SYMPLECTIC GEOMETRY OF RATIONALLY CONNECTED THREEFOLDS

ZHIYU TIAN

Abstract
We study the symplectic geometry of rationally connected 3*-folds. The first result shows that rational connectedness is a symplectic deformation invariant in dimension* 3*. If a rationally connected* 3*-fold* X *is Fano or has Picard number* 2*, we prove that there is a nonzero Gromov–Witten invariant with two insertions being the class of a point. That is,* X *is symplectic rationally connected. Finally we prove that many rationally connected* 3*-folds are birational to a symplectic rationally connected variety.*

1. Introduction

In this paper we study the symplectic geometry of smooth projective rationally connected 3-folds over the complex numbers. First recall the following definitions.

Definition 1.1
A variety X is called *rationally connected* if two general points in X can be connected by a rational curve.

A related notion is uniruledness.

Definition 1.2
A variety X is called *uniruled* if there exists a rational curve through a general point.

The motivation of this paper is the following theorem, proved independently by Kollár and Ruan.

THEOREM 1.3 (Kollár [6, Theorem 4.2.10], Ruan [16, Proposition G, Proposition 4.9])
Let X *be a smooth projective uniruled variety. Then there is a nonzero Gromov–Witten invariant of the form* $\langle [pt], \dots \rangle^X_{0,\beta}$.

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The proof of this theorem is given at the end of this section. We now discuss the two generalizations of this theorem to the case of rationally connected varieties.

1.1. Symplectic topology of rationally connected varieties

In dimensions 1 and 2, rational connectedness is a topological property in the sense that a smooth projective variety of dimension at most 2 is rationally connected if and only if it is *diffeomorphic* to a rationally connected variety. However this fails in higher dimensions. For more details and many other interesting problems about rationally connected varieties, see [6].

Given Theorem 1.3, we could ask if rational connectedness is a symplectic topological property. If X and X' are two smooth projective varieties, then they can also be considered symplectic manifolds with symplectic form ω and ω' given by the polarizations. We say that X and X' are *symplectic deformation equivalent* if there is a family of symplectic manifolds (X_t, ω_t) diffeomorphic to each other such that (X_0, ω_0) (resp., (X_1, ω_1)) is isomorphic to (X, ω) (resp., (X', ω')) as a symplectic manifold. Since Gromov–Witten invariants are symplectic deformation invariants, Kollár's and Ruan's result implies that uniruledness is a symplectic deformation invariant. Motivated by this result, Kollár conjectured the following.

CONJECTURE 1.4 (Kollár, [6])
Let X and X' be two smooth projective varieties which are symplectic deformation equivalent. Then X is rationally connected if and only if X' is.

The first evidence of this conjecture in higher dimensions is the following theorem of Voisin [19].

THEOREM 1.5 ([19, Theorem 0.8])
Let X and X' be two smooth projective 3-folds which are symplectic deformation equivalent. If X is Fano or rationally connected with Picard number 2, then X' is also rationally connected.

The idea of the proof in [19] is the following. It suffices to show that the *maximal rationally connected quotient (MRC quotient)* of X' is a point. By the result of Kollár and Ruan (Theorem 1.3), the MRC quotient is either a surface, a curve, or a point. For topological reasons, it cannot be a curve. If it is a surface, then X is birational to a conic bundle over a rational surface. Then the condition of being Fano or having Picard number 2 enables us to show that X is actually a conic bundle over a surface. Then Voisin shows that there is a nonzero (higher genus) Gromov–Witten invariant of the form

$$\langle \underbrace{[C], \ldots, [C]}_{\geq (g+1)[C]}, [A]^2, \ldots, [A]^2 \rangle^X_{g,\beta},$$

where $[C]$ is the Poincaré dual of the curve class of a general fiber and $[A]$ is the class of a very ample divisor. Thus the MRC quotient surface has to be uniruled, which is impossible by results in [2].

In this paper we attack this problem in dimension 3 by using the same strategy as in [19] together with techniques of handling Gromov–Witten invariants under blow-ups and blow-downs. We first construct a smooth birational model Y which is *almost* a conic bundle over a *smooth* rational surface. On this new 3-fold, there is a nonzero *genus zero* Gromov–Witten invariant of the form $\langle [C], \ldots \rangle^Y_{0,\beta}$. By weak factorization, we can factorize the birational map from Y to X by a number of blow-ups and blow-downs. Then using the ideas developed in [14] and [3], we show that there is a similar nonzero descendant Gromov–Witten invariant on X, and hence on X'. So the MRC quotient of X' cannot be a surface. In this way we have verified Conjecture 1.4 for 3-folds.

THEOREM 1.6
Let X and X' be two smooth projective 3-folds which are symplectic deformation equivalent. Then X is rationally connected if and only if X' is.

1.2. Symplectic birational geometry

Next we would like to mention the so-called "symplectic birational geometry program." The ultimate goal of this program is to carry out a "birational" classification of symplectic manifolds. In this paper we restrict ourselves to the study of a particular class of symplectic manifolds.

Definition 1.7
A symplectic manifold is *symplectic uniruled* (resp., *symplectic rationally connected*) if there is a nonzero Gromov–Witten invariant of the form $\langle [pt], \ldots \rangle^X_{0,\beta}$ (resp., $\langle [pt], [pt], \ldots \rangle^X_{0,\beta}$).

There are two basic questions about these definitions.

(1) Are these conditions symplectic birational invariant?
(2) Is a smooth projective uniruled (resp., rationally connected) variety symplectic uniruled (resp., symplectic rationally connected)?

For symplectic uniruledness, the answer is positive (see [3], [6], [16]). It is not known if symplectic rational connectedness is a (symplectic) birational invariant, although we do expect this to be true. And it is not known if rationally connected projective manifolds are symplectic rationally connected. Note that Kollár's conjecture would follow if we could show that rational connectedness implies symplectic rational connectedness.

Every one-dimensional rationally connected variety is just $\mathbb{P}^1$, thus symplectic rationally connected. It is also easy to prove rational connectedness is equivalent to symplectic rational connectedness in dimension 2 (cf. Proposition 2.2).

In general, this question is very difficult since the moduli space of stable maps might be reducible and there might be components whose dimensions are higher than the expected dimension. Then one has to introduce the *virtual fundamental class* in order to define the Gromov–Witten invariants. The components of dimension higher than the expected dimension can contribute negatively, thus making the Gromov–Witten invariant zero. For an example where this does happen, see [7, Example 42].

Our second theorem addresses this question in some special cases.

THEOREM 1.8
Let X be a smooth projective rationally connected 3-fold. If X is Fano or has Picard number 2, then there is a nonzero Gromov–Witten invariant of the form $\langle [pt], [pt], \dots \rangle^X_{0,\beta}$.

Here is one possible way to prove that every rationally connected variety is symplectic rationally connected. First show that symplectic rational connectedness is a birational invariant, and then find in each birational class a "good" representative which is symplectic rationally connected. In this paper, we try to carry out the second part in dimension 3.

By the minimal model program (MMP) in dimension 3, every rationally connected variety is birational to one of the following:

(1) a conic bundle over a rational surface,

(2) a fibration over $\mathbb{P}^1$ with general fiber a Del Pezzo surface, or

(3) a $\mathbb{Q}$-Fano 3-fold.

Here is our third theorem.

THEOREM 1.9
Let X be a $\mathbb{Q}$-factorial rationally connected 3-fold with at worst terminal singularities, and let $\pi : X \to S$ be a fiber-type contraction of some K_X-negative extremal face (equivalently, X is a Mori fiber space). Assume that one of the following holds:

(1) $\dim S \geq 1$*, that is, X is a conic bundle or a Del Pezzo fibration;*

(2) $\dim S = 0$*, and the smooth locus of X is rationally connected.*

Then there is a resolution of singularities $\tilde{X} \to X$ such that $\tilde{X}$ is symplectic rationally connected.

Note that we do not assume that the relative Picard number $\rho(X/S)$ is 1 in the above theorem.

In general, it is difficult to determine if the smooth locus of a singular variety is rationally connected. However in this paper we note that Gorenstein $\mathbb{Q}$-Fano 3-folds satisfy this condition.

1.3. Symplectic analogues of a theorem of Graber, Harris, and Starr ([2])

We want to mention a related interesting question. By the theorem of Graber, Harris, and Starr [2], a rationally connected fibration over a curve always has a section. As a corollary, the total space of a rationally connected fibration over a rationally connected variety is itself rationally connected. It would be interesting to know if a similar result holds in the symplectic category.

Question 1.10

Let $X \to C$ be a projective morphism from a smooth projective variety to a smooth curve such that a general fiber is rationally connected (or symplectic rationally connected). Is there a nonzero Gromov–Witten invariant given by some curve whose class is a section?

Question 1.11

Let $X \to Y$ be a morphism between smooth projective varieties. Assume that Y and a general fiber are symplectic rationally connected. Is X symplectic rationally connected?

A first step might be to analyze the case of rationally connected fibrations over a rational curve. If a general fiber is $\mathbb{P}^1$, then (1) is true (cf. Proposition 2.2). Part of Theorem 1.9 is also an attempt to analyze a special case: a fibration in Del Pezzo surfaces coming from a contraction. We prove that there is a section on the resolution which gives the nonzero Gromov–Witten invariant with two insertions being the class of a point.

Understanding this symplectic version of the Graber–Harris–Starr theorem may help solve the question whether a rationally connected variety is symplectic rationally connected since, in many cases, the MMP produces birational models which are fibrations.

1.4. Proof of Theorem 1.3

We conclude this introduction by giving the proof of Theorem 1.3 from [6] and [16] and by explaining why the same strategy does not work for the case of rationally connected varieties.

We need to introduce some terminology.

Definition 1.12
Let X be a smooth variety. A curve $f : \mathbb{P}^1 \to X$ is called *free* (resp., *very free*) if $f^*T_X \cong \oplus\mathcal{O}_{\mathbb{P}^1}(a_i)$ with $a_i \geq 0$ (resp., $a_i \geq 1$).

A smooth projective variety is uniruled (resp., rationally connected) if and only if there is a free (resp., very free) curve.

Proof of Theorem 1.3
We first choose a polarization of X. Then there exists a free curve C of minimal degree with respect to the polarization. Note that every rational curve through a very general point p in X is free. So if we choose such a point and consider all the curves mapping to X of class $[C]$ and passing through p, then we get a proper family (by minimality) of expected dimension. (The deformation is unobstructed.) Therefore the Gromov–Witten invariant $\langle [pt], [A]^2, \ldots, [A]^2 \rangle^X_{0,[C]}$ is nonzero, where $[A]$ is the class of a very ample divisor. Clearly this is the number of curves meeting all the constraints. □

One may want to prove the similar result for rationally connected varieties by choosing the minimal curve class such that a curve in this curve class connects two general points. However it may happen that every such curve is reducible and disappears even after an algebraic deformation. For example, in a Hirzebruch surface $\mathbb{F}_n$ with n large, we can choose a suitable polarization such that the minimal such curve is the union of two general fibers and the section at infinity. If n is large, this curve disappears after an algebraic deformation. Thus it cannot give a nonzero Gromov–Witten invariant.

If one insists on choosing a minimal very free curve, then there may be reducible curves in the same curve class that lie in a component whose dimension is higher than the expected dimension. For example, in the case of the Hirzebruch surface $\mathbb{F}_n$, the curve class of the minimal very free curve is $[S_\infty + nF]$, where S_∞ is the section at infinity, and F is a general fiber. Then the union of the multicovers of two general fibers and S_∞ gives a component whose dimension is higher than the expected dimension. In general there is no way to determine the contribution of such components other than an explicit computation.

2. Conic bundles

Our main goal in this section is to prove the following theorem.

