{n+1})^{g_3}$.

Again by functorialities and the base change formula, we have

$$q_* \circ p^* \circ \delta_*(\mathbb{1}_{K_n(A)}) = i^* \circ q'_* \circ p'^* \circ \delta'_*(\mathbb{1}_{A^{[n+1]}}).$$

Since i is a morphism of ATTSs on each component (Lemma 6.6), one concludes that $q_* \circ p^* \circ \delta_*(\mathbb{1}_{K_n(A)})$ is symmetrically distinguished on each component. Hence W, being a linear combination of such cycles, is also symmetrically distinguished.

For Z, as in Case (A), it is easy to see that all the obstruction bundles F_{g_1, g_2} are (at least virtually) trivial vector bundles because according to Definition 2.5, there are only tangent/normal bundles of/between abelian varieties involved. Therefore the only nonzero case is the pushforward of the fundamental class of $M^{<g_1, g_2>}$ by the inclusion into $M^{g_1} \times M^{g_2} \times M^{g_1 g_2}$, which is obviously symmetrically distinguished. □

6.3 Step (iii): Cohomological realizations

We keep the notation as before. To finish the proof of Proposition 6.11, hence Theorem 1.5, it remains to show that the cohomology classes of the symmetrizations of W and Z are the same. In other words, they have the same realization for Betti cohomology.

Proposition 6.13 *The cohomology realization of the* (a priori additive) *isomorphism in Proposition 6.4,*

$$\phi\colon \mathfrak{h}(K_n(A)) \xrightarrow{\simeq} \left(\bigoplus_{g \in G} \mathfrak{h}((A_0^{n+1})^g)(-\operatorname{age}(g)) \right)^{\mathfrak{S}_{n+1}},$$

is an isomorphism of $\mathbb{Q}$*–algebras*

$$\bar{\phi}\colon H^*(K_n(A)) \xrightarrow{\simeq} H^*_{\operatorname{orb,dt}}([A_0^{n+1}/\mathfrak{S}_{n+1}])$$

$$= \left(\bigoplus_{g \in \mathfrak{S}_{n+1}} H^{*-2\operatorname{age}(g)}((A_0^{n+1})^g), \star_{\operatorname{orb,dt}} \right)^{\mathfrak{S}_n}.$$

In other words, $\operatorname{Sym}(W)$ *and* $\operatorname{Sym}(Z)$ *are homologically equivalent.*

Proof We use Nieper-Wißkirchen's following description [47] of the cohomology ring $H^*(K_n(A), \mathbb{C})$. Let $s \colon A^{[n+1]} \to A$ be the composition of the Hilbert–Chow morphism followed by the summation map. Recall that s is an isotrivial fibration. In the sequel, if not specified, all cohomology groups are with complex coefficients. We have a commutative diagram

$$
\begin{array}{ccc}
H^*(A) & \xrightarrow{\ s^*\ } & H^*(A^{[n]}) \\
{\scriptstyle \epsilon}\downarrow & & \downarrow{\scriptstyle \text{restr.}} \\
\mathbb{C} & \longrightarrow & H^*(K_n(A))
\end{array}
$$

where the upper arrow s^* is the pullback by s, the lower arrow is the unit map sending 1 to the fundamental class $\mathbb{1}_{K_n(A)}$, the map ϵ is the quotient by the ideal consisting of elements of strictly positive degree and the right arrow is the restriction map. The commutativity comes from the fact that $K_n(A) = s^{-1}(O_A)$ is a fiber. Thus one has a ring homomorphism

$$
R \colon H^*(A^{[n]}) \otimes_{H^*(A)} \mathbb{C} \to H^*(K_n(A)).
$$

Then [47, Theorem 1.7] asserts that this is an isomorphism of $\mathbb{C}$–algebras.

Now consider the following diagram:

$$
\begin{array}{ccc}
H^*(A^{[n+1]}) \otimes_{H^*(A)} \mathbb{C} & \xrightarrow[\simeq]{\ \Phi\ } & \big(\bigoplus_{g \in \mathfrak{S}_{n+1}} H^{*-2\,\mathrm{age}(g)}((A^{n+1})^g)\big)^{\mathfrak{S}_{n+1}} \otimes_{H^*(A)} \mathbb{C} \\
{\scriptstyle R}\Big\downarrow {\scriptstyle\simeq} & & \Big\downarrow{\scriptstyle r} \\
H^*(K_n(A)) & \xrightarrow[\bar{\phi}]{\simeq} & \big(\bigoplus_{g \in \mathfrak{S}_{n+1}} H^{*-2\,\mathrm{age}(g)}((A_0^{n+1})^g)\big)^{\mathfrak{S}_{n+1}}
\end{array}
$$

(32)

• As just stated, the left arrow is an isomorphism of $\mathbb{C}$–algebras, by Nieper-Wißkirchen [47, Theorem 1.7].

• The upper arrow Φ comes from the ring isomorphism (which is exactly the CHRC Conjecture 1.1 for Case (A); see Section 5.4)

$$
H^*(A^{[n+1]}) \xrightarrow{\simeq} \Big(\bigoplus_{g \in \mathfrak{S}_{n+1}} H^{*-2\,\mathrm{age}(g)}((A^{n+1})^g)\Big)^{\mathfrak{S}_{n+1}},
$$

established in [26] based on [41]. By (the proof of) Proposition 5.8, this isomorphism is actually induced by $\sum_g (-1)^{\mathrm{age}(g)} \cdot U_*^g \colon H(A^{[n+1]}) \to \bigoplus_g H((A^{n+1})^g)$, with U^g the incidence subvariety defined in (17). Note that on the upper-right term of the diagram, the ring homomorphism $H^*(A) \to \big(\bigoplus_{g \in \mathfrak{S}_{n+1}} H^{*-2\,\mathrm{age}(g)}((A^{n+1})^g)\big)^{\mathfrak{S}_{n+1}}$

lands in the summand indexed by $g = \mathrm{id}$, and the map $H^*(A) \to H^*(A^{n+1})^{\mathfrak{S}_{n+1}}$ is simply the pullback by the summation map $A^{(n+1)} \to A$.

