

WEAK APPROXIMATION FOR ISOTRIVIAL FAMILIES

ZHIYU TIAN, HONG RUNHONG ZONG

ABSTRACT. We prove weak approximation for isotrivial families of rationally connected varieties defined over the function field of a smooth projective complex curve.

1. INTRODUCTION

Let X be a variety over a number field or function field K , it has been several decades both for number theorists and geometers to investigate into the set of rational points $X(K)$; when is $X(K)$ non-empty? and how many of them if $X(K)$ is non-empty? etc. Elementary observations suggest a good candidate class of varieties where the most rational points emerge as the *rational varieties*, which belong to a broader class as *rationally connected* varieties. A variety X is *rationally connected* if for two general points $x, y \in X$, there is a rational curve connecting them, e.g. there is an irreducible component V of $Hom(\mathbb{P}^1, X)$ and a family of maps $f : \mathbb{P}^1 \times V \rightarrow X$ such that the double evaluation $f^{(2)} : \mathbb{P}^1 \times \mathbb{P}^1 \times V \rightarrow X \times X$ is dominant. For a characteristic free version, to be *separably rationally connected* is to further require $f^{(2)}$ generic étale. We recall the pioneering work by Prof. Graber, Prof. Harris and Prof. Starr ([GHS03])

Theorem 1.1. *Let $\pi : \mathcal{X} \rightarrow B$ be a flat surjective morphism from a projective variety to a smooth projective complex curve such that a general fiber is smooth and rationally connected. Then π has a section $s : B \rightarrow \mathcal{X}$.*

Denote the generic fiber of $\mathcal{X} \rightarrow B$ by \mathcal{X}_η and the function field of B by $K = \mathbb{C}(B)$. Theorem 1.1 said that $\mathcal{X}_\eta(K) = \mathcal{X}_\eta(\mathbb{C}(B)) \neq \emptyset$. Furthermore, let $\hat{\mathcal{O}}_{B,b}$ be the completion of the local ring at $b \in B$ and let $Frac \hat{\mathcal{O}}_{B,b}$ denote its fraction field. One can consider the ring of *Adèles*

$$\mathbb{A}(B) := \prod_{b \in B}^{\circ} Frac \hat{\mathcal{O}}_{B,b}$$

of $\mathbb{C}(B)$ with weak product topology. Prof. Hassett and Prof. Tschinkel proposed the following conjecture in [HT06]

Conjecture 1.2. Notation as in Theorem 1.1. Then $\mathcal{X} \rightarrow B$ satisfies *weak approximation* at all places, namely the embedding $\mathcal{X}_\eta(K) = \mathcal{X}_\eta(\mathbb{C}(B)) \subset \mathcal{X}_\eta(\mathbb{A}(B))$ has dense image in $\mathcal{X}_\eta(\mathbb{A}(B))$. Equivalently, for every finite sequence (b_1, \dots, b_m) of distinct closed points of B , for every sequence $(\hat{s}_1, \dots, \hat{s}_m)$ of formal power series sections of π over b_i , and for every positive integer N , there exists a regular section s of π which is congruent to \hat{s}_i modulo \mathfrak{m}_{B,b_i}^N for every $i = 1, \dots, m$.

Date: October 31, 2018.

Some special cases of the conjecture are known, e.g.

- \mathbb{P}^n , conic bundles over \mathbb{P}^1 , del Pezzo surfaces of degree at least 4,
- low degree complete intersections of degree (d_1, \dots, d_c) such that $\sum d_i^2 \leq n + 1$ [Has10],
- smooth cubic hypersurfaces in $\mathbb{P}^n, n \geq 6$ [HT09],
- at places of good reduction (for any family) [HT06],
- a general family of del Pezzo surfaces of degree at most 3, [HT08], [Kne13], [Xu12],
- a smooth hypersurface with square-free discriminant [HT09].

And the starting point of the current work is the the following question by Prof. Starr

Question 1.3. Assume $k = \bar{k}$. Let X be a smooth projective separably rationally connected k -variety, and G be a cyclic subgroup of $Aut(X)$. Then is X G -equivariantly rationally connected? Namely, for a pair x, y as fixed points of G , is there a G -equivariant rational curve connecting them?

Over complex numbers, Question 1.3 follows from Conjecture 1.2 so can be seen as a geometric obstruction to the latter(c.f. Theorem 2.11). Now recalling the proof of Theorem 1.1 in [GHS03]: for a rationally connected fibration $\mathcal{X} \rightarrow B$, by the powerful smoothing of comb argument initiated by Prof. Kollár, Prof. Miyaoka and Prof. Mori in [KMM92], and a specialization argument cancelling monodromy of $C^* \rightarrow B$ around multiple fibers, one can find a “flexible” curve C^* where the forgetful map

$$\mathcal{F}_{g,0} : \overline{\mathcal{M}}_{g,0}(\mathcal{X}, [C^*]) \rightarrow \overline{\mathcal{M}}_{g,0}(B, \pi_*[C^*])$$

is smooth and surjective. Degenerate $(C^* \rightarrow B)$ in the Hurwitz scheme $\mathcal{H}_{g,B} = \overline{\mathcal{M}}_{g,0}(B, \pi_*[C^*])$ to contain a component isomorphic to B , a preimage of this component will be a section. Quite unexpectedly, we make an *Observation* that the situation of Question 1.3 inherits a variant of the main argument of in [GHS03]; based on a G -equivariant curve $f : C \hookrightarrow X$ connecting x to y we consider a moduli compactification of the rational *forgetful* map

$$\begin{array}{ccc} \mathcal{F}_{g,2}^G : \overline{\mathcal{M}}_{g,2}^G(X, [C]) \{f(x_1) = x, f(x_2) = y\} & \xrightarrow{\dots\dots\dots} & \mathcal{H}_{g,2}(G) \\ \uparrow & \nearrow & \\ (f : C \hookrightarrow X) \in \mathcal{U} & & \end{array}$$

-the left hand side is the G -fixed point set, and an open neighbourhood \mathcal{U} of C in it parametrizes G -equivariantly embedded curves connecting x to y -so it maps to $\mathcal{H}_{g,2}(G)$ as the Hurwitz scheme of Galois covers $C \rightarrow C/G$ with two specified ramifications. Taking a good functorial compactification of $\mathcal{F}_{g,2}^G$, then similar to [GHS03], tracing back the forgetful map, the preimage of a good component of some degenerated G -cover of $C \rightarrow C/G$ will give a G -equivariant rational curve connecting x to y . We note that for the compactification, theory of *twisted stable maps* by Prof. Abramovich, Prof. Olsson and Prof. Vistoli is used, and an explicit pencil of twisted curves cancels the monodromy of $C \rightarrow C/G$ which is similar to the second main construction in [GHS03]. This finally leads to

Theorem 1.4. *Notation as in Question 1.3. Assume that $G \cong \mathbb{Z}/l\mathbb{Z}$ with l divisible in k , and that G acts on \mathbb{P}^1 by the canonical action $z \mapsto \zeta z$, where ζ is a primitive l -th root of unity. Then there is a G -equivariant map $f : \mathbb{P}^1 \rightarrow X$ such that $f(0) = x$ and $f(\infty) = y$.*

Back to Conjecture 1.2. Though the vanishing of this geometric obstruction does not necessarily implies Conjecture 1.2, we prove that it already suffices for isotrivial families; a family $\mathcal{X} \rightarrow B$ is isotrivial if there is an étale (but not necessarily surjective) morphism $B' \rightarrow B$ such that there is a B' isomorphism

$$\mathcal{X}' = \mathcal{X} \times_B B' \cong X \times B',$$

for some fixed variety X . Our methods uses a relative version of G -equivariant smoothing of combs which works for families with G -actions. And more generally, our argument also works for families at places satisfying the following Hypothesis 1.5, which covers a large part of all the cases currently known for Conjecture 1.2.

Hypothesis 1.5. For any $b \in B$ with \mathcal{X}_b , let K be the fraction field $\text{Frac } \widehat{\mathcal{O}}_{b,B}$ of $\widehat{\mathcal{O}}_{b,B}$ and \mathcal{X}_K be the base change to $\text{Spec } K$. Assume there is a Galois extension K'/K with cyclic Galois group G_b such that $\mathcal{X}_K \times_{\text{Spec } K} \text{Spec } K'$ extends to a smooth proper family $\mathcal{X}' \rightarrow \text{Spec } \widehat{\mathcal{O}}'$, where $\widehat{\mathcal{O}}'$ is the ring of integers in K' . Further assume the action of G_b extends to \mathcal{X}' and is compatible with the projection to $\text{Spec } \widehat{\mathcal{O}}'$.

Theorem 1.6. *Notation as in Theorem 1.1. Then Conjecture 1.2 holds for the family $\mathcal{X} \rightarrow B$ at places $b \in B$ which satisfy Hypothesis 1.5. In particular, Conjecture 1.2 holds for isotrivial families (Proposition 2.10).*

An example satisfying the above hypothesis 1.5 is the family of cubic surfaces $F(X_1, X_2, X_3) + tX_0^3 + \dots$, whose central fiber is a cone over a smooth elliptic curve. One can make a degree 3 base change $t = s^3$ and change of variables

$$Y_0 = sX_0, Y_1 = X_1, Y_2 = X_2, Y_3 = X_3.$$

The new family has a $\mathbb{Z}/3\mathbb{Z}$ action and the central fiber $F(Y_1, Y_2, Y_3) + Y_0^3$ is smooth. As proposed by Prof. Hassett in [Has10] this is expected to be an exceptionally hard case of Conjecture 1.2. We remark that technics and ideas here, esp. around this example, together with some observations and ideas which deal with much subtler singularities, finally led to the solution of Conjecture 1.2 for cubic hypersurfaces, esp. for cubic surfaces in [T13]. This is currently the best known low dimensional result to Conjecture 1.2.

Remark 1.7. In spite of the classical relative equivariant smoothing in Section 4 which culminates all the smoothing technics, we present another proof of Theorem 1.6 in the Appendix, which uses some notion from derived algebraic geometry and is more conceptual and straightforward. Following the idea of lifting by forgetful maps in the proof of Theorem 1.1 and our Theorem 1.4, for a general rationally connected fibration $\mathcal{X} \rightarrow B$ with bad reduction at b , we use the *derived* forgetful map initiated by Prof. Starr and Roth

$$\mathcal{F}_D : \mathcal{Hilb}_{\mathcal{X}/B} \longrightarrow \mathcal{Pseudo}_{\mathcal{X}_b}$$

here $\mathcal{Pseudo}_{\mathcal{X}_b}$ is the moduli stack of pseudo-ideal sheaves, or equivalently-differential graded subscheme of \mathcal{X}_b with amplitude $(1, 0)$. All sections lie in $\mathcal{Hilb}_{\mathcal{X}/B}$, and \mathcal{F}_D is the *derived* intersection product

$$C \longmapsto C \times_{\mathcal{X}}^L \mathcal{X}_b$$

introduced by Prof. Lurie ([L04], [L11]). As one of the applications of *derived Algebraic Geometry*, the prominent feature of \mathcal{F}_D is that it has a well-behaved tangent

obstruction $\mathcal{R}\Gamma(C, \mathcal{N}_{C/\mathcal{X}} \otimes \mathcal{O}_{\mathcal{X}_b})$ at $C \subset X$ when C is locally complete intersection- \mathcal{F}_D remembers the jet datum of $(C \subset \mathcal{X}) \rightarrow B$ and so once the first term of $\mathcal{R}\Gamma(C, \mathcal{N}_{C/\mathcal{X}} \otimes \mathcal{O}_{\mathcal{X}_b})$ is cancelled at some “flexible” curve, then a well-behaved degeneration and the forgetful-map type argument applies to get the approximating section as desired. To explore for wider situations than in Hypothesis 1.5 where this machinery technically applies is believed by the authors to be a possible way towards a full solution to Conjecture 1.2.

Acknowledgment: The authors would first like to thank Professor Jason Starr for introducing them to the question and innumerous helpful discussions; Professor János Kollár for his constant support of the second named author and encouraging comments on the proof of Theorem 1.4; Professor Chenyang Xu for helpful discussions, sharing us with his ideas and generous support in Beijing International Center of Mathematics Research where the authors partially carried out this project; Professor Tommaso de Fernex for helpful comments on Theorem 1.6; Professor Martin Olsson, Professor Vivek Shende, and Professor Xinyi Yuan for enlightening comments and suggestions on the first version of this paper; Doctor Qile Chen and Doctor Yi Zhu for reading part of the first draft and helpful comments.

2. PRELIMINARY RESULTS

In this section we collect some preliminary results for later reference. The results proved here are slightly stronger than what are actually needed. In this section we will assume that G is a finite group except in Lemma 3.7, and k is an algebraically closed field whose characteristic is not divisible by the order of G .

2.1. Everything with a group action. Firstly, we are concerned with the following infinitesimal lifting problem. Let S and R be k -algebras with a G -action and $f : S \rightarrow R$ be a k -algebra homomorphism compatible with the action. Also assume that R is a finite type S -algebra via the map f . Let A be an Artinian k -algebra with a G -action, $I \subset A$ an invariant ideal such that $I^2 = 0$. Consider the following commutative diagram, where p is a G -equivariant k -algebra homomorphism

$$\begin{array}{ccc} S & \xrightarrow{f} & R \\ \downarrow & & \downarrow p \\ A & \xrightarrow{\pi} & A/I \longrightarrow 0 \end{array}$$

We want to know when one can find a G -equivariant lifting $h : R \rightarrow A$. The following lemma completely answers this question.

Lemma 2.1. *If we can lift the map p to a k -algebra homomorphism $h : R \rightarrow A$ such that $\pi \circ h = p$, then we can find an equivariant lifting $\tilde{h} : R \rightarrow A$ with the same property.*

Proof. For every element g in G , define a map $h_g : R \rightarrow A$ by $h_g(r) = g \cdot h(g^{-1} \cdot r)$. This is an S -algebra homomorphism and also a lifting of the map $p : R \rightarrow A/I$. The map h is G -equivariant if and only if $h_g(r) = h(r)$ for every $g \in G$ and every $r \in R$. The difference of any two such liftings is an element in $\text{Hom}(\Omega_{R/S}, I)$, where $\Omega_{R/S}$ is the module of relative differentials. Therefore one has $\theta(g)(r) = h_g(r) - h(r)$ in

$\text{Hom}(\Omega_{R/S}, I)$. Notice that $\text{Hom}(\Omega_{R/S}, I)$ is naturally a G -module with the action of G on $\text{Hom}(\Omega_{R/S}, I)$ given by

$$\begin{aligned} G \times \text{Hom}(\Omega_{R/S}, I) &\rightarrow \text{Hom}(\Omega_{R/S}, I) \\ (g, \eta) &\mapsto g \cdot \eta = (\omega \mapsto g \cdot \eta(g^{-1} \cdot \omega)). \end{aligned}$$

It is easy to check that

$$\theta(gh) = g \cdot \theta(h) + \theta(g).$$

Thus θ defines an element $[\theta]$ in $H^1(G, \text{Hom}(\Omega_{R/S}, I))$. The existence of an equivariant lifting is equivalent to the existence of an element $\Theta \in \text{Hom}(\Omega_{R/S}, I)$ such that $g\Theta - \Theta = \theta$, i.e, the class defined by θ is zero in $H^1(G, \text{Hom}(\Omega_{R/S}, I))$. But the characteristic of the field is relatively prime to the order of G , so the higher cohomologies of G vanish ([Wei94], Proposition 6.1.10, Corollary 6.5.9). \square

Corollary 2.2. *Let X and Y be two k -schemes with a G -action and $f : X \rightarrow Y$ be a finite type morphism compatible with the action. Let $x \in X$ be a fixed point, and $y = f(x)$ (hence also a fixed point). Assume that f is smooth at x . Then there exists a G -equivariant section $s : \text{Spec } \widehat{\mathcal{O}}_{y,Y} \rightarrow X$.*

Proof. Let S be the local ring at y , and R be the local ring at x . There is an obvious G -action on both of these k -algebras. We start with the section

$$\begin{aligned} s_0 : \text{Spec } k(y) &\rightarrow f^{-1}(y), \\ \text{Spec } k(y) &\mapsto x, \end{aligned}$$

which is clearly G -equivariant. By the smoothness assumption, a section from $\text{Spec } (\widehat{\mathcal{O}}_{y,Y}/\mathfrak{m}_y^n)$ always lifts to a section from $\text{Spec } (\widehat{\mathcal{O}}_{y,Y}/\mathfrak{m}_y^{n+1})$. Now apply Lemma 2.1 inductively to finish the proof. \square

2.2. Comb and equivariant comb construction. Let X be an algebraic variety over an algebraically closed field k , a *very free curve* (map) is an $f : \mathbb{P}^1 \rightarrow X$ with $f^*\mathcal{T}_X$ ample. Let X_{sm} be the smooth locus of X , and *very free locus* $X_{vf} \subset X_{sm}$ be the open locus of points $x \in X$ which lies in the image of a *very free curve*. Then it is well known that X_{sm} is separably rationally connected if and only if $X_{vf} \neq \emptyset$ ([KMM92], [Kol96]).