THEOREM 2.1
Let $X \to \Sigma$ be a surjective morphism from a smooth projective rationally connected 3-fold X to a smooth projective surface Σ such that a general fiber is isomorphic

to $\mathbb{P}^1$. Then X is symplectic rationally connected. There is also a nonzero Gromov–Witten invariant of the form $\langle [C], \ldots, [C], [A]^2, \ldots, [A]^2\rangle^X_{0,\beta}$, where $[C]$ is the class of a general fiber and $[A]$ is the class of a very ample divisor.

As preparation, we prove the following proposition. There are easier ways to prove the result. But here we present a proof which only depends on MMP on surfaces and requires no further knowledge about the classification of rational surfaces. This proof actually motivates the results in Section 6.

PROPOSITION 2.2
Let Σ be a rationally connected surface. Then there is a nonzero Gromov–Witten invariant of the form $\langle [pt], [pt], \ldots, [pt]\rangle^\Sigma_{0,\beta}$.

Proof
We can run the MMP for Σ. Then we have a sequence of contractions of (-1)-curves:

$$\Sigma = X_0 \to X_1 \to \cdots \to X_n,$$

where X_n is either a geometrically ruled surface over $\mathbb{P}^1$ or a Fano surface of Picard number 1 (thus is $\mathbb{P}^2$, but we do not need this).

In the former case, we may choose a section s_0 of the ruled surface and take the curve class to be $[s_0] + kF$, where F is a fiber class. If we take k large enough, a general curve in this class is an embedded very free curve passing through m (≥ 3) general points. We want to show that all the curves in this curve class meeting m general points are necessarily irreducible when m is chosen to be large enough. Note that any reducible curve in this class is the union of a section and some fibers. Passing through a point is a codimension 1 condition while any fiber curve has anticanonical degree at least 2. Thus if the section is not free, then there cannot be too many fiber curves once we have a lower bound of the anticanonical degree of the section and m is large. But $\pi^*\mathcal{O}_{\mathbb{P}^1}(1) - \epsilon K_{X_n}$ is ample for some small positive number ϵ; thus the $-K_{X_n}$-degree is bounded below. Once we know that the section is free, it is easy to see that we can choose the points to be general to avoid any reducible curves. For details and the generalization to dimension 3 of this argument, see Section 6.

In the latter case, first choose a minimal free rational curve. If it is already very free, then we are done. If not, then notice that we can take the union of two such general curves and a general deformation is a very free curve of $-K_{X_n}$ degree 4. Note that any irreducible rational curve passing through k general points in a surface X_n has $-K_{X_n}$-degree at least $k+1$. The consideration of anticanonical degree shows that any curve that can contribute to the Gromov–Witten invariant has to be irreducible, and thus contributes positively.

Then the proposition follows by comparing Gromov–Witten invariants on X and X_n. Indeed, our argument above shows that the Gromov–Witten invariants considered here are enumerative, and if we choose the points to be general enough, every curve with nonzero contribution is away from the blow-up centers. So the corresponding Gromov–Witten invariant of X is the same as that of X_n (for more details, see the proof of Corollary 4.9). □

The proof of Theorem 2.1 basically follows the same line as [19, Theorem 2.4], although the setup and statements are slightly different. We only point out necessary changes.

Proof of Theorem 2.1
By Proposition 2.2, there is a nonzero enumerative Gromov–Witten invariant of the form

$$\langle \underbrace{[pt], [pt], \ldots, [pt]}_{r[pt]} \rangle^{\Sigma}_{0,\beta}.$$

We can choose β such that a general curve of class β is an embedded curve. Here we need to know that a Del Pezzo surface of Picard number 1 is $\mathbb{P}^2$. The curve class corresponds to a free linear system. So we may choose the constraints in Σ to be general such that if Γ is the curve through these points, then $Z = \pi^{-1}(\Gamma)$ is a smooth surface.

Let s_0 be a section of $Z \to \Gamma$. Choose k large enough. Then it is easy to see that any curve in X in the class $[s_0 + kC]$ which meets 2 general points and $r-2$ general fibers (or r general fibers C) has to be mapped to an irreducible curve in Σ through r general points. Thus the curve lies in Z. We may take other constraints to be curves meeting the surface Z at finitely many points. There are some positive contributions to the Gromov–Witten invariant coming from these irreducible section curves.

The issue here is that the map $i_* : H_2(Z, \mathbb{Z}) \to H_2(X, \mathbb{Z})$ is not injective. So we have to consider the contributions coming from curves classes in Z whose image under i_* is also $s_0 + kC$. This has been done in [19]. By deforming Z to be the blowup of some Hirzebruch surface at *distinct* points, it is shown there that all such contributions are nonnegative. □

Remark 2.3
We need this theorem in three situations, in the proof of Theorem 1.6, the Picard number 2 case of Theorem 1.8, and the conic bundle case of Theorem 1.9. For our purpose, we can avoid doing such a deformation as in [19].

In the first and third case, we are allowed to make a birational modification. And we can use relative MMP to get a *better* birational model in the following way. In the first case, first run a relative MMP to get a conic bundle over the base (which is smooth). In the third case, first take a resolution of singularities of the base, then a base change and a resolution of singularities of the total space, and finally a relative MMP over the base. Thus we are in the following situation: Y is a rationally connected 3-fold with terminal singularities, $Y \to \Sigma$ is a conic bundle coming from the contraction of a K_Y-negative extremal ray, and Σ is smooth. The 3-fold Y has only isolated singularities. Take a strong resolution of singularities $X \to Y$ which is isomorphic over the smooth locus of Y. The exceptional divisors of $X \to Y$ are mapped to isolated points in Σ. Then Z is the blow-up of some Hirzebruch surface at distinct point in distinct fibers.

In the second case X is a smooth 3-fold and $X \to \Sigma$ is a contraction of type (C); then Σ is smooth. In this case Z is also the blow-up of some Hirzebruch surface at *distinct* point.

3. Symplectic deformation invariance for rationally connected 3-folds

In this section we prove symplectic deformation invariance for rationally connected 3-folds. As indicated in the introduction, we need to study the change of Gromov–Witten invariants under blow-ups/blow-downs. We first review the degeneration formula and relative Gromov–Witten invariants needed in the proof. Then we prove a blow-up/blow-down correspondence similar to the one in [3], and finally we give the proof of the symplectic deformation invariance.

3.1. Descendant Gromov–Witten invariants, relative Gromov–Witten invariants, and the degeneration formula

In this section we recall some variants of Gromov–Witten invariants.

Definition 3.1
Let $\overline{\mathcal{M}}_{0,n}^{X,\beta}$ be the moduli stack of genus zero n-pointed stable maps to X whose curve class is β. Let $\mathcal{L}_i$ be the line bundle on $\overline{\mathcal{M}}_{0,n}^{X,\beta}$ whose fiber over each point $(C, p_1, \ldots, p_n)$ is the restriction of the cotangent line bundle of the curve C to the point p_i. Let ψ_i be the first Chern class of $\mathcal{L}_i$. Then the descendant Gromov–Witten invariant is defined as

$$\langle \tau_{k_1} A_1, \ldots, \tau_{k_m} A_m \rangle_{0,\beta}^X = \int_{[\overline{\mathcal{M}}_{0,n}^{X,\beta}]^{\mathrm{virt}}} \prod_i \psi_i^{k_i} \mathrm{ev}_i^* A_i,$$

where ev_i is the evaluation map given by the ith marked point, and $A_i \in H^*(X, \mathbb{Q})$.

The relative Gromov–Witten invariants were first introduced in the symplectic category by Li and Ruan [11] and in the algebraic category by Li [9] and [10]. We do not recall the precise definition here since it is not needed. The reader should refer to the above-mentioned papers for more details.

Intuitively, the relative Gromov–Witten invariants count the number of stable maps satisfying certain incidence constraints and having prescribed tangency condition with a given divisor. Let X be a smooth projective variety, and let $D \subset X$ be a smooth divisor. Fix a curve class β such that the intersection number $D \cdot \beta = n$ is nonnegative. Note that relative Gromov–Witten invariants are not defined if the number $D \cdot \beta$ is negative. Also choose a partition $\{m_i\}$ of n. Then the relative Gromov–Witten invariants count the number of stable maps $f : (C, p_1, \dots, p_r, q_1, \dots, q_r) \to X$ with $r + s$ marked points such that the first r points (absolute marked points) are mapped to cycles in X and the last s points (relative marked points) are mapped to some cycles in D with $f^*D = \sum m_i q_i$. We can also define descendant relative Gromov–Witten invariants. We write such invariants as

$$\langle \tau_{k_1} A_1, \dots, \tau_{k_r} A_r \mid (m_1, B_1), \dots, (m_s, B_s) \rangle_{0,\beta}^{(X,D)},$$

where $A_i \in H^*(X, \mathbb{Q})$, $B_j \in H^*(D, \mathbb{Q})$. We also use the abbreviation

$$\big\langle \Gamma\big\{(d_i, A_i)\big\} \bigm| \mathcal{T}_k \big\rangle_\beta^{X,D}$$

following [3]. In the degeneration formula, we have to consider stable maps from disconnected domains. The corresponding relative invariants are defined to be the product of those of stable maps from connected domains. Such invariants are denoted by

$$\big\langle \Gamma^\bullet\big\{(d_i, A_i)\big\} \bigm| \mathcal{T}_k \big\rangle_\beta^{X,D}.$$

Finally we note that we can represent these invariants by decorated dual graphs (cf. [3, Section 3.2]).

Now we describe the degeneration formula. Let $W \to S$ be a projective morphism from a smooth variety to a pointed curve $(S, 0)$ such that a general fiber is smooth and connected and the fiber over zero is the union of two smooth irreducible varieties (W^+, W^-) intersecting transversely at a smooth subvariety Z. Let A_i be cohomology classes in a general fiber. Assume that the specialization of A_i in W_0 can be written as $A_i(0) = A_i^+ + A_i^-$, where $A_i^+ \in H^*(W^+, \mathbb{Q})$ and $A_i^- \in H^*(W^-, \mathbb{Q})$. Let $\{\beta_i\}$ be a self-dual basis of Z. Also let $\mathcal{T}_k = \{(t_j, \beta_{a_j})\}$ be a weighted partition, and let $\check{\mathcal{T}}_k = \{(t_j, \check{\beta}_{a_j})\}$ be the dual partition; that is, $\check{\beta}_{a_j}$ is the Poincaré dual of β_{a_j}. Then the degeneration formula expresses the Gromov–Witten invariants of a general fiber in terms of the relative Gromov–Witten invariants of the degeneration in the following way:

$$\langle \Pi_i \tau_{d_i} A_i \rangle^{W_t} = \sum \Delta(\mathcal{T}_k) \big\langle \Gamma^\bullet\{(d_i, A_i^+)\} \mid \mathcal{T}_k \big\rangle^{W^+, Z} \big\langle \Gamma^\bullet\{(d_i, A_i^-)\} \mid \check{\mathcal{T}}_k \big\rangle^{W^-, Z},$$

where the summation is taken over all possible degenerations and

$$\Delta(\mathcal{T}_k) = \Pi_j t_j \,|\, \mathrm{Aut}(T_k)|.$$

In this paper we are mainly interested in the following special case of such degenerations: the deformation to the normal cone. Namely, let X be a smooth projective variety, and let $S \subset X$ be a smooth subvariety. Then we take W to be the blow-up of $X \times \mathbb{A}^1$ with blow-up center $S \times 0$. In this case, $W^- \cong \tilde{X}$, the blow-up of X along S, and $W^+ \cong \mathbb{P}_S(\mathcal{O} \oplus N_{S/X})$.

