Submodular Function Optimization

http://bicmr.pku.edu.cn/~wenzw/bigdata2024.html

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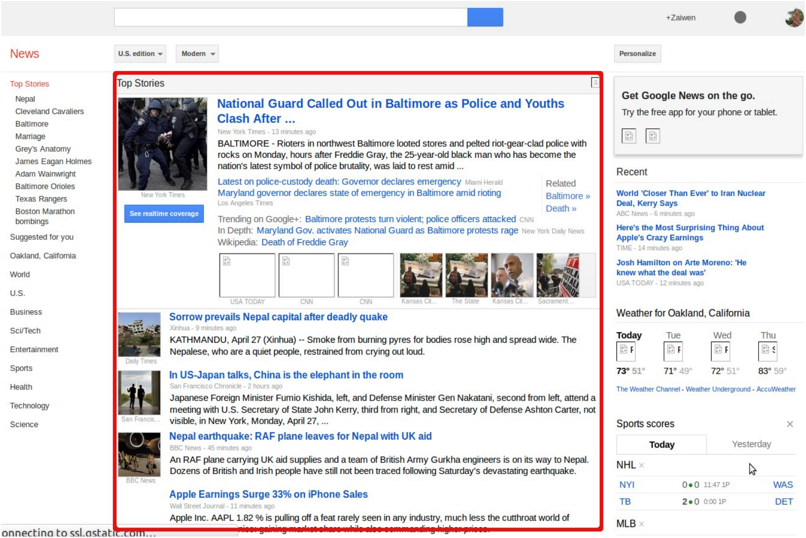
Outline

Interactive recommendation

- Number of recommendations k to choose from large data.
  - Similar articles → similar click-through rates!
- Performance depends on query / context.
  - Similar users → similar click-through rates!
- Need to compile sets of k recommendations(instead of only one).
  - Similar sets→ similar click-through rates!

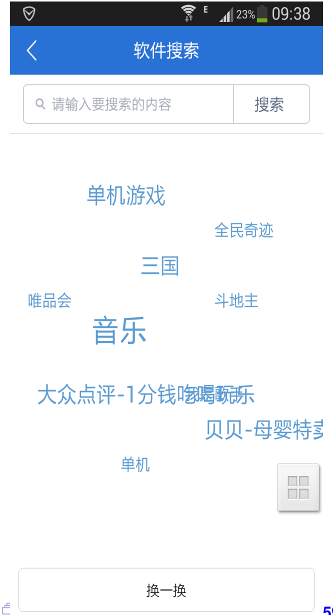
News recommendation



Which set of articles satisfies most users?

Relevance vs. Diversity

- Users may have different interests / queries may be ambiguous.
  - E.g., "jaguar", "squash",· · · .
- Want to choose a set that is relevant to as many users as possible.
  - Users may choose from the set the article they’re most interested in.
- Want to optimize both relevance and diversity.



Simple abstract model

- Given a set W of users and a collection V of articles/ads.
- Each article i is relevant to a set of users S_i.
  - For now suppose this is known!
- For each set A of articles, define

$$F(A) = |\cup_{i \in A} S_i|.$$

- Want to select k articles from V to maximize "users covered"

$$\max_{A \subseteq V, |A| < k} F(A).$$

- Number of sets A grows exponential in k!
- Finding optimal A is NP-hard.

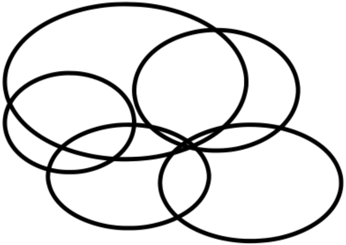
Maximum coverage

- **Given:** Collection V of sets, utility function $F(.)$.

Want: $A^* \subseteq V$ such that

$$\mathcal{A}^* = \text{argmax}_{|\mathcal{A}| \le k} F(\mathcal{A})$$

NP-hard!



Set Functions

- Ground set $X := \{x_1, x_2, ..., x_n\}$ is the domain of interest or the universe of elements.
  - In sensor network, the ground set might consist of all possible locations where sensors could be placed.
- The solution space $V := 2^X = \{A \mid A \subseteq X\}$.
- A set function takes as input a set, and outputs a real number.
  - Inputs are some subsets of ground set X.
  - $F : 2^X \to \mathbb{R}$.
- It is common in the literature to use either X or V as the ground set.
- We will follow this inconsistency in the literature and will inconsistently use either X or V as our ground set (hopefully not in the same equation, if so, please point this out).

Modular Functions

- If F is a modular function, then for any $A, B \subseteq X$, we have

$$F(A) + F(B) = F(A \cap B) + F(A \cup B).$$

- If F is a modular function, it may be written as

$$F(A) = F(\emptyset) + \sum_{a \in A} \left(F(\{a\}) - F(\emptyset) \right).$$

- Modular set functions
  - Associate a weight w_i with each $i \in X$, and set $F(S) = \sum_{i \in S} w_i$.
  - Discrete analogue of linear functions.
- Other possibly useful properties a set function may have:
  - Monotone: if $A \subseteq B \subseteq X$, then $F(A) \leq F(B)$.
  - Nonnegative: $F(S) \geq 0$ for all $S \subseteq X$.
  - Normalized: $F(\emptyset) = 0$.

