FINAL EXAM: ALGEBRAIC GEOMETRY II, SPRING 2023

You may use the textbook (Hartshorne), your notes, and lecture notes, and watch the recorded lecture videos.

Any other form of assistance is strictly prohibited.

If you think that there are mistakes in the statements, explain the mistakes and prove what you think should be the correct statements.

You may use the statements in other problems to solve a particular problem, even if you cannot prove these statements.

1. Preliminary

Problem 1(10 points) Let $f : X \to Y$ be a birational morphism between two projective surfaces and assume that X is smooth. Let $y \in Y$ be such that the set $f^{-1}(y)$ consists of curves $E_i, i = 1, \ldots, n$. Prove that the intersection number matrix $(a_{ij} = E_i \cdot E_j)_{1 \le i,j \le n}$ is negative definite ($E_i \cdot E_j$ is the intersection number of E_i and E_j in X).

As an application, prove the following negativity lemma: If $\sum a_i E_i$ is relatively (very) ample, then $a_i < 0, 1 \le i \le n$.

Problem 2(10 points) Prove the following Rigidity lemma: Every scheme is noetherian. Let $p : X \to Z$ be a proper morphism such that $p_* \mathcal{O}_X = \mathcal{O}_Z$. Let $f : X \to Y$ be a finite type morphism. Assume that there is a point $z \in Z$ such that $f(X_z)$ is a point where X_z is the fiber of X over z. Then there is a non-empty open neighborhood U of z and a morphism $g : U \to Y$ such that $f : p^{-1}(U) \to Y$ factors through $g : U \to Y$.

Problem 3(10 points) Let Y be a connected noetherian scheme and $p : X \to Y$ be a flat proper morphism. Assume that for every algebraically closed field k and every morphism Spec $k \to Y$, the fiber product X_k is isomorphic to $\mathbb{P}^n_k$ for a fixed n. Prove that $\mathrm{Pic}(X) \cong \mathrm{Pic(Y)} \times \mathbb{Z}$. Note that there are examples where X is not of the form $\mathbb{P}(\mathcal{E})$.

As a corollary, prove that if B is a non-singular projective curve over an algebraically closed field and $S \to B$ is a dominant morphism from a projective variety such that all fibers over closed points are isomorphic to $\mathbb{P}^1$, then S is a smooth projective surface, the morphism $S \to B$ is smooth, and $\mathrm{Pic}(S)/\mathrm{num}$ (i.e. Picard group modulo numerical equivalence) is isomorphic to $\mathbb{Z} \times \mathbb{Z}$.

In the following fix a noetherian scheme S, a smooth projective morphism $\pi : C \to S$ of relative dimension 1 with geometrically connected fibers, sections $\sigma_i : S \to C$ (i.e. $\pi \circ \sigma_i = id_S$), and a flat projective morphism $X \to S$, and a relative very ample invertible sheaf $\mathcal{O}_{X/S}(1)$. Fix S-morphisms $g_i : S \to X$. Consider the following functor:

$$\mathrm{Hom}_d(C, X, g_i) : (\mathrm{Sch}/S)^{\mathrm{op}} \to S$$

$$T \mapsto \{f_T : T \times_S C \to T \times_S X, f_T \circ \sigma_i = g_i, f_T^* \mathcal{O}_{X/T}(1) \text{ has degree } d \text{ with fibers of } C_T \to T\}.$$

You will need the following theorem later.

Theorem 1.1. *The functor $Hom_d(C, X, g_i)$ is represented by a quasi-projective scheme H_d over S. Here for simplicity of notations, we omit g_i's.*

Theorem 1.2. *Let S be the spectrum of a finitely generated $\mathbb{Z}$-algebra, $[f] \in H_d$ be a point, and k the residue field of the image of $[f]$ in S (Thus $[f]$ corresponds to a morphism $f : C_k \to X_k$). Assume that $X_k = X \times_S$ Spec k is smooth. The k-scheme $H_d \times_S$ Spec k has dimension at least $-\deg_{C_k} \omega_{X_k} + \dim X_k(1 - g(C_k) - n)$ at $[f]$. The Zariski tangent space of $H_d \times_S$ Spec k at $[f]$ is isomorphic to $H^0(C_k, f^*T_X \otimes \mathcal{O}_{C_k}(-\sum c_i))$, where T_X is the dual of Ω_X, $c_i \in C_k(k)$ are the images of g_i(Spec k). Here C_k is the fiber over Spec k and n is the number of sections.*

Problem 4 (10 points) Prove Theorem 1.1 when $S =$ Spec $k, C = \mathbb{P}^1_k, X \subset \mathbb{P}^n$ a hypersurface (not necessarily smooth) k being a field, and there are no sections. In fact, give an explicit construction of a quasi-projective scheme over Spec k that represents the functor.