3.2. A partial ordering

Let X be a smooth projective 3-fold, and let $S \subset X$ be a smooth subvariety of codimension k. Denote by $\tilde{X}$ the blow-up of X along S, and denote by E the exceptional divisor. Here we allow S to be a codimension 1 subvariety, that is, a divisor. In this case $\tilde{X}$ is isomorphic to X and E is isomorphic to S.

In the following we define a partial ordering on certain Gromov–Witten invariants. The partial ordering is basically the same as the one in [3]. The major difference is that we choose a different self-dual basis of E (cf. Remark 3.2).

Let $\theta_1 = 1$, and let $\theta_2, \dots, \theta_{m_S} = \omega \in H^*(S, \mathbb{Q})$ be a self-dual basis of S, where 1 (resp., ω) is the generator of the degree zero (resp., $2(3-k)$) cohomology. We now describe a self-dual basis of the exceptional divisor E. Note that $E = \mathbb{P}_S(N_{S/X})$ is a $\mathbb{P}^{k-1}$-bundle over S. Let $[E]$ be the first Chern class of the relative $\mathcal{O}(-1)$-bundle over $\mathbb{P}_S(N_{S/X})$. If k is 2, that is, S is a smooth curve in X, then $\pi_S : E \to S$ is a ruled surface over S. In this case, define

$$\lambda = [E] - \frac{[E]\cdot[E]}{2} \pi_S^* \omega$$

so that $\lambda^2 = 0$. Otherwise just take λ to be $[E]$. Then the cohomology classes

$$\pi_S^* \theta_i \cup \lambda^j, \quad 1 \le i \le m_S,\ 0 \le j \le k-1$$

form a self-dual basis of E. Denote them by $\Theta = \{\delta_i\}$.

Remark 3.2

In the paper [3], the authors claim $\pi_S^* \theta_i \cup [E]^j$ to be self-dual, which is not always true if $N_{S/X}$ is not a trivial bundle over S. However, this has been fixed, and the proof is essentially the same since only the degree of the $[E]$-part is important in the proof. The authors take a different basis.

Definition 3.3
A *standard (relative) weighted partition* μ is a partition

$$\mu = \{(\mu_1, \delta_{d_1}), \ldots, (\mu_{l(\mu)}, \delta_{d_{l(\mu)}})\},$$

where μ_i and d_i are positive integers with $d_i \leq km_S$; $l(\mu)$ is called the *length* of the partition.

Definition 3.4
For $\delta = \pi_S^* \theta \cup \lambda^j \in H^*(E, \mathbb{Q})$, define

$$\deg_S(\delta) = \deg \theta, \qquad \deg_f(\delta) = 2j.$$

For a standard weighted partition μ, define

$$\deg_S(\mu) = \sum_{i=1}^{l(\mu)} \deg_S(\delta_{d_i}),$$
$$\deg_f(\mu) = \sum_{i=1}^{l(\mu)} \deg_f(\delta_{d_i}).$$

Definition 3.5
We define a partial ordering, the *size relation*, on the set of pairs (m, δ) where $m \in \mathbb{Z}_{>0}$ and $\delta \in H^*(E, \mathbb{Q})$ as follows:

$$(m, \delta) > (m', \delta')$$

if
(1) $m > m'$, or
(2) $m = m'$ and $\deg_S(\delta) > \deg_S(\delta')$, or
(3) equality in the above and $\deg_f(\delta) > \deg_f(\delta')$.

Definition 3.6
A *lexicographic ordering* on weighted partitions is defined as

$$\mu > \mu'$$

if after we place the pairs of μ and μ' in decreasing order, the first pair for which μ and μ' are not equal is larger for μ.

Let $\sigma_1, \ldots, \sigma_{m_X}$ be a basis of $H^*(X, \mathbb{Q})$. Then the set of cohomology classes

$$\gamma_j = \pi^* \sigma_j, \quad 1 \leq j \leq m_X,$$

$$\gamma_{j+m_X} = \iota_*(\delta_j), \quad 1 \leq j \leq km_S,$$

generate a $\mathbb{Q}$-basis of $\tilde{X}$, where $\iota : E \to \tilde{X}$ is the inclusion and ι_* is the induced Gysin map.

Definition 3.7
A *connected standard relative Gromov–Witten invariant* of $(\tilde{X}, E)$ is of the form

$$\langle \omega \mid \mu \rangle_{0,A}^{\tilde{X},E} = \langle \tau_{k_i}\gamma_{L_1}, \ldots, \tau_{k_n}\gamma_{L_n} \mid \mu \rangle_{0,A}^{\tilde{X},E},$$

where A is an effective curve class, μ is a standard weighted partition with $\sum \mu_j = E \cdot A$, and $\gamma_{L_i} = \pi^* \sigma_{L_i}$.

We write $\Gamma(\omega) \mid \mu$ for the decorated graph of such invariants.

Definition 3.8
We define the partial ordering on all the decorated graphs of the standard relative invariants as follows:

$$\Gamma(\omega') \mid \mu' < \Gamma(\omega) \mid \mu$$

if
(1) $\pi_*(A') < \pi_*(A)$, that is, the difference $\pi_*(A) - \pi_*(A')$ is an effective curve class in X;
(2) equality in (1) and the arithmetic genus satisfies $g' < g$;
(3) equality in (1) and (2) and $\|\omega'\| < \|\omega\|$;
(4) equality in (1)–(3) and $\deg_S(\mu') > \deg_S(\mu)$;
(5) equality in (1)–(4) and $\mu' > \mu$;
where $\|\omega\|$ is the number of insertions of ω.

We have the following observation.

LEMMA 3.9
Given a standard relative invariant, there are only finitely many standard relative invariants smaller than it in the partial ordering defined above.

Proof
For a curve class $[A]$ of $\tilde{X}$, there are only finitely many curve classes lower than it. Thus there are only finitely many relative invariants lower than the given invariants. However, in our definition, we are comparing the curve classes by their images in X. It is easy to see that two different curve classes in $\tilde{X}$ define the same curve class

in X if and only if the difference is a multiple of L, where L is a line or a ruling in the exceptional divisor. So we have to consider all the possible standard relative invariants whose corresponding curve class is smaller than $A+kL$ for some k. But k is bounded below since $A+kL$ has to be an effective curve class. It is also bounded above since $E \cdot (A+kL)$ is nonnegative and $E \cdot L$ is -1. □

3.3. *From relative to absolute*

In this subsection, we discuss how to obtain an absolute invariant of X from a relative invariant of $\tilde{X}$.

Definition 3.10

For a relative insertion (m, δ) with $\delta = \pi_S^* \theta_i \cup \lambda^j$, we associate to it the absolute insertion $\tau_{d(m,\delta)}(\tilde{\delta})$, where

$$\tilde{\delta} = \iota_*(\theta_i),$$

$$d(m,\delta) = km - k + j.$$

Given a weighted partition $\mu = \{(\mu_i, \delta_{k_i})\}$, we define

$$d_i(\mu) = d(\mu_i, \delta_{k_i}) = k\mu_i - k + \frac{1}{2}\deg_f(\delta_{k_i}),$$

$$\tilde{\mu} = \left\{\tau_{d_1(\mu)}(\tilde{\delta}_{k_1}), \ldots, \tau_{d_{l(\mu)}(\mu)}(\tilde{\delta}_{k_{l(\mu)}})\right\}.$$

Given a standard relative invariant $\langle \Gamma^\bullet(\omega) \mid \mu \rangle^{\tilde{X},E}$, we define the absolute descendant invariant associated to the relative invariant to be

$$\langle \tilde{\Gamma}^\bullet(\omega, \tilde{\mu}) \rangle^X.$$

Here all the insertions ω in the relative invariants are of the form $\pi^*\sigma_i$, and the corresponding insertions in the absolute invariants are just σ_i.

Definition 3.11

An absolute descendant invariant of X is called a *colored absolute descendant invariant relative to* S if it can be written in the form $\langle \Gamma(\omega, \tilde{\mu}) \rangle$ such that each insertion in ω is of the form $\tau_{d_i}\sigma_i$ and each insertion in $\tilde{\mu}$ is of the form $\tau_{d_k}\tilde{\delta}_k$.

Definition 3.12

If $k=1$, then a colored absolute descendant invariant of X relative to S is called *admissible* if $\sum \mu_j = E \cdot A$.

The following lemma is essentially [3, Lemma 5.14]. Note that in their paper they only consider the case of primary Gromov–Witten invariants. But the proof is actually the same.

LEMMA 3.13
If $\mu \neq \mu'$, then $\tilde{\mu} \neq \tilde{\mu}'$. Therefore there is a natural bijection between the set of colored weighted absolute graphs relative to S and the set of weighted relative graphs in $\tilde{X}$ relative to E if $k > 1$. The same is true if we restrict to the admissible ones when $k = 1$.

Remark 3.14
Notice that different relative invariants may give the same absolute invariants. But these absolute invariants are different as colored absolute invariants.

Finally, let C be a curve in $\tilde{X}$ which does not intersect E. Then the image of C under the map $\tilde{X} \to X$ is a curve in X, also denoted by C. Note that $[C]$ as an element of $H^4(\tilde{X})$ is just the pullback of $[C]$ in $H^4(X)$. Let I be the partially ordered set of standard weighted relative graphs $\Gamma^\bullet([C], \omega) \mid \mu$.

Define $\mathbb{R}^I_{\tilde{X},E}$ to be an infinite-dimensional vector space whose coordinates are ordered in the same way as the partial ordering in I. A standard weighted relative invariant $\langle \Gamma^\bullet([C], \omega) \mid \mu \rangle^{\tilde{X},E}$ gives rise to a vector in $\mathbb{R}^I_{\tilde{X},E}$. By Lemma 3.13, I is also the set of colored standard weighted absolute graphs relative to S. Thus we also have an infinite-dimensional vector space $\mathbb{R}^I_{X,S}$ whose coordinates are also ordered by the partial ordering in I. Similarly, an absolute invariant $\langle \Gamma^\bullet([C], \omega, \tilde{\mu}) \rangle^X$ gives a vector in this vector space.

3.4. The correspondence

In this section we sketch the proof of the following theorem.

THEOREM 3.15
Let $\pi : \tilde{X} \to X$ be the blow-up of a 3-fold along a smooth center S. Then there is an invertible lower triangular linear map

$$A_S : \mathbb{R}^I_{\tilde{X},E} \to \mathbb{R}^I_{X,S},$$

given by the degeneration formula, where $\mathbb{R}^I_{\tilde{X},E}$ and $\mathbb{R}^I_{X,S}$ are the vector spaces defined at the end of Section 3.3.

Proof
In the setup of the degeneration formula, we take W to be the blow-up of $X \times \mathbb{A}^1$

along the smooth subvariety $S \times 0$. Then $W^- \cong \tilde{X}$, $W^+ \cong \mathbb{P}_S(\mathcal{O} \oplus N_{S/X})$, and they intersect transversely at E. We apply the degeneration formula in this setting.

We start with a connected standard weighted relative invariant

$$\langle \Gamma([C], \omega) \mid \mu \rangle^{(\tilde{X}, E)}$$

with the unique vertex decorated by $(0, \beta)$. Then the associated absolute invariant is

$$\langle [C], \omega, \tilde{\mu} \rangle^{X}.$$

To apply the degeneration formula, we have to specify the specialization of the cohomology classes. Since C does not intersect E, we may specialize C to lie entirely in W^-; that is, we set $[C]^- = [C]$ and $[C]^+ = 0$. The Poincaré dual of the cohomology classes in $\tilde{\mu}$ are supported in S. So we specialize these cohomology classes to the W^+-side and set the cohomology classes in the W^--side to be zero. Finally, classes in ω are of the form σ_i. Therefore we set $\sigma_i^- = \gamma_i = \pi^* \sigma_i$ with appropriate classes σ^+ on the W^+-side.