Definition 2.3. Let k be an arbitrary field. A *comb* with n teeth over k is a curve C with $n + 1$ components D, C_1, \dots, C_n over \bar{k} satisfying:

- (1) D and $C_1 \cup \dots \cup C_n$ are defined over k
- (2) $C_i \cong \mathbb{P}^1$, $C_i \cap C_j = \emptyset$ for $i \neq j$, and each C_i intersects D transversely in a single smooth point over \bar{k} .

D is the *handle* of the comb, and C_1, \dots, C_n the *teeth*.

Definition 2.4 (Smoothing). Given $f : C \rightarrow X$ from a *comb* C with *handle* D , a smoothing of f is a family $\Sigma \rightarrow T$ over a pointed curve $(T, 0)$ with $F : \Sigma \rightarrow X$ such that $\Sigma_0 \cong C$, $F|_{\Sigma_0} = f$ and Σ_t is a smooth curve for general t .

We have a general procedure for *smoothing of comb* using *very free curves*, the most general form is presented in [TZ12].

Theorem 2.5 (Proposition 2.4 of [TZ12]). *Given $f_0 : C_0 \rightarrow X$ with C_0 a curve and X a smooth quasi-projective k -variety, and an integer d . Suppose X is separably rationally connected and $f_0(C) \cap X_{v,f} \neq \emptyset$, then there are $q \gg 0$ very free curves $f_i : C_i \rightarrow X, 1 \leq i \leq q$, such that:*

- (1) $C = C_0 \cup C_1 \cup \dots \cup C_q$ is a comb with q teeth. And there is a k -morphism $f : C \rightarrow X$ with a smoothing $\Sigma \rightarrow T, G : \Sigma \rightarrow X$.
- (2) $H^1(\Sigma_t, G_t^* \mathcal{T}_X \otimes M) = 0$ for a general member $G_t : \Sigma_t \rightarrow X$ and any line bundle M with $|\deg M| \leq d$.

For proper family $\mathcal{X} \rightarrow B$ where \mathcal{X} can be a *Deligne-Mumford stack*, we need a relative version of *smoothing of comb* which can be proved using the same method for Theorem 2.5. And it was implicitly stated in [GHS03], [DS03] and [HT06].

Theorem 2.6 (Relative smoothing). *Let $\pi : \mathfrak{X} \rightarrow B$ be a family of smooth quasi-projective k -varieties over a curve B with separably rationally connected general fibers. Given: a compactification $\bar{\pi} : \bar{\mathfrak{X}} \rightarrow B$ where $\bar{\mathfrak{X}}$ can be an algebraic space, Deligne-Mumford stack or Artin stack; a multisection $(C' \subset \bar{\mathfrak{X}}) \rightarrow B$ which lies in the smooth locus of $\bar{\pi}$ and intersects with $\bigcup_{b \in B} \bar{\mathfrak{X}}_{b,v,f}$; an integer d . Then there are $q \gg 0$ morphism $f_i : C_i \cong \mathbb{P}^1 \rightarrow \bar{\mathfrak{X}}, 1 \leq i \leq q$, such that:*

- (1) For $1 \leq i \leq q$, $g_i : C_i \rightarrow \bar{\mathfrak{X}}$ factors through a separably rationally connected fiber $\bar{\mathfrak{X}}_{c_i}$ for some $c_i \in C$, and $g_i : C_i \rightarrow \bar{\mathfrak{X}}_{c_i}$ is very free in $\bar{\mathfrak{X}}_{c_i}$, namely, C_i lies in the smooth locus of $\bar{\pi}$ with $f_i^* \mathcal{T}_{\bar{\mathfrak{X}}/B}$ ample.
- (2) $\hat{C} = C' \cup C_0 \cup C_1 \cup \dots \cup C_q$ is a comb with q teeth with a natural k -morphism $f : \hat{C} \rightarrow \bar{\mathfrak{X}}$ and a smoothing $\Sigma \rightarrow T, G : \Sigma \rightarrow \bar{\mathfrak{X}}$ which lies in the smooth locus of $\bar{\pi}$.
- (3) $H^1(\Sigma_t, G_t^* \mathcal{T}_{\bar{\mathfrak{X}}/B} \otimes M) = 0$ for a general member $G_t : \Sigma_t \rightarrow \bar{\mathfrak{X}}$ and any line bundle M with $|\deg M| \leq d$, where $G_t : \Sigma_t \rightarrow \bar{\mathfrak{X}}$ denotes the restriction of G to the fiber Σ_t .

Proof. The proof is basically the same as [TZ12, Proposition 2.4], so we only sketch the proof. By [HT06, Lemma 25] and [HT06, Proposition 24], one can attach sufficiently many very free curves $C_i, i = 1, \dots, q$ on the fibers of π to C' such that there is a smoothing of the comb $\hat{C} = C' \cup C_1 \cup \dots \cup C_q$ whose handle is C' . Then there is a natural morphism $g : \hat{C} \rightarrow \bar{\mathfrak{X}}$. Moreover, one can choose q large enough such that $q - h^1(C', (g^* \mathcal{T}_{\bar{\mathfrak{X}}/B})|_{C'}) \gg 0$. Then $H^1(\Sigma_t, G_t^* \mathcal{T}_{\bar{\mathfrak{X}}/B} \otimes M) = 0$ for any line bundle M on Σ_t with $|\deg M|$ bounded d by [TZ12, Lemma 2.5] and [TZ12, Lemma 2.6]. \square

We also need a G -equivariant version of Theorem 2.5.

Theorem 2.7 (G -equivariant smoothing). *Assume $k = \bar{k}$. Let C_0 be a curve and X a smooth quasi-projective k -variety both with G actions, with a G -equivariant embedding $f_0 : C_0 \hookrightarrow X$ and an integer d . Suppose X is separably rationally connected and $f_0(C) \cap X_{v,f} \neq \emptyset$, then there are $q \gg 0$ very free curves $f_i : C_i \rightarrow X, 1 \leq i \leq q$, such that:*

- (1) $C = C_0 \cup C_1 \cup \dots \cup C_q$ is a comb with q teeth invariant by G . And there is a G -equivariant morphism $f : C \rightarrow X$ with G -equivariant smoothing $\Sigma \rightarrow T, G : \Sigma \rightarrow X, T$ with trivial G action.
- (2) $H^1(\Sigma_t, G_t^* \mathcal{T}_X \otimes M)^G = 0$ for a general member $G_t : \Sigma_t \rightarrow X$ and any line bundle M with a G linearization and $|\deg M| \leq d$.

Proof. One can assume $\dim X \geq 3$ by replacing X with $X \times \mathbb{P}^M$, $M \gg 0$. Let C_0 be the image of f_0 . Apply Theorem 2.5, pick $p \gg 0$ very free curves C_i intersecting with C transversely at general point p_i with general tangent direction. Form the *comb* $C = C_0 \cup_{g \in G} \cup_{i=1, \dots, p} g C_i$ the latter is G -invariant and have $H^1(C, \mathcal{T}_X|_C \otimes M) = 0$ for any line bundle M with $|\deg M| \leq d$. The vanishing implies that Since $|G|$ is divisible in k we have $H^1(C, \mathcal{T}_X|_C \otimes M)^G = 0$ since higher Galois cohomologies of G vanish (c.f. beginning of this section and the proof of Lemma 2.1). So it remains to prove the existence of a G -equivariant smoothing of C , which is equivalent to find a G -equivariant section of $H^0(C, \mathcal{N}_{C/X})$ which has general direction at the nodes of C and hence can smooth them.

Let $D_1 = C_0 \cup C_1 \dots \cup C_p$, by Theorem 2.5 again, $\mathcal{N}_{D_1/X}$ is globally generated if $p \gg 0$ and $H^1(D_1, \mathcal{N}_{D_1/X} \otimes L_1) = 0$, where L_1 is a line bundle on D_1 which has degree $-l$ on C and 0 on all the other irreducible components C_i 's. After attaching all G -conjugates of C_i 's, the new nodal curve is simply C . We have $H^1(C, \mathcal{N}_{C/X} \otimes L) = 0$, where L is an extension of the line bundle L_1 on C_0 which has degree $-l$ on C_0 and 0 on all the other irreducible components.

We have the following two exact sequences

$$0 \rightarrow \oplus \mathcal{N}_{C/X}|_{C_j}(-p_j) \rightarrow \mathcal{N}_{C/X} \rightarrow \mathcal{N}_{C/X}|_{C_0} \rightarrow 0,$$

$$0 \rightarrow \mathcal{N}_{C_0/X} \rightarrow \mathcal{N}_{C/X} \rightarrow \oplus_j Q_j \rightarrow 0,$$

where Q_j is a torsion sheaf supported on the point $p_j \in C_0$. Every sheaf has a natural G action and the G -equivariant deformations are given by G -invariant sections of $\mathcal{N}_{C/X}$. One just need to find a G -invariant section in $H^0(C, \mathcal{N}_{C/X})$ which is not mapped to 0 under the composition of maps

$$H^0(C, \mathcal{N}_{C/X}) \rightarrow H^0(C_0, \mathcal{N}_{C/X}|_{C_0}) \rightarrow Q_j$$

for all j .

Since $H^1(C, \mathcal{N}_{C/X} \otimes L) = 0$, we also have $H^1(C_0, \mathcal{N}_{C/X} \otimes L \otimes \mathcal{O}_{C_0}) = 0$. Let c_1, \dots, c_l be an orbit of the G action on C_0 . Then there is a section of $\mathcal{N}_{C/X}|_{C_0}$ which vanishes on c_1, \dots, c_{l-1} but not on c_l . Taking average over G gives a G -invariant section of $\mathcal{N}_{C/X}|_{C_0}$ which does not vanish on any of c_1, \dots, c_l . In particular, for any l nodes on C_0 which lie in a G -orbit, we can find a G -invariant section of $\mathcal{N}_{C/X}$ which does not vanish on them. Then a general G -invariant section of $\mathcal{N}_{C/X}|_{C_0}$ does not vanish on any of the nodes p_i 's. We have the surjection

$$H^0(C, \mathcal{N}_{C/X}) \rightarrow H^0(C_0, \mathcal{N}_{C/X}|_{C_0}) \rightarrow 0$$

and hence the surjection

$$H^0(C, \mathcal{N}_{C/X})^G \rightarrow H^0(C_0, \mathcal{N}_{C/X}|_{C_0})^G \rightarrow 0$$

by the vanishing of higher *Galois* cohomologies of G again. So a general G -invariant section in $H^0(C, \mathcal{N}_{C/X})^G$ does not vanish on the nodes. We take the G -equivariant deformation given by this section, which necessarily smooths all the nodes of C . \square

Remark 2.8. In Theorem 2.5, Theorem 2.6 and Theorem 2.7, by taking resolution of syzygies one can replace M by a specific coherent sheaf supported on the *comb*, and it is left to reader to find a universal degree bound.

2.3. Iterated blow-up. Let $\pi : \mathcal{X} \rightarrow C$ be a flat proper family over a smooth projective connected curve C . Let $c \in C$ be a closed point and $\widehat{s}_0 : \text{Spec } \widehat{\mathcal{O}}_{c,C} \rightarrow \mathcal{X}$ be a formal section. Assume that \widehat{s}_0 lies in the smooth locus of $\mathcal{X} \rightarrow C$. The N -th iterated blow-up associated to \widehat{s}_0 is defined inductively as follows.

The 0-th iterated blow-up \mathcal{X}_0 is \mathcal{X} itself. Assume the i -th iterated blow-up \mathcal{X}_i has been defined. And let \widehat{s}_i be the strict transform of \widehat{s}_0 in \mathcal{X}_i . Then \mathcal{X}_{i+1} is defined as the blow-up of \mathcal{X}_i at the point $\widehat{s}_i(c)$.

We remark that if both \mathcal{X} and C has a G action such that

- the map $\pi : \mathcal{X} \rightarrow C$ is G -equivariant.
- the point c is the fixed point of G . And \widehat{s}_0 is G -equivariant.

Then each \mathcal{X}_i has a G action such that both the natural morphism $\mathcal{X}_{i+1} \rightarrow \mathcal{X}_i$ and the formal section \widehat{s}_i are G -equivariant. In particular, the intersection of \widehat{s}_i with the central fiber is a fixed point of G .

One can also do this at fibers over a G -orbit in C , provided the formal sections over these points are mapped to each other via the G action. Then the iterated blow-up still has a G action and every morphism is compatible with the action.

On \mathcal{X}_N , the fiber over the point c are the strict transform of \mathcal{X}_c and exceptional divisors E_1, \dots, E_N and

- $E_i, i = 1, \dots, N - 1$, is the blowup of \mathbb{P}^d at $r_i (= \widehat{s}_i(c))$, the point where the proper transform of \widehat{s}_0 (i.e. \widehat{s}_i) meets the fiber over c of the $(i - 1)$ -th iterated blow-up;
- $E_N \cong \mathbb{P}^d$, where d is the dimension of the fiber.

The intersection $E_i \cap E_{i+1}$ is the exceptional divisor $\mathbb{P}^{d-1} \subset E_{i+1}$, and a proper transform of a hyperplane in E_{i+1} , for $i = 0, \dots, N - 1$.

Furthermore, to find a section that agrees with \widehat{s}_0 up to the N -th order is the same as finding a section in \mathcal{X}_{N+1} intersecting the fiber over c at E_{N+1} , or equivalently, a section in \mathcal{X}_N which intersects the exceptional divisor E_N at the point $r_N = \widehat{s}_N(c)$ (Proposition 11, [HT06]).

2.4. Isotrivial family and geometric obstruction.

Definition 2.9. Let $\pi : \mathcal{X} \rightarrow B$ be a flat proper family of k -varieties over arbitrary base B . It is isotrivial if there is an étale morphism $B' \rightarrow B$ such that there is a B' isomorphism $\mathcal{X}' = \mathcal{X} \times_B B' \cong X \times B'$ for some k -variety X .

