QUIZ 2: ALGEBRAIC GEOMETRY II, SPRING 2023

To be finished in consecutive 24 hours.

If you think that there are mistakes in the statements, explain the mistakes and prove what you think should be the correct statements.

Problem 1(10 points)

(1) (7 points) Let $C \subset X$ be a non-singular curve contained in a non-singular projective variety, both defined over an algebraically closed field. Assume that $\dim X \geq 3$. Fix a very ample line bundle L. Prove that for every m large enough, $L^{\otimes m}$ has a section whose zero locus defines a non-singular hypersurface $D \subset X$ containing C.

(2) (3 points) Let $\mathbb{P}^2 \subset \mathbb{P}^4$ be a linear space. Prove that there is no non-singular hypersurface $D \subset \mathbb{P}^4$ that contains $\mathbb{P}^2$ except when D is a hyperplane. In view of this example, you should double check that your argument in the previous question only works for curves.

Problem 2 (10 points) Let X be a non-singular non-hyperelliptic curve of genus 5. Let us study the presentation of X as a plane curve with only nodes. Prove the followings.

(1) (5 points) X has a g^1_3 if and only if X is the normalization of a plane curve of degree 5 with only one node.

(2) (5 points) If X has no g^1_3, then there is a degree 6 divisor D on X such that $\mathcal{O}_X(D)$ is base point free and induces a morphism $X \to \mathbb{P}^2$.

(3) (extra credit 10 points) Prove that for X and a general degree 6 divisor as in (2), the image is a curve with only nodes as singularities.

Problem 3(10 points) Let X be a projective geometrically integral k-variety. For any coherent sheaf F, we write F^* as the dual sheaf $\mathbf{Hom}(F, \mathcal{O}_X)$ (i.e. take the sheaf Hom). Let ω_X be the dualizing sheaf of X. Prove that $\omega_X \cong \omega_X^{**}$, that is ω_X is a *reflexive sheaf*. Hint: both ω_X and ω_X^{**} are torsion free.