Then the degeneration formula in this case gives

$$\begin{aligned}\langle [C], \omega, \tilde{\mu} \rangle^{X} \\ &= \sum \big\langle \Gamma_-^{\bullet}([C], \omega_1) \bigm| \eta \big\rangle^{(\tilde{X}, E)} \Delta(\eta) \langle \Gamma_+^{\bullet}(\omega_2, \tilde{\mu}) \mid \check{\eta} \rangle^{\mathbb{P}_S(\mathcal{O} \oplus N_{S/X}), E}.\end{aligned}$$

We view $\Delta(\eta) \langle \Gamma_+^{\bullet}(\omega_2, \tilde{\mu}) \mid \check{\eta} \rangle^{\mathbb{P}_S(\mathcal{O} \oplus N_{S/X}), E}$ as the coefficients of the linear map A_S.

Notice that we have the term

$$\big\langle \Gamma_-^{\bullet}([C], \omega) \bigm| \mu \big\rangle^{(\tilde{X}, E)} \Delta(\eta) \langle \Gamma_+^{\bullet}(\tilde{\mu}) \mid \check{\mu} \rangle^{\mathbb{P}_S(\mathcal{O} \oplus N_{S/X}), E}$$

on the right-hand side. We need to show that $\langle \Gamma_-^{\bullet}([C], \omega) \mid \mu \rangle^{(\tilde{X}, E)}$ is the largest term with nonzero coefficient on the right-hand side. We show that the coefficient

$$\langle \Gamma_+^{\bullet}(\tilde{\mu}) \mid \check{\mu} \rangle^{\mathbb{P}_S(\mathcal{O} \oplus N_{S/X}), E},$$

that is, entries on the diagonal, is nonzero. This is basically step II in the proof of [3, Theorem 5.15]. One can check that under our choice of the self-dual basis, the coefficient is the product of relative invariants

$$\langle \tau_{nd-1-j}[pt] \mid H^j \rangle_{0, dL}^{\mathbb{P}^k, \mathbb{P}^{k-1}},$$

where $j = \deg_f(\delta_{k_i})$ and H is the hyperplane class. These types of invariants are computed in [3] via virtual localization and are shown to be nonzero. So the diagonal of the linear map A_S is nowhere zero.

Notice that this is the only step in [3] where the form of self-dual basis matters since we need to know the diagonal is nowhere zero. For the rest, the proof proceeds exactly as the proof of [3, Theorem 5.15]. □

3.5. *Birational invariance*

In this subsection we prove the following theorem.

THEOREM 3.16

Let $\pi : \tilde{X} \to X$ be the blow-up of a smooth projective 3-fold along a smooth subvariety S. Also let C be a curve in $\tilde{X}$ which does not intersect the exceptional divisor E. Then there is a nonzero descendant Gromov–Witten invariant of $\tilde{X}$ of the form

$$\langle [C], \tau_{d_1}\tilde{A}_1, \dots, \tau_{d_n}\tilde{A}_n \rangle^{\tilde{X}}_{0,\tilde{\beta}}$$

if and only if there is a nonzero descendant Gromov–Witten invariant of X of the form

$$\langle [C], \tau_{d_1}A_1, \dots, \tau_{d_m}A_m \rangle^{X}_{0,\beta}.$$

Here we use C to denote both the curve in $\tilde{X}$ and its image in X.

Proof

Suppose that there is a nonzero descendant Gromov–Witten invariant of X of the form

$$\langle [C], \tau_{d_1}A_1, \dots, \tau_{d_n}A_n \rangle^{X}_{0,\beta}.$$

We may assume that all the A_i are of the form σ_i. We degenerate X into $\tilde{X}$ and $\mathbb{P}_S(\mathcal{O} \oplus N_{S/X})$ and apply the degeneration formula. So there is a nonzero relative invariant

$$\langle [C], \tau_{d_1}\tilde{A}_1, \dots, \tau_{d_k}\tilde{A}_k \mid \mu \rangle^{\tilde{X},E}_{0,\tilde{\beta}'}.$$

Then apply Theorem 3.15 to $(\mathrm{Bl}_E\,\tilde{X}, E)$ and $(\tilde{X}, E)$. Note that the blow-up of $\tilde{X}$ with center E is $\tilde{X}$ itself. So Theorem 3.15 gives a nonzero absolute invariant of $\tilde{X}$ of the desired form.

Conversely, suppose that there is a nonzero descendant Gromov–Witten invariant of $\tilde{X}$ of the form

$$\langle [C], \tau_{d_1}\tilde{A}_1, \dots, \tau_{d_n}\tilde{A}_n \rangle^{\tilde{X}}_{0,\tilde{\beta}}.$$

We may assume that $\tilde{A}_i, 1 \le i \le m$ are of the form $\pi^*\sigma_{i_j}$ and $\tilde{A}_i, m+1 \le i \le n$ are of the form $\iota_*(\delta_{i_j})$. Then we degenerate $\tilde{X}$ into $\tilde{X}$ and $\mathbb{P}_E(\mathcal{O} \oplus N_{E/X})$ and specialize $\tilde{A}_i$, for $m+1 \le i \le n$, to the projective bundle side. Then there is a nonzero relative invariant of the form

$$\langle [C], \tau_{d_1} \tilde{A}_1, \ldots, \tau_{d_k} \tilde{A}_k \mid \mu \rangle^{\tilde{X}, E}_{0, \tilde{\beta}'},$$

with $k \leq m$. In particular, all the $A_i, 1 \leq i \leq k$, are of the form $\pi^* \sigma_{i_j}$. Again apply Theorem 3.15 to $(\tilde{X}, E)$ and (X, S). We get a nonzero absolute descendant invariant of the desired form. □

Remark 3.17

We may get a nonvanishing Gromov–Witten invariant of a *disconnected* curve in using the invertible map A_S. However this is sufficient for our purpose since this invariant is the product of the Gromov–Witten invariants of each of the connected components, and we only need to keep track of one insertion. This is also the reason that the same argument cannot prove the birational invariance of rational connectedness because then we need to keep track of two insertions.

3.6. Symplectic deformation invariance

First we recall the setup. Let X and X' be two smooth projective 3-folds which are symplectic deformation equivalent. Assume that X is rationally connected. We want to show that X' is also rationally connected.

Proof of symplectic deformation invariance

By Theorem 1.3, X' is uniruled. So we can look at the *maximal rationally connected (MRC) quotient* of X': a rational map $X' \dashrightarrow S$ such that the closure of a general fiber (in X') is the equivalence class of points in X' that are connected by a chain of rational curves. X' is rationally connected if and only if S is a point. In our case, the dimension of S is at most 2. And by the theorem of Graber, Harris, and Starr [2], S is not uniruled.

We use proof by contradiction. So assume that S is not a point.

If S is a curve, it cannot be a rational curve. Then we get a nonzero section of $H^0(X', \Omega_{X'})$ by pulling back a nonzero section of $H^0(S, \Omega_S)$. But X' is simply connected since X and X' are diffeomorphic and X is simply connected. Then by Hodge decomposition, $H^0(X', \Omega_{X'}) = 0$. This is a contradiction.

So S has to be a nonuniruled surface, and the closure of a general fiber of the rational map $X' \dashrightarrow S$ is a rational curve passing through a general point in X' and thus is free. The rational map can be extended to the complement of a codimension 2 locus. Therefore it is actually well defined on the closure of a general fiber since a free rational curve can be moved away from any codimension 2 locus.

Furthermore, a general fiber has to be the minimal free curve of X'; otherwise S is uniruled. So we see that on X' we have

$$\langle [pt] \rangle^{X'}_{0,[C]} = 1, \qquad -K_{X'} \cdot C = 2.$$

Both of these conditions are symplectic deformation invariant. So we get on X:

$$\langle [pt] \rangle^X_{0,[C]} = 1, \qquad -K_X \cdot C = 2.$$

Clearly the curve class $[C]$ is also the minimal free curve class on X; otherwise the minimal free curves in X give rise to rational curves in X' not in the fiber of $X' \dashrightarrow S$. We have seen that this Gromov–Witten invariant is enumerative. So there is exactly one minimal free curve passing through a general point in X. Let

$$\pi : \mathcal{C} \to \Sigma$$

be the universal family of the minimal free curves, and let

$$f : \mathcal{C} \to X$$

be the universal map. Then the morphism f is birational. Thus there is a rational map $X \dashrightarrow \Sigma$. That is, X is birational to a conic bundle over a surface. The surface Σ is dominated by a rationally connected 3-fold X and thus is also rationally connected, that is, a rational surface. Since we only need a birational model of X in the sequel, we assume the surface Σ is smooth and projective by taking a resolution of singularities.

Let $\Gamma \subset X \times \Sigma$ be the closure of the rational map $X \dashrightarrow \Sigma$. By the same argument proving that the map $X' \dashrightarrow S$ is well defined along a general fiber, the map $X \dashrightarrow \Sigma$ is also well defined along a general fiber. Then the exceptional divisors of $\Gamma \to X$ do not dominate Σ. So there is an open subset U of X and a smooth open subset V of Σ such that $U \to V$ is a well-defined proper morphism and a general fiber is $\mathbb{P}^1$. We can choose smooth projective compactifications of U and V, denoted by Y and Σ', together with a morphism $Y \to \Sigma'$ such that a general fiber is $\mathbb{P}^1$. By the weak factorization theorem in [1], there is a sequence of blow-ups/blow-downs

$$X = X_0 \dashrightarrow X_1 \dashrightarrow \cdots \dashrightarrow X_n = Y$$

such that every birational map is an isomorphism over U. In particular, there is a free curve C in every X_i away from every exceptional divisor.

By Theorem 2.1, there is a nonzero Gromov–Witten invariant on Y of the form $\langle [C], \ldots, [C], [A]^2, \ldots, [A]^2 \rangle^Y_{0,\beta}$ with $[A] \in H^2(Y, \mathbb{Q})$ being the class of a very ample divisor of Y. Then by the proof of Theorem 3.16, there is a descendant Gromov–Witten invariant on X of the form $\langle [C], \tau_{d_1}\gamma_1, \ldots, \tau_{d_n}\gamma_n \rangle^X_{0,\beta'}$ with $\gamma_j \in H^{\geq 4}(X, \mathbb{Q})$. Since X' is symplectic deformation equivalent to X, $\langle [C], \tau_{d_1}\gamma_1, \ldots, \tau_{d_n}\gamma_n \rangle^{X'}_{0,\beta'} \neq 0$. If a curve of class $[\beta']$ in X' were supported in a general fiber of $X' \dashrightarrow S$, we would have that $[\beta']$ is a multiple of $[C]$ and $-K_{X'} \cdot \beta' \geq 2$. So there are other insertions in the descendant invariant. But we can choose representatives of the cycles γ_i disjoint from a general fiber C. Then the invariant should be zero since a curve supported in a fiber cannot meet the cycles representing γ_i. Thus the curves with curve class $[\beta']$ are not supported in the fibers of the rational map $X' \dashrightarrow S$, and S is uniruled by their images. This is a contradiction. □

4. Fano threefolds

The main result in this section is the following theorem.

THEOREM 4.1
If X is a Fano threefold, then there is a nonzero Gromov–Witten invariant of the form $\langle [pt], [pt], [A]^2, \ldots, [A]^2 \rangle^X_{0,\beta}$, where $[A]$ is the class of a very ample divisor.

We begin by reviewing some results from birational geometry which allow us to construct low-degree very free curves in a Fano 3-fold. Then we prove that these low-degree curves give nonzero Gromov–Witten invariants we want. The key observation is that bend-and-break allows one to get some control of the deformation of low-degree curves in a Fano variety (see Lemma 4.7).

4.1. Some results from birational geometry

In this section we collect some results on the classification of K_X-negative extremal contractions on a smooth projective 3-fold.