Submodular Functions

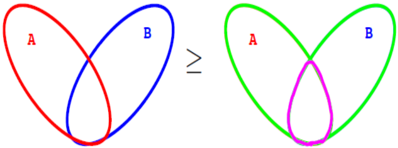
Definition 1

A set function $F : 2^X \to \mathbb{R}$ is submodular if and only if

$$F(A) + F(B) \geq F(A \cap B) + F(A \cup B)$$

for all $A, B \subseteq X$.

- "Uncrossing" two sets reduces their total function value.



Definition

A set function $F : 2^X \to \mathbb{R}$ is supmodular if and only if $-F$ is submodular.

Submodular Functions

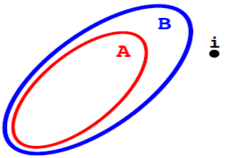
Definition 2 (diminishing returns)

A set function $F : 2^X \to \mathbb{R}$ is submodular if and only if

$$\underbrace{F(B \cup \{s\}) - F(B)}_{\text{Gain of adding an element } s \text{ to a large set}} \leq \underbrace{F(A \cup \{s\}) - F(A)}_{\text{Gain of adding an element } s \text{ to a small set}}$$

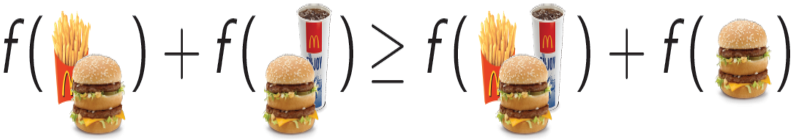
for all $A \subseteq B \subseteq X$ and $s \in X \setminus B$.

- The marginal value of the added element exhibits “diminishing marginal returns”.
- This means that the incremental “value”, “gain”, or “cost” of s decreases (diminishes) as the context in which s is considered grows from A to B.

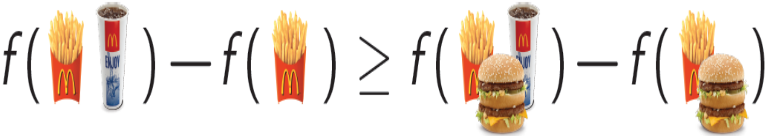


Submodular: Consumer Costs of Living

- Consumer costs are very often submodular.
  - For example:



  - When seen as diminishing returns:



Submodular Functions

Definition 3 (group diminishing returns)

A set function $F : 2^X \to \mathbb{R}$ is submodular if and only if

$$F(B \cup C) - F(B) \leq F(A \cup C) - F(A)$$

for all $A \subseteq B \subseteq X$ and $C \subseteq X \backslash B$.

- This means that the incremental "value", "gain", or "cost" of set C decreases (diminishes) as the context in which C is considered grows from A to B.

Equivalence of Definitions

Definition 2 $\Longrightarrow$ Definition 3

Let $C = \{c_1, \ldots, c_k\}$. The Definition 2 implies

$$\begin{aligned}
& F(A \cup C) - F(A) \\
= \; & F(A \cup C) - \sum_{i=1}^{k-1} (F(A \cup \{c_1, \ldots, c_i\}) - F(A \cup \{c_1, \ldots, c_i\})) - F(A) \\
= \; & \sum_{i=1}^{k} (F(A \cup \{c_1, \ldots, c_i\}) - F(A \cup \{c_1, \ldots, c_{i-1}\})) \\
\geq \; & \sum_{i=1}^{k} (F(B \cup \{c_1, \ldots, c_i\}) - F(B \cup \{c_1, \ldots, c_{i-1}\})) \\
= \; & F(B \cup C) - F(B)
\end{aligned}$$

Equivalence of Definitions

Definition 1 $\Longrightarrow$ Definition 2

Let $A' = A \cup \{s\}$, $B' = B$, from Definition 1, we have

$$\begin{aligned} F(A \cup \{s\}) + F(B) &= F(A') + F(B') \\ &\geq F(A' \cap B') + F(A' \cup B') \\ &= F(A) + F(B \cup \{s\}) \end{aligned}$$

Definition 2 $\Longrightarrow$ Definition 1

Assume $A \neq B$. Define $A' = A \cap B$, $C = A \backslash B$ and $B' = B$. Then

$$\begin{aligned} & F(A' \cup C) - F(A') \geq F(B' \cup C) - F(B') \\ \Longleftrightarrow \quad & F((A \cap B) \cup (A \backslash B)) + F(B) \geq F(B \cup (A \backslash B)) + F(A') \\ \Longleftrightarrow \quad & F(A) + F(B) \geq F(A \cup B) + F(A \cap B) \end{aligned}$$

Submodularity

- Submodular functions have a long history in economics, game theory, combinatorial optimization, electrical networks, and operations research.
- They are gaining importance in machine learning as well.
- Arbitrary set functions are hopelessly difficult to optimize, while the minimum of submodular functions can be found in polynomial time, and the maximum can be constant-factor approximated in low-order polynomial time.
- Submodular functions share properties in common with both convex and concave functions.