Prove that the dimension estimate in Theorem 1.2 holds even in this case. Here you have to interpret ω_X as the dualizing sheaf of X.

Problem 5 (10 points) Let $S =$ Spec $k, c_i \in C(k), x_i \in X(k), i = 1, \ldots, n$. Fix an ample line bundle L on X. Let T be a smooth not necessarily proper curve. Consider the family

$$f : C \times T \to X \times T, f(c_i \times T) = x_i.$$

Consider the following two cases:

(1) Assume $g(C) \geq 1$, and $n = 1$. A;so assume that there is a point $t_0 \in T$ such that for a general $t \in T$, $f_t \neq f_{t_0}$
(2) Assume that $g(C) = 0$, and $n = 2$. Also assume that the image of f is two dimensional, then T cannot be proper.

In both cases, let $\bar{T}$ be the smooth projective compactification of T. Prove that the rational map $\bar{T} \times C \dashrightarrow X$ (induced by $T \times C \to X$) cannot be extended to a morphism in a neighborhood of $\bar{T} \times c_1$.

Prove that in both cases, one can find a non-constant morphism $g : \mathbb{P}^1 \to X$ such that $g(0) = f(c_1), \deg g^*L < \deg_C(f|_{C \times t})^*L$.

Hint: apply the following theorem on resolving the indeterminancy by repeatedly blowing-up smooth points to $\bar{T} \times C \dashrightarrow X$: Let $S \dashrightarrow \mathbb{P}^n$ be a rational map from a smooth surface S. Then one can resolve the indeterminancy by repeatedly blowing-up smooth points. That is there is a sequence of blow-ups at smooth points $S_n \to S_{n-1} \to \ldots \to S_0 = S$ and commuting rational maps $S_i \dashrightarrow \mathbb{P}^n$, such that $S_n \to \mathbb{P}^n$ is a morphism.

2. Mori's theorem

The goal is to prove the following celebrated theorem of Mori:

Theorem 2.1 (Mori)**.** *Let X be a smooth projective variety defined over an algebraically closed field. Assume that ω_X^*, the dual invertible sheaf of the dualizing sheaf ω_X, is ample. Then for any point $x \in X$, there is a non-constant morphism $f : \mathbb{P}^1 \to X$ such that $f(0) = x, \deg f^*\omega_X^* \leq \dim X + 1$.*

Up until the time of this final exam, the only available proof of Mori's theorem for X over the complex numbers $\mathbb{C}$ is via the following "reduction mod p" argument relying on Grothendieck's general machinary of Hom schemes over a base, even if the statement is purely complex analytic.

Problem 6 (10 points) Everything is over an algebraically closed field. Let $f : C \to X$ be a morphism and $c \in C(k)$.

(1) If $-\deg f^*\omega_X > g(C)\dim X$ and $g(C) \geq 1$, then there is morphism $g : \mathbb{P}^1 \to X$ such that $g(0) = f(c)$.
(2) If $-\deg f^*\omega_X > \dim X + 1$ and $C \cong \mathbb{P}^1$, then there is morphism $g : \mathbb{P}^1 \to X$ such that $g(0) = f(c), -\deg g^*\omega_X \leq \dim X + 1$.

Problem 7(10 points) Prove the theorem of Mori when k has positive characteristic. Hint: let $F : Y \to Y$ be the absolute Frobenius and L and invertible sheaf. What is F^*L?

Problem 8(10 points) Use the technique of spreading-out to prove Mori's theorem in characteristic 0. Hint: find a finitely generated $\mathbb{Z}$-algebra $A \subset k$, a family of curves $C \to \text{Spec } A$, varieties $\mathcal{X} \to \text{Spec } A$, whose geometric generic fiber is X/k, and fibers over closed points in Spec A are defined over positive characteristic.