Homework 4 for “Algorithms for Big-Data Analysis”

Acknowledgement:

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Note: Please write up your solutions independently. If you get significant help from others, write down the source of references. A formal mathematical proof for all your claims is required.

1. Let F be a nondecreasing submodular set function on X, and q be a real number.
   (a) Prove that the function
   $$G(U) = \min(q, F(U)), \forall U \subseteq X$$
   is submodular.
   (b) Is monotonicity required? If so, show that monotonicity cannot be removed. If not, show that monotonicity is not critical for this property to be true.
   (c) What about non-increasing (decreasing) functions in min? That is, is submodularity preserved in this case as well when F is monotone non-increasing? Prove or give a counterexample.
2. Properties of Submodular Functions
   (a) Prove that any non-negative submodular function is also subadditive, i.e. if $F : 2^X \to \mathbb{R}_+$ is submodular then $F(S \cup T) \leq F(S) + F(T)$ for any $S, T \subseteq X$. Here, $\mathbb{R}_+ = \{x \in \mathbb{R}, x \geq 0\}$.
   (b) Prove that a function $F : 2^X \to \mathbb{R}_+$ is submodular if and only if for any $S, T \subseteq X$, the marginal contribution function $F_S(T) = F(S \cup T) - F(S)$ is subadditive.
3. Given finite ground set X , and given $w_d \in [0, 1]$ for all $d \in X$, define
   $$F(S) = \Pi_{d \in S} w_d,$$
   where $F(\emptyset) = 1$. Is this submodular, supermodular, modular, or neither?