THEOREM 4.2 ([4], [8])
Let X be a smooth 3-fold, and let contr : $X \to Y$ *be a contraction of a K_X-negative extremal ray. Then one of the following holds.*

- (E1) *Y is smooth and X is the blow-up of Y along a smooth curve.*
- (E2) *Y is smooth and X is the blow-up of Y along a point.*
- (E3) *Y is singular and locally analytically isomorphic to $x^2+y^2+z^2+w^2=0$. X is the blow-up at the singular point.*
- (E4) *Y is singular and locally analytically isomorphic to $x^2+y^2+z^2+w^3=0$. X is the blow-up at the singular point.*
- (E5) contr *contracts a smooth $\mathbb{P}^2$ with normal bundle $\mathcal{O}_{\mathbb{P}^2}(-2)$ to a point of multiplicity* 4 *in Y.*
- (C) *Y is a smooth surface and X is a conic bundle over Y.*
- (D) *Y is a smooth curve and X is a fibration in Del Pezzo surfaces.*
- (F) *X is a Fano* 3*-fold with Picard number one and Y is a point.*

It is very easy to work-out what the exceptional divisors are in the cases of exceptional contractions, and we have the following corollary.

COROLLARY 4.3

In the case of (E2)–(E5), the exceptional divisor is rationally connected, and the following is the list of very free curves of minimal degree in the exceptional divisor and their normal bundles in X:

- (E2) *a line L in $\mathbb{P}^2$, $N_{L/X} \cong \mathcal{O}(1) \oplus \mathcal{O}(-1)$;*
- (E3) *a conic C in a smooth quadric hypersurface, $N_{C/X} \cong \mathcal{O}(2) \oplus \mathcal{O}(-2)$;*
- (E4) *a conic C in a quadric cone, $N_{C/X} \cong \mathcal{O}(2) \oplus \mathcal{O}(-2)$;*
- (E5) *a line L in $\mathbb{P}^2$, $N_{L/X} \cong \mathcal{O}(1) \oplus \mathcal{O}(-2)$.*

We also need the following result from [12] and [13].

PROPOSITION 4.4

Let X be a Fano threefold, and let $\mathrm{contr} : X \to Y$ *be the blow-up along a smooth curve in Y. Then Y is Fano unless we are in the following case: X is the blow-up along a smooth $\mathbb{P}^1$ whose normal bundle in Y is $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$.*

In this case, the exceptional divisor E is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ and the normal bundle of the curve of bidegree $(1,1)$ is $\mathcal{O}(2) \oplus \mathcal{O}(-2)$.

Definition 4.5

Let X be a Fano 3-fold. We say that X is *primitive* if it is not the blow-up along a smooth curve of another smooth Fano 3-fold.

4.2. *Construction of low-degree very free curves*

We first show the following theorem.

THEOREM 4.6

Let X be a Fano 3*-fold. Then there is very free curve whose $-K_X$-degree is at most* 6*.*

Proof

We first consider the case $\rho(X) = 1$.

Fix a polarization on X. Let C be a minimal free rational curve. If $-K_X \cdot C = 4$, then $N_{C/X} \cong \mathcal{O}(1) \oplus \mathcal{O}(1)$. We are done in this case.

If $-K_X \cdot C = 3$, then we can fix a general point in X and the deformation of C fixing that point sweeps out a surface S. Since $\rho(X) = 1$, the divisor S is ample. If we take a minimal free curve through another general point, S intersects that curve at a finite number of points. So we have a reducible curve, which is the union of two free curves passing through two general points in X. A general deformation of this curve is an irreducible curve passing through two general points and thus is very free. The anticanonical degree of the very free curve is 6.

If $-K_X \cdot C = 2$, then it is proved in [5, Corollary 4.14, Chapter IV; Proposition 2.6, Chapter V] that there is a chain of free curves of length at most 3 (i.e., $\dim X$) connecting two general points. Thus a deformation of this chain gives a very free curve whose anticanonical degree is at most 6.

Next assume $\rho(X) \geq 2$. We may further assume that X is a primitive Fano 3-fold. Otherwise there is a birational morphism $X \to Y$ such that Y is a primitive Fano 3-fold. Then the very free curve in Y can be moved away from the blow-up centers and gives a very free curve in X with the same anticanonical degree. Under this assumption, every exceptional divisor is rationally connected by Corollary 4.3 and Proposition 4.4.

It is proved in [13] that there is an extremal ray R1 corresponding to a contraction of type (C) (cf. [17, Theorem 7.1.6]). Let C be a general fiber.

The discussion is divided into three parts according to the type of contractions given by other extremal rays.

If there is an extremal ray corresponding to a divisorial contraction with exceptional divisor E, then $E \cdot C > 0$. In fact, we may find a possibly reducible curve of class $[C]$ which intersects E by specializing a general curve C. But then $E \cdot C > 0$ since $[C]$ spans an extremal ray and the class of every irreducible component of the specialization of C lies in that extremal ray. So we can assemble a chain of rational curves in the following way. Take two minimal free curves each through a general point and a minimal very free curve in E connecting the two minimal free curves. It is easy to see that this chain of rational curves deforms to a very free rational curve with $-K_X$-degree at most 6.

If there is an extremal ray of type (D), X is a Del Pezzo fibration over $\mathbb{P}^1$. We can assemble a connected reducible curve by gluing a free curve C to a minimal very free curve in a general fiber. It is easy to see that a general deformation of this curve is very free with $-K_X$-degree at most 6.

Finally, assume that all the extremal rays are of type (C). Then there are at least two free curves C_1 and C_2 through a general point. A general deformation of the union of these two curves is a free curve of $-K_X$-degree 4. If it is not very free, we may deform this curve by fixing a point and get a divisor H. We are done if H intersects either C_1 or C_2. If $H \cdot C_1 = H \cdot C_2 = 0$, then $\rho(X) \geq 3$. Note that H is nef since we can move H by changing the point so that it does not contain any prespecified curve. Thus some multiple of H is base point–free and defines a morphism $\pi : X \to \mathbb{P}^1$. This morphism contracts both C_1 and C_2, and a general fiber is a Del Pezzo surface. There is a third extremal ray, necessarily of type (C) by our assumption. Then we get a very free curve with anticanonical degree no more than 6 by the same argument as above. □

4.3. Fano 3-folds are symplectic rationally connected

We begin with a simple but important observation.

LEMMA 4.7

Let X be a Fano 3-fold. Let $C = C_1 \cup C_2 \cup C_3$ be a chain of $\mathbb{P}^1$s, and let $f : C \to X$ be a stable map. Assume that $-f^* K_X \cdot C_2 = 1$ and C_i passes through a very general point for $i = 1, 3$. Then a general point of any irreducible component of the Kontsevich moduli space containing (C, f) corresponds to an irreducible very free curve.

Proof

Notice that C_2 only deforms in a surface S. Both C_1 and C_3 intersect the surface S at finitely many points, and C_2 has to pass through at least two of them. Then C_2 does not move once we make the choice of the two points. Otherwise we can deform C_2 by fixing two points, and, by bend-and-break, C_2 breaks into a reducible or nonreduced curve. But this cannot happen since $-K_X \cdot C_2 = 1$ and $-K_X$ is ample.

We claim that a general deformation of the stable map $f : C \to X$ smooths at least one of the nodes. If not, then a general deformation is given by the deformation of the two free curves and the deformation of the degree 1 curve. By the above argument, up to finitely many choices, the deformation of the degree 1 curve is determined by the deformation of the two free curves. So the deformation space has dimension equal to the sum of the dimension of the deformation space of C_1 and C_3, which is $(-f^* K_X) \cdot C_1 + (-f^* K_X) \cdot C_3$. But every irreducible component containing the point $(f : C \to X)$ has dimension at least $-f^* K_X \cdot C$, which is greater than $(-f^* K_X) \cdot C_1 + (-f^* K_X) \cdot C_3$. So at least one of the nodes can be smoothed out, and we get a reducible curve, which is the union of two irreducible curves each passing through a very general point. Then it is easy to see that a general deformation of this new curve is very free. □

Proof of Theorem 4.1

By Theorem 4.6, there is a very free curve with $-K_X$-degree no more than 6. Take such a curve C with minimal degree. We consider a Gromov–Witten invariant of the form $\langle [\mathrm{pt}], [\mathrm{pt}], [A]^2, \dots, [A]^2 \rangle^X_{0,C}$, where $[A]$ is the class of a very ample divisor. Note that in any case, the components whose general points parameterize very free curves contribute positively to this Gromov–Witten invariant. Most of the time we do not distinguish the stable map and the curve it maps to.

We first choose two very general points such that any irreducible rational curve through them is very free and any irreducible rational curve through one of them is free. In particular, the anticanonical degree of any irreducible rational curve through one of them is at least 2.

We want to show that if we choose the constraints to be general enough, then no reducible curve can meet all the constraints. Clearly we only need to consider reducible curves that can pass through the two general points.

The following observation is immediate. If a reducible curve passes through the two general points and is smoothable, then it lies in an irreducible component, whose general points parameterize irreducible very free curves passing through two general points. Thus that irreducible component has expected dimension, and we could choose the constraints to be general such that this reducible curve, whose corresponding point necessarily lies in the boundary of that component, cannot meet all of the constraints.

We use this observation in the following two situations. If the reducible curve is a union of free curves, then it is certainly smoothable. Another case is the curve in Lemma 4.7. Note that the two points are not connected by an irreducible curve by minimality of the very free curve. We discuss three different cases according to the anticanonical degree of C.

(1) $-K_X \cdot C = 4$. Then the only reducible curve that can pass through two general points is the union of two free curves. We are done in this case.

(2) $-K_X \cdot C = 5$. Then we may add one more constraint, a curve which is the complete intersection of two very ample divisors. Notice that for degree reasons, every possible reducible curve is either a union of free curves or a curve as in Lemma 4.7. So we are done in this case.

(3) $-K_X \cdot C = 6$. In this case we add two more constraints, both of which are curves coming from complete intersections of very ample divisors. By Lemma 4.7, we only need to consider two cases: a curve with four irreducible components whose $-K_X$-degrees are $2, 2, 1, 1$, and a curve with three irreducible components with $-K_X$-degree 2.

First consider the case where the curve has four irreducible components whose $-K_X$-degrees are $2, 2, 1, 1$.

If the two curves of degree 1 deform in two different surfaces, then each of the degree 1 curves connects one (and only one by Lemma 4.7) free curve and the other degree 1 curve. We first choose one of our constraint curves to avoid the degree 2 curves. Then one of the degree 1 curves has to pass through an intersection point of this constraint curve with the surface swept out by itself. Notice that there are only finitely many such choices by bend-and-break (cf. the first paragraph of the proof of Lemma 4.7). Also once we make the choice for one curve, there are only finitely many choices for the other degree 1 curve since it has to pass through the intersection of the degree 2 and the degree 1 curves with the surface. So we can choose the last constraint to avoid all of the four components.

Now assume that the two degree 1 curves deform in the same surface. We may choose the constraint curves to intersect the surface at general points. Note that the degree 2 curves cannot deform once the two points are fixed. Thus they cannot meet the other constraints. If the two degree 1 curves meet all the other constraints, then one of them has to pass through both the intersection of the degree 2 free curve with the surface and a general point in the surface (i.e., the intersection of one constraint curve with the surface). Thus we get a smoothable chain of rational curves as in Lemma 4.7. Then we get a very free curve with $-K_X$-degree 5. This is a contradiction to our choice of C.

Then consider the case where the curve consists of three irreducible components with $-K_X$-degree 2 and one of them is not free.

If the nonfree curve is the specialization of a free curve, then it lies in a component of dimension 2. But if we choose the two points to be general, the two free curves of $-K_X$-degree 2 cannot both meet this nonfree curve. So the nonfree curve only deforms in a surface.

