Homework 7 for "Algorithms for Big-Data Analysis"

March 12, 2018

Note: Please write up your solutions independently. If you get significant help from others, write down the source of references. A formal mathematical proof for all your claims is required.

1. Given a $n \times p$ matrix X. Derive the optimal solution for the following problem:

$$\min_{Z,V} \|X - ZV\|_F^2, \text{ s.t. } \quad V^T V = I, Z^T 1 = 0,$$

   where Z is a $n \times q$ matrix and V is a $q \times p$ matrix.

2. Derive the dual optimization problem for

$$\begin{aligned} \min_{w,b,\xi} \quad & \frac{1}{2}\|w\|_2^2 + C_1 \sum_{i=1}^n \xi_i + C_2 \sum_{i=1}^n \xi_i^2 \\ \text{s.t.} \quad & y_i \cdot (x_i \cdot w + b) \geq 1 - \xi_i, \forall i = 1, \ldots, n \\ & \xi_i \geq 0, \forall i = 1, \ldots, n \end{aligned}$$

3. Properties of Submodular Functions

   (a) Prove that any non-negative submodular function is also subadditive, i.e. if $F : 2^X \to \mathbb{R}_+$ is submodular then $F(S \cup T) \leq F(S) + F(T)$ for any $S, T \subseteq X$. Here, $\mathbb{R}_+ = \{x \in \mathbb{R}, x \geq 0\}$.

   (b) Prove that a function $F : 2^X \to \mathbb{R}_+$ is submodular if and only if for any $S, T \subseteq X$, the marginal contribution function $F_S(T) = F(S \cup T) - F(S)$ is subadditive.

4. Given finite ground set X , and given $w_d \in [0, 1]$ for all $d \in X$, define

$$F(S) = \Pi_{d \in S} w_d,$$

   where $F(\emptyset) = 1$. Is this submodular, supermodular, modular, or neither?