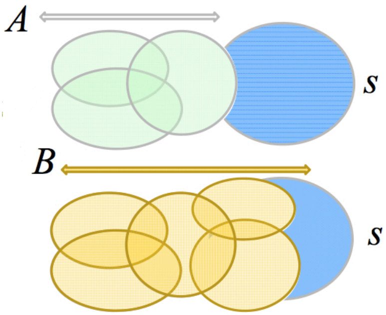
Example: Set cover

- F is submodular: $A \subseteq B$

$$\underbrace{F(A \cup \{s\}) - F(A)}_{\text{Gain of adding an element } s \text{ to a small set}} \geq \underbrace{F(B \cup \{s\}) - F(B)}_{\text{Gain of adding an element } s \text{ to a large set}}$$

- Natural example:
  - Set $S_1, S_2, \cdots, S_n$
  - F(A)=size of union of S_i (e.g., number of satisfied users)

$$F(A) = |\cup_{i \in A} S_i|$$



Closedness properties

- $F_1, \cdots, F_m$ are submodular functions on V and $\lambda_1, \cdots, \lambda_m \geq 0$.
- Then: $F(A) = \sum_i \lambda_i F_i(A)$ is submodular!

- Submodularity closed under nonnegative linear combinations
- Extremely useful fact:
  - $F_\theta(A)$ is submodular $\Rightarrow \sum_\theta P(\theta) F_\theta(A)$ is submodular!
  - Multi-objective optimization:
    $F_1, \cdots, F_m$ are submodular, $\lambda_i > 0 \Rightarrow \sum_i \lambda_i F_i(A)$ is submodular.

Probabilistic set cover

- Document coverage function:
  cover$_d(c)$=probability document d covers concept c, e.g., how strongly d covers c.
  It can model how relevant is concept c for user u.

- Set coverage function:

$$\text{cover}_A(c) = 1 - \Pi_{d \in A}(1 - \text{cover}_d(c)).$$

  Probability that at least one document in A covers c.

- Objective:

$$\max_{|A| \leq k} \quad F(A) = \sum_c w_c.\text{cover}_A(c)$$

  w_c is the concept weights.

- The objective function is submodular.

The value of a friend

- Let X be a group of individuals. How valuable to you is a given friend $x \in X$?
- It depends on how many friends you have.
- Given a group of friends $S \subseteq X$, can you valuate them with a function $F(S)$ and how?
- Let $F(S)$ be the value of the set of friends S. Is submodular or supermodular a good model?

Information and Summarization

- Let X be a set of information containing elements
  - X might say be either words, sentences, documents, web pages, or blogs.
  - Each $x \in X$ is one element, so x might be a word, a sentence, a document, etc.
  - The total amount of information in X is measure by a function $F(X)$; subset $S \subseteq X$ measures the amount of information in S, given by $F(S)$.
- How informative is any given item x in different sized contexts? Any such real-world information function would exhibit diminishing returns, i.e., the value of x decreases when it is considered in a larger context.
- So a submodular function would likely be a good model.

Restriction

Restriction

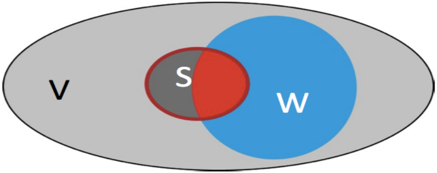
If $F(S)$ is submodular on V and $W \subseteq V$. Then $F'(S) = F(S \cap W)$ is submodular.

Proof: Given $A \subseteq B \subseteq V \backslash \{i\}$, prove:

$$F((A \cup \{i\}) \cap W) - F(A \cap W) \geq F((B \cup \{i\}) \cap W) - F(B \cap W).$$

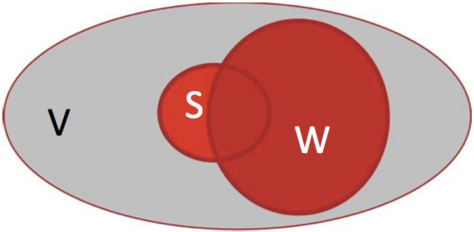
If $i \notin W$, then both differences on each size are zero.
Suppose that $i \in W$, then $(A \cup \{i\}) \cap W = (A \cap W) \cup \{i\}$ and $(B \cup \{i\}) \cap W = (B \cap W) \cup \{i\}$. We have $A \cap W \subseteq B \cap W$, the submodularity of F yields

$$F((A \cap W) \cup \{i\}) - F(A \cap W) \geq F((B \cap W) \cup \{i\}) - F(B \cap W).$$



Conditioning

If $F(S)$ is submodular on V and $W \subseteq V$. Then $F'(S) = F(S \cup W)$ is submodular



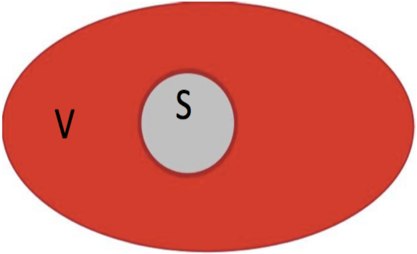
Reflection

Reflection

If $F(S)$ is submodular on V. Then $F'(S) = F(V \setminus S)$ is submodular.

Proof: Since $V \backslash (A \cup B) = (V \backslash A) \cap (V \backslash B)$ and $V \backslash (A \cap B) = (V \backslash A) \cup (V \backslash B)$, then

$$F(V \backslash A) + F(V \backslash B) \geq F(V \backslash (A \cup B)) + F(V \backslash (A \cap B)))$$



Contraction

Let $F : 2^X \to \mathbb{R}$ and $A \subseteq X$. Define $F_A(S) = F(A \cup S) - F(A)$.
Lemma: If F is monotone and submodular, then F_A is monotone, submodular, and normalized for any A.

- Proof: Monotone:
  - Let $S \subseteq T$, then $F_A(S) = F(A \cup S) - F(A) \leq F(A \cup T) - F(A) = F_A(T)$
- Submodular. Let $S, T \subseteq X$:

$$\begin{aligned} F_A(S) + F_A(T) &= F(S \cup A) - F(A) + F(T \cup A) - F(A) \\ &\geq F(S \cup T \cup A) - F(A) + F((S \cap T) \cup A) - F(A) \\ &= F_A(S \cup T) + F_A(S \cap T) \end{aligned}$$

Lemma

If F is normalized and submodular, and $A \subseteq X$, then there is $j \in A$ such that $F(\{j\}) \geq \frac{1}{|A|} F(A)$

- Proof. If A_1 and A_2 partition A, i.e., $A = A_1 \cup A_2$ and $A_1 \cap A_2 = \emptyset$, then

$$F(A_1) + F(A_2) \geq F(A_1 \cup A_2) + F(A_1 \cap A_2) = F(A)$$

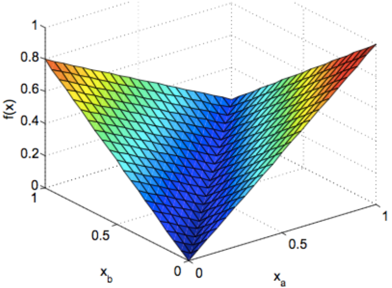
- Applying recursively, we get

$$\sum_{j \in A} F(\{j\}) \geq F(A)$$

- Therefore, $\max_{j \in A} F(\{j\}) \geq \frac{1}{|A|} F(A)$

Convex aspects

- Submodularity as discrete analogue of convexity



- Convex extension
- Duality
- Polynomial time minimization!

$$A^* = \arg\min_{A \subseteq V} F(A)$$

- Many applications (computer vision,ML, $\cdots$)

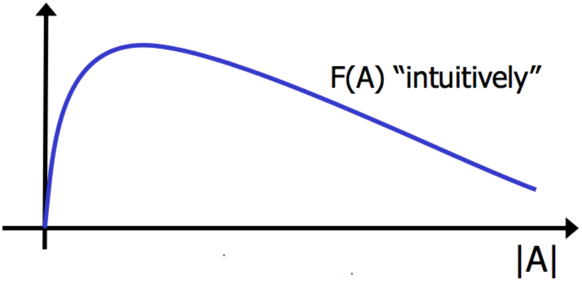
Concave aspects

- Marginal gain $\triangle_F(s|A) = F(\{s\} \cup A) - F(A)$
- Submodular:

$$\forall A \subseteq B, s \notin B: \quad F(A \cup \{s\}) - F(A) \geq F(B \cup \{s\}) - F(B)$$

- Concave:

$$\forall a \leq b, s > 0 \qquad g(a+s) - g(a) \geq g(b+s) - g(b)$$



$$\forall a \leq b, s > 0 \qquad g(a+s) - g(a) \geq g(b+s) - g(b)$$

- Suppose that $a + s \in [a, b]$
- Apply the concavity of $g(x)$ to $[a, a+s, b+s]$:

$$\begin{aligned} g(a+s) &\geq \frac{b-a}{b+s-a} g(a) + \frac{s}{b+s-a} g(b+s) \\ \iff g(a+s) - g(a) &\geq \frac{-s}{b+s-a} g(a) + \frac{s}{b+s-a} g(b+s) \end{aligned}$$

- Apply the concavity of $g(x)$ to $[a+s, b, b+s]$:

$$\begin{aligned} g(b) &\geq \frac{s}{b+s-a} g(a) + \frac{b-a}{b+s-a} g(b+s) \\ \iff g(b+s) - g(b) &\leq \frac{-s}{b+s-a} g(a) + \frac{s}{b+s-a} g(b+s) \end{aligned}$$

Submodularity and Concavity

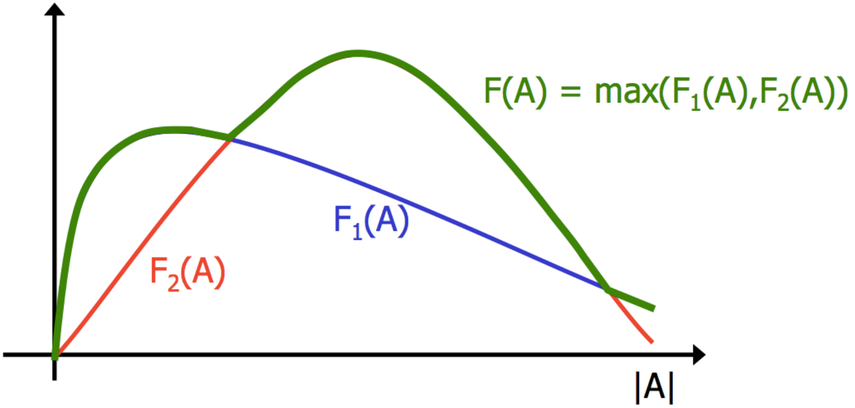
Let $m \in \mathbb{R}_+^X$ be a modular function, and g a concave function over $\mathbb{R}$. Define $F(A) = g(m(A))$. Then F(A) is submodular.

Proof: Given $A \subseteq B \subseteq X \backslash v$, we have $0 \leq a = m(A) \leq b = m(B)$, and $0 \leq s = m(v)$. For g concave, we have $g(a+s) - g(a) \geq g(b+s) - g(b)$, which implies

$$g(m(A) + m(v)) - g(m(A)) \geq g(m(B) + m(v)) - g(m(B))$$

Maximum of submodular functions

Suppose $F_1(A)$ and $F_2(A)$ submodular.
Is $F(A) = \max(F_1(A), F_2(A))$ submodular?



$\max(F_1, F_2)$ not submodular in general!

Minimum of submodular functions

Well,maybe $F(A) = \min(F_1(A), F_2(A))$ instead?

	$F_1(A)$	$F_2(A)$
{}	0	0
{a}	1	0
{b}	0	1
{a,b}	1	1

$$F(\{b\}) - F(\{\}) = 0$$
$$<$$
$$F(\{a,b\}) - F(\{a\}) = 1$$

$\min(F_1, F_2)$ not submodular in general!

Max - normalized

Given V, let $c \in \mathbb{R}_+^V$ be a given fixed vector. Then $F : 2^V \to \mathbb{R}_+$, where

$$F(A) = \max_{j \in A} c_j$$

is submodular and normalized (we take $F(\emptyset) = 0$).

Proof: Since

$$\max(\max_{j \in A} c_j, \max_{j \in B} c_j) = \max_{j \in A \cup B} c_j$$

and

$$\min(\max_{j \in A} c_j, \max_{j \in B} c_j) \geq \max_{j \in A \cap B} c_j,$$

we have

$$\max_{j \in A} c_j + \max_{j \in B} c_j \geq \max_{j \in A \cup B} c_j + \max_{j \in A \cap B} c_j$$

Monotone difference of two functions

Let F and G both be submodular functions on subsets of V and let $(F-G)(\cdot)$ be either monotone increasing. Then $h : 2^V \to R$ defined by $h(A) = \min(F(A), G(A))$ is submodular.

- If $h(A)$ agrees with either f or g on both X and Y , the result follows since

$$\begin{matrix} F(X)+F(Y) \\ G(X)+G(Y) \end{matrix} \geq \min(F(X\cup Y), G(X\cup Y)) + \min(F(X\cap Y), G(X\cap Y))$$

- otherwise, w.l.o.g., $h(X) = F(X)$ and $h(Y) = G(Y)$, giving

$$h(X)+h(Y) = F(X)+G(Y) \geq F(X\cup Y)+F(X\cap Y)+G(Y)-F(Y)$$

Assume $F-G$ is monotonic increasing. Hence, $F(X\cup Y)+G(Y)-F(Y) \geq G(X\cup Y)$ giving

$$h(X)+h(Y) \geq G(X\cup Y)+F(X\cap Y) \geq h(X\cup Y)+h(X\cap Y)$$

Min

- Let $F : 2^V \to \mathbb{R}$ be an increasing or decreasing submodular function and let k be a constant. Then the function $h : 2^V \to \mathbb{R}$ defined by

$$h(A) = \min(k; F(A))$$

  is submodular

- In general, the minimum of two submodular functions is not submodular. However, when wishing to maximize two monotone non-decreasing submodular functions, we can define function $h : 2^V \to R$ as

$$h(A) = \frac{1}{2}(\min(k, F) + \min(k, G))$$

  then h is submodular, and $h(A) \geq k$ if and only if both $F(A) \geq k$ and $G(A) \geq k$

Outline

1. What is submodularity?
2. Submodular maximization
3. Submodular minimization

Submodular maximization with Cardinality Constraint

Problem Definition

Given a non-decreasing and normalized submodular function $F: 2^X \to \mathbb{R}^+$ on a finite ground set X with $|X| = n$, and an integer $k \leq n$:

$$\max \quad F(A), \text{ s.t. } |A| \leq k$$

Greedy Algorithm

- $A_0 \leftarrow \emptyset$, set $i = 0$
- While $|A_i| \leq k$
  - Choose $s \in X$ maximizing $F(A_i \cup \{s\})$
  - $A_{i+1} \leftarrow A_i \cup \{s\}$

Greedy maximization is near-optimal

Theorem[Nemhauser, Fisher& Wolsey'78]

For monotonic submodular functions, Greedy algorithm gives constant factor approximation

$$F(A_{\text{greedy}}) \geq \underbrace{(1 - 1/e)}_{\sim 63\%} F(A^*)$$

- Greedy algorithm gives near-optimal solution!
- For many submodular objectives: Guarantees best possible unless P=NP
- Can also handle more complex constraints.

Greedy maximization is near-optimal

Theorem[Nemhauser, Fisher& Wolsey'78]

For monotonic submodular functions, Greedy algorithm gives constant factor approximation

$$F(A_{\text{greedy}}) \geq (1 - 1/e)F(A^*)$$

- Proof: Let A_i be the working set in the algorithm
- Let A^* be optimal solution.
- We will show that the suboptimality $F(A^*) - F(A)$ shrinks by a factor of $(1 - 1/k)$ each iteration
- After k iterations, it has shrunk to $(1 - 1/k)^k \leq 1/e$ from its original value
- The algorithm choose $s \in X$ maximizing $F(A_i \cup \{s\})$. Hence:

$$F(A_{i+1}) = F(A_i) + F(A_i \cup \{s\}) - F(A_i) = F(A_i) + \max_j F_{A_i}(\{j\})$$

- By our lemmas, there is $j \in A^*$ s.t.

$$
\begin{aligned}
F_{A_i}(\{j\}) &\geq \frac{1}{|A^*|} F_{A_i}(A^*) \quad \text{(apply lemma to } F_{A_i}\text{)} \\
&= \frac{1}{k}(F(A_i \cup A^*) - F(A_i)) \\
&\geq \frac{1}{k}(F(A^*) - F(A_i))
\end{aligned}
$$

- Therefore

$$
\begin{aligned}
F(A^*) - F(A_{i+1}) &= F(A^*) - F(A_i) - \max_j F_{A_i}(\{j\}) \\
&\leq \left(1 - \frac{1}{k}\right)(F(A^*) - F(A_i)) \\
&\leq \left(1 - \frac{1}{k}\right)^k (F(A^*) - F(\emptyset))
\end{aligned}
$$

Scaling up the greedy algorithm [Minoux'78]

In round i+1,

- have picked $A_i = s_1, \cdots, s_i$
- pick $s_{i+1} = \arg\max_s F(A_i \cup \{s\}) - F(A_i)$.
- Update the gain of other elements affected by the addition of s_{i+1}.

The core of the algorithm is maximize "marginal benefit" $\triangle(s|A_i)$

$$\triangle(s|A_i) = F(A_i \cup \{s\}) - F(A_i)$$

Key observation: Submodularity implies

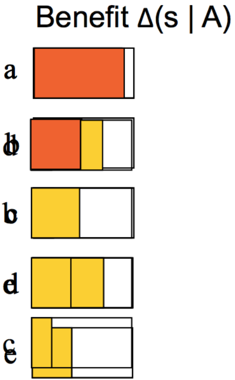
$$\Delta(s \mid A_i) \geq \Delta(s \mid A_{i+1})$$

Marginal benefits can never increase!

"Lazy" greedy algorithm [Minoux'78]

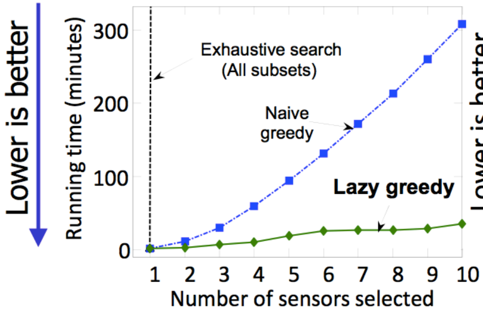
Lazy greedy algorithm:

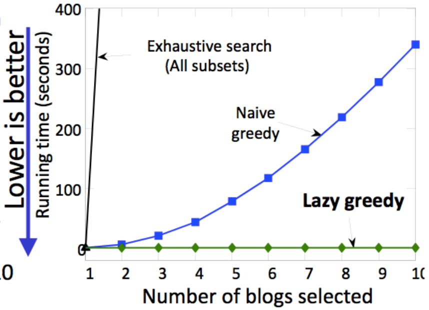
- First iteration as usual
- Keep an ordered list of marginal benefits $\triangle_i$ from previous iteration
- Re-evaluate $\triangle_i$ only for top element
- If $\triangle_i$ stays on top, use it, otherwise re-sort



Note: Very easy to compute online bounds, lazy evaluations, etc. [Leskovec,Krause et al.'07]

Empirical improvements [Leskovec, Krause et al'06]





Sensor placement

Blog selection

30x speedup

700x speedup

Stochastic-greedy algorithm [[Mirzasoleimanet al'14]

In round i+1,

- have picked $A_i = s_1, \cdots, s_i$.
- R is a random subset obtained by sampling s random elements from $X \setminus A$.
- pick $s_{i+1} = \arg\max_{s \in R} F(A_i \cup \{s\}) - F(A_i)$.

The algorithm at each step selects a random subset R of size $s = \frac{n}{k} \log \frac{1}{\epsilon}$, choosing the element from R that provides the maximum marginal gain to the current solution A.
It achieves a $(1 - \frac{1}{e} - \epsilon)$ approximation guarantee with $O(n \log \frac{1}{\epsilon})$ function evaluations, where ϵ is an acceptable error bound for the algorithm.

Outline

Optimizing Submodular Functions

- As our examples suggest, optimization problems involving submodular functions are very common
- These can be classified on two axes: constrained/unconstrained and maximization/minimization

	Maximization	Minimization
Unconstrained	NP-hard $\frac{1}{2}$ approximation	Polynomial time via convex opt
Constrained	Usually NP-hard $1 - 1/e$ (mono, matroid) O(1) (“nice” constraints)	Usually NP-hard to apx. Few easy special cases

Representation

In order to generalize all our examples, algorithmic results are often posed in the value oracle model. Namely, we only assume we have access to a subroutine evaluating $F(S)$.

Problem Definition

Given a submodular function $f : 2^X \to \mathbb{R}$ on a finite ground set X,

$$\begin{array}{ll} \min & F(S) \\ \text{s.t.} & S \subseteq X \end{array}$$

- We denote $n = |X|$
- We assume $F(S)$ is a rational number with at most b bits
- Representation: in order to generalize all our examples, algorithmic results are often posed in the value oracle model. Namely, we only assume we have access to a subroutine evaluating $F(S)$ in constant time.

Goal

An algorithm which runs in time polynomial in n and b.

Some more notations

- $E = \{1, 2, \ldots, n\}$
- $\mathbb{R}^E = \{x = (x_j \in \mathbb{R} : j \in E)\}$
- $\mathbb{R}^E_+ = \{x = (x_j \in \mathbb{R} : j \in E) : x \geq 0\}$
- Any vector $x \in \mathbb{R}^E$ can be treated as a normalized modular function, and vice verse. That is

$$x(A) = \sum_{a \in A} x_a.$$

  Note that x is said to be normalized since $x(\emptyset) = 0$.
- Given $A \subseteq E$, define the vector $1_A \in \mathbb{R}^E_+$ to be

$$1_A(j) = \begin{cases} 1 & \text{if } j \in A \\ 0 & \text{if } j \notin A \end{cases}$$

- given modular function $x \in \mathbb{R}^E$, we can write $x(A)$ in a variety of ways, i.e., $x(A) = x \cdot 1_A = \sum_{i \in A} x_i$

Continuous Extensions of a Set Function

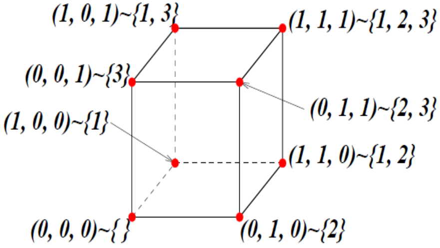
- A set function F on $X = \{1, \ldots, n\}$ can be thought of as a map from the vertices $\{0, 1\}^n$ of the n-dimensional hypercube to the real numbers.

Extension of a Set Function

Given a set function $F : \{0, 1\}^n \to \mathbb{R}$, an extension of F to the hypercube $[0, 1]^n$ is a function $g : [0, 1]^n \to \mathbb{R}$ satisfying $g(x) = F(x)$ for every $x \in \{0, 1\}^n$.

$$\min_{w \in \{0,1\}^n} \quad F(w)$$

with $\forall A \subseteq X$, $F(1_A) = F(A)$



Choquet integral - Lovász extension

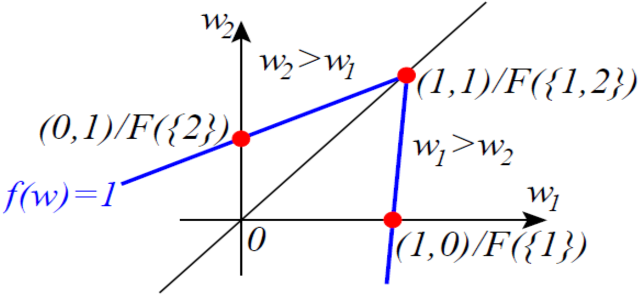
- Subsets may be identified with elements of $\{0,1\}^n$
- Given any set-function F and w such that $w_{j_1} \geq \ldots \geq w_{j_n}$, define

$$
\begin{aligned}
f(w) &= \sum_{k=1}^{n} w_{j_k}[F(\{j_1,\ldots,j_k\}) - F(\{j_1,\ldots,j_{k-1}\})\\
&= \sum_{k=1}^{n-1} (w_{j_k} - w_{j_{k+1}}) F(\{j_1,\ldots,j_k\}) + w_{j_n} F(\{j_1,\ldots,j_n\})
\end{aligned}
$$

- If $w = 1_A$, $f(w) = F(A) \Longrightarrow$ extension from $\{0,1\}^n$ to $\mathbb{R}^n$

Choquet integral - Lovász extension, example: $p = 2$

- If $w_1 \geq w_2, f(w) = F(\{1\})w_1 + [F(\{1,2\}) - F(\{1\})]w_2$
- If $w_1 \leq w_2, f(w) = F(\{2\})w_2 + [F(\{1,2\}) - F(\{2\})]w_1$



level set $\{w \in \mathbb{R}^2, f(w) = 1\}$ is displayed in blue

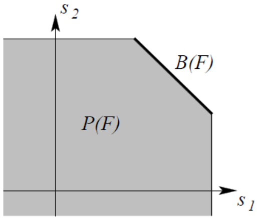
- Compact formulation: $f(w) = [F(\{1,2\}) - F(\{1\}) - F(\{2\})] \min(w_1, w_2) + F(\{1\})w_1 + F(\{2\})w_2$

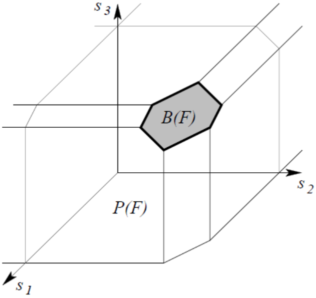
Links with convexity

Theorem (Lovász, 1982)

F is submodular if and only if f is convex

- Proof requires: Submodular and base polyhedra
- Submodular polyhedron: $P(F) = \{s \in \mathbb{R}^n, \forall A \subseteq V, s(A) \leq F(A)\}$
- Base polyhedron: $B(F) = P(F) \cap \{s(V) = F(V)\}$





Submodular and base polyhedra

- $P(F)$ has non-empty interior
- Many facets (up to 2^n), many extreme points (up to $n!$)

Fundamental property (Edmonds, 1970): If F is submodular, maximizing linear functions may be done by a "greedy algorithm"

- Let $w \in \mathbb{R}_+^n$ such that $w_{j_1} \geq \ldots \geq w_{j_n}$
- Let $s_{j_k} = F(\{j_1, \ldots, j_k\}) - F(\{j_1, \ldots, j_{k-1}\})$ for $k \in \{1, \ldots, n\}$
- Then

$$f(w) = \max_{s \in P(F)} w^\top s = \max_{s \in B(F)} w^\top s$$

- Both problems attained at s defined as above.
- proofs: pages 41-44 in `http://bicmr.pku.edu.cn/~wenzw/bigdata/submodular_fbach_mlss2012.pdf`

Links with convexity

Theorem (Lovász, 1982)

F is submodular if and only if f is convex

- If F is submodular, f is the maximum of linear functions. Then f is convex
- If f is convex, let $A, B \subseteq V$
  - $1_{A\cup B} + 1_{A\cap B} = 1_A + 1_B$ has components equal to 0 (on $V\backslash(A\cup B)$), 2 (on $A\cap B$) and 1 (on $A\Delta B = (A\backslash B)\cup(B\backslash A)$)
  - Thus $f(1_{A\cup B} + 1_{A\cap B}) = F(A\cup B) + F(A\cap B)$. Proof by writing out $f(1_{A\cup B} + 1_{A\cap B})$ and the definition of $f(w)$.
  - By homogeneity and convexity, $f(1_A + 1_B) \leq f(1_A) + f(1_B)$, which is equal to $F(A) + F(B)$, and thus F is submodular.

Links with convexity

Theorem (Lovász, 1982)

If F is submodular, then

$$\min_{A\subseteq V} F(A) = \min_{w\in\{0,1\}^n} f(w) = \min_{w\in[0,1]^n} f(w)$$

- Since f is an extension of F,

$$\min_{A\subseteq V} F(A) = \min_{w\in\{0,1\}^n} f(w) \geq \min_{w\in[0,1]^n} f(w)$$

- Any $w \in [0,1]^n$ can be decomposed as $w = \sum_{i=1}^m \lambda_i 1_{B_i}$, where $B_1 \subseteq \ldots \subseteq B_m = V$, where $\lambda \geq 0$ and $\lambda(V) \leq 1$:
  - Since $\min_{A\subseteq V} F(A) \leq 0$ ($F(\emptyset) = 0$),

$$f(w) = \sum_{i=1}^m \lambda_i F(B_i) \geq \sum_{i=1}^m \lambda_i \min_{A\subseteq V} F(A) \geq \min_{A\subseteq V} F(A)$$

  - Thus $\min_{w\in[0,1]^n} f(w) \geq \min_{A\subseteq V} F(A)$.

Links with convexity

- Any $w \in [0,1]^n$, sort $w_{j_1} \geq \ldots \geq w_{j_n}$. Find λ such that

$$\sum_{k=1}^{n} \lambda_{j_k} = w_{j_1}, \sum_{k=2}^{n} \lambda_{j_k} = w_{j_2}, \ldots, \lambda_{j_n} = w_{j_n},$$
$$B_1 = \{j_1\}, B_2 = \{j_1, j_2\}, \ldots, B_n = \{j_1, j_2, \ldots, j_n\}$$

Then we have $w = \sum_{i=1}^{n} \lambda_i 1_{B_i}$, where $B_1 \subseteq \ldots \subseteq B_n = V$, where $\lambda \geq 0$ and $\lambda(V) = \sum_{i \in V} \lambda_i \leq 1$.

Submodular function minimization

- Let $F : 2^V \to \mathbb{R}$ be a submodular function (such that $F(\emptyset) = 0$)
- **convex duality**:

$$\begin{aligned}
\min_{A \subseteq V} F(A) &= \min_{w \in [0,1]^n} f(w) \\
&= \min_{w \in [0,1]^n} \max_{s \in B(F)} w^\top s \\
&= \max_{s \in B(F)} \min_{w \in [0,1]^n} w^\top s = \max_{s \in B(F)} s_-(V)
\end{aligned}$$

Submodular function minimization

Convex optimization

If F is submodular, then

$$\min_{A \subseteq V} F(A) = \min_{w \in \{0,1\}^n} f(w) = \min_{w \in [0,1]^n} f(w)$$

Using projected subgradient descent to minimize f on $[0,1]^n$

- Iteration: $w_t = \Pi_{[0,1]^n}(w_{t-1} - \frac{C}{\sqrt{t}} s_t)$, where $s_t \in \partial f(w_{t-1})$
- $f(w) = \max_{s \in B(F)} w^\top s$
- Standard convergence results from convex optimization

$$f(w_t) - \min_{w \in [0,1]^n} f(w) \leq \frac{C}{\sqrt{t}}$$

Summary

- Many problems of recommending sets can be cast as submodular maximization
- Greedy algorithm gives best set of size k
- Can use lazy evaluations to speed up
- Approximate submodular maximization possible under a variety of constraints:
    - Matroid
    - Knapsack
    - Multiple matroid and knapsack constraints
    - Path constraints (Submodular orienteering)
    - Connectedness (Submodular Steiner)
    - Robustness (minimax)