• The lower arrow is the morphism $\bar{\phi}$ in question. It is already shown in Step (i) Proposition 6.4 to be an isomorphism of vector spaces. The goal is to show that it is also multiplicative.

• The right arrow r is defined as follows. On the one hand, let the image of the unit $1 \in \mathbb{C}$ be the fundamental class of $A_0^{(n+1)}$ in the summand indexed by $g = \mathrm{id}$. On the other hand, for any $g \in \mathfrak{S}_{n+1}$, we have a natural restriction map $H^{*-2\,\mathrm{age}(g)}((A^{n+1})^g) \to H^{*-2\,\mathrm{age}(g)}((A_0^{n+1})^g)$. These will induce a ring homomorphism $H^*(A^{n+1}, \mathfrak{S}_{n+1})_{\mathbb{C}} \to H^*(A_0^{n+1}, \mathfrak{S}_{n+1})_{\mathbb{C}}$ by Lemma 6.14, which is easily seen to be compatible with the $\mathfrak{S}_{n+1}$–action and the ring homomorphisms from $H^*(A)$, hence r is a well-defined homomorphism of $\mathbb{C}$–algebras.

• To show the commutativity of diagram (32), the case for the unit $1 \in \mathbb{C}$ is easy to check. For the case of $H^*(A^{[n+1]})$, it suffices to remark that for any g the diagram

$$
\begin{array}{ccc}
H^*(A^{[n+1]}) & \xrightarrow{\ \mathrm{restr.}\ } & H^*(K_n(A)) \\
U_*^g \downarrow & & V_*^g \downarrow \\
H((A^{n+1})^g) & \xrightarrow[\mathrm{restr.}]{} & H((A_0^{n+1})^g)
\end{array}
$$

is commutative, where V^g is the incidence subvariety defined in (26).

In conclusion, since, in the commutative diagram (32), Φ and R are isomorphisms of $\mathbb{C}$–algebras, r is a homomorphism of $\mathbb{C}$–algebras and $\bar{\phi}$ is an isomorphism of vector spaces, we know that they are all isomorphisms of algebras. Thus Proposition 6.13 is proved assuming the following:

Lemma 6.14 *The natural restriction maps*

$$H^{*-2\,\mathrm{age}(g)}((A^{n+1})^g) \to H^{*-2\,\mathrm{age}(g)}((A_0^{n+1})^g) \quad \text{for all } g \in \mathfrak{S}_{n+1}$$

induce a ring homomorphism $H^*(A^{n+1}, \mathfrak{S}_{n+1}) \to H^*(A_0^{n+1}, \mathfrak{S}_{n+1})$, *where their product structures are given by the orbifold product (see Definition 2.5 or 2.7).*

Proof This is straightforward from the definitions. Indeed, for any $g_1, g_2 \in \mathfrak{S}_{n+1}$, together with $\alpha \in H((A^{n+1})^{g_1})$ and $\beta \in H((A^{n+1})^{g_2})$, since the obstruction bundle F_{g_1, g_2} is a trivial vector bundle, we have

$$
\alpha \star_{\mathrm{orb}} \beta = \begin{cases} i_*(\alpha|_{(A^{n+1})^{<g_1, g_2>}} \cup \beta|_{(A^{n+1})^{<g_1, g_2>}}) & \text{if } \mathrm{rk}\, F_{g_1, g_2} = 0, \\ 0 & \text{if } \mathrm{rk}\, F_{g_1, g_2} \neq 0, \end{cases}
$$

where $i\colon (A^{n+1})^{<g_1,g_2>} \hookrightarrow (A^{n+1})^{g_1 g_2}$ is the natural inclusion. Therefore by the base change for the cartesian diagram without excess intersection,

$$
\begin{array}{ccc}
(A_0^{n+1})^{<g_1,g_2>} & \xhookrightarrow{\ i_0\ } & (A_0^{n+1})^{g_1 g_2} \\
\Big\uparrow & & \Big\uparrow \\
(A^{n+1})^{<g_1,g_2>} & \xhookrightarrow{\ i\ } & (A^{n+1})^{g_1 g_2}
\end{array}
$$

we have

$$
\begin{aligned}
&\alpha \star_{\mathrm{orb}} \beta\big|_{(A_0^{n+1})^{g_1 g_2}} \\
&= \begin{cases}
i_{0*}\big((\alpha|_{(A^{n+1})^{<g_1,g_2>}} \cup \beta|_{(A^{n+1})^{<g_1,g_2>}})|_{(A_0^{n+1})^{<g_1,g_2>}}\big) \\
\qquad\qquad = i_{0*}(\alpha|_{(A_0^{n+1})^{<g_1,g_2>}} \cup \beta|_{(A_0^{n+1})^{<g_1,g_2>}}) & \text{if } \operatorname{rk} F_{g_1,g_2} = 0, \\
0 & \text{if } \operatorname{rk} F_{g_1,g_2} \neq 0
\end{cases} \\
&= \alpha|_{(A_0^{n+1})^{g_1}} \star_{\mathrm{orb}} \beta|_{(A_0^{n+1})^{g_2}},
\end{aligned}
$$

which means that the restriction map is a ring homomorphism. $\qquad\square$

The proof of Proposition 6.13 is finished. $\qquad\square$

Now the proof of Theorem 1.5 is complete: by Proposition 6.12 and Proposition 6.13, we know that, thanks to Proposition 6.9(iii), the symmetrizations of Z and W in Proposition 6.11 are rationally equivalent, which proves Proposition 4.1 in Case (B). Hence the isomorphism ϕ in Proposition 6.4 is an isomorphism of algebra objects between the motive of the generalized Kummer variety $\mathfrak{h}(K_n(A))$ and the orbifold Chow motive $\mathfrak{h}_{\mathrm{orb}}([A_0^{n+1}/\mathfrak{S}_{n+1}])$. $\qquad\square$