Once the two points are chosen, the two free curves cannot meet any more constraints. Thus if this reducible curve can meet all the constraints, then the nonfree curve passes through at least four general points coming from the intersections of the two free curves and the constraints with the surface. So after we fix the two intersection points and a third general point, the curve deforms in a positive-dimensional family. Then it can break into two irreducible components or a nonreduced curve. In this way we get a rational curve with $-K_X$-degree 1 and passing through two general points in the surface. In particular, there is a chain of curves of $-K_X$-degree 5 as in Lemma 4.7. Then we can smooth them and get a very free curve of $-K_X$-degree 5. This is a contradiction. $\square$

Remark 4.8
For another proof, see Section 5.

COROLLARY 4.9
Let X be a Fano 3-fold, and let $f : Y \to X$ be a birational morphism. Then Y is symplectic rationally connected.

Proof

The images under f of the exceptional divisors have codimension at least 2 in X. Thus the minimal very free curve in X can be deformed away from them. We can also choose the constraints in Theorem 4.1 to be away from the centers. Then the very free curves in X meeting all the constraints are all away from the images of exceptional divisors. Observe that the image of any curve satisfying the constraints in Y also satisfies the constraints in X. Thus the images are irreducible curves not intersecting the exceptional locus. Then it follows that no components are contracted by the map $f : Y \to X$ and the curves in Y that can meet these constraints are again irreducible very free curves. $\square$

5. $\mathbb{Q}$-Fano variety

Observe that the proof of symplectic rational connectedness of Fano 3-folds could be greatly simplified if we could choose the constraints to be three points. But this requires knowing that a general very free curve constructed in the proof has normal bundle $\mathcal{O}(2) \oplus \mathcal{O}(2)$. This idea works in a more general context as below.

We begin with some preliminary definitions and observations.

Definition 5.1

A normal projective variety X is a $\mathbb{Q}$-*Fano variety* if X is $\mathbb{Q}$-factorial, has terminal singularities, and $-K_X$ is ample.

LEMMA 5.2

Let X be a projective variety of dimension n with terminal singularities. Then for any irreducible rational curve C passing through r very general points, the intersection number $-K_X \cdot C$ is at least $(n-1)(r-1)+2$. And if this lower bound is achieved, then C is a curve contained in the smooth locus of X.

Proof

Let $f : Y \to X$ be a resolution of singularities such that X and Y are isomorphic over the smooth locus of X. We have $K_Y = f^* K_X + \sum b_i E_i$ where E_i's are exceptional divisors of f, and b_i are positive rational numbers by definition of terminal singularity. The normal bundle of an irreducible rational curve C through r very general points is

$$N_{C/Y} \cong \bigoplus_i \mathcal{O}(a_i), \quad a_i \geq r-1.$$

So

$$-K_X \cdot C \geq -K_Y \cdot C \geq (n-1)(r-1)+2.$$

The last statement follows from the fact that $E_i \cdot C \geq 0$ and equality holds if and only if C does not intersect E_i, or equivalently, $f(C)$ is contained in the smooth locus of X. □

PROPOSITION 5.3

Let X be $\mathbb{Q}$-Fano 3-fold, and let $f : Y \to X$ be a resolution of singularities. Assume that there is a very free curve in the smooth locus of X whose normal bundle is $\mathcal{O}(a) \oplus \mathcal{O}(a), a \geq 1$. Then Y is symplectic rationally connected.

Proof

Let C be such a very free curve in the smooth locus of X. We can move C away from the locus where f is not an isomorphism. So we get a very free curve C in Y with normal bundle $\mathcal{O}(a) \oplus \mathcal{O}(a)$, for $a \geq 1$.

We choose the constraints to be $a+1$ general points. Any irreducible curve in Y through k general points has $-f^*K_X$-degree at least $2k$ by Lemma 5.2, and equality holds if and only if its image in X is contained in the locus where X and Y are isomorphic, which is contained in the smooth locus of X. Note that $-f^*K_X$ is nef and the $-f^*K_X$-degree of C is $2a$. This forces every irreducible component passing through one of the general points to lie in the locus where X and Y are isomorphic. Then no components can be contracted by the map $Y \to X$. Thus every irreducible component contains one of the chosen points and is free. Now the proposition follows immediately. □

Therefore the problem is reduced to the existence of such curves. Fortunately there is a way to show the existence in dimension 3.

Definition 5.4

Let $C_i \subset X_i$ be a curve on a variety X_i, $i = 1, 2$. We say that (X_1, C_1) is *equivalent to* (X_2, C_2) if there is an open neighborhood V_i of C_i in X_i and an isomorphism $f : V_1 \to V_2$ such that $f|_{C_1} : C_1 \to C_2$ is also an isomorphism.

Our goal is to prove the following.

THEOREM 5.5

Let X be a $\mathbb{Q}$-Fano 3-fold. Assume that the smooth locus of X is rationally connected; then there is a very free curve in the smooth locus with normal bundle $\mathcal{O}(a) \oplus \mathcal{O}(a), a \geq 1$.

We start with the following construction in [18] for smooth rationally connected 3-folds.

Construction 5.6
Let $f : Y \to X$ be a resolution of singularities which is isomorphic over the smooth locus of X. Let C be a very free curve in the smooth locus and general in an irreducible component of moduli space of very free curve. We may assume that $-K_X \cdot C$ is an even number (otherwise take a 2-fold cover and a general deformation). Assume that the normal bundle of C is $\mathcal{O}(a + 2b) \oplus \mathcal{O}(a), a, b \geq 1$. In the following we do not distinguish between $-K_X$ and $-f^*K_X$.

Choose $a + 1$ points in C and deform C with these points fixed. Then the deformation of C sweeps out a surface Σ in Y. Let Σ' be the normalization, and let $\tilde{\Sigma}$ be the minimal resolution of Σ'.

The following results are proved in [18, Sections 2.2, 2.3].

PROPOSITION 5.7 ([18, Corollary 2.2.7, Proposition 2.3.3, Lemma 2.3.13])
Notations are as above.

(1) *Σ is independent of the choice of the points. Σ' is smooth along C and $N_{C/\Sigma'} \cong \mathcal{O}(a + 2b)$.*
(2) *There is a neighborhood U of C such that the map $\phi : \tilde{\Sigma} \to X$ has injective tangent map. And the normal sheaf $N_{\tilde{\Sigma}/X}$ is locally free along C and $N_{\tilde{\Sigma}/X}|_C \cong \mathcal{O}(a)$.*
(3) *The pair $(\tilde{\Sigma}, C)$ is equivalent to $(\mathbb{P}^2, \text{conic})$ or $(\mathbb{F}_n, \sigma)$, where $\mathbb{F}_n$ is the nth Hirzebruch surface and σ is a section of $\mathbb{F}_n \to \mathbb{P}^1$.*
(4) *If the pair is equivalent to $(\mathbb{F}_n, \sigma)$, where σ is a section of $\mathbb{F}_n \to \mathbb{P}^1$, then there is a (reducible) curve D in $\tilde{\Sigma}$ such that $D^2 = -n$ and $D \cdot F = 1$, where F is a fiber of $\mathbb{F}_n \to \mathbb{P}^1$. If $C \cdot D > 0$, then a general fiber F lies in U. And the sheaf $N_{F/Y}$ cannot be $\mathcal{O} \oplus \mathcal{O}(1)$.*

Our goal is to start with this curve C and produce another very free curve with balanced normal bundle. The case where the pair $(\tilde{\Sigma}, C)$ is equivalent to $(\mathbb{P}^2, \text{conic})$ is straightforward.

LEMMA 5.8
If the pair $(\tilde{\Sigma}, C)$ is equivalent to $(\mathbb{P}^2, \text{conic})$, then there is a very free rational curve C in the smooth locus of X such that $N_{C/X} \cong \mathcal{O}(1) \oplus \mathcal{O}(1)$.

Proof
Note that in this case $N_{C/Y} \cong \mathcal{O}(4) \oplus \mathcal{O}(2)$. So $-K_X \cdot C = -K_Y \cdot C = 8$. C may degenerate into two "lines" which can pass through two very general points and thus is very free with normal bundle (in Y) $\mathcal{O}(1) \oplus \mathcal{O}(1)$. Note that a general such "line" is necessarily contained in the smooth locus since the intersection number with $-K_X$ is 4, the same as the intersection number with $-K_Y$. So we are done. $\square$

Now assume that the pair is equivalent to $(\mathbb{F}_n, \sigma)$, where σ is a section. Let D in $\tilde{\Sigma}$ be the (reducible) curve such that $D^2 = -n$, and let $D \cdot F = 1$ be as in Proposition 5.7. Then $C = D + cF$.

LEMMA 5.9
Notations are as above. Then

$$n \leq c \leq a + b$$

and

$$c - n = a + 2b - c \geq b.$$

Proof
First note that $0 \leq C \cdot D = c - n$.

We know that $N_{C/\tilde{\Sigma}} \cong \mathcal{O}(a + 2b)$ and $N_{\tilde{\Sigma}/Y}|_C \cong \mathcal{O}(a)$. Thus,

$$C \cdot C = 2c - n = a + 2b,$$

$$c(-K_X \cdot F) = 2 + 2a + 2b - (-K_X \cdot D) \leq 2 + 2a + 2b.$$

Note that $-K_X \cdot F$ is at least two since a general fiber F passes through a very general point. Thus

$$n \leq c \leq 1 + a + b.$$

If $c = 1 + a + b$, then

$$-K_X \cdot F = 2$$

and

$$-K_X \cdot D = 0.$$

The first equality implies that a general fiber F is mapped to a free curve in the smooth locus. The second one implies that D is either mapped to a point in Y or into the exceptional divisors of $f : Y \to X$. But F and D intersect in $\tilde{\Sigma}$. So do their images in X. Thus D is contracted to a point in the smooth locus of X. Then there are two free curves (i.e., images of two general fibers) that meet in the smooth locus of X, each having $-K_X$-degree 2 and passing through a very general point. But if we choose the two points to be general enough, any two irreducible curve of $-K_X$-degree 2 through these points cannot not meet each other. Therefore

$$c \leq a + b$$

and thus

$$c - n = a + 2b - c \geq b. \qquad \square$$

LEMMA 5.10

Notations are as above. Then $-K_Y \cdot F = 2$ and $N_{F/Y} = \mathcal{O} \oplus \mathcal{O}$ for a general fiber F.

Proof

Note that

$$-K_Y \cdot F \leq -K_X \cdot F \leq \frac{2 + 2a + 2b}{c} = \frac{2(2 + 2a + 2b)}{a + 2b + n} \leq \frac{4(a + b + 1)}{a + b + 1} = 4.$$

Thus $-K_Y \cdot F$ is at most 4. On the other hand it is at least 2 since a general fiber F passes through a very general point.

If $-K_Y \cdot F = 4$, then every inequality above is an equality. Thus,

$$n = 0, \qquad b = 1,$$

$$-K_Y \cdot F = -K_X \cdot F, \qquad -K_X \cdot D = 0.$$

So F is a free curve in the smooth locus of X, and D is contracted to a point in the smooth locus. Furthermore the pair $(\tilde{\Sigma}, \sigma)$ is equivalent to $\mathbb{P}^1 \times \mathbb{P}^1$ with one ruling. Then D is a moving curve in $\tilde{\Sigma}$ and $-K_X \cdot D > 0$ since some deformation of D is not contracted. This is a contradiction.

Since $C \cdot D > 0$, a general fiber F is contained in U and the normal bundle $N_{F/Y}$ is not $\mathcal{O} \oplus \mathcal{O}(1)$ by Proposition 5.7. Thus $-K_Y \cdot F$ is not 3. $\square$

Now we are ready to finish the proof of Theorem 5.5.