We have to show that an isotrivial family over complex numbers satisfies Hypothesis 1.5, as follows:

Proposition 2.10. *Let $\mathbb{C}((t))$ be the Laurent field over complex numbers. And let $\pi : \mathcal{X} \rightarrow \text{Spec } \mathbb{C}((t))$ be an projective family of complex varieties. If π is isotrivial, then there is a finite cyclic group G of order l , a smooth projective variety X , and a group homomorphism from G to the automorphism group of X , such that the family $\mathcal{X} \rightarrow \text{Spec } \mathbb{C}((t))$ is isomorphic to the quotient $(X \times \text{Spec } \mathbb{C}((t')))/G$, where the action of G on $\text{Spec } \mathbb{C}((t'))$ is given by the choice of a primitive l -th root of unity ζ and $t \mapsto \zeta \cdot t'$. In particular, Hypothesis 1.5 is satisfied for any isotrivial family over complex numbers.*

Proof. By definition, after an étale base change, the family becomes trivial. Since the only connected étale cover of $\text{Spec } \mathbb{C}((t))$ is of the form

$$\text{Spec } \mathbb{C}((t')) \rightarrow \text{Spec } \mathbb{C}((t)), t = t'^l,$$

we have a trivial family $X \times \text{Spec } \mathbb{C}((t'))$ together with an action of a cyclic group G of order l .

Since the family $\mathcal{X} \rightarrow \text{Spec } \mathbb{C}((t))$ is projective, there is a relative very ample line bundle \mathcal{L} on \mathcal{X} . Thus there is a G -invariant line bundle L on $X \times \text{Spec } \mathbb{C}((t'))$. The line bundle L is the pull-back of a line bundle L_0 on X via the first projection. Choose a G -linearization on L . Then the group G acts on the space of sections $H^0(X \times \text{Spec } \mathbb{C}((t')), L) = H^0(X, L_0) \otimes \mathbb{C}((t'))$. This action naturally extends to an action of $H^0(X, L_0) \otimes \mathbb{C}[[t']]$. Thus there is an extension of the G action to $X \times \text{Spec } \mathbb{C}[[t']]$. And there is a natural action of G on X (by restricting the action to the closed central fiber) so that the family $\mathcal{X} \rightarrow \text{Spec } \mathbb{C}((t))$ is isomorphic to the quotient $(X \times \text{Spec } \mathbb{C}((t')))/G$. \square

And now it comes to the starting point of this paper:

Theorem 2.11. *Over complex numbers, Question 1.3 follows from Conjecture 1.2 so geometrically obstructs the latter.*

Proof. Let X be a smooth projective complex rationally connected variety with a G action, where $G \cong \mathbb{Z}/l\mathbb{Z}$, and $C \cong \mathbb{P}^1$. Let G act on $C \cong \mathbb{P}^1$ by $z \mapsto \zeta \cdot z$ where ζ is a primitive l -th root of unity. Take the diagonal action of G on $X \times C$ and form the quotient $q : X \times C \rightarrow \mathcal{X} = X \times C/G$ with $B = C/G \cong \mathbb{P}^1$. Then we have projection $\pi : \mathcal{X} \rightarrow C/G$ and diagram

$$\begin{array}{ccc} X \times C & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \\ C & \longrightarrow & B \end{array}$$

If Conjecture 1.2 holds for the family $\pi : \mathcal{X} \rightarrow B$ with rationally connected general fibers, one can choose two fixed points x, y in X and find a section $s : \mathbb{P}^1 \rightarrow \mathcal{X}$ which satisfies $s(0) = q(x, 0)$ and $s(\infty) = q(y, \infty)$. Now s gives a G -equivariant section \tilde{s} of $\pi_2 : X \times C \rightarrow C$ such that $\tilde{s}(0) = (x, 0)$, $\tilde{s}(\infty) = (y, \infty)$. Then the projection onto X gives a G -equivariant rational curve connecting x to y . This leads to Question 1.3. \square

2.5. Twisted curves and Twisted Stable Maps. In this subsection, we give a short introduction to the theory of twisted curves and n -pointed twisted stable maps. We refer to [AV02] and [Ol07] for more details.

Definition 2.12. A *twisted nodal n -pointed curve* over S is a diagram

$$\begin{array}{ccc} \Sigma_i^{\mathcal{C}} & \subset & \mathcal{C} \\ & \searrow & \downarrow \\ & & C \\ & & \downarrow \\ & & S \end{array}$$

where:

- \mathcal{C} is a tame Deligne-Mumford stack, proper and of finite presentation over S , and étale locally is a nodal curve over S ;
- $\Sigma_i^{\mathcal{C}} \subset \mathcal{C}$ are disjoint closed substacks in the smooth locus of $\mathcal{C} \rightarrow S$;

- $\Sigma_i^{\mathcal{C}} \rightarrow S$ are étale gerbes;
- The morphism $\mathcal{C} \rightarrow \mathcal{C}$ exhibits \mathcal{C} as the coarse moduli scheme of \mathcal{C} ;
- $\mathcal{C} \rightarrow \mathcal{C}$ is an isomorphism over C_{gen} .

And we can define twisted stable maps (for full categorical definition see [AV02]). We consider a proper tame Deligne-Mumford stack \mathcal{M} admitting a projective coarse moduli scheme \mathbf{M} . We fix an ample invertible sheaf on \mathbf{M} .

Definition 2.13. A *Twisted Stable n -pointed map of genus g and degree d over S*

$$(\mathcal{C} \rightarrow S, \Sigma_i^{\mathcal{C}} \subset \mathcal{C}, f: \mathcal{C} \rightarrow \mathcal{M})$$

consists of a commutative diagram

$$\begin{array}{ccc} \mathcal{C} & \longrightarrow & \mathcal{M} \\ \downarrow & & \downarrow \\ \mathcal{C} & \longrightarrow & \mathbf{M} \\ \downarrow & & \\ S & & \end{array}$$

along with n closed substacks $\Sigma_i^{\mathcal{C}} \subset \mathcal{C}$, satisfying:

- (1) $\mathcal{C} \rightarrow \mathcal{C} \rightarrow S$ along with $\Sigma_i^{\mathcal{C}}$ is a twisted nodal n -pointed curve over S ;
- (2) the morphism $\mathcal{C} \rightarrow \mathcal{M}$ is representable; and
- (3) $(\mathcal{C} \rightarrow S, \Sigma_i^{\mathcal{C}}, f: \mathcal{C} \rightarrow \mathbf{M})$ is a stable n -pointed map of degree d .

We recall the main theorem of [AV02].

Theorem 2.14. *Let $\overline{\mathcal{M}}_{g,n}(\mathcal{M}, d)$ be fibered over $\mathcal{S}ch/S$, the category of the twisted stable n -pointed maps $\mathcal{C} \rightarrow \mathcal{M}$ of genus g and degree d .*

- (1) *The category $\overline{\mathcal{M}}_{g,n}(\mathcal{M}, d)$ is a proper algebraic stack.*
- (2) *The coarse moduli space $\overline{\mathbf{M}}_{g,n}(\mathcal{M}, d)$ of $\overline{\mathcal{M}}_{g,n}(\mathcal{M}, d)$ is projective.*
- (3) *There is a commutative diagram*

$$\begin{array}{ccc} \overline{\mathcal{M}}_{g,n}(\mathcal{M}, d) & \longrightarrow & \overline{\mathcal{M}}_{g,n}(\mathbf{M}, d) \\ \downarrow & & \downarrow \\ \overline{\mathbf{M}}_{g,n}(\mathcal{M}, d) & \longrightarrow & \overline{\mathbf{M}}_{g,n}(\mathbf{M}, d), \end{array}$$

where the top arrow is proper, quasi-finite and relatively of Deligne-Mumford type, and the bottom arrow is finite.

Remark 2.15. For a morphism between tame Deligne-Mumford stacks $f: \mathcal{X} \rightarrow \mathcal{Y}$, we have a natural morphism

$$\overline{\mathcal{M}}_{g,n}(\mathcal{X}, d) \longrightarrow \overline{\mathcal{M}}_{g,n}(\mathcal{Y}, f_*d).$$

This map will contract the components which become non-stable after composed with the map f -from the definition one can show that all such components are isomorphic to $[\mathbb{P}^1/G]$ with 2 stacky points $[0]$ and $[\infty]$, where G is some cyclic group with $|G|$ divisible in k and the G -action is canonically defined by multiplying a primitive $|G|$ -th root of unity.

3. FINDING G -EQUIVARIANT RATIONAL CURVES

This section is devoted to prove

Theorem 3.1 (Theorem 1.4). *Let X be a smooth projective separably rationally connected variety over k with $\bar{k} = k$. Assume $G \cong \mathbb{Z}/l\mathbb{Z}$ with l divisible in k , and G acts on X and on \mathbb{P}^1 by $z \mapsto \zeta z$, where ζ is a primitive l -th root of unity. Then for x, y as 2-fixed points of the G action on X , there is a G -equivariant map $f : \mathbb{P}^1 \rightarrow X$ with $f(0) = x$ and $f(\infty) = y$.*

Now recalling Theorem 1.1 again: for a rationally connected fibration $\mathcal{X} \rightarrow B$, by the powerful smoothing of comb argument initiated by Prof. Kollár, Prof. Miyaoka and Prof. Mori in [KMM92], and a specialization argument cancelling monodromy of $C^* \rightarrow B$ around multiple fibers, one can find a “flexible” curve C^* where the forgetful map

$$\mathcal{F}_{g,0} : \overline{\mathcal{M}}_{g,0}(\mathcal{X}, [C^*]) \rightarrow \overline{\mathcal{M}}_{g,0}(B, \pi_*[C^*])$$

is smooth and surjective. Degenerate $(C^* \rightarrow B)$ in the Hurwitz scheme $\mathcal{H}_{g,B} = \overline{\mathcal{M}}_{g,0}(B, \pi_*[C^*])$ to contain a component isomorphic to B , a preimage of this component will be a section. Quite unexpectedly, we have the following main observation

Observation 3.2. The situation of Question 1.3 inherits a variant of the above *Graber-Harris-Starr* argument.

By taking Lefschetz pencils, it is easy to have a high genus G -equivariant curve C embedded in X connecting x to y , an open neighbourhood \mathcal{U} of C in the G -fixed locus of the Kontsevich’s moduli space of stable map

$$\overline{\mathcal{M}}_{g,2}(X, [C])\{f(p) = x, f(q) = y\}$$

also parametrizes G -equivariant embedded maps marked twice connecting x to y , and so it maps to $\mathcal{H}_{g,2}(G)$ as the Hurwitz scheme of Galois covers $C \rightarrow C/G$ with two specified ramifications. So we have a rational map

$$\mathcal{F}_{g,2}^G : \overline{\mathcal{M}}_{g,2}^G(X, [C])\{f(x_1) = x, f(x_2) = y\} \dashrightarrow \mathcal{H}_{g,2}(G)$$

If one can find a good compactification of this rational map, then similar to [GHS03], the preimage of a good component ($\mathbb{P}^1 \rightarrow \mathbb{P}^1/G \cong \mathbb{P}^1$ totally ramified at 2 points) of some degenerated G -cover might give a G -equivariant rational curve connecting x to y .

3.1. Moduli compactification and tangent obstruction of $\mathcal{F}_{g,2}^G$. Instead of working with G -equivariant stable maps, passing to the stacky quotient map $[C/G] \rightarrow [X/G]$ helps to find such a compactification: applying Remark 2.15 to the natural morphism $[X/G] \rightarrow \mathcal{B}G$ we get the forgetful map between proper algebraic stacks

$$[\mathcal{F}]_{g',2}^G : \overline{\mathcal{M}}_{g',2}([X/G], [C/G])\{f'([x_1]) = [x], f'([x_2]) = [y]\} \longrightarrow \overline{\mathcal{M}}_{g',2}(\mathcal{B}G).$$

Here g' is the genus of the stacky curve $[C/G]$, in order to apply the lift argument of *Graber-Harris-Starr*’s paper [GHS03], the remained problem is to analysis the tangent obstruction of this map and to deal with possible monodromy problems which arise in the cover $C \rightarrow C/G$. As we will discuss in the following: for the first problem, we restrict to a special type of twisted map where the deformation theory is easier to describe; and for the second we construct a special high genus equivariant curve C admitting a special degeneration that cancels the monodromy.

We restrict to a special type of G -equivariant stable map or twisted stable map that has simpler deformation and tangent obstruction for $[\mathcal{F}]_{g',2}^G$.

Definition 3.3. For a stable nodal curve C with a G -action, containing p, q as two marked points, we say (C, p, q) is G -simple if the follows are satisfied:

- The G action on C is effective.
- p, q are fixed points of G .
- For any node n of C , the stabilization subgroup of G that fixed n as $Stab_G(n)$ is trivial.

Let (C, p, q) be a G -simple nodal curve, and we assume that there is a G -equivariant morphism $f : C \rightarrow X$. We say $f : (C, p, q) \rightarrow (X, f(p), f(q))$ is a G -simple map if $f : (C, p, q) \rightarrow X$ is a stable map and that f is an immersion.

Remark 3.4. We note that a G -simple curve can be G -equivariantly smoothed to a smooth and irreducible curve with marking p, q which is again G -simple. Actually by taking quotients $[C/G]$, they form a special class of *balanced* twisted stable nodal 2 point curves as defined in [AV02] and [O107], since $[C/G]$ has no stacky structure at the nodes. And for a G -simple map $f : (C, p, q) \rightarrow X$, clearly it gives a twisted stable map

$$[f] : ([C/G], [p/G], [q, G]) \rightarrow ([X/G], [f(p)/G], [f(q)/G]).$$

The normal sheaf $\mathcal{N}_{f/X}$ is locally free with a natural G -linearization. In particular, it gives a special type of *balanced* twisted stable 2 point maps in the sense of [AV02] and [O107].

Proposition 3.5. *Let $f : (C, p, q) \rightarrow (X, f(p), f(q))$ be a G -simple equivariant map. Assume $f(p) = x, f(q) = y$, and $H^1(C, f^*\mathcal{T}_X(-p - q))^G = 0$. Then*

$$[\mathcal{F}]_{g',2}^G : \overline{\mathcal{M}}_{g',2}([X/G], [C/G])\{f'([p]) = [x], f'([q]) = [y]\} \longrightarrow \overline{\mathcal{M}}_{g',2}(BG)$$

will be smooth and surjective at the point $([f] : [C/G] \rightarrow [X/G])$.

Proof. We have the following long exact sequence of cohomology groups

$$\begin{aligned} 0 &\rightarrow Ext^0(\Omega_C(p+q), \mathcal{O}_C) \rightarrow Ext^0(f^*\Omega_X(p+q), \mathcal{O}_C) \rightarrow \\ &\rightarrow \mathbb{H}^1(\mathbb{R}Hom_{\mathcal{O}_C}(\Omega_f(p+q), \mathcal{O}_C)) \rightarrow Ext^1(\Omega_C(p+q), \mathcal{O}_C) \rightarrow \\ &\rightarrow Ext^1(f^*\Omega_X(p+q), \mathcal{O}_C) \rightarrow \mathbb{H}^2(\mathbb{R}Hom_{\mathcal{O}_C}(\Omega_f(p+q), \mathcal{O}_C)) \rightarrow 0 \end{aligned}$$

The deformation and obstruction space of the stable map $(f : (C, p, q) \rightarrow (X, x, y))$ are

$$\begin{aligned} Def(f) &= \mathbb{H}^1(C, \mathbb{R}Hom_{\mathcal{O}_C}(\Omega_f(p+q), \mathcal{O}_C)) \\ Obs(f) &= \mathbb{H}^2(C, \mathbb{R}Hom_{\mathcal{O}_C}(\Omega_f(p+q), \mathcal{O}_C)) \end{aligned}$$

where $\Omega_f(p+q)$ is the complex

$$\begin{array}{ccc} -1 & & 0 \\ & & \downarrow \\ & f^*\Omega_X(p+q) & \xrightarrow{df^\dagger} \Omega_C(p+q). \end{array}$$

Since $(f : (C, p, q) \rightarrow (X, x, y))$ is G -simple, by Remark 3.4, $\Omega_f(p+q)$ is quasi-isomorphic to $Hom_{\mathcal{O}_C}(\Omega_C, \mathcal{N}_{f/X})(p+q)$ which is locally free with a natural G -linearization, and the deformation and obstruction space of the stable map

$$(f : ([C/G], [p/G], [q/G]) \rightarrow ([X/G], [x/G], [y/G]))$$

is the G -invariant part of $Def(f), Obs(f)$. Since C admits G -equivariant smoothings, the deformation space of the stacky curve $([C/G], [p/G], [q/G])$ will be the G -invariant part of $Ext^1(\Omega_C(p+q))$. And the tangent map between

$$\overline{\mathcal{M}}_{g',2}([X/G], [C/G])\{[f]([p]) = [x], [f]([q]) = [y]\}$$

and $\overline{\mathcal{M}}_{g',2}(\mathcal{B}G)$ is simply

$$\mathbb{H}^1(\mathbb{R}Hom_{\mathcal{O}_C}(\Omega_f(p+q), \mathcal{O}_C))^G \rightarrow Ext^1(\Omega_C(p+q), \mathcal{O}_C)^G$$

So if $H^1(C, f^*T_X(-p-q))^G = 0$, the morphism $[\mathcal{F}]_{g',2}^G$ is smooth at the point represented by $[f] : ([C/G], [p/G], [q/G]) \rightarrow ([X/G], [x/G], [y/G])$ and the forgetful map is smooth at this point. Thus it is surjective when restricted to the unique irreducible component containing this stable map since the forgetful map is also proper. \square

3.2. A pencil in $\overline{\mathcal{M}}_{g',2}(\mathcal{B}G)$. Even with Proposition 3.5 in hand, we still meet with delicate monodromy problem in order to find a twisted curve containing $\mathbb{P}^1 \rightarrow \mathbb{P}^1/G \cong \mathbb{P}^1$ totally ramified at 2 points in $\overline{\mathcal{M}}_{g',2}(\mathcal{B}G)$.