We would like to note the following corollary obtained by applying the cohomological realization functor to Theorem 1.5.

Corollary 6.15 (CHRC: Kummer case) *The cohomological hyper-Kähler resolution conjecture is true for* Case (B), *namely, one has an isomorphism of* $\mathbb{Q}$–*algebras,*

$$
H^*(K_n(A), \mathbb{Q}) \simeq H^*_{\mathrm{orb,dt}}([A_0^{n+1}/\mathfrak{S}_{n+1}]).
$$

Remark 6.16 This result has never appeared in the literature. It is presumably not hard to check the CHRC in the case of generalized Kummer varieties directly based on the cohomology result of Nieper-Wißkirchen [47], which is of course one of the key ingredients used in our proof. It is also generally believed that the main result of

Britze's PhD thesis [14] should also imply this result. However, the proof of its main result [14, Theorem 40] seems to be flawed: the linear map Θ constructed in the last line of page 60, which is claimed to be the desired ring isomorphism, is actually the zero map. Nevertheless, the authors believe that it is feasible to check the CHRC in this case with the very explicit description of the ring structure of $H^*(K_n(A) \times A)$ obtained in [14].

7   Application 1: Toward Beauville's splitting property

In this section, a *holomorphic symplectic* variety is always assumed to be smooth projective unless stated otherwise and we require neither the simple connectedness nor the uniqueness up to scalar of the holomorphic symplectic 2–form. Hence examples of holomorphic symplectic varieties include projective deformations of Hilbert schemes of K3 or abelian surfaces, generalized Kummer varieties, etc.

7.1   Beauville's splitting property

Based on [8] and [11], Beauville envisages in [10] the following *splitting property* for all holomorphic symplectic varieties.

Conjecture 7.1   (splitting property: Chow rings)   *Let X be a holomorphic symplectic variety of dimension $2n$. Then one has a canonical bigrading of the rational Chow ring* $\mathrm{CH}^*(X)$, *called a **multiplicative splitting of $\mathrm{CH}^\bullet(X)$ of Bloch–Beilinson type**: for any $0 \leq i \leq 4n$,*

$$(33) \qquad\qquad \mathrm{CH}^i(X) = \bigoplus_{s=0}^{i} \mathrm{CH}^i(X)_s,$$

which satisfies:

- **Multiplicativity** $\mathrm{CH}^i(X)_s \bullet \mathrm{CH}^{i'}(X)_{s'} \subset \mathrm{CH}^{i+i'}(X)_{s+s'}$.

- **Bloch–Beilinson**   *The associated ring filtration*

$$F^j \mathrm{CH}^i(X) := \bigoplus_{s \geq j} \mathrm{CH}^i(X)_s$$

  satisfies the Bloch–Beilinson conjecture (see e.g. [57, Conjecture 11.21]). In particular:

- **$F^1 = \mathrm{CH}_{\mathrm{hom}}$**   *The restriction of the cycle class map*

$$\mathrm{cl}: \bigoplus_{s>0} \mathrm{CH}^i(X)_s \to H^{2i}(X, \mathbb{Q})$$

 is zero.

- **Injectivity**   *The restriction of the cycle class map*

$$\mathrm{cl}: \mathrm{CH}^i(X)_0 \to H^{2i}(X, \mathbb{Q})$$

 is injective.

Following Shen–Vial [53], we would like to strengthen Conjecture 7.1 by using the language of Chow motives, which is, we believe, more fundamental. The following notion, which was introduced in [53], avoids any mention of the Bloch–Beilinson conjecture.

Definition 7.2 (multiplicative Chow–Künneth decomposition)   Given a smooth projective variety X of dimension n, a *self-dual multiplicative Chow–Künneth decomposition* is a direct sum decomposition in the category CHM of Chow motives with rational coefficients,

$$(34) \qquad\qquad \mathfrak{h}(X) = \bigoplus_{i=0}^{2n} \mathfrak{h}^i(X),$$

satisfying the following three properties:

- **Chow–Künneth**   The cohomology realization of the decomposition gives the Künneth decomposition: for each $0 \le i \le 2n$, $H^*(\mathfrak{h}^i(X)) = H^i(X)$.

- **Self-duality**   The dual motive $\mathfrak{h}^i(X)^\vee$ identifies with $\mathfrak{h}^{2n-i}(X)(n)$.

- **Multiplicativity**   The product $\mu: \mathfrak{h}(X) \otimes \mathfrak{h}(X) \to \mathfrak{h}(X)$ given by the small diagonal $\delta_X \subset X \times X \times X$ respects the decomposition: the restriction of μ on the summand $\mathfrak{h}^i(X) \otimes \mathfrak{h}^j(X)$ factorizes through $\mathfrak{h}^{i+j}(X)$.

