

# Lecture: Introduction to LP, SDP and SOCP

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# Linear Programming (LP)

## Primal

$$\min \quad c_1x_1 + \dots + c_nx_n$$

$$\text{s.t.} \quad a_{11}x_1 + \dots + a_{1n}x_n = b_1$$

...

$$a_{m1}x_1 + \dots + a_{mn}x_n = b_m$$

$$x_i \geq 0$$

## Dual

$$\max \quad b_1y_1 + \dots + b_my_m$$

$$\text{s.t.} \quad a_{11}y_1 + \dots + a_{m1}y_m \leq c_1$$

...

$$a_{1n}y_1 + \dots + a_{mn}y_m \leq c_n$$

# Linear Programming (LP)

more succinctly

**Primal (P)**

$$\begin{array}{ll} \min & c^\top x \\ \text{s.t.} & Ax = b \\ & x \geq 0 \end{array}$$

**Dual (D)**

$$\begin{array}{ll} \max & b^\top y \\ \text{s.t.} & A^\top y + s = c \\ & s \geq 0 \end{array}$$

# Weak duality

Suppose

- $x$  is feasible to (P)
- $(y, s)$  is feasible to (D)

Then

$$\begin{aligned} 0 &\leq x^\top s \quad \text{because } x_i s_i \geq 0 \\ &= x^\top (c - A^\top y) \\ &= c^\top x - (Ax)^\top y \\ &= c^\top x - b^\top y \\ &= \text{duality gap} \end{aligned}$$

# Key Properties of LP

- Strong duality: If both Primal and Dual are feasible then at the optimum

$$c^T x = b^T y \iff x^T s = 0$$

- complementary slackness: This implies

$$\begin{aligned}x^T s &= x_1 s_1 + \dots + x_n s_n = 0 \quad \text{and therefore} \\x_i s_i &= 0\end{aligned}$$

# complementarity

- Putting together primal feasibility, dual feasibility and complementarity together we get a square system of equations

$$\begin{aligned}Ax &= b, & x &\geq 0 \\A^\top y + s &= c, & s &\geq 0 \\x_i s_i &= 0 & \text{for } i = 1, \dots, n\end{aligned}$$

- At least in principle this system determines the primal and dual optimal values

# Algebraic characterization

- We can define  $x \circ s = (x_1s_1, \dots, x_ns_n)^\top$  and

$$\mathbf{L}_x : y \rightarrow (x_1y_1, \dots, x_ny_n)^\top \text{ i.e. } \mathbf{L}_x = \text{Diag}(x)$$

- We can write complementary slackness conditions as

$$x \circ s = \mathbf{L}_x s = \mathbf{L}_x \mathbf{L}_s \mathbf{1} = 0$$

- $\mathbf{1}$ , the vector of all ones, is the identity element:

$$x \circ \mathbf{1} = x$$

# Semidefinite Programming (SDP)

- $X \succeq Y$  means that the the symmetric matrix  $X - Y$  is positive semidefinite
- $X$  is positive semidefinite

$$a^\top X a \geq 0 \text{ for all vector } a \iff X = B^\top B \iff$$

all eigenvalues of  $X$  is nonnegative



# Semidefinite Programming (SDP)

- $\langle X, Y \rangle = \sum_{ij} X_{ij}Y_{ij} = \text{Tr}(XY)$

**Primal (P)**

$$\begin{aligned} \min \quad & \langle C_1, X_1 \rangle + \dots + \langle C_n, X_n \rangle \\ \text{s.t.} \quad & \langle A_{11}, X_1 \rangle + \dots + \langle A_{1n}, X_n \rangle = b_1 \\ & \dots \\ & \langle A_{m1}, X_1 \rangle + \dots + \langle A_{mn}, X_n \rangle = b_m \\ & X_i \succeq 0 \end{aligned}$$

**Dual (D)**

$$\begin{aligned} \max \quad & b_1 y_1 + \dots + b_m y_m \\ \text{s.t.} \quad & A_{11} y_1 + \dots + A_{m1} y_m + S_1 = c_1 \\ & \dots \\ & A_{1n} y_1 + \dots + A_{mn} y_m + S_n = c_n \\ & S_i \succeq 0 \end{aligned}$$

# Simplified SDP

- For simplicity we deal with single variable SDP:

**Primal (P)**

$$\begin{aligned} \min \quad & \langle C, X \rangle \\ \text{s.t.} \quad & \langle A_1, X \rangle = b_1 \\ & \dots \\ & \langle A_m, X \rangle = b_m \\ & X \succeq 0 \end{aligned}$$

**Dual (D)**

$$\begin{aligned} \max \quad & b^\top y \\ \text{s.t.} \quad & \sum_i y_i A_i + S = C \\ & S \succeq 0 \end{aligned}$$

- A single variable LP is trivial
- But a