As in Proposition 3.5, one would like to degenerate the nodal curve C so that the two points p, q mapped to the fixed points come together and lie in an irreducible component which is isomorphic to \mathbb{P}^1 with the canonical G action. It is always possible to degenerate the curve with the points coming together. But in order that the two points lie in a \mathbb{P}^1 with a G action, the monodromy around p, q has to be inverse to each other. The following proposition gives such a kind of degeneration in $\overline{\mathcal{M}}_{g',2}(\mathcal{B}G)$.

Proposition 3.6. *There is a G -simple nodal curve \hat{C} marked twice at fixed points p, q , and a pencil of G -simple curves $\mathcal{C} \rightarrow \mathbb{P}^1$ which deforms $(\mathcal{C}_0 \cong \hat{C}, p, q)$ to smooth G -simple curve $(\mathcal{C}_t, p_t, q_t)$ for general $t \in \mathbb{P}^1$. And $\mathcal{C}_\infty = C_1 \cup C_2$ where both C_1 and C_2 are isomorphic to \mathbb{P}^1 with the canonical G -action, and the marking p_∞ and q_∞ of \mathcal{C}_∞ is concentrated on the 0 and ∞ in C_1 .*

In particular, by taking quotients $[\mathcal{C}/G] \rightarrow \mathbb{P}^1$ gives a pencil of twisted maps in $\overline{\mathcal{M}}_{g',2}(\mathcal{B}G)$, which deforms a high genus twisted curve marked twice to contain a component isomorphic to $([\mathbb{P}^1/G], [0/G], [\infty/G])$.

Proof. Let G acts on $\mathbb{P}^1 \times \mathbb{P}^1$ by $g \cdot ([X_0, X_1], [Y_0, Y_1]) \mapsto ([X_0, \zeta \cdot X_1], [Y_0, Y_1])$, where ζ is a primitive l -th root of unity. Consider the reducible nodal curve

$$\hat{C} = V((X_0^{2l} - X_1^{2l})Y_0Y_1) \subset \mathbb{P}^1 \times \mathbb{P}^1.$$

There is a natural G action on \hat{C} by restricting the action on $\mathbb{P}^1 \times \mathbb{P}^1$. We take two marked points $p = ([1, 0], [1, 0]), q = ([0, 1], [0, 1])$.

One can directly write down such a pencil. In $\mathbb{P}^1 \times \mathbb{P}^1$, take the pencil spanned by the curves $(X_0^{2l} - X_1^{2l})Y_0Y_1 = 0$ and $(X_0^lY_1 + X_1^lY_0)(X_1^lY_1 + X_0^lY_0) = 0$. Note that the curve defined by $(X_0^lY_1 + X_1^lY_0)(X_1^lY_1 + X_0^lY_0) = 0$ is the union of two smooth rational curve meeting transversely at $2l$ points, both of which has a natural G action. Moreover, the irreducible component $X_0^lY_1 + X_1^lY_0 = 0$ contains p, q as the fixed points of the G action. A general member of the pencil is a stable curve since C is stable. And every member of the family has a natural G action and contains p, q as two fixed points. So set $p_t = p, q_t = q$, this gives a family of G -simple curves satisfying out hypothesis.

□

3.3. Final proof of Theorem 1.4. With Proposition 3.5 and Proposition 3.6 at hand, we need one more lemma to prove Theorem 1.4, which is an application of the equivariant smoothing of comb-Theorem 2.7.

Lemma 3.7. *Assume $G \cong \mathbb{Z}/l\mathbb{Z}$. Let X be a smooth quasi-projective separably rationally connected k -variety with a G action. Let G actions on \mathbb{P}^1 by $z \mapsto \zeta z$, where ζ is a primitive l -th root of unity.*

- (1) *Let $f : \mathbb{P}^1 \rightarrow X$ be a G -equivariant map. Then there is a G -equivariant very free curve $\tilde{f} : \mathbb{P}^1 \rightarrow X$ with $\tilde{f}(0) = f(0)$, $\tilde{f}(\infty) = f(\infty)$.*
- (2) *Let $f_i : C_i \rightarrow X$, $1 \leq i \leq n$ be a chain of equivariant maps, i.e. $f_i(\infty) = f_{i+1}(0)$ and $C_i \cong \mathbb{P}^1$ for all i 's. Then there is a G -equivariant map $\tilde{f} : \mathbb{P}^1 \rightarrow X$ with $\tilde{f}(0) = f_1(0)$ and $\tilde{f}(\infty) = f_n(\infty)$.*

Proof. For part (1), Let G acts on \mathbb{P}^M by $[x_0, \dots, x_M] \rightarrow [x_0, \dots, \zeta x_M]$ (or any other effective action) with an embedded G -equivariant rational curve $f' : \mathbb{P}^1 \rightarrow \mathbb{P}^M$, e.g. $x_1 = \dots = x_{M-1} = 0$. For $M \gg 0$, we may assume that the equivariant map f is an embedding and $\dim X \geq 3$ by replacing X with $X \times \mathbb{P}^M$ and replacing $f : \mathbb{P}^1 \rightarrow X$ with the diagonal G -equivariant map

$$\hat{f} : \mathbb{P}^1 \rightarrow X \times \mathbb{P}^M$$

defined by $\hat{f}(x) = (f(x), f'(x))$. We note there that for $\dim X \geq 2$, $M = 1$ is enough. Let C_0 be the image of the morphism f . Then (1) follows by applying Theorem 2.7 to $f_0 = f$, $d = 2$ and $M = \mathcal{O}_{C_0}(-0 - \infty)$.

For part (2), we may assume that all the f_i 's are very free by the first part. Let f be the G -equivariant map obtained by gluing the f_i 's. Let (T, o) be a pointed smooth curve with a trivial G -action. And let $\tilde{\Sigma}$ be $\mathbb{P}^1 \times T$ with the natural diagonal action. There are two G -equivariant sections, $s_0 = 0 \times T$, $s_\infty = \infty \times T$. Blow up $s_\infty(o)$ with an extension of the G -action and equate s_0, s_∞ with their strict transforms. Doing this n times, we get a smooth surface Σ with a chain of n rational curves as the fiber over $o \in T$. Let $h_0 : s_0 \rightarrow X \times T$ and $h_\infty : s_\infty \rightarrow X \times T$ be T -morphisms such that $h_0(s_0) = f_1(0) \times T$ and $h_\infty(s_\infty) = f_n(\infty) \times T$. Consider the map

$$\mu : \text{Hom}_T(\Sigma, X \times T, h_0, h_\infty) \rightarrow T$$

where $\text{Hom}_T(\Sigma, X \times T, h_0, h_\infty)$ parameterizes T -morphisms from Σ to $X \times T$ fixing h_0 and h_∞ , it has a natural G action and the map μ is G -equivariant. Now it is easy to show that μ is smooth at f since f_i 's are very free. So there is a G -equivariant smoothing of the morphism f by Corollary 2.2. □

Proof of Theorem 1.4. Given two fixed points x and y , there is a very free rational curve $f_1 : C_1 \cong \mathbb{P}^1 \rightarrow X$ such that $f(0) = x, f(\infty) = y$. The G -orbit of the morphism f_1 consists of l very free curves $f_i : C_i \cong \mathbb{P}^1 \rightarrow X, f_i(0) = x, f_i(\infty) = y, i = 1, \dots, l$. Take another G -orbit of very free curves $g_i : D_i \cong \mathbb{P}^1 \rightarrow X, g_i(0) = x, g_i(\infty) = y, i = 1, \dots, l$.

Now consider the special G -nodal curve as in Proposition 3.6

$$\hat{C} = V((X_0^{2l} - X_1^{2l})Y_0Y_1) \subset \mathbb{P}^1 \times \mathbb{P}^1.$$

One can define a G -equivariant morphism $f : \hat{C} \rightarrow X$ whose restriction to the curve $Y_0 = 0$ (resp. $Y_1 = 0$) is the constant map to x (resp. y), to the curve

$V(X_0 - \zeta^i X_1 = 0) \cong C_i$ the map f_i , to the curve $V(X_0 - \zeta^i \xi X_1 = 0) \cong D_i$ the map g_i , where ξ is a root of the equation $T^l + 1 = 0$. Take the marking $p = ([1, 0], [1, 0]), q = ([0, 1], [0, 1])$ as in Proposition 3.6. The triple $(f : \hat{C} \rightarrow X, f(p) = x, f(q) = y)$ is a G -equivariant stable map of genus g with two marked points.

We note there that although \hat{C} is a G -simple nodal curve, $(f : \hat{C} \rightarrow X, f(p) = x, f(q) = y)$ is not necessarily G -simple. But we claim that there is a smooth projective separably rationally connected variety \hat{X} with a G action, 2 fixed points \hat{x}, \hat{y} , and a G -equivariant morphism $X' \rightarrow X$ which maps \hat{x}, \hat{y} to x, y respectively, such that there is a lifting

$$\begin{array}{ccc} (\hat{C}, p, q) & \longrightarrow & (\hat{X}, \hat{x}, \hat{y}) \\ & \searrow & \downarrow \\ & & (X, x, y) \end{array}$$

where $(\hat{f} : \hat{C} \rightarrow \hat{X}, \hat{f}(p) = \hat{x}, \hat{f}(q) = \hat{y})$ is G -simple. To see this first we take the product $X \times \mathbb{P}^M$ for $M \gg 0$ and take suitable preimages x', y' of x, y . Here \mathbb{P}^M has a suitable G action similar to the proof of part 1 in Lemma 3.7. So we can assume that the very free curves C_i 's are all immersed rational curves which are distinct with each other and has different tangent directions at x' and y' . Then we blow-up along the preimages x', y' of x, y on $X \times \mathbb{P}^M$ to get \hat{X} as desired.

Replacing X by \hat{X} , we may assume there is a G -simple map $(f : \hat{C} \rightarrow X, f(p) = x, f(q) = y)$ which also gives rise to a twisted stable map $[f] : ([\hat{C}/G], [p/G], [q/G]) \rightarrow ([X/G], [x/G], [y/G])$. By Theorem 2.6 and part 1 of Lemma 3.7, up to adding very free curves and smoothing, we may assume that $H^1(\hat{C}, f^* \mathcal{T}_X(-p-q))^G = 0$. Then by Proposition 3.5, there is a surjection

$$\mathcal{S} : \mathcal{U}_{([f]:[\hat{C}/G] \rightarrow [X/G])} \rightarrow \mathcal{V}_{\hat{C}}$$

where $\mathcal{U}_{([f]:[\hat{C}/G] \rightarrow [X/G])}$ and $\mathcal{V}_{\hat{C}}$ are the unique components in the moduli stacks $\overline{\mathcal{M}}_{g', 2}([X/G], [\hat{C}/G]) \{f'([x_1]) = [x], f'([x_2]) = [y]\}$ and $\overline{\mathcal{M}}_{g', 2}(\mathcal{B}G)$ containing $[f]$ and \hat{C} respectively.

Now as in Proposition 3.6, one can deform the curve $[\hat{C}/G]$ in such a way that the two stacky points lie in a stacky $[\mathbb{P}^1/G]$, where the G action on \mathbb{P}^1 is $[X_0, X_1] \mapsto [X_0, \zeta X_1]$ -since the general member $[C_t/G]$ this deformation is parametrized by an irreducible curve (actually \mathbb{P}^1), the resulting curve C_t is also in the component $\mathcal{V}_{\hat{C}}$ containing \hat{C} .

Thus by surjectivity of \mathcal{S} there is a preimage as a twisted nodal curve \mathcal{C}_t^p mapped to $[C_t/G]$ with a representable morphism which maps the two marked points to $[x/G]$ and $[y/G]$ in $[X/G]$. By Remark 2.15,

$$\mathcal{C}_t^p \mapsto [C_t/G]$$

will contract non-stable components which are again all isomorphic to $[\mathbb{P}^1/G']$ marked at the two total ramifications where G' is a cyclic group with order divisible in k . By replacing with a suitable cover, one can assume that $G' = G$, so this is a chain of stacky curves $[\mathbb{P}^1/G]$ connected at the total ramification points, which is equivalent to a chain of G -equivariant morphisms for \mathbb{P}^1 to X connecting

x to y . So finally by part 2 of Lemma 3.7, we have a G -equivariant morphism from \mathbb{P}^1 to X mapping 0 and ∞ to x, y . \square

4. PROOF OF THEOREM 1.6 USING RELATIVE G -EQUIVARIANT SMOOTHING

By Lemma 2.10, for an isotrivial family, at least in the formal neighborhood, we can find a ramified base change and a birational modification so that the new central fiber becomes smooth and the Galois group acts on the total space of the formal neighborhood, namely, it satisfies Hypothesis 1.5.

The first goal is to show that we can make the cyclic base change globally on the curve B . Then to get back to the original family, one just need to remember the Galois group action and do things in a G -equivariant way.

Given finitely many points x_1, x_2, \dots, x_n in B , and any positive integer l , there is a cyclic cover of degree l of B which is totally ramified over x_1, \dots, x_n (and other points). To see this, take a general Lefschetz pencil which maps x_1, \dots, x_n (and other points) to $0 \in \mathbb{P}^1$ and is unramified over these points. Take a degree l map $B_1 = \mathbb{P}^1 \rightarrow \mathbb{P}^1, [X_0, X_1] \mapsto [X_0^l, X_1^l]$ and let $C = B \times_{\mathbb{P}^1} B_1$ be the fiber product. Then C is the desired cyclic cover. Note that we have the freedom to increase the number of branched points so that the genus of C can be arbitrarily large. We could also choose the cover $C \rightarrow B$ so that the preimages of b_1, \dots, b_k are l distinct points.