Such a decomposition induces a (multiplicative) bigrading of the rational Chow ring $\mathrm{CH}^*(X) = \bigoplus_{i,s} \mathrm{CH}^i(X)_s$ by setting

$$(35) \qquad \mathrm{CH}^i(X)_s := \mathrm{CH}^i(\mathfrak{h}^{2i-s}(X)) := \mathrm{Hom}_{\mathrm{CHM}}(\mathbb{1}(-i), \mathfrak{h}^{2i-s}(X)).$$

Conjecturally (see [35]), the associated ring filtration $F^j\mathrm{CH}^i(X) := \bigoplus_{s \ge j} \mathrm{CH}^i(X)_s$ satisfies the Bloch–Beilinson conjecture.

By the definition of motives (see Definition 2.1), a multiplicative Chow–Künneth decomposition is equivalent to a collection of self-correspondences $\{\pi^0, \dots, \pi^{2\dim X}\}$, where $\pi^i \in \mathrm{CH}^{\dim X}(X \times X)$, satisfying

- $\pi^i \circ \pi^i = \pi^i$ for all i;
- $\pi^i \circ \pi^j = 0$ for all $i \neq j$;
- $\pi^0 + \cdots + \pi^{2\dim X} = \Delta_X$;
- $\mathrm{Im}(\pi^i_*: H^*(X) \to H^*(X)) = H^i(X)$;
- $\pi^k \circ \delta_X \circ (\pi^i \otimes \pi^j) = 0$ for all $k \neq i + j$.

The induced multiplicative bigrading on the rational Chow ring $\mathrm{CH}^*(X)$ is given by

$$\mathrm{CH}^i(X)_s := \mathrm{Im}(\pi^{2i-s}_*: \mathrm{CH}^i(X) \to \mathrm{CH}^i(X)).$$

The above Chow–Künneth decomposition is self-dual if the transpose of π^i is equal to $\pi^{2\dim X - i}$.

For later use, we need to generalize the previous notion for Chow motive algebras:

Definition 7.3 Let $\mathfrak{h}$ be an (associative but not necessarily commutative) algebra object in the category CHM of rational Chow motives. Denote by $\mu: \mathfrak{h} \otimes \mathfrak{h} \to \mathfrak{h}$ its multiplication structure. A *multiplicative Chow–Künneth decomposition* of $\mathfrak{h}$ is a direct sum decomposition

$$\mathfrak{h} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{h}^i,$$

such that:

- **Chow–Künneth** The cohomology realization gives the Künneth decomposition: $H^i(\mathfrak{h}) = H^*(\mathfrak{h}^i)$ for all $i \in \mathbb{Z}$.
- **Multiplicativity** The restriction of μ to $\mathfrak{h}^i \otimes \mathfrak{h}^j$ factorizes through $\mathfrak{h}^{i+j}$ for all $i, j \in \mathbb{Z}$.

Now one can enhance Conjecture 7.1 to the following:

Conjecture 7.4 (Conjecture 1.9: motivic splitting property) *Let X be a holomorphic symplectic variety of dimension $2n$. Then we have a canonical (self-dual) multiplicative Chow–Künneth decomposition of $\mathfrak{h}(X)$,*

$$\mathfrak{h}(X) = \bigoplus_{i=0}^{4n} \mathfrak{h}^i(X),$$

*which is moreover **of Bloch–Beilinson–Murre type**, that is, for any $i, j \in \mathbb{N}$,*

(i) $\mathrm{CH}^i(\mathfrak{h}^j(X)) = 0$ *if* $j < i$;

(ii) $\mathrm{CH}^i(\mathfrak{h}^j(X)) = 0$ *if* $j > 2i$;

(iii)   *the realization induces an injective map*

$$\mathrm{Hom}_{\mathrm{CHM}}(\mathbb{1}(-i), \mathfrak{h}^{2i}(X)) \to \mathrm{Hom}_{\mathbb{Q}-HS}(\mathbb{Q}(-i), H^{2i}(X)).$$

One can deduce Conjecture 7.1 from Conjecture 7.4 via (35). Note that the range of s in (33) follows from the first two Bloch–Beilinson–Murre conditions in Conjecture 7.4.

7.2 Splitting property for abelian varieties

Recall that for an abelian variety B of dimension g, using the Fourier transform [6], Beauville [8] constructs a multiplicative bigrading on $\mathrm{CH}^*(B)$,

$$(36) \qquad \mathrm{CH}^i(B) = \bigoplus_{s=i-g}^{i} \mathrm{CH}^i(B)_s \quad \text{for any } 0 \le i \le g,$$

where

$$(37) \qquad \mathrm{CH}^i(B)_s := \{\alpha \in \mathrm{CH}^i(B) \mid \boldsymbol{m}^*\alpha = m^{2i-s}\alpha \text{ for all } m \in \mathbb{Z}\}$$

is the simultaneous eigenspace for all $\boldsymbol{m} \colon B \to B$, the multiplication by $m \in \mathbb{Z}$ map.

Using ideas similar to those of [8], Deninger and Murre [22] constructed a multiplicative Chow–Künneth decomposition (Definition 7.2)

$$(38) \qquad \mathfrak{h}(B) = \bigoplus_{i=0}^{2g} \mathfrak{h}^i(B),$$

with (by [39])

$$(39) \qquad \mathfrak{h}^i(B) \simeq \mathrm{Sym}^i(\mathfrak{h}^1(B)).$$

Moreover, one may choose such a multiplicative Chow–Künneth decomposition to be symmetrically distinguished; see [53, Chapter 7]. This Chow–Künneth decomposition induces, via (35), Beauville's bigrading (37). That such a decomposition satisfies the Bloch–Beilinson condition is the following conjecture of Beauville [6] on $\mathrm{CH}^*(B)$, which is still largely open.

