QUIZ 1: ALGEBRAIC GEOMETRY II, SPRING 2023

To be finished in consecutive 3 hours.

Problem 1(10 points) Resolve the singularities of Whitney umbrella $x^2 = zy^2$ (in characteristic 0) by giving a sequence of blowing-ups along *non-singular centers*. In each blow-up, write down the singular locus before the blow-up, the blow-up center, and the defining equation of the blow-up in charts.

Problem 2 (10 points) Let X be a non-singular variety defined over an algebraically closed field, $Y \subset X$ a non-singular subvariety of codimension $r \geq 2$, and $\pi : \tilde{X} \to X$ the blow-up along Y. Write $E = \pi^{-1}(Y)$ as the exceptional divisor.

(1) Prove that $\mathrm{Pic}(\tilde{X}) \cong \mathrm{Pic}(X) \oplus \mathbb{Z}$.
(2) Prove that $\omega_{\tilde{X}} \cong \pi^* \omega_X \otimes \mathcal{O}_{\tilde{X}}((r-1)E)$.
(3) Prove that $\Omega_{\tilde{X}/X}$ is isomorphic to $i_*(\Omega_{E/Y})$, where $i : E \to \tilde{X}$ is the inclusion.

Problem 3 (10 points) Let Y be a noetherian scheme and E a locally free sheaf of rank $r+1$ on Y. Denote $\mathbb{P}(E)$ by X and let $\pi : X \to Y$ be the natural morphism.

(1) Prove that there is an Euler sequence of $\Omega_{X/Y}$ as follows:
$$0 \to \Omega_{X/Y} \to \pi^* E \otimes \mathcal{O}_X(-1) \to \mathcal{O}_X \to 0.$$
(2) Denote by $\omega_{X/Y}$ the relative dualizing sheaf, i.e. $\Lambda^r \Omega_{X/Y}$. Prove that there is a natural isomorphism $R^r \pi_*(\omega_{X/Y}) \cong \mathcal{O}_Y$.
(3) Compute all the higher direct images of $\mathcal{O}(n)$ for all n.