For different points on the base B , the base change we need may have different degrees. But we can approximate the formal sections one by one (again using the iterated blow-up to fix jet data) so that each time we only need to deal with a single base change. So Theorem 1.6 is reduced to

Theorem 4.1. *Let G be a cyclic group of order l and let \mathcal{X} (resp. C) be a smooth proper variety (resp. a smooth projective curve) with a G -action. Let $\pi : \mathcal{X} \rightarrow C$ be a flat family of rationally connected varieties. Assume the following:*

- (1) *The morphism π is G -equivariant.*
- (2) *There is a G -equivariant section $s : C \rightarrow \mathcal{X}$.*
- (3) *The G -action on C has a fixed point p and the action of G near p is given by $t \mapsto \zeta t$, where t is a local parameter and ζ is a primitive l -th root of unity.*
- (4) *The fiber of $\pi : \mathcal{X} \rightarrow C$ over the point p is smooth.*

Then for any positive integer N , and any G -equivariant formal section $\widehat{s} : \text{Spec } \widehat{\mathcal{O}}_{p,C} \rightarrow \mathcal{X}$, there is a G -equivariant section s' which agrees with the formal section \widehat{s} to order N .

4.1. Idea and formal set-up. The idea of the proof goes back to [HT06]. Namely, we would like to add suitable rational curves to the given section and make G -equivariant deformations to produce a new section with prescribed jet data. The only subtlety in the proof is that in general we cannot choose the rational curves to be immersed. So instead of working with the normal sheaf as is done in [HT06], we work with the complex Ω_f defined as

$$\begin{array}{ccc} -1 & & 0 \\ f^*\Omega_X & \xrightarrow{df^\dagger} & \Omega_C. \end{array}$$

and its derived dual in the derived category. All the tensor products, duals, pull-backs, and push-forwards in the proof should also be taken as the derived functors in the derived category.

The following is a general form of the commonly used short exact sequences (of normal sheaves) which govern the deformation of a stable map from a nodal domain.

Lemma 4.2. *Let $f : C \cup D \rightarrow X$ be a morphism from a nodal curve $C \cup D$ with a single node to a smooth variety X and f_0 (resp. f_1) the restriction of f to C (resp. D). Then*

(1) *We have the following distinguished triangles:*

$$\begin{aligned} \Omega_f^\vee \otimes \mathcal{O}_D(-n) &\rightarrow \Omega_f^\vee \rightarrow \Omega_f^\vee \otimes \mathcal{O}_C \rightarrow \Omega_f^\vee \otimes \mathcal{O}_D(-n)[1] \\ \Omega_{f_0}^\vee &\rightarrow \Omega_f^\vee \otimes \mathcal{O}_C \rightarrow \epsilon[-1] \rightarrow \Omega_{f_0}^\vee[1] \end{aligned}$$

where n is the preimage of the node in D , and ϵ is a skyscraper sheaf supported at the preimage of the node in C .

(2) *Let G be a cyclic group of order l . Assume that there is a G -action on $C \cup D$ fixing each irreducible component. Then the node is a fixed point of the action and there is a natural G -action on all the complexes above. If locally around the node, the action is given by*

$$\begin{aligned} \mathbb{C}[x, y]/xy &\longrightarrow \mathbb{C}[x, y]/xy \\ (x, y) &\longrightarrow (\zeta x, \zeta^{-1}y), \end{aligned}$$

where ζ is a primitive l -th roots of unity, then the G -action on ϵ is trivial.

Proof. The first distinguished triangle comes from restriction to the component C .

For the second distinguished triangle, consider the following distinguished triangles and the map between them:

$$\begin{array}{ccccccc} \Omega_{C \cup D} \otimes \mathcal{O}_C & \longrightarrow & \Omega_f \otimes \mathcal{O}_C & \longrightarrow & f^* \Omega_X \otimes \mathcal{O}_C[1] & \longrightarrow & \Omega_{C \cup D} \otimes \mathcal{O}_C[1] \\ \downarrow & & \downarrow & & \parallel & & \downarrow \\ \Omega_C & \longrightarrow & \Omega_{f_0} & \longrightarrow & f_0^* \Omega_X[1] & \longrightarrow & \Omega_C[1] \end{array}$$

Therefore we have distinguished triangles

$$\Omega_{C \cup D} \otimes \mathcal{O}_C \rightarrow \Omega_C \rightarrow Q[1] \rightarrow \Omega_{C \cup D} \otimes \mathcal{O}_C[1]$$

$$(1) \quad \begin{aligned} \Omega_f \otimes \mathcal{O}_C &\rightarrow \Omega_{f_0} \rightarrow Q'[1] \rightarrow \Omega_f \otimes \mathcal{O}_C[1], \\ Q[1] &\rightarrow Q'[1] \rightarrow 0 \rightarrow Q[2]. \end{aligned}$$

where Q is a skyscraper sheaf supported at the node. The last distinguished triangle shows that $Q \cong Q'$. Taking dual of the distinguished triangle (1) gives the second triangle in the lemma.

Part 2 of the lemma can be proved by a local computation. Or we can argue that the sheaf ϵ corresponds to a G -equivariant smoothing of the node. Therefore it has to be G -invariant. \square

Now we begin the proof.

Proof of Theorem 4.1. The proof is divided into two steps.

4.2. Step 1: Approximation at 0-th order. We may assume that

$$H^1(C, \mathcal{N}_{C/\mathcal{X}}(-p)) = 0$$

by the same argument as in Lemma 3.7.

The section s and the formal section \widehat{s} intersect the fiber \mathcal{X}_p at two fixed points of the G -action. Take a rational curve $D \cong \mathbb{P}^1$ with a G -action as $[X_0, X_1] \mapsto [\zeta X_0, X_1]$, where ζ is the primitive l -th root of unity in the assumptions. By Theorem 1.4, there is a G -equivariant very free curve $D \cong \mathbb{P}^1 \rightarrow \mathcal{X}_p \rightarrow \mathcal{X}$ which maps $0 = [1, 0]$ to $s(p)$ and $\infty = [0, 1]$ to $\widehat{s}(p)$. Let $f : C \cup D \rightarrow X$ be the nodal curve by combining the section and the curve D and f_0 (resp. f_1) the restriction of f to C (resp. D).

By Lemma 4.2, we have the following distinguished triangles:

$$\begin{aligned} \Omega_f^\vee(-\infty) \otimes \mathcal{O}_C(-p) &\rightarrow \Omega_f^\vee(-\infty) \rightarrow \Omega_f^\vee(-\infty) \otimes \mathcal{O}_D \rightarrow \Omega_f^\vee \otimes \mathcal{O}_C(-p)[1] \\ \Omega_{f_0}^\vee(-p) &\rightarrow \Omega_f^\vee \otimes \mathcal{O}_C(-p) \rightarrow \epsilon[-1] \rightarrow \Omega_{f_0}^\vee(-p)[1] \\ \Omega_{f_1}^\vee \otimes \mathcal{O}_D(-\infty) &\rightarrow \Omega_f^\vee \otimes \mathcal{O}_D(-\infty) \rightarrow \epsilon'[-1] \rightarrow \Omega_{f_1}^\vee \otimes \mathcal{O}_D(-\infty)[1] \end{aligned}$$

where ϵ and ϵ' are torsion sheaves supported at the node of C and D . Every complex has a natural G -action, and the G -actions on ϵ and ϵ' are trivial. Also note that

$$\Omega_f(-\infty) \otimes \mathcal{O}_C \cong \Omega_f \otimes \mathcal{O}_C.$$

Taking hypercohomology gives long exact sequences

$$(2) \quad \begin{aligned} 0 &\rightarrow \mathcal{H}(x, y)^1(\Omega_f^\vee \otimes \mathcal{O}_C(-p)) \rightarrow \mathcal{H}(x, y)^1(\Omega_f^\vee(-\infty)) \rightarrow \mathcal{H}(x, y)^1(\Omega_f^\vee(-\infty) \otimes \mathcal{O}_D) \\ &\rightarrow \mathcal{H}(x, y)^2(\Omega_f^\vee \otimes \mathcal{O}_C(-p)) \rightarrow \mathcal{H}(x, y)^2(\Omega_f^\vee(-\infty)) \rightarrow \mathcal{H}(x, y)^2(\Omega_f^\vee(-\infty) \otimes \mathcal{O}_D) \rightarrow \dots, \end{aligned}$$

$$(3) \quad \begin{aligned} 0 &\rightarrow \mathcal{H}(x, y)^1(\Omega_{f_0}^\vee \otimes \mathcal{O}_C(-p)) \rightarrow \mathcal{H}(x, y)^1(\Omega_f^\vee \otimes \mathcal{O}_C(-p)) \rightarrow \epsilon \\ &\rightarrow \mathcal{H}(x, y)^2(\Omega_{f_0}^\vee \otimes \mathcal{O}_C(-p)) \rightarrow \mathcal{H}(x, y)^2(\Omega_f^\vee \otimes \mathcal{O}_C(-p)) \rightarrow 0, \end{aligned}$$

and

$$(4) \quad \begin{aligned} 0 &\rightarrow \mathcal{H}(x, y)^1(\Omega_{f_1}^\vee \otimes \mathcal{O}_D(-\infty)) \rightarrow \mathcal{H}(x, y)^1(\Omega_f^\vee \otimes \mathcal{O}_D(-\infty)) \rightarrow \epsilon' \\ &\rightarrow \mathcal{H}(x, y)^2(\Omega_{f_1}^\vee \otimes \mathcal{O}_D(-\infty)) \rightarrow \mathcal{H}(x, y)^2(\Omega_f^\vee \otimes \mathcal{O}_D(-\infty)) \rightarrow 0. \end{aligned}$$

Note that $\Omega_{f_0}^\vee$ is quasi-isomorphic to $\mathcal{N}_{C/\mathcal{X}}[-1]$. Thus by the second long exact sequence,

$$\mathcal{H}(x, y)^2(\Omega_{f_0}^\vee \otimes \mathcal{O}_C(-p)) = \mathcal{H}(x, y)^2(\Omega_f^\vee \otimes \mathcal{O}_C(-p)) = 0.$$

Note that $\Omega_{f_1}^\vee$ is quasi-isomorphic to a shifted sheaf $\mathcal{N}[-1]$, where \mathcal{N} is defined as the quotient in

$$0 \rightarrow T_D \rightarrow f^*T_X \rightarrow \mathcal{N} \cong f^*T_X/T_D \rightarrow 0.$$

Since f^*T_X is globally generated, $\mathcal{H}(x, y)^2(\Omega_{f_1}^\vee \otimes \mathcal{O}_D(-\infty)) = 0$. Then by the third long exact sequence,

$$\mathbb{H}^2(\Omega_f^\vee \otimes \mathcal{O}_D(-\infty)) = 0.$$

Therefore by the long exact sequence (2),

$$\mathcal{H}(x, y)^2(\Omega_f^\vee(-\infty)) = 0,$$

and thus the G -equivariant deformation of the nodal curve $C \cup D$ with the point ∞ fixed is unobstructed.

Then by the long exact sequences (2), (4) and the vanishing, the composition of maps

$$\mathcal{H}(x, y)^1(\Omega_f^\vee(-\infty))^G \rightarrow \mathcal{H}(x, y)^1(\Omega_f^\vee(-\infty) \otimes \mathcal{O}_D)^G \rightarrow \epsilon'$$

is surjective. Thus there is a G -equivariant deformation with ∞ fixed which smooths the node between C and D .

4.3. Step 2: Approximation at higher order. Assume that we have a section, still denoted by s , which agrees with \widehat{s} to the $k(\geq 0)$ -th order. We want to find a section agreeing with \widehat{s} to order $k + 1$.

Now let \mathcal{X}_{k+1} be the $(k + 1)$ -th iterated blow-up of \mathcal{X} associated to the formal section \widehat{s} . Then G also acts on \mathcal{X}_{k+1} and the projective to C is G -equivariant. By abuse of notations, still denote the strict transforms of s and \widehat{s} by s and \widehat{s} . Then they both intersect the exceptional divisor $E_{k+1} \cong \mathbb{P}^d$ at fixed points of G . Assume the intersection points are different, otherwise there is nothing to prove.

Again we assume that $H^1(C, \mathcal{N}_{C/\mathcal{X}_{k+1}}(-p)) = 0$.

The key lemma is the following.

Lemma 4.3. *There is a comb $f : C \cup D \rightarrow \mathcal{X}_{k+1}$ from a nodal domain consisting of the given section $s(C)$ and suitable rational curves in the fiber such that*

- $D = D_{k+1} \cup \cup_{j=1}^l R_j$, where $D_{k+1} \cong \mathbb{P}^1$ and $R_j = \cup_{i=1}^k D_{ij}$ is a chain of rational curves. Denote by x_j the node that connects D_{k+1} to R_j .
- There is a G -action on D in the following way. The G -action on D_{k+1} is given by

$$[X_0, X_1] \mapsto [X_0, \zeta^{-1} X_1].$$

The group G acts on $R_j, j = 1, \dots, l$ via a cyclic permutation among them. In particular, the points $x_j \in D_{k+1}$ are conjugate to each other under the G -action.

- The morphism $f : C \cup D \rightarrow X$ is G -equivariant.
- The G -fixed point $\infty = [0, 1]$ on D_{k+1} is mapped to $\widehat{s}(p)$, and $0 = [1, 0]$ on D_{k+1} connects C .
- The morphism $f : C \cup D$ is an immersion except at 0 and ∞ in D_{k+1} .
- The complex Ω_f^\vee satisfies the following vanishing conditions.

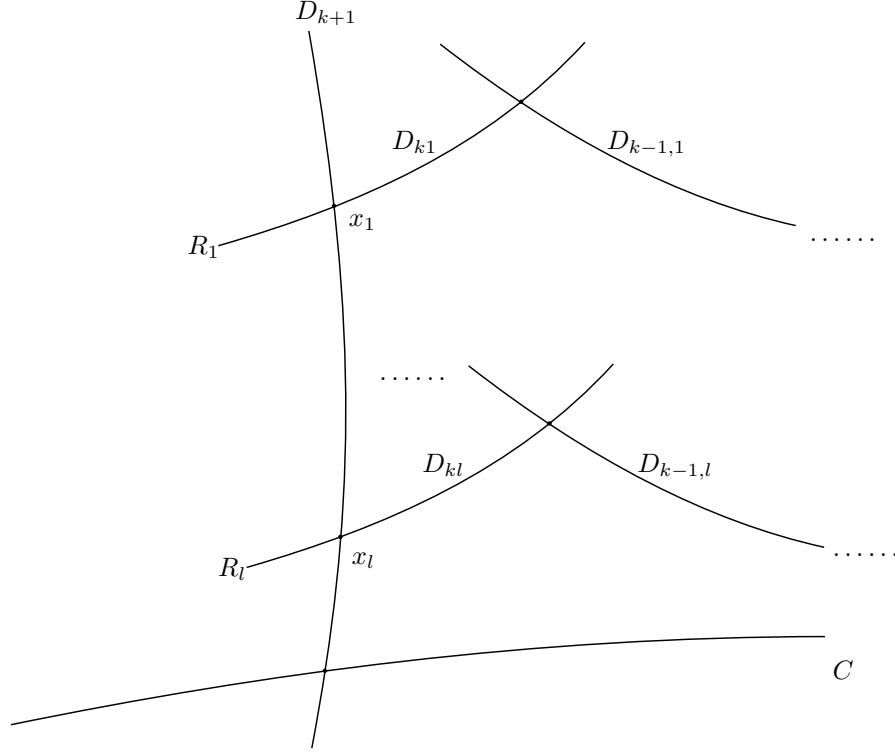
$$(5) \quad \mathcal{H}(x, y)^2(\Omega_f^\vee \otimes \mathcal{O}_C(-p)) = \mathcal{H}(x, y)^2(\Omega_f^\vee \otimes \mathcal{O}_{D_{k+1}}(-0-\infty)) = \mathcal{H}(x, y)^2(\Omega_f^\vee \otimes \mathcal{O}_{D_{ij}}(-1)) = 0,$$

$$(6) \quad \mathcal{H}(x, y)^2(\Omega_f^\vee(-\infty)) = 0,$$

$$(7) \quad \mathcal{H}(x, y)^2(\Omega_f^\vee \otimes \mathcal{O}_{D_{k+1}}(-\infty - x_1 - \dots - x_l))^G = 0.$$

The construction is essentially the same as the one in [HT06], with the only difference coming from the consideration of the G -action. For an illustration of the comb $C \cup D$, see Figure. 1 below and for the configuration of the comb with respect to the iterated blow-up \mathcal{X}_{k+1} , see Figure. 2.