Conjecture 7.5 (Beauville's conjecture on abelian varieties)   *Our notation is as above. For all* $i \in \mathbb{N}$*:*

- $\mathrm{CH}^i(B)_s = 0$ *for* $s < 0$.

- *The restriction of the cycle class map* $\mathrm{cl}\colon \mathrm{CH}^i(B)_0 \to H^{2i}(B, \mathbb{Q})$ *is injective.*

Remark 7.6   As torsion translations act trivially on the Chow rings of abelian varieties (Lemma 6.7), the Beauville–Deninger–Murre decompositions (37) and (38) naturally extend to the slightly broader context of *abelian torsors with torsion structure* (see Definition 6.5).

We collect some facts about the Beauville–Deninger–Murre decomposition (38) for the proof of Theorem 7.9 in the next section. By choosing markings for ATTSs, thanks to Lemma 6.7, we see that ATTSs can be endowed with multiplicative Chow–Künneth decompositions consisting of Chow–Künneth projectors that are symmetrically distinguished, and enjoying the properties embodied in the two following lemmas. Their proofs are reduced immediately to the case of abelian varieties, which are certainly well known.

Lemma 7.7   (Künneth)   *Let B and B' be two abelian varieties (or more generally* ATTS*s). Then the natural isomorphism $\mathfrak{h}(B) \otimes \mathfrak{h}(B') \simeq \mathfrak{h}(B \times B')$ identifies the summand $\mathfrak{h}^i(B) \otimes \mathfrak{h}^j(B)$ as a direct summand of $\mathfrak{h}^{i+j}(B \times B')$ for any $i, j \in \mathbb{N}$.*   □

Lemma 7.8   *Let $f\colon B \to B'$ be a morphism of abelian varieties (or more generally* ATTS*s) of dimensions g and g' respectively. Then:*

- *The pullback $f^* := {}^t\Gamma_f\colon \mathfrak{h}(B') \to \mathfrak{h}(B)$ sends $\mathfrak{h}^i(B')$ to $\mathfrak{h}^i(B)$.*
- *The pushforward $f_* := \Gamma_f\colon \mathfrak{h}(B) \to \mathfrak{h}(B')$ sends $\mathfrak{h}^i(B)$ to $\mathfrak{h}^{i+2g'-2g}(B')$.* □

7.3   Candidate decompositions in Cases (A) and (B)

In the sequel, let A be an abelian surface and we consider the holomorphic symplectic variety X, which is either $A^{[n]}$ or $K_n(A)$. We construct a canonical Chow–Künneth decomposition of X and show that it is self-dual and multiplicative. In Remark 7.12, we observe that this decomposition can be expressed in terms of the Beauville–Deninger–Murre decomposition of the Chow motive of A, and as a consequence we note that Conjecture 7.5 (Beauville) for powers of A implies the Bloch–Beilinson conjecture for X.

We start with the existence of a self-dual multiplicative Chow–Künneth decomposition:

Theorem 7.9   *Given an abelian surface A, let X be*

Case (A)   *the $2n$–dimensional Hilbert scheme $A^{[n]}$; or*

Case (B)   *the nth generalized Kummer variety $K_n(A)$.*

Then X has a canonical self-dual multiplicative Chow–Künneth decomposition.

Remark 7.10 The existence of a self-dual multiplicative Chow–Künneth decomposition of $A^{[n]}$ is not new: it was previously obtained by Vial in [55]. As for the generalized Kummer varieties, if one ignores the multiplicativity of the Chow–Künneth decomposition, which is of course the key point here, then it follows rather directly from de Cataldo and Migliorini's result [17] as explained in Section 6.1 (see Corollary 6.3) and is explicitly written down by Ze Xu [62].

Proof of Theorem 7.9 The following proof works for both cases. Let $M := A^n$, $G := \mathfrak{S}_n$ and $X := A^{[n]}$ in Case (A) and $M := A_0^{n+1}$, $G := \mathfrak{S}_{n+1}$ and $X := K_n(A)$ in Case (B). Thanks to Theorem 1.4 and Theorem 1.5, we have an isomorphism of motive algebras,

$$\mathfrak{h}(X) \xrightarrow{\cong} \left(\bigoplus_{g \in G} \mathfrak{h}(M^g)(-\operatorname{age}(g)), \star_{\mathrm{orb,dt}} \right)^G,$$

whose inverse on each direct summand $\mathfrak{h}(M^g)(-\operatorname{age}(g))$ is given by a rational multiple of the transpose of the induced morphism $\mathfrak{h}(X) \to \mathfrak{h}(M^g)(-\operatorname{age}(g))$. It thus suffices to prove that each direct summand has a self-dual Chow–Künneth decomposition in the sense of Definition 7.2, and that the induced Chow–Künneth decomposition on the motive algebra

$$\mathfrak{h} := \bigoplus_{g \in G} \mathfrak{h}(M^g)(-\operatorname{age}(g)), \quad \text{with } \star_{\mathrm{orb,dt}} \text{ as the product,}$$

is multiplicative in the sense of Definition 7.3. To this end, for each $g \in G$, an application of the Deninger–Murre decomposition (38) to M^g, which is an abelian variety in Case (A) and a disjoint union of ATTSs in Case (B), gives us a self-dual multiplicative Chow–Künneth decomposition

$$\mathfrak{h}(M^g) = \bigoplus_{i=0}^{2 \dim M^g} \mathfrak{h}^i(M^g).$$

Now we define, for each $i \in \mathbb{N}$,

(40) $$\mathfrak{h}^i := \bigoplus_{g \in G} \mathfrak{h}^{i-2\operatorname{age}(g)}(M^g)(-\operatorname{age}(g)).$$

Here, by convention, $\mathfrak{h}^j(M^g) = 0$ for $j < 0$, hence in (40), $\mathfrak{h}^i = 0$ if $i - 2\operatorname{age}(g) > 2\dim(M^g)$ for any $g \in G$, that is, when $i > \max_{g \in G}\{4n - 2\operatorname{age}(g)\} = 4n$.