Proof of Theorem 5.5

We only need to consider the case where the pair $(\tilde{\Sigma}, C)$ is equivalent to $(\mathbb{F}_n, \sigma)$. By Lemma 5.9, $c - b \geq n$. Thus C may specialize to the union of a section C' whose curve class is $D + (c - b)F$ and b general fibers. Also note that C' passes through $a + 1$ general points in $\tilde{\Sigma}$ since its normal bundle in $\tilde{\Sigma}$ is $\mathcal{O}(a)$. Since C passes through $a + 1$ general points in X, the same is true for C'. Also notice that

$$-K_X \cdot C' = -K_X \cdot C - b(-K_X) \cdot F \leq -K_X \cdot C - b(-K_Y) \cdot F = 2 + 2a.$$

For the last equality we use Lemma 5.10. Then the equality has to hold by Lemma 5.2, and C' is a very free curve in the smooth locus with normal bundle $\mathcal{O}(a) \oplus \mathcal{O}(a)$. □

COROLLARY 5.11
On every smooth Fano 3-fold, there is an embedded very free curve with normal bundle $\mathcal{O}(a) \oplus \mathcal{O}(a), a \geq 1$.

Combining these results we get a new proof that every smooth Fano 3-fold is symplectic rationally connected.

In general it is not an easy task to determine if the smooth locus of a $\mathbb{Q}$-Fano variety is rationally connected. In the following we prove that this is true for a large class of $\mathbb{Q}$-Fano varieties we are interested in.

PROPOSITION 5.12
Let X be a Gorenstein $\mathbb{Q}$-Fano 3-fold. Then the smooth locus of X is rationally connected.

Proof
By a result of Namikawa [15], there is a smoothing of X, $\pi : \mathcal{X} \to S$ such that a general fiber is a smooth Fano 3-fold and the central fiber $\mathcal{X}_0$ is X.

By Corollary 5.11, there is a very free curve D in a general fiber whose relative normal bundle is $\mathcal{O}(a) \oplus \mathcal{O}(a)$, for $a \geq 1$. So this curve can pass through $a+1$ general points in a general fiber. Now choose $a+1$ general points in X. We can find $a+1$ sections of $\mathcal{X} \to S$ passing through these points in X, possibly after a base change. Then consider the specialization of the curve D passing through these sections in the relative Kontsevich moduli space, and we get a stable map to X whose image contains the chosen $a+1$ general points. But as observed in Proposition 5.3, the domain of the stable map has to be irreducible and the image is contained in the smooth locus. □

Remark 5.13
A 3-fold Gorenstein terminal singularity is an isolated hypersurface singularity, in particular, a locally complete intersection singularity. Then the proposition can be proved easily by comparing the deformation space of a very free curve in a resolution and that of its image in X. But the proof presented here gives more information about the very free curves in the smooth locus. For example, we know that the anticanonical degree is no more than 8 by looking at the construction of low-degree very free curves in the proof of Theorem 4.6.

6. Del Pezzo fibrations

Let X be a normal projective 3-fold, with at worst terminal singularities. And let $\pi : X \to \mathbb{P}^1$ be a contraction of some K_X-negative extremal face. Then a general fiber of f is a smooth Del Pezzo surface. Let $f : Y \to X$ be a resolution of singularities that is isomorphic near a general fiber. Note that all the exceptional divisors are supported in special fibers of $\pi \circ f : Y \to X \to \mathbb{P}^1$. The main result in this section is the following.

THEOREM 6.1
There is a nonzero Gromov–Witten invariant $\langle [pt], [pt], \ldots \rangle_{0,\beta}^{Y}$ for some class β which is a section of the fibration $Y \to \mathbb{P}^1$.

Proof
We proceed in two steps. First we construct a section satisfying certain properties and then show that this section gives rise to the nonzero Gromov–Witten invariant.

Step 1: Construction. By definition of terminal singularities, we have $K_Y \sim_{\mathbb{Q}} f^*K_X + \sum a_i E_i, a_i > 0$.

By [2, Theorem 1.1], there exists a section of $Y \to \mathbb{P}^1$.

Let

$$A = \min\left\{ d \;\middle|\; \left(\sum a_i E_i\right) \cdot s = d, s \text{ is a section} \right\}.$$

Once we have a section, we can attach very free curves in general fibers to it and deform the reducible curve to get a section which is a free curve. This operation does not change the intersection numbers with the exceptional divisors as long as we attach very free curves in general fibers not containing the exceptional divisors. So there is a free section s such that $\left(\sum a_i E_i\right) \cdot s = A$.

Define

$$B_1 = \min\left\{ b \geq 0 \;\middle|\; s \text{ is a section}, s \cdot \left(\sum a_i E_i\right) = A, \right.$$
$$\left. N_{s/Y} \cong \mathcal{O}(a) \oplus \mathcal{O}(a+b), a, b \geq 0 \right\}.$$

A general fiber of $Y \to \mathbb{P}^1$ is a Del Pezzo surface. So it is either $\mathbb{P}^1 \times \mathbb{P}^1$, $\mathbb{P}^2$, or the blow-up of $\mathbb{P}^2$ at d $(1 \leq d \leq 8)$ general points.

CLAIM 6.2
If a general fiber is not $\mathbb{P}^1 \times \mathbb{P}^1$, then there is a section s whose normal bundle is $\mathcal{O}(a) \oplus \mathcal{O}(a)$ with a arbitrarily large.

If a general fiber is $\mathbb{P}^1 \times \mathbb{P}^1$*, and* $B_1 > 0$*, then there is very free section* s *whose normal bundle is* $\mathcal{O}(a) \oplus \mathcal{O}(a + B_1)$ *with* a *arbitrarily large.*

If a general fiber is $\mathbb{P}^1 \times \mathbb{P}^1$*, and* $B_1 = 0$*, then there is very free section* s *whose normal bundle is* $\mathcal{O}(a) \oplus \mathcal{O}(a + b)$ *with* a *arbitrarily large and* b *is at most* 2.

Proof

First consider the case where a general fiber is $\mathbb{P}^2$ or the blow-up of $\mathbb{P}^2$ at d $(1 \leq d \leq 8)$ general points. Take a section s with normal bundle $\mathcal{O}(a) \oplus \mathcal{O}(a + b)$, for some $a \geq 0, b > 0$. We attach a line L in a general fiber to s along a general direction. Let $\mathcal{N}$ be the normal bundle of this reducible curve in Y. Choose a point p in the line L and a divisor $D = q_1 + \cdots + q_{a+2}$ in s. Let $\mathcal{E} = \mathcal{N}(-p - D)$. Then

$$\mathcal{E}|_L \cong \mathcal{O} \oplus \mathcal{O}, \qquad \mathcal{E}|_s \cong \mathcal{O}(-1) \oplus \mathcal{O}(b - 2).$$

We have the short exact sequence of sheaves

$$0 \to \mathcal{E}|_L(-p) \to \mathcal{E} \to \mathcal{E}|_s \to 0,$$

where p is the node of $L \cup s$. Then $H^1(\mathcal{E}) = 0$. The same is true for a general deformation by semicontinuity. Thus a general deformation is again a free section s' with $N_{s'/Y} \cong \mathcal{O}(a') \oplus \mathcal{O}(a' + b'), a' \geq a + 2, b' < b$. Continuing with this process shows that there are free sections whose normal bundle is $\mathcal{O}(a) \oplus \mathcal{O}(a)$ with a arbitrarily large.

When a general fiber is $\mathbb{P}^1 \times \mathbb{P}^1$, we can run a similar argument as above. We start with a free section s whose normal bundle is $\mathcal{O}(a) \oplus \mathcal{O}(a + B_1)$ and attach curves of bidegree $(1, 0)$ or $(0, 1)$ in a general fiber to s along a general normal direction. The tangent directions of these two types of curves at a point span the tangent space of $\mathbb{P}^1 \times \mathbb{P}^1$ at that point. So the above analysis of the normal bundle is still valid provided that we choose the appropriate type of curve. If $B_1 > 0$, in the end we can find very free sections whose normal bundle is $\mathcal{O}(a) \oplus \mathcal{O}(a + B_1)$ with a arbitrarily large. However if $B_1 = 0$, we can only guarantee a very free curve with normal bundle $\mathcal{O}(a) \oplus \mathcal{O}(a + b)$ with $b \leq 2$. □

In any case, there is a number B_2 such that we have a very free section s whose normal bundle is $\mathcal{O}(a) \oplus \mathcal{O}(a + b)$ with a arbitrarily large and b bounded above by B_2.

There exists a positive rational number ϵ such that $\pi^*\mathcal{O}(1) - \epsilon K_X$ is ample on X. So there are rational numbers b_i such that

$$\begin{aligned} H &= f^*\big(\pi^*\mathcal{O}(1) - \epsilon K_X\big) - \sum b_i E_i \\ &= f^*\pi^*\mathcal{O}(1) - \epsilon K_Y + \sum (\epsilon a_i - b_i) E_i \end{aligned}$$

is ample on Y.

Let $\tilde{s}$ be a section such that $(-K_Y)\cdot\tilde{s}$ is at most B_2. Then

$$H\cdot\tilde{s}\le 1+\epsilon B_2+\sum|\epsilon a_i-b_i|$$

since $E_i\cdot\tilde{s}$ is either 0 or 1. So there are only finitely many such curve classes. Then there is an integer M such that any such section can meet at most M general free rational curves with $-K_Y$-degree 2, and there are only finitely many such sections. We also have a lower bound N of $-K_Y\cdot\tilde{s}$ for all such sections.

The above discussion shows that there is a very free section s such that

$$\Big(\sum a_i E_i\Big)\cdot s=A, \tag{1}$$

$$N_{s/Y}\cong\mathcal{O}(a)\oplus\mathcal{O}(a+b), \tag{2}$$

$$2M+3(a+1-M)+N>2+2a+B_2, \tag{3}$$

$$b\le B_2. \tag{4}$$

Define

$$B=\min\big\{b\ge 0\;\big|\; s \text{ is a section which satisfies the conditions (1), (2), and (3) above}\big\}.$$

Clearly $B\le B_2$.

Denote by s the section of minimal degree with respect to some polarization in the set of all the very free sections with the above properties and whose normal bundle is of the form $\mathcal{O}(a)\oplus\mathcal{O}(a+B)$.

Step 2: Analyzing reducible curves in the curve class $[s]$. We prove that this curve class $[s]$ gives a nonzero Gromov–Witten invariant with two point insertions.

A general such section passes through $a+1$ general points. So we start with $a+1$ point constraints. If $B>0$, then we need to add B curve constraints. We take these curves to be a complete intersection of a very ample divisor and the pullback of $\mathcal{O}_{\mathbb{P}^1}(1)$. In particular, they lie in general fibers not containing any exceptional divisors. Then a general section meets all of these constraints and contributes positively to the Gromov–Witten invariant

$$\langle\underbrace{[pt],\dots,[pt]}_{a+1},\underbrace{[\text{Curve}],\dots,[\text{Curve}]}_{B}\rangle^X_{0,s}.$$

Now we show that any reducible curve in this curve class cannot meet all the constraints. Write the reducible curve as $C\cup C_e\cup C_g$, where C is a section, C_e the vertical components supported in the fibers containing exceptional divisors, and C_g all the other vertical components.

First notice that $E_i\cdot C_g=0$. Therefore

$$\begin{aligned}-K_Y\cdot C_e &= \Big(-f^*K_X-\sum a_iE_i\Big)\cdot C_e\\ &= -f^*K_X\cdot C_e-\Big(\sum a_iE_i\Big)\cdot(s-C-C_g)\\ &= -f^*K_X\cdot C_e-\Big(\sum a_iE_i\Big)\cdot(s-C)\geq 0,\end{aligned}$$

with equality if and only if $\left(\sum a_iE_i\right)\cdot C=A$ and C_e is mapped to a point in X. There are three different cases according to what kind of curve C is.

Case 1: The section C is a free curve. Suppose that C meets a' general points and b' $(0\leq b'\leq B)$ general curves in the fiber. Then

$$-K_Y\cdot C=\dim\overline{\mathcal{M}}(X,[C])\geq 2a'+b'.$$

We also have

$$-K_Y\cdot C_g\geq 2(a+1-a')+(B-b')$$

and

$$-K_Y\cdot C_e\geq 0.$$

Add all of these together; we get

$$-K_Y\cdot(C+C_g+C_e)\geq 2a+2+b'+(B-b')=2a+2+B.$$

So equality has to hold in each inequality. This implies that $\left(\sum a_iE_i\right)\cdot C=A$ and C_g is the union of $(a+1-a')$ free curves with $-K_Y$-degree 2 and $(B-b')$ curves of $-K_Y$-degree 1. Then the reducible curve consisting of C and $(a+1-a')$ free curves in C_g deforms to an irreducible section curve $\tilde{C}$, which may pass $a+1$ general points.

Therefore the normal bundle of the new section curve $\tilde{C}$ has to be $\mathcal{O}(a'')\oplus(a''+b'')$, for some $a''\geq a$. Note that $\left(\sum a_iE_i\right)\cdot\tilde{C}=A$. Thus $\tilde{C}$ satisfies the conditions (1), (2), and (3). So $b''\geq B$. But we also have the reverse inequality since

$$2a+B+2=-K_Y\cdot C\geq -K_Y\cdot\tilde{C}=2a''+b''+2.$$

Hence equality holds and $b'=b''=B$. Then by the minimality of the section curve we start with, the curve class $[C_e]$ is zero. In other words, $C\cup C_e\cup C_g=C\cup C_g$ is in the boundary of an irreducible component of expected dimension. So we can choose the constraints to miss such configurations.

Case 2: The section C is not a free curve and $-K_Y\cdot C>B$. We may choose the $a+1$ points to lie in different general fibers and such that any curve through them is free. Then neither C nor C_e passes through any of them. So

$$-K_Y\cdot C_g\geq 2(a+1)$$

and

$$-K_Y \cdot (C_e + C + C_g) > 0 + 2a + 2 + B.$$

This is impossible.

Case 3: The section C is not free and $-K_Y \cdot C \leq B$. Again C does not meet any point constraints. So C_g has at least $a+1$ curves D_i in different fibers and $-K_Y \cdot D_i \geq 2$. If $-K_Y \cdot D_i = 2$, D_i is an irreducible free curve. There can be at most M such curves, and only finitely many sections can meet all of these curves. So if we choose other points to be general, then every curve through those points with $-K_X$-degree 2 will not meet these sections. Thus for all the other D_i's (which are possibly reducible),

$$-K_Y \cdot D_i \geq 3.$$

But then

$$\begin{aligned} -K_Y \cdot (C_e + C + C_g) &\geq 2M + 3(a + 1 - M) + N \\ &> 2 + 2a + B_2 \\ &\geq 2 + 2a + B \\ &= -K_Y \cdot s. \end{aligned}$$

This is impossible. □

7. Rationally connected 3-folds with Picard number 2

In this section we prove the following theorem.

THEOREM 7.1

Let X be a smooth projective rationally connected 3-fold X with Picard number 2. Then X is symplectic rationally connected.

There is at least one K_X-negative extremal ray of X since X is rationally connected. Let $f : X \to Y$ be the corresponding contraction. Then previous results of Theorems 2.1 and 6.1, Corollary 4.9, Theorem 5.5, and Propositions 5.3 and 5.12 cover the cases where the contraction is of type (E1)–(E4), (C), (D). The only remaining case is the (E5)-type contraction, where the exceptional divisor E is a smooth $\mathbb{P}^2$ with normal bundle $\mathcal{O}(-2)$. In this case the variety Y is a non-Gorenstein $\mathbb{Q}$-Fano 3-fold of Picard number 1.

We introduce some notations first. Let $A = -f^*K_Y - \epsilon E$ with $0 < \epsilon \ll 1$ be an ample $\mathbb{Q}$-divisor. Let C be a minimal free curve with respect to $-f^*K_Y$ and A. By abuse of notation we write K_Y instead of f^*K_Y in the following.

If $E \cdot C = 0$, then C gives rise to a free curve in the smooth locus of Y. Thus the smooth locus of Y is rationally connected, and the result follows from Proposition 5.3.

In the following we assume that $E \cdot C > 0$. First we have the following observation.

LEMMA 7.2
If $E \cdot C > 0$, then $-K_X \cdot C = 2$.

Proof
If $-K_X \cdot C \geq 3$, then we can deform such a curve with one point fixed. The deformation of the curve C in X also gives a deformation of its image in Y. Since $E \cdot C \geq 1$, the deformation in Y fixes the chosen point and the unique singular point of Y. So by bend-and-break, we get a curve through the fixed point with smaller $-K_Y$-degree. If the fixed point is chosen to be a very general point, then this curve with smaller $-K_Y$-degree gives a free curve in X. This is a contradiction to our choice. $\square$

Take a reducible curve Γ to be a union of two such free curves passing through two very general points and a line in $E \cong \mathbb{P}^2$. It is easy to see that we can smooth the nodes of Γ and get a very free curve Γ' with $-K_X \cdot \Gamma' = 5$.

LEMMA 7.3
Let D be a reducible curve passing through two general points such that the two points are not connected by an irreducible component of the curve D. If $-K_Y \cdot D \leq -K_Y \cdot \Gamma'$ and $A \cdot D \leq A \cdot \Gamma'$, then one of the following holds.
(1) *The curve D consists of three irreducible components, two of which are the free curves of class $[C]$ and one of which is a line in the exceptional divisor E.*
(2) *The curve D consists of two irreducible components, which are free curves with $-K_X$ degree 2 and 3.*
Furthermore, the curve classes $[D]$ and $[\Gamma']$ are the same.

Proof
Write $D = D_1 \cup D_2 \cup D_3$, where D_1 and D_2 are the irreducible components through the two general points and D_3 is the rest of the irreducible components. Then

$$-K_Y \cdot (D_1 + D_2) \leq -K_Y \cdot D \leq -K_Y \cdot \Gamma' = 2(-K_Y \cdot C).$$

We also have the reverse inequality by our choice of C. Thus

$$-K_Y \cdot D_1 = -K_Y \cdot D_2 = -K_Y \cdot C,$$

and all the irreducible components of D_2 are supported in the exceptional divisor E.

Write $D_1 = C + \lambda_1 L$ and $D_2 = C + \lambda_2 L$, where L is the class of a line in the exceptional divisor E. Note that λ_1 and λ_2 are nonnegative rational numbers. Since $-K_X \cdot L = 1$, they are actually integers. Since $A \cdot D \leq A \cdot \Gamma'$, we have $\lambda_1 + \lambda_2 \leq 1$.

If $\lambda_1 + \lambda_2 = 0$, then D_1 and D_2 have the same curve class as C and we can choose the two points so that D_1 and D_2 do not intersect each other. In this case it is easy to see that D_3 is a line in the exceptional divisor E.

If $\lambda_1 + \lambda_2 = 1$, then D_1 and D_2 are the only components of D.

Notice that we have $-K_Y \cdot D = -K_Y \cdot \Gamma'$ and $A \cdot D = A \cdot \Gamma'$ in these cases. But X has Picard number 2. Therefore $[D] = [\Gamma']$. □

PROPOSITION 7.4

If there is a very free irreducible curve C' such that $-K_X \cdot C' = 4$, $-K_Y \cdot C' \leq -K_Y \cdot \Gamma'$, and $A \cdot C' \leq A \cdot \Gamma'$, then there is a nonzero Gromov–Witten invariant of the form $\langle [\mathrm{pt}], [\mathrm{pt}] \rangle^X_{0,D}$.

Proof

Choose a very free curve D whose $-f^*K_Y$-degree is minimal among all very free curves satisfying the conditions

$$-K_X \cdot C' = 4,$$
$$-K_Y \cdot C' \leq -K_Y \cdot \Gamma',$$
$$A \cdot C' \leq A \cdot \Gamma'.$$

Note that any curve satisfying these conditions and having minimal A-degree has the same curve class as D since $b_2(X) = 2$. We want to show that this curve D gives a nonzero Gromov–Witten invariant of the form $\langle [\mathrm{pt}], [\mathrm{pt}] \rangle^X_{0,D}$.

First notice that there does not exist a reducible curve F passing through two general points such that

(1) the two points are not connected by an irreducible component,

(2) $-K_Y \cdot F \leq -K_Y \cdot D$, and

(3) $A \cdot F \leq A \cdot D$.

In fact, if there is such a curve, we can apply Lemma 7.3 since $-K_Y \cdot C' \leq -K_Y \cdot \Gamma'$ and $A \cdot C' \leq A \cdot \Gamma'$. Then the curve classes $[F]$, $[D]$, and $[\Gamma']$ are all the same. This is impossible since $-K_X \cdot D = 4$ but $-K_X \cdot \Gamma' = 5$.

We claim that if there is an irreducible curve F through the two points such that $-f^*K_Y \cdot F \leq -f^*K_Y \cdot D$, then the class $[F]$ is the same as $[D]$. Actually, if $-K_X \cdot F = 4$, then this follows from our choice of D. If $-K_X \cdot F \geq 5$, then we may deform this curve with two points fixed. F specializes to a reducible curve. If there is again an irreducible curve F_1 passing through two general points and $-K_X \cdot F_1 \geq 5$, then continue deforming it with two points fixed. This process terminates at some point, and we get an irreducible curve F_n passing through the two very general points (thus very free) such that $-K_X \cdot F_n = 4$.

Then we have

$$-K_Y \cdot F_n \leq -K_Y \cdot D \leq -K_Y \cdot F_n.$$

Here we get the first inequality by the construction of F_n and the second inequality by the choice of D. Thus the curve F_n has the same curve class as D. This is a contradiction since by construction $A \cdot F_n < A \cdot D$.

The above discussion shows that no reducible curve of class $[D]$ can meet all the constraints. Thus we are done. $\square$

Next assume that there is no irreducible curve C' such that $-K_X \cdot C' = 4$, $-K_Y \cdot C' \leq -K_Y \cdot \Gamma'$, and $A \cdot C' \leq A \cdot \Gamma'$. Choose a very free curve D with $-K_X$-degree 5 and minimal with respect to A. Choose the constraints to be two very general points and a moving curve G which meets every divisor but E (e.g., the strict transform of the intersection of two very ample divisors in Y).

The following proposition concludes the proof of Theorem 7.1.

PROPOSITION 7.5
Assumptions are as above. Then the Gromov–Witten invariant $\langle [\mathrm{pt}], [\mathrm{pt}], G \rangle^X_{D,0}$ is nonzero.

Proof
The proof is similar to that of Proposition 7.4.

First we claim that there cannot be any irreducible curve D' connecting the two general points with smaller A-degree and $-K_Y$-degree. In fact, if there is such a curve, then $-K_X \cdot D'$ is at least 5 by assumption. So we can deform D' with the points fixed and break D' into a reducible or nonreduced curves. Continue this process until we end up with a reducible/nonreduced curve with no irreducible components containing the two chosen points. Thus the curve has two components which are free curves, and each passes through one chosen point. Then application of Lemma 7.3 gives a contradiction.

Thus we only need to consider the case where there are two irreducible components, which are free curves passing through the two chosen general points. Thus we are in the situation of Lemma 7.3. Clearly we can choose G to avoid all such configurations of curves. $\square$

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Department of Mathematics, California Institute of Technology, Pasadena, California, 91125, USA; tian@caltech.edu