Proof of Lemma 4.3. The line L in $E_{k+1} \cong \mathbb{P}^d$ joining $s(p)$ and $\widehat{s}(p)$ is invariant and intersects the exceptional divisor E_k of \mathcal{X}_{k+1} at a unique point y_k , which is necessarily a fixed point of G . Then there are 3 fixed points in the line L and thus

FIGURE 1. The comb $C \cup D$

all points are fixed points of G . Take a curve $D_{k+1} \cong \mathbb{P}^1$. We impose a G -action on it by

$$[X_0, X_1] \mapsto [X_0, \zeta^{-1} X_1].$$

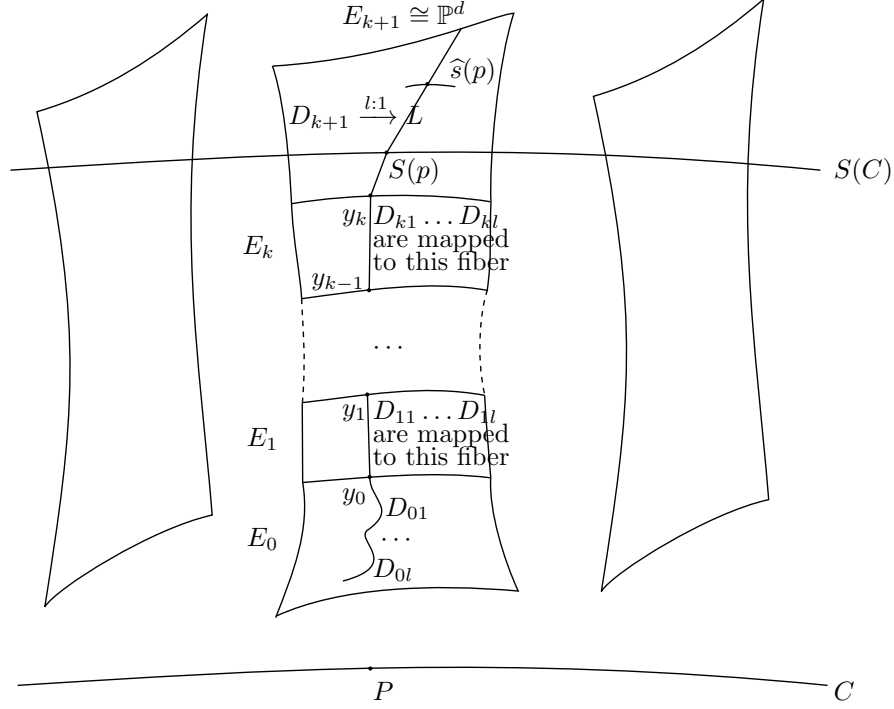
Take an l -to-1 G -equivariant map from D_{k+1} to the line L such that $0 = [1, 0]$ is mapped to $s(p)$ and $\infty = [0, 1]$ is mapped to $\widehat{s}(p)$. There are l points x_1, \dots, x_l , which lie in the same orbit of G , being mapped to the point $y_k \in E_k \cap E_{k+1}$, where E_k and E_{k+1} are exceptional divisors of the $(k+1)$ -th iterated blow-up.

The exceptional divisor E_k is isomorphic to the blow-up of \mathbb{P}^d at a point, thus is a \mathbb{P}^1 -bundle over \mathbb{P}^{d-1} . Let $D_{k,1}, \dots, D_{k,l}$ be l copies of \mathbb{P}^1 each mapped isomorphically to the fiber curve \mathbb{P}^1 containing the point y_k .

Inductively, let y_i be the intersection point of $D_{i+1,1}$ with E_i and $D_{i,1}, \dots, D_{i,l}$ be l copies of \mathbb{P}^1 each mapped isomorphically to the fiber \mathbb{P}^1 containing the point y_i for all $i = k-1, \dots, 1$.

Finally let y_0 be the point of the intersection of $D_{1,1}$ with the strict transform of $\mathcal{X}|_p$ and let $D_{0,1}, \dots, D_{0,l}$ be l copies of \mathbb{P}^1 mapped to a very free curve in the strict transform of \mathcal{X}_p intersecting E_1 at the point y_0 . We may also assume that the maps are immersions.

Let R_j be the chain of rational curves $\cup_{i=1}^k D_{i,j}$ connected to D_{k+1} at the point x_j for $j = 1, \dots, l$, and let D be the curve $D_{k+1} \cup \cup_{j=1}^l R_j$. There is a natural G -action on D , which permutes the l -chains of rational curves R_j and acts on the irreducible component D_{k+1} as specified above.

FIGURE 2. Construction of the comb $C \cup D$

The restriction of the complex Ω_f^\vee to each curve $D_{i,j}$ is quasi-isomorphic to the normal sheaf with a shift $N_f[-1]$ (since the comb is an immersion along such curves). One can compute the restriction of N_f to each curve $D_{i,j}$ as follows (see the proof of Sublemma 27, [HT06]).

$$(8) \quad \mathcal{N}_f|_{D_{i,j}} = \begin{cases} \mathcal{O}^{\oplus d}, & 1 \leq i \leq k, \\ \bigoplus_{n=1}^{d-1} \mathcal{O}(a_n) \oplus \mathcal{O}, a_n \geq 1 & i = 0. \end{cases}$$

We now compute $\Omega_{f_{k+1}}^\vee$ on D_{k+1} , where f_{k+1} is the restriction of the map to D_{k+1} (i.e. the degree l multiple cover of the line in \mathbb{P}^{d-1}). This complex is quasi-isomorphic to the complex

$$0 \quad \quad \quad 1$$

$$T_{D_{k+1}} \cong \mathcal{O}(2) \longrightarrow f_{k+1}^* T_{\mathcal{X}_{k+1}} \cong \mathcal{O}(2l) \oplus \bigoplus_{i=1}^{d-2} \mathcal{O}(l) \oplus \mathcal{O}(-l),$$

Also note that the sheaf map $T_{D_{k+1}} \rightarrow f_{k+1}^* T_{\mathcal{X}_{k+1}}$ is injective and is the composition of maps $\mathcal{O}(2) \rightarrow \mathcal{O}(2l) \rightarrow f_{k+1}^* T_{\mathcal{X}_{k+1}}$.

We have a distinguished triangle

$$(9) \quad \Omega_{f_{k+1}}^\vee \rightarrow \Omega_f^\vee \otimes \mathcal{O}_{D_{k+1}} \rightarrow \epsilon[-1] \oplus \bigoplus_{j=1}^l \epsilon_j[-1] \rightarrow \Omega_{f_{k+1}}^\vee[1],$$

where ϵ is a torsion sheaf supported at the node connecting D_{k+1} and C , and ϵ_j is a torsion sheaf supported at the node connecting D_{k+1} and $D_{k,j}$. The group G acts on ϵ by the trivial action and acts on ϵ_j by permutation.

So the restriction of Ω_f^\vee to D_{k+1} is quasi-isomorphic to the complex

$$\begin{array}{ccc} 0 & & 1 \\ \mathcal{O}(2) & \longrightarrow & \mathcal{O}(2l) \oplus \bigoplus_{i=1}^{d-2} \mathcal{O}(l) \oplus \mathcal{O}(1). \end{array}$$

Since the above map maps the sheaf $\mathcal{O}(2)$ injectively into the sheaf $\mathcal{O}(2l)$, this complex is quasi-isomorphic to the shifted sheaf

$$Q \oplus \bigoplus_{i=1}^{d-2} \mathcal{O}(l) \oplus \mathcal{O}(1)[-1],$$

where Q is the torsion sheaf defined as the quotient of $\mathcal{O}(2) \rightarrow \mathcal{O}(2l)$. Note that the $\mathcal{O}(1)$ direction is the normal direction of the fiber.

Finally, the restriction of Ω_f^\vee to C fits into the distinguished triangle

$$\mathcal{N}_{C/\mathcal{X}_{k+1}}[-1] \rightarrow \Omega_f^\vee \otimes \mathcal{O}_C \rightarrow \epsilon_0 \rightarrow \mathcal{N}_{C/\mathcal{X}_{k+1}},$$

where ϵ_0 is a torsion sheaf supported at the node.

Then the vanishing conditions (5) are immediate from the identifications above.

By the distinguished triangle

$$\Omega_f^\vee \otimes \mathcal{O}_C(-p) \rightarrow \Omega_f^\vee(-\infty) \rightarrow \Omega_f^\vee \otimes \mathcal{O}_D(-\infty) \rightarrow \Omega_f^\vee \otimes \mathcal{O}_C(-p)[1]$$

and the three vanishing results in 5, we know that

$$\mathcal{H}(x, y)^2(\Omega_f^\vee(-\infty)) = 0.$$

This is the vanishing in (6).

The vanishing in (7) needs a little bit more work since it is only the G -invariant part of the hypercohomology group that vanishes. First notice the following.

Lemma 4.4. *Assume only that the comb $C \cup D$ satisfies vanishing results (5) and (6). Then a general G -equivariant deformation of $C \cup D$ with ∞ fixed is unobstructed and smooths the node connecting C and D_{k+1} .*

Proof. The vanishing result (6) implies that the G -equivariant deformation of $C \cup D$ with ∞ fixed is unobstructed.

We first consider the following distinguished triangles

$$(10) \quad \Omega_f^\vee \otimes \mathcal{O}_D(-\infty - 0) \rightarrow \Omega_f^\vee(-\infty) \rightarrow \Omega_f^\vee(-\infty) \otimes \mathcal{O}_C \rightarrow \Omega_f^\vee \otimes \mathcal{O}_D(-\infty - 0)[1]$$

and

$$(11) \quad \begin{aligned} \bigoplus_{j=1}^l \Omega_f^\vee \otimes \mathcal{O}_{R_j}(-x_j) &\rightarrow \Omega_f^\vee \otimes \mathcal{O}_D(-\infty - 0) \rightarrow \Omega_f^\vee \otimes \mathcal{O}_{D_{k+1}}(-\infty - 0) \\ &\rightarrow \bigoplus_{j=1}^l \Omega_f^\vee \otimes \mathcal{O}_{R_j}(-x_j)[1]. \end{aligned}$$

Recall that $\Omega_f^\vee \otimes \mathcal{O}_{R_j}$ is quasi-isomorphic to a shifted normal sheaf $\mathcal{N}_f \otimes \mathcal{O}_{R_j}[-1]$, and the sheaves $\mathcal{N} \otimes \mathcal{O}_{R_j}$ are locally free and globally generated by (8). Therefore

$$\mathcal{H}(x, y)^2(\bigoplus_{j=1}^l \Omega_f^\vee \otimes \mathcal{O}_{R_j}(-x_j)) = 0,$$

and thus by the distinguished triangle (11),

$$\mathcal{H}(x, y)^2(\Omega_f^\vee \otimes \mathcal{O}_D(-\infty - 0)) = 0,$$

which, combined with the long exact sequence of hypercohomology of the distinguished triangle (10), implies that the map

$$(12) \quad \mathcal{H}(x, y)^1(\Omega_f^\vee(-\infty))^G \rightarrow \mathcal{H}(x, y)^1(\Omega_f^\vee(-\infty) \otimes \mathcal{O}_C)^G$$

is surjective.

Then we look at the distinguished triangle

$$\Omega_{f_0}^\vee \rightarrow \Omega_f^\vee \otimes \mathcal{O}_C \rightarrow \epsilon_0[-1] \rightarrow \Omega_{f_0}^\vee[1],$$

where f_0 is the restriction of f to C and ϵ is a skyscraper sheaf supported at the point p .

By the vanishing results (5), the map

$$(13) \quad \mathcal{H}(x, y)^1(\Omega_f^\vee(-\infty) \otimes \mathcal{O}_C)^G \rightarrow (\epsilon_0)^G = \epsilon_0$$

is surjective.

Note that $\Omega_{f_{k+1}}^\vee \otimes \mathcal{O}_C \cong \Omega_f^\vee(-\infty) \otimes \mathcal{O}_C$. Combining this identification and the surjectivity of maps in (13) and (12), we have proved that a general G -equivariant deformation with ∞ fixed smooths the node connecting C and D_{k+1} . \square

We have a distinguished triangle

$$\Omega_{f_{k+1}}^\vee(-\infty) \rightarrow \Omega_f^\vee \otimes \mathcal{O}_{D_{k+1}}(-\infty) \rightarrow \epsilon[-1] \oplus \bigoplus_{j=1}^l \epsilon_j[-1] \rightarrow \Omega_{f_{k+1}}^\vee(-\infty)[1],$$

where ϵ is a torsion sheaf supported at $0 \in D_{k+1}$. This induces a map

$$(14) \quad \mathcal{H}(x, y)^1(\Omega_f^\vee \otimes \mathcal{O}_{D_{k+1}}(-\infty))^G \rightarrow \epsilon$$

By Lemma 4.4, a general deformation of $C \cup D$ with ∞ fixed is unobstructed and smooths the node connecting C and D_{k+1} (note that the proof of this result is independent of the vanishing (7)). Thus the composition

$$\mathcal{H}(x, y)^1(\Omega_f^\vee(-\infty) \rightarrow \mathcal{H}(x, y)^1(\Omega_f^\vee \otimes \mathcal{O}_{D_{k+1}}(-\infty))^G \rightarrow \epsilon$$

is surjective. So the map in (14) is also surjective.

Recall that $\Omega_f^\vee \otimes \mathcal{O}_{D_{k+1}}(-\infty)$ is quasi-isomorphic to the shifted sheaf

$$(Q \oplus \bigoplus_{i=1}^{d-2} \mathcal{O}(l) \oplus \mathcal{O}(1)) \otimes \mathcal{O}_{D_{k+1}}(-\infty)[-1],$$

and the $\mathcal{O}(1)$ direction is the normal direction of the fiber.

Moreover the map in (9) is can be written as

$$\begin{aligned} Q \oplus \bigoplus_{i=1}^{d-2} \mathcal{O}(l) \oplus \mathcal{O}(-l)[-1] &\rightarrow Q \oplus \bigoplus_{i=1}^{d-2} \mathcal{O}(l) \oplus \mathcal{O}(1)[-1] \\ \rightarrow \epsilon[-1] \oplus \bigoplus_{j=1}^l \epsilon_j[-1] &\rightarrow Q \oplus \bigoplus_{i=1}^{d-2} \mathcal{O}(l) \oplus \mathcal{O}(-l) \end{aligned}$$

Thus only the $\mathcal{O}(1) \otimes \mathcal{O}_{D_{k+1}}(-\infty)$ summand may have a non-zero map to ϵ in the above evaluation map in (14). Thus the unique section in this summand (i.e. the section of $H^0(\mathcal{O}(1) \otimes \mathcal{O}_{D_{k+1}}(-\infty)) = H^0(\mathcal{O}_{D_{k+1}})$) is mapped to a non-zero element in ϵ . Furthermore, this unique section, thought of as a section in

$$\mathcal{H}(x, y)^1(\Omega_f^\vee \otimes \mathcal{O}_{D_{k+1}})^G$$

via the inclusion

$$\mathcal{H}(x, y)^1(\Omega_f^\vee \otimes \mathcal{O}_{D_{k+1}}(-\infty))^G \rightarrow \mathcal{H}(x, y)^1(\Omega_f^\vee \otimes \mathcal{O}_{D_{k+1}})^G$$

only vanishes at $\infty \in D_{k+1}$. Therefore the map

$$(15) \quad H^0(\mathcal{O}(1) \otimes \mathcal{O}_{D_{k+1}}(-\infty))^G \rightarrow (\bigoplus_{j=1}^l \epsilon_j)^G$$

is surjective.