Then obviously, as a direct sum of Chow–Künneth decompositions,

$$\mathfrak{h} = \bigoplus_{i=0}^{4n} \mathfrak{h}^i$$

is a Chow–Künneth decomposition. It is self-dual because each M^g has dimension $2n - 2 \operatorname{age}(g)$. It remains to show the multiplicativity condition that $\mu \colon \mathfrak{h}^i \otimes \mathfrak{h}^j \to \mathfrak{h}$ factorizes through $\mathfrak{h}^{i+j}$, which is equivalent to saying that for any $i, j \in \mathbb{N}$ and $g, h \in G$, the orbifold product $\star_{\mathrm{orb}}$ (discrete torsion only changes a sign and thus is irrelevant here) restricted to the summand $\mathfrak{h}^{i-2\operatorname{age}(g)}(M^g)(-\operatorname{age}(g)) \otimes \mathfrak{h}^{j-2\operatorname{age}(h)}(M^h)(-\operatorname{age}(h))$ factorizes through $h^{i+j-2\operatorname{age}(gh)}(M^{gh})(-\operatorname{age}(gh))$. Thanks to the fact that the obstruction bundle $F_{g,h}$ is always a trivial vector bundle in both of our cases, we know that (see Definition 2.5) $\star_{\mathrm{orb}}$ either is zero when $\operatorname{rk}(F_{g,h}) \neq 0$ or, when $\operatorname{rk}(F_{g,h}) = 0$, is defined as the correspondence from $M^g \times M^h$ to M^{gh} given by the composition

$$(41) \quad \mathfrak{h}(M^g) \otimes \mathfrak{h}(M^h) \xrightarrow{\simeq} \mathfrak{h}(M^g \times M^h) \xrightarrow{\iota_1^*} \mathfrak{h}(M^{<g,h>}) \xrightarrow{\iota_{2*}} \mathfrak{h}(M^{gh})(\operatorname{codim}(\iota_2)),$$

where

$$M^{gh} \xleftarrow{\iota_2} M^{<g,h>} \xhookrightarrow{\iota_1} M^g \times M^h$$

are morphisms of abelian varieties in Case (A) and morphisms of ATTSs in Case (B). Therefore, one can suppose further that $\operatorname{rk}(F_{g,h}) = 0$, which implies by using (8) that the Tate twists match:

$$\operatorname{codim}(\iota_2) - \operatorname{age}(g) - \operatorname{age}(h) = -\operatorname{age}(gh).$$

Now Lemma 7.7 applied to the first isomorphism in (41) and Lemma 7.8 applied to the last two morphisms in (41) show that, omitting the Tate twists, the summand $\mathfrak{h}^{i-2\operatorname{age}(g)}(M^g) \otimes \mathfrak{h}^{j-2\operatorname{age}(h)}(M^h)$ is sent by μ inside the summand $h^k(M^{gh})$, with index

$$k = i + j - 2\operatorname{age}(g) - 2\operatorname{age}(h) + 2\dim(M^{gh}) - 2\dim(M^{<g,h>}) = i + j - 2\operatorname{age}(gh),$$

where the last equality is by (8) together with the assumption $\operatorname{rk}(F_{g,h}) = 0$.

In conclusion, we get a multiplicative Chow–Künneth decomposition $\mathfrak{h} = \bigoplus_{i=0}^{4n} \mathfrak{h}^i$ with $\mathfrak{h}^i$ given in (40); hence a multiplicative Chow–Künneth decomposition for its G–invariant part, which is isomorphic to $\mathfrak{h}(X)$ as motive algebras. $\square$

The decomposition in Theorem 7.9 is supposed to be Beauville's splitting of the Bloch–Beilinson–Murre filtration on the rational Chow ring of X. In particular:

Conjecture 7.11 (Bloch–Beilinson for X) *Our notation is as in Theorem 7.9. For all $i \in \mathbb{N}$:*

- $\mathrm{CH}^i(X)_s = 0$ *for $s < 0$.*

- *The restriction of the cycle class map* $\mathrm{cl}\colon \mathrm{CH}^i(X)_0 \to H^{2i}(X, \mathbb{Q})$ *is injective.*

As a first step toward this conjecture, let us make the following remark:

Remark 7.12 *Conjecture 7.5 (Beauville's conjecture on abelian varieties) implies Conjecture 7.11.* Indeed, keep the same notation as before. From (40) (together with the canonical isomorphisms (21) and (27)), we obtain

$$\mathrm{CH}^i(A^{[n]})_s = \mathrm{CH}^i(\mathfrak{h}^{2i-s}(A^{[n]}))$$
$$= \left(\bigoplus_{g \in \mathfrak{S}_n} \mathrm{CH}^{i-\mathrm{age}(g)}(\mathfrak{h}^{2i-s-2\,\mathrm{age}(g)}(A^{O(g)})) \right)^{\mathfrak{S}_n}$$
$$= \bigoplus_{\lambda \in \mathscr{P}(n)} \mathrm{CH}^{i+|\lambda|-n}(A^\lambda)_s^{\mathfrak{S}_\lambda},$$

$$\mathrm{CH}^i(K_n A)_s = \mathrm{CH}^i(\mathfrak{h}^{2i-s}(K_n A))$$
$$= \left(\bigoplus_{g \in \mathfrak{S}_{n+1}} \mathrm{CH}^{i-\mathrm{age}(g)}(\mathfrak{h}^{2i-s-2\,\mathrm{age}(g)}(A_0^{O(g)})) \right)^{\mathfrak{S}_{n+1}}$$
$$= \bigoplus_{\lambda \in \mathscr{P}(n+1)} \mathrm{CH}^{i+|\lambda|-n-1}(A_0^\lambda)_s^{\mathfrak{S}_\lambda},$$

in our two cases, respectively, whose vanishing ($s < 0$) and injectivity into cohomology by cycle class map ($s = 0$) follow directly from those of A^λ or A_0^λ.

In fact, [56, Theorem 3] proves more generally that the second point of Conjecture 7.5 (the injectivity of the cycle class map $\mathrm{cl}\colon \mathrm{CH}^i(B)_0 \to H^{2i}(B, \mathbb{Q})$ for all complex abelian varieties B) implies Conjecture 7.11 for all smooth projective complex varieties X whose Chow motive is of abelian type, which is the case for a generalized Kummer variety by Proposition 6.4. Of course, one has to check that our definition of $\mathrm{CH}^i(X)_0$ here coincides with the one in [56], which is quite straightforward.

The Chern classes of a (smooth) holomorphic symplectic variety X are also supposed to be in $\mathrm{CH}^i(X)_0$ with respect to Beauville's conjectural splitting. We can indeed check this in both cases considered here:

Proposition 7.13 *The setup is the same as in Theorem 7.9. The Chern class $c_i(X)$ belongs to $\mathrm{CH}^i(X)_0$ for all i.*

Proof In Case (A), that is, in the case where X is the Hilbert scheme $A^{[n]}$, this is proved in [55]. Let us now focus on Case (B), that is, on the case where X is the generalized Kummer variety $K_n(A)$. Let $\{\pi^i \mid 0 \le i \le 2n\}$ be the Chow–Künneth decomposition of $K_n(A)$ given by (40). We have to show that $c_i(K_n(A)) = (\pi^{2i})_* c_i(K_n(A))$, or equivalently that $(\pi^j)_* c_i(K_n(A)) = 0$ as soon as $(\pi^j)_* c_i(K_n(A))$ is homologically trivial. By Proposition 6.4, it is enough to show that for any $g \in G$, $(\pi^j_{M^g})_*((V^g)_* c_i(K_n(A))) = 0$ as soon as $(\pi^j_{M^g})_*((V^g)_* c_i(K_n(A)))$ is homologically trivial. Here, recall that (28) makes M^g a disjoint union of ATTSs and that $\pi^j_{M^g}$ is a Chow–Künneth projector on M^g which is symmetrically distinguished on each component of M^g. By Proposition 6.9, it is enough to show that $(V^g)_*(c_i(K_n(A)))$ is symmetrically distinguished on each component of M^g. As in the proof of Proposition 6.12, we have for any $g \in G$ the following commutative diagram, whose squares are cartesian and without excess intersections:

$$
(42) \qquad
\begin{array}{ccccc}
A^{[n+1]} & \xleftarrow{\ p'\ } & U^g & \xrightarrow{\ q'\ } & (A^{n+1})^g \\
\big\uparrow & \square & \big\uparrow & \square & \big\uparrow{\scriptstyle i} \\
K_n(A) & \xleftarrow{\ p\ } & V^g & \xrightarrow{\ q\ } & M^g
\end{array}
$$

where the incidence subvariety U^g is defined in (17) in Section 5.2 (with n replaced by $n+1$) and the bottom row is the base change by $O_A \hookrightarrow A$ of the top row. Note that $c_i(K_n(A)) = c_i(A^{[n+1]})|_{K_n(A)}$, since the tangent bundle of A is trivial. Therefore, by functorialities and the base change formula (see [32, Theorem 6.2]), we have

$$(V^g)_*(c_i(K_n(A)) := q_* \circ p^*(c_i(K_n(A))) = i^* \circ q'_* \circ p'^*(c_i(A^{[n+1]})).$$

By Voisin's result [60, Theorem 5.12], $q'_* \circ p'^*(c_i(A^{[n+1]}))$ is a polynomial of big diagonals of $A^{|O(g)|}$, thus symmetrically distinguished in particular. It follows from Proposition 6.9 that $(V^g)_*(c_i(K_n(A)))$ is symmetrically distinguished on each component of M^g. This concludes the proof of the proposition. □

8 Application 2: Multiplicative decomposition theorem of rational cohomology

Deligne's decomposition theorem states the following:

Theorem 8.1 (Deligne [21]) *Let $\pi\colon \mathcal{X} \to B$ be a smooth projective morphism. In the derived category of sheaves of $\mathbb{Q}$–vector spaces on B, there is a decomposition (which is noncanonical in general)*

$$(43) \qquad\qquad R\pi_*\mathbb{Q} \cong \bigoplus_i R^i\pi_*\mathbb{Q}[-i].$$

Both sides of (43) carry a cup product: on the right-hand side the cup product is the direct sum of the usual cup products $R^i\pi_*\mathbb{Q} \otimes R^j\pi_*\mathbb{Q} \to R^{i+j}\pi_*\mathbb{Q}$ defined on local systems, while on the left-hand side the derived cup product $R\pi_*\mathbb{Q} \otimes R\pi_*\mathbb{Q} \to R\pi_*\mathbb{Q}$ is induced by the (derived) action of the relative small diagonal $\delta \subset \mathcal{X} \times_B \mathcal{X} \times_B \mathcal{X}$ seen as a relative correspondence from $\mathcal{X} \times_B \mathcal{X}$ to $\mathcal{X}$. As explained in [59], the isomorphism (43) does not respect the cup product in general. Given a family of smooth projective varieties $\pi\colon \mathcal{X} \to B$, Voisin [59, Question 0.2] asked if there exists a decomposition as in (43) which is multiplicative, i.e. which is compatible with the cup product, maybe over a nonempty Zariski open subset of B. By Deninger–Murre [22], there does exist such a decomposition for an abelian scheme $\pi\colon \mathcal{A} \to B$. The main result of [59] is:

Theorem 8.2 (Voisin [59]) *For any smooth projective family $\pi\colon \mathcal{X} \to B$ of K3 surfaces, there exist a decomposition isomorphism as in (43) and a nonempty Zariski open subset U of B such that this decomposition becomes multiplicative for the restricted family $\pi|_U\colon \mathcal{X}|_U \to U$.*

As implicitly noted in [55, Section 4], Voisin's result (Theorem 8.2) holds more generally for any smooth projective family $\pi\colon \mathcal{X} \to B$ whose generic fiber admits a multiplicative Chow–Künneth decomposition (K3 surfaces do have a multiplicative Chow–Künneth decomposition; this follows by suitably reinterpreting, as in [53, Proposition 8.14], the vanishing of the modified diagonal cycle of Beauville–Voisin [11] as the multiplicativity of the Beauville–Voisin Chow–Künneth decomposition):

Theorem 8.3 *Let $\pi\colon \mathcal{X} \to B$ be a smooth projective family, and assume that the generic fiber X of π admits a multiplicative Chow–Künneth decomposition. Then there exist a decomposition isomorphism as in (43) and a nonempty Zariski open subset U of B such that this decomposition becomes multiplicative for the restricted family $\pi|_U\colon \mathcal{X}|_U \to U$.*

Proof By spreading out a multiplicative Chow–Künneth decomposition of X, there exist a sufficiently small but nonempty Zariski open subset U of B and relative correspondences $\Pi^i \in \mathrm{CH}^{\dim_B \mathcal{X}}(\mathcal{X}|_U \times_U \mathcal{X}|_U)$, for $0 \le i \le 2\dim_B \mathcal{X}$, forming a relative Chow–Künneth decomposition, which means that $\Delta_{\mathcal{X}|_U / U} = \sum_i \Pi^i$, $\Pi^i \circ \Pi^i = \Pi^i$, $\Pi^i \circ \Pi^j = 0$ for $i \ne j$, and Π^i acts as the identity on $R^i(\pi^{[n]}|_U)_* \mathbb{Q}$ and as zero on $R^j(\pi^{[n]}|_U)_* \mathbb{Q}$ for $j \ne i$. By [59, Lemma 2.1], the relative idempotents Π^i induce a decomposition in the derived category,

$$R(\pi|_U)_* \mathbb{Q} \cong \bigoplus_{i=0}^{4n} H^i(R(\pi|_U)_* \mathbb{Q})[-i] = \bigoplus_{i=0}^{4n} R^i(\pi|_U)_* \mathbb{Q}[-i],$$

with the property that Π^i acts as the identity on the summand $H^i(R(\pi|_U)_* \mathbb{Q})[-i]$ and acts as zero on the summands $H^j(R(\pi|_U)_* \mathbb{Q})[-j]$ for $j \ne i$. In order to establish the existence of a decomposition as in (43) that is multiplicative and hence to conclude the proof of the theorem, we thus have to show that $\Pi^k \circ \delta \circ (\Pi^i \times \Pi^j)$ acts as zero on $R(\pi|_U)_* \mathbb{Q} \otimes R(\pi|_U)_* \mathbb{Q}$, after possibly further shrinking U, whenever $k \ne i+j$. But more is true: being generically multiplicative, the relative Chow–Künneth decomposition $\{\Pi^i\}$ is multiplicative, that is, $\Pi^k \circ \delta \circ (\Pi^i \times \Pi^j) = 0$ whenever $k \ne i+j$, after further shrinking U if necessary. The theorem is now proved. □

As a corollary, we can extend Theorem 8.2 to families of generalized Kummer varieties:

Corollary 8.4 *Let $\pi\colon \mathcal{A} \to B$ be an abelian surface over B. Consider*

Case (A) $\mathcal{A}^{[n]} \to B$ *the relative Hilbert scheme of length-n subschemes on $\mathcal{A} \to B$; or*

Case (B) $K_n(\mathcal{A}) \to B$ *the relative generalized Kummer variety.*

Then, in both cases, there exist a decomposition isomorphism as in (43) and a nonempty Zariski open subset U of B such that this decomposition becomes multiplicative for the restricted family over U.

Proof The generic fiber of $\mathcal{A}^{[n]} \to B$ (resp. $K_n(\mathcal{A}) \to B$) is the $2n$–dimensional Hilbert scheme (resp. generalized Kummer variety) attached to the abelian surface that is the generic fiber of π. By Theorem 7.9, it admits a multiplicative Chow–Künneth decomposition. (Strictly speaking, we only established Theorem 7.9 for Hilbert schemes of abelian surfaces and generalized Kummer varieties over the complex numbers; however, the proof carries through over any base field of characteristic zero.) We conclude by invoking Theorem 8.3. □

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