To prove the vanishing in (7), we only need to consider the $\mathcal{O}(1)$ summand since all the other summands have enough positivity to kill the higher cohomology $\mathcal{H}(x, y)^2$. For the $\mathcal{O}(1)$ summand, consider the short exact sequence

$$0 \rightarrow \mathcal{O}(1) \otimes \mathcal{O}_{D_{k+1}}(-\infty - x_1 - \dots - x_l) \rightarrow \mathcal{O}(1) \otimes \mathcal{O}_{D_{k+1}}(-\infty) \rightarrow \bigoplus_{j=1}^l \epsilon_j \rightarrow 0,$$

which induces a map on the G -invariant part of cohomology

$$\begin{aligned} H^0(\mathcal{O}(1) \otimes \mathcal{O}_{D_{k+1}}(-\infty))^G &\rightarrow (\oplus_{j=1}^l \epsilon_j)^G \\ \rightarrow H^1(\mathcal{O}(1) \otimes \mathcal{O}_{D_{k+1}}(-\infty - x_1 - \dots - x_l))^G &\rightarrow H^1(\mathcal{O}(1) \otimes \mathcal{O}_{D_{k+1}}(-\infty))^G. \end{aligned}$$

Since the map (15) is surjective and $H^1(\mathcal{O}(1) \otimes \mathcal{O}_{D_{k+1}}(-\infty))^G$ vanishes, we have

$$H^1(\mathcal{O}(1) \otimes \mathcal{O}_{D_{k+1}}(-\infty - x_1 - \dots - x_l))^G = 0,$$

and thus

$$\mathcal{H}(x, y)^2(\Omega_f^\vee \otimes \mathcal{O}_{D_{k+1}}(-\infty - x_1 - \dots - x_l))^G = 0.$$

□

We now finish the proof of step 2. Consider the distinguished triangles

$$\begin{aligned} \Omega_f^\vee(-\infty) \otimes \mathcal{O}_C(-p) &\rightarrow \Omega_f^\vee(-\infty) \rightarrow \Omega_f^\vee(-\infty) \otimes \mathcal{O}_D \rightarrow \Omega_f^\vee \otimes \mathcal{O}_C(-p)[1] \\ \Omega_f^\vee \otimes \mathcal{O}_{D_{k+1}}(-\infty - x_1 - \dots - x_l) &\rightarrow \Omega_f^\vee \otimes \mathcal{O}_D(-\infty) \\ \rightarrow \oplus_{j=1}^l \Omega_f^\vee \otimes \mathcal{O}_{R_j} &\rightarrow \Omega_f^\vee \otimes \mathcal{O}_{D_{k+1}}(-\infty - x_1 - \dots - x_l)[1]. \end{aligned}$$

The vanishings in (5), (7) imply that the map

(16)

$$\mathcal{H}(x, y)^1(\Omega_f^\vee(-\infty))^G \rightarrow \mathcal{H}(x, y)^1(\Omega_f^\vee \otimes \mathcal{O}_D(-\infty))^G \rightarrow \mathcal{H}(x, y)^1(\oplus_{j=1}^l \Omega_f^\vee \otimes \mathcal{O}_{R_j})^G$$

is surjective (note that $\Omega_f^\vee \otimes \mathcal{O}_{R_j} \cong \Omega_f^\vee(-\infty) \otimes \mathcal{O}_{R_j}$).

Since the G -action on the chain of rational curves R_j is permutation. There is a section of

$$\mathcal{H}(x, y)^1(\oplus_{j=1}^l \Omega_f^\vee \otimes \mathcal{O}_{R_j})^G$$

which is mapped to a non-zero element in the G -invariant part of the torsion sheaf supported at the nodes on R_j , $j = 1, \dots, l$ if and only if there is a section of

$$\mathcal{H}(x, y)^1(\Omega_f^\vee \otimes \mathcal{O}_{R_j})$$

which is mapped to a non-zero element in the torsion sheaf supported at the nodes on R_j and for some (and hence for all) j .

Since the restriction of Ω_f^\vee to R_j is quasi-isomorphic to $\mathcal{N}_f \otimes \mathcal{O}_{R_j}[-1]$ and $\mathcal{N}_f \otimes \mathcal{O}_{R_j}$ is locally free and globally generated by the vanishing (5) or (8), this follows from the same argument as in [HT06] (in particular, the bottom of P. 187 and P. 188).

So combining this observation with the surjectivity of the map in (16) and Lemma 4.4, we have proved that a general G -equivariant deformation with ∞ fixed smooths all the nodes and produces a new section which agrees with \widehat{s} to order $k + 1$. □

5. APPENDIX: A CONCEPTUAL PROOF OF THEOREM 1.6

In this section, using the tool of pseudo-ideal sheaves, which is invented by Professor Jason Starr and Michael Roth in [RS09], and observed by Professor Dan Abramovich and Professor Chenyang Xu to be equivalent to *differential graded subscheme* of aplitude (1, 0) ([AX02]), we are going for a conceptual proof of Theorem 1.6.

Definition 5.1. Let Y be an algebraic space and $f : X \rightarrow Y$ be a flat, locally finitely presented, proper algebraic stack over Y . For every morphism of algebraic spaces $g : Y' \rightarrow Y$, a flat family of *pseudo-ideal sheaves of X/Y over Y'* is a pair (\mathcal{F}, u) consisting of

- (i) a Y' -flat, locally finitely presented, quasi-coherent $\mathcal{O}_{X_{Y'}}$ -module \mathcal{F} , and
- (ii) an $\mathcal{O}_{X_{Y'}}$ -homomorphism $u : \mathcal{F} \rightarrow \mathcal{O}_{X_{Y'}}$

such that the following induced morphism is zero

$$u' : \bigwedge^2 \mathcal{F} \rightarrow \mathcal{F}, \quad f_1 \wedge f_2 \mapsto u(f_1)f_2 - u(f_2)f_1.$$

Let D be an effective Cartier divisor in X , considered as a closed subscheme of X , and assume D is flat over Y . Denote by \mathcal{I}_D the pullback

$$\mathcal{I}_D := \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{O}_D$$

on $\mathrm{Hilb}_{X/Y} \times_X D$. And denote by

$$u_D : \mathcal{I}_D \rightarrow \mathcal{O}_{\mathrm{Hilb}_{X/Y} \times_X D}$$

the restriction of u . Then we have

Proposition 5.2. [RS09] *The locally finitely presented, quasi-coherent sheaf \mathcal{I}_D is flat over $\mathrm{Hilb}_{X/Y}$. Thus the pair (\mathcal{I}_D, u_D) is a flat family of pseudo-ideal sheaves of D/Y over $\mathrm{Hilb}_{X/Y}$.*

Denote by

$$\iota_D : \mathrm{Hilb}_{X/Y} \longrightarrow \mathcal{P}\mathrm{seudo}_{D/Y}$$

the 1-morphism associated to the flat family (\mathcal{I}_D, u_D) of pseudo-ideal sheaves of D/Y over $\mathrm{Hilb}_{X/Y}$. This is the *divisor restriction map*.

We have the following theorem, due to M. Roth and J. Starr.

Theorem 5.3. [RS09] *Let X be a Deligne-Mumford stack over \mathbb{C} and let $C_\kappa \subset X$ be a regularly immersed proper substack of X . If both*

$$H^1(C_\kappa, \mathcal{O}_X(-D) \cdot \mathrm{Hom}_{\mathcal{O}_{C_\kappa}}(\mathcal{I}_\kappa/\mathcal{I}_\kappa^2, \mathcal{O}_{C_\kappa}))$$

and

$$H^1(C_\kappa, \mathrm{Tor}_{\mathcal{O}_X}(\mathcal{O}_{C_\kappa}, \mathcal{O}_D) \cdot \mathrm{Hom}_{\mathcal{O}_{C_\kappa}}(\mathcal{I}_\kappa/\mathcal{I}_\kappa^2, \mathcal{O}_{C_\kappa}))$$

equal 0, then the divisor restriction map ι_D is smooth at $[C_\kappa]$.

Remark 5.4. When C_κ is locally complete intersection, and as if one looks at the derived intersection product

$$C_\kappa \longmapsto C_\kappa \times_X^L D$$

which has the well-behaved tangent obstruction $\mathcal{R}\Gamma(C, \mathcal{N}_{C/X} \otimes \mathcal{O}_{\mathcal{X}_b})$ ([L04], [L12]). Then assuming Proposition 5.2, Theorem 5.3 will follow directly from the statement that ι_D is smooth at $[C_\kappa]$ if $\mathcal{R}\Gamma(C, \mathcal{N}_{C/X} \otimes \mathcal{O}_{\mathcal{X}_b})$ has vanishing first term-this is a standard result due to the theory of tangent complex. To see this, we have the short exact sequence

$$\begin{aligned} 0 \rightarrow \mathrm{Tor}_{\mathcal{O}_X}(\mathcal{O}_{C_\kappa}, \mathcal{O}_D) \cdot \mathrm{Hom}_{\mathcal{O}_{C_\kappa}}(\mathcal{I}_\kappa/\mathcal{I}_\kappa^2, \mathcal{O}_{C_\kappa}) \rightarrow \mathcal{O}_X(-D) \otimes \mathrm{Hom}_{\mathcal{O}_{C_\kappa}}(\mathcal{I}_\kappa/\mathcal{I}_\kappa^2, \mathcal{O}_{C_\kappa}) \\ \rightarrow \mathcal{O}_X(-D) \cdot \mathrm{Hom}_{\mathcal{O}_{C_\kappa}}(\mathcal{I}_\kappa/\mathcal{I}_\kappa^2, \mathcal{O}_{C_\kappa}) \rightarrow 0 \end{aligned}$$

Taking the long exact sequence, then vanishing of the two first cohomology groups in Theorem 5.3 will imply that

$$H^1(C_\kappa, \mathcal{O}_X(-D) \otimes \mathrm{Hom}_{\mathcal{O}_{C_\kappa}}(\mathcal{I}_\kappa/\mathcal{I}_\kappa^2, \mathcal{O}_{C_\kappa})) = 0$$

and the rest is to remember that the first term of $\mathcal{R}\Gamma(C, \mathcal{N}_{C/X} \otimes \mathcal{O}_{\mathcal{X}_b})$ is exactly $H^1(C_\kappa, \mathcal{O}_X(-D) \otimes \mathrm{Hom}_{\mathcal{O}_{C_\kappa}}(\mathcal{I}_\kappa/\mathcal{I}_\kappa^2, \mathcal{O}_{C_\kappa}))$.

Here is the idea of the proof of Theorem 1.6 using pseudo-ideal sheaves. We first show that we can deform the restriction of a section s to a formal neighborhood of the singular fiber deforms to another formal section which agrees with the given formal section \widehat{s} to any pre-specified order. In particular, this deformation gives a deformation of the corresponding pseudo-ideal sheaves. Then we use Theorem 5.3 to show that this deformation of pseudo-ideal sheaves lifts to a deformation of global sections. In order to apply Theorem 5.3, we need to control the cohomology of the curve in the central fiber. To do this, we replace the formal neighborhood of a singular fiber by a smooth Deligne-Mumford stack. This is possible by Proposition 2.10.

5.1. A remark on R -equivalence. We firstly recall

Definition 5.5. Let X be a variety over an arbitrary field k . Let x, y be 2 points in $X(k)$, they are R -equivalent if there is a k -morphism $f : \mathbb{P}^1 \rightarrow X$ such that $f(0) = x, f(\infty) = y$.

Proposition 5.6. Let $\mathcal{X} \rightarrow \text{Spec } \mathbb{C}((t))$ be an family of rationally connected varieties satisfying Hypothesis 1.5, esp. isotrivial family (Proposition 2.10). Then the R -equivalence class of rational points consists of a unique class.

Note that it is not known in general that the R -equivalence class of rational points on a rationally connected variety defined over the Laurent field $\mathbb{C}((t))$ is finite.

Proof. By the assumption of Hypothesis 1.5, there is a Galois extension of the field $K \subset \widehat{K} = K((t'))/(t'^l - t)$ with Galois group $G \cong \mathbb{Z}/l\mathbb{Z}$ such that $\mathcal{X}_{\widehat{K}} = \mathcal{X}_K \otimes_K \text{Spec } \widehat{K} \cong \mathcal{X}'$ where \mathcal{X}' is a smooth family with a G action such that all compatibilities are satisfied. In other words, we have the following Cartesian diagram

$$\begin{array}{ccc} \mathcal{X}_{\widehat{K}} & \longrightarrow & \mathcal{X}_K \\ \downarrow & & \downarrow \\ \text{Spec } \widehat{K} & \longrightarrow & \text{Spec } K \end{array}$$

Let $\widehat{\mathcal{O}} = \mathbb{C}[[t']]$. The group G acts on $\text{Spec } \widehat{\mathcal{O}}$ and the smooth family \mathcal{X}' extends to a smooth family $\mathcal{X}' \times \text{Spec } \widehat{\mathcal{O}}$ which we denote \mathcal{X}' again, in such a way that the projection onto the second factor is G -equivariant. In particular, G acts on the central fiber \mathcal{X}'_0 naturally.

Let s_1, s_2 be two rational points of \mathcal{X} . They induce two \widehat{K} -rational points $\widehat{s}_1, \widehat{s}_2$ of $\mathcal{X}_K \otimes_K \text{Spec } \widehat{K}$, invariant under the action of the Galois group G .

By the valuative criterion of properness, we have two $\widehat{\mathcal{O}}$ -points of \mathcal{X}' , still denoted by $\widehat{s}_1, \widehat{s}_2$. Let $\widehat{s}_1^0, \widehat{s}_2^0$ be the intersection points of these two section with the central fiber. Since $\widehat{s}_1, \widehat{s}_2$ are invariant under the Galois group, the two points $\widehat{s}_1^0, \widehat{s}_2^0$ are fixed points of the G -action on \mathcal{X}'_0 .

By Theorem 1.4 and Lemma 3.7, there exists a G -equivariant very free curve $f : (\mathbb{P}^1, 0, \infty) \rightarrow (\mathcal{X}'_0, \widehat{s}_1^0, \widehat{s}_2^0)$. Since the morphism is very free, we have $H^1(\mathbb{P}^1, f^*T_{\mathcal{X}'_0}(-\infty - \infty)) = 0$. Thus the map

$$p : \text{Hom}(\mathbb{P}^1 \times \text{Spec } \widehat{\mathcal{O}}, f|_{0 \times \text{Spec } \widehat{\mathcal{O}}} = \widehat{s}_1, f|_{\infty \times \text{Spec } \widehat{\mathcal{O}}} = \widehat{s}_2) \rightarrow \text{Spec } \widehat{\mathcal{O}}$$

is smooth at $[f]$. Notice that both spaces have a natural G action such that this map is equivariant and $[f]$ is a fixed point of the action. So by Corollary 2.2, there is a G -equivariant section

$$\sigma : \text{Spec } \widehat{\mathcal{O}} \rightarrow \text{Hom}(\mathbb{P}^1 \times \text{Spec } \widehat{\mathcal{O}}, \mathcal{X}').$$

i.e., we have a G -equivariant map

$$f : \mathbb{P}^1 \times \text{Spec } \widehat{\mathcal{O}} \rightarrow \mathcal{X}'$$

such that

$$f|_{0 \times \text{Spec } \widehat{\mathcal{O}}} = \widehat{s}_1, f|_{\infty \times \text{Spec } \widehat{\mathcal{O}}} = \widehat{s}_2.$$

Restricting to the generic fiber gives a Galois invariant \mathbb{P}^1 connecting the two rational points $\widehat{s}_1, \widehat{s}_2$. Thus the two rational points s_1, s_2 are connected by a rational curve defined over $\mathbb{C}((t))$. \square

5.2. Framework of the proof. It is proved in [HT06] that weak approximation is satisfied if the points are chosen such that the fiber over them are smooth. Their proof is a deformation argument. And the proof given below is also a deformation argument. So using the iterated blow-up construction, it suffices to prove weak approximation at a single place of bad reduction $b \in B$.

Let \widehat{s} be a formal section in the formal neighborhood of b we want to approximate. There is a regular section s , which we assume to lie in the smooth locus of \mathcal{X} by taking a resolution of singularities.

As indicated above, we will work with the *smooth* Deligne-Mumford stack obtained by replacing the formal neighborhood of the singular fiber over b with the smooth Deligne-Mumford stack based on Hypothesis 1.5.

To be more precise, recall that there is a Galois extension of the field $K \subset \widehat{K} = K((t'))/(t' - t)$ with Galois group $G \cong \mathbb{Z}/l\mathbb{Z}$ such that $\mathcal{X}_{\widehat{K}} = \mathcal{X}_K \otimes_K \text{Spec } \widehat{K} \cong \mathcal{X}'$. And we have the following Cartesian diagram

$$\begin{array}{ccc} \mathcal{X}_{\widehat{K}} & \longrightarrow & \mathcal{X}_K \\ \downarrow & & \downarrow \\ \text{Spec } \widehat{K} & \longrightarrow & \text{Spec } K \end{array}$$

together a natural action of G on \mathcal{X}' and $\text{Spec } \widehat{K}$. Let $\widehat{\mathcal{O}} = \mathbb{C}[[t']]$. The group G acts on $\text{Spec } \widehat{\mathcal{O}}$ and the smooth family \mathcal{X}' extends smoothly over $\text{Spec } \widehat{\mathcal{O}}$ in such a way that the projection onto the second factor is G -equivariant. So G acts on the central fiber \mathcal{X}'_0 .

Denote by $\mathfrak{X}' \rightarrow B'$ again the new stacky family obtained by replacing the formal neighborhood of b of the original family $\mathcal{X} \rightarrow B$ with the quotient stack $[\mathcal{X}'/G]$.

The section s and the formal section \widehat{s} give two $\widehat{\mathcal{O}}$ -points of \mathcal{X}' , invariant under the action of the Galois group G . We still denote the corresponding invariant section and formal section by s and \widehat{s} . Furthermore, they induce a section s' of the new family $\mathcal{X}' \rightarrow B'$ and a formal section \widehat{s}' of $[\mathcal{X}'/G] \rightarrow [\text{Spec } \widehat{\mathcal{O}}/G]$.

The weak approximation problem for the two families are equivalent.

The proof of Proposition 5.6 shows that we have a G -equivariant map

$$f : \mathbb{P}^1 \times \text{Spec } \widehat{\mathcal{O}} \rightarrow \mathcal{X}'$$

such that

$$f|_{0 \times \text{Spec } \widehat{\mathcal{O}}} = s|_{\text{Spec } \widehat{\mathcal{O}}}, f|_{\infty \times \text{Spec } \widehat{\mathcal{O}}} = \widehat{s}$$

We may assume that the morphism is a closed immersion up to replacing the family with product and taking the graph of f .

This gives a closed immersion

$$i : [\mathbb{P}^1 \times \text{Spec } \widehat{\mathcal{O}}/G] \rightarrow [\mathcal{X}'/G].$$

Let D be the divisor in $[\mathcal{X}'/G]$ over the closed point of $[\text{Spec } \widehat{\mathcal{O}}/G]$ and E its restriction to this ruled surface.

Let N be the order to which we want the regular section to agree with the given formal section. Denote the stacky curve in D by C_0 . Let C_s be the union of the subscheme consisting of curve $N \cdot C_0$ and the section $s'(B')$. Let C_κ be a comb obtained by attaching very free rational curves in general fibers along general normal directions to C_s . Notice that the pseudo-ideal sheaf obtained by restricting C_κ to $N \cdot D$ is the same as that obtained by restricting C_s .

Now we need to use Proposition 5.6 to produce a pencil of pseudo-ideal sheaves.

5.3. A non-separated pencil in $\mathcal{P}seudo_{\mathcal{X}_b}$.

Lemma 5.7. *There is a family of pseudo-ideal sheaves $\mathcal{I}_t, t \in \mathbb{P}^1$ in $N \cdot D$ such that \mathcal{I}_0 is isomorphic to the pseudo-ideal of the restriction of C_κ to $N \cdot D$, and a general member of the family is isomorphic to the restriction of \widehat{s}' to $N \cdot D$.*

Proof. It is easy to see that there is a ruled surface

$$\pi : P = [\mathbb{P}^1 \times C/G] \rightarrow [C/G]$$

for some curve C which has a G action (for example, take C to be a cyclic cover of B totally ramified over the point we want to approximate), together with two divisors 0_P and ∞_P , such that

- (1) the base change of this ruled surface to $\text{Spec } \widehat{\mathcal{O}}_{0, [C/G]}$ is isomorphic to

$$[\mathbb{P}^1 \times \text{Spec } \widehat{\mathcal{O}}/G] \rightarrow [\text{Spec } \widehat{\mathcal{O}}/G];$$

- (2) the restriction of 0_P (resp. ∞_P) to $\text{Spec } \widehat{\mathcal{O}}$ is congruent to $C_\kappa|_{\text{Spec } \widehat{\mathcal{O}}}$ (resp. \widehat{s}') modulo $(N+1) \cdot E$.

On P consider the invertible sheaf $\mathcal{L} = \mathcal{O}_P(\infty_P - 0_P - \pi^*(N \cdot E))$, where N is the order to which we want the regular section to agree with the given formal section. Since the restriction to the generic fiber of $P \rightarrow [C/G]$ is a degree 0 invertible sheaf on $[C/G]$, it has a 1-dimensional space of global sections. Thus the push-forward of \mathcal{L} to $[C/G]$ is a torsion-free, coherent $\mathcal{O}_{[C/G]}$ -module of rank 1, i.e., it is an invertible sheaf. Thus there exists an effective divisor Δ in C , not intersecting E such that $\mathcal{O}_P(\infty_P - 0_P + \pi^*\Delta - \pi^*(N \cdot E))$ has a global section. In other words, there is an effective divisor F in P , necessarily vertical, such that

$$\infty_P + \pi^*\Delta = 0_P + \pi^*(N \cdot E) + F.$$

Let the curves G_t be the members of the pencil spanned by $0_P + \pi^*(N \cdot E) + F$ and $\infty_P + \pi^*\Delta$. All but finitely many members of this pencil are comb-like curves with handle a section of $P \rightarrow [C/G]$. Since the base locus of the pencil contains $\infty_P \cap \pi^*(N \cdot E)$, these section curves agree with ∞_P over $(N \cdot E)$. Restricting to $(N \cdot E)$ we get a one parameter family of pseudo-ideal sheaves in $N \cdot E$, hence also in $N \cdot D$ such that they agree with the given formal section \widehat{s} to a given order.

□

With this family of pseudo-ideal sheaves in hand, all we need to do is to show that the obstruction groups of lifting the deformation to the Hilbert scheme vanish.

5.4. Vanishing of the tangent obstruction. The sheaf

$$\mathcal{O}_{\mathcal{X}'}(-N \cdot D) \cdot \text{Hom}_{\mathcal{O}_{C_\kappa}}(\mathcal{I}_\kappa/\mathcal{I}_\kappa^2, \mathcal{O}_{C_\kappa})$$

is supported in the union of the section $s'(B')$ and the very free curves since C_0 is contained in D . If we attach enough very free curves, we can make

$$H^1(C_\kappa, \mathcal{O}_{\mathcal{X}'}(-N \cdot D) \cdot \text{Hom}_{\mathcal{O}_{C_\kappa}}(\mathcal{I}_\kappa/\mathcal{I}_\kappa^2, \mathcal{O}_{C_\kappa}))$$

zero (by Remark 2.8, applying the *relative smoothing* Theorem 2.6).

Now we want to show that

$$H^1(C_\kappa, \text{Tor}_{\mathcal{O}_X}(\mathcal{O}_{C_\kappa}, \mathcal{O}_D) \cdot \text{Hom}_{\mathcal{O}_{C_\kappa}}(\mathcal{I}_\kappa/\mathcal{I}_\kappa^2, \mathcal{O}_{C_\kappa}))$$

is also 0. First, using the exact sequence

$$0 \rightarrow \mathcal{O}(-N \cdot D) \rightarrow \mathcal{O}_{\mathcal{X}'} \rightarrow \mathcal{O}_{N \cdot D} \rightarrow 0$$

we see that

$$\text{Tor}_{\mathcal{O}_{\mathcal{X}'}}(\mathcal{O}_{C_\kappa}, \mathcal{O}_{N \cdot D}) \cong \mathcal{O}_{N \cdot C_0}(-p),$$

where p is the intersection of section and the stacky curve C_0 in D . Thus the sheaf is supported on C_0 . We have the short exact sequence of sheaves

$$0 \rightarrow \mathcal{N}_{N \cdot C_0/\mathcal{X}'}(-p) \rightarrow \mathcal{N}_{C_\kappa/\mathcal{X}'} \rightarrow Q \rightarrow 0,$$

where Q is a torsion sheaf supported at p . Thus it suffices to show that

$$H^1(C_0, \mathcal{N}_{C_0/\mathcal{X}'}(-p)) = H^1(C_0, \mathcal{N}_{C_0/[\mathcal{X}'/G]}(-p))$$

is 0. Let C'_0 and p' be the preimages of C_0 and p in \mathcal{X}' and $q : C'_0 \rightarrow C_0$ be the restriction of the quotient map $\mathcal{X}' \rightarrow [\mathcal{X}'/G]$. Then the sheaf

$$\mathcal{N}_{C_0/[\mathcal{X}'/G]}(-p)$$

is a direct summand of

$$q_*(q^* \mathcal{N}_{C_0/[\mathcal{X}'/G]}(-p)) = q_*(\mathcal{N}_{C'_0/\mathcal{X}'}(-p'))$$

Therefore, the cohomology groups is also a direct summand of the corresponding cohomology. But we have chosen the curve C'_0 to be a very free curve in the fiber in the proof of Proposition 5.6. Thus we have $H^1(C'_0, \mathcal{N}_{C'_0/\mathcal{X}'}(-p')) = 0$ and so is

$$H^1(C_\kappa, \text{Tor}_{\mathcal{O}_{\mathcal{X}'}}(\mathcal{O}_{C_\kappa}, \mathcal{O}_D) \cdot \text{Hom}_{\mathcal{O}_{C_\kappa}}(\mathcal{I}_\kappa/\mathcal{I}_\kappa^2, \mathcal{O}_{C_\kappa})).$$

Therefore the divisor restriction map $\iota_{N \cdot D} : \text{Hilb}_{\mathcal{X}'} \rightarrow \text{Pseudo}_{N \cdot D}$ is smooth at $[C_\kappa]$. And we get a one parameter family of section curves whose restriction to the divisor $N \cdot D$ contains general members of the family of pseudo-ideal sheaves \mathcal{I}_t in Lemma 5.7 (c.f. Remark 5.8). In particular, we get a section of the actual family which agrees with the formal section to the given order N .

Remark 5.8. As we noted in the beginning of this section, the space of pseudo-ideals is highly non-separated. Thus even if we know the divisor restriction map is dominant and there is a family of pseudo-ideal sheaves deforming from one to another, all we can conclude is that we have a family of sections whose restriction to the divisor is isomorphic to the family of pseudo-ideal sheaves at general points

of the family. Therefore in our situations, it is essential to have a family of pseudo-ideal sheaves whose general members are all isomorphic to the pseudo-ideal sheaf coming from the formal section (e.g. the family G_t coming from a pencil).

REFERENCES

- [AV02] Dan Abramovich, Angelo Vistoli, Compactifying the Space of Stable Maps, *J. Amer. Math. Soc.* 15 (2002), 27-75.
- [AX02] Dan Abramovich, Chenyang Xu, Pseudo-ideal sheaves as differential graded subschemes, preprint.
- [DS03] Aise Johan de Jong and Jason Starr. Every Rationally Connected Variety over the Function Field of a Curve Has a Rational Point. *American Journal of Mathematics*, **125**, 567–580 (2003).
- [GHS03] Tom Graber, Joe Harris, Jason Starr. Families of Rationally Connected Varieties. *J. Amer. Math. Soc.*, 16(1):57–67 (electronic), 2003.
- [Has10] Brendan Hassett. Weak Approximation and Rationally Connected Varieties over Function Fields of Curves. In *Variétés rationnellement connexes: aspects géométriques et arithmétiques*, volume 31 of *Panor. Synthèses*, pages 115–153. Soc. Math. France, Paris, 2010.
- [HT06] Brendan Hassett, Yuri Tschinkel. Weak Approximation over Function Fields. *Invent. Math.*, 163(1):171–190, 2006.
- [HT08] Brendan Hassett, Yuri Tschinkel. Approximation at Places of Bad Reduction for Rationally Connected Varieties. *Pure Appl. Math. Q.*, 4(3, Special Issue: In honor of Fedor Bogomolov. Part 2):743–766, 2008.
- [HT09] Brendan Hassett, Yuri Tschinkel. Weak Approximation for Hypersurfaces of Low Degree. In *Algebraic geometry—Seattle 2005. Part 2*, volume 80 of *Proc. Sympos. Pure Math.*, pages 937–955. Amer. Math. Soc., Providence, RI, 2009.
- [Kne13] Amanda Knecht. Weak Approximation for General Degree Two *Del Pezzo* Surfaces. *Proc. Amer. Math. Soc.*, 141(3):801–811, 2013.
- [Kol96] János Kollár. *Rational Curves on Algebraic Varieties*, volume 32 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, 1996.
- [KMM92] János Kollár, Yoichi Miyaoka, Shigefumi Mori. Rationally Connected Varieties. *J. Algebraic Geom.*, 1(3):429–448, 1992.
- [KM98] János Kollár, Shigefumi Mori. *Birational Geometry of Algebraic Varieties*. Cambridge Tract. in Math. **134**, Cambridge University Press, Cambridge, 1998.
- [L04] Jacob Lurie. Thesis: Derived Algebraic Geometry. Massachusetts Institute of Technology. 2004.
- [L11] Jacob Lurie. Derived Algebraic Geometry X: Formal Moduli Problems. <http://www.math.harvard.edu/~lurie/>
- [L12] Jacob Lurie. Higher Algebra. <http://www.math.harvard.edu/~lurie/>
- [OI07] Martin Olsson. On (log) Twisted Curves, *Comp. Math.* **143** (2007), 476-494.
- [RS09] Mike Roth, Jason Starr. Weak Approximation and R-Equivalence over Function Fields of Curves. Preprint, 2009.
- [T10] Zhiyu Tian. Weak Approximation for Iso-trivial Families (v1), arXiv:1003.3502.
- [T13] Zhiyu Tian. Weak Approximation for Cubic Hypersurfaces. to appear in *Duke. Math. J.*, arXiv:1303.7273, 2013.
- [TZ12] Zhiyu Tian, Hong R. Zong. One Cycles on Rationally Connected Varieties. to appear in *Comp. Math.*, arXiv:1209.4342, 2012.
- [Wei94] Charles A. Weibel. *An Introduction to Homological Algebra*. volume 38 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1994.
- [Xu12] Chenyang Xu. Weak Approximation for Low Degree *Del Pezzo* Surfaces. *J. Algebraic Geom.*, 21(4):753–767, 2012.

DEPARTMENT OF MATHEMATICS, PRINCETON UNIVERSITY, PRINCETON, NJ, 08544-1000
E-mail address: rzong@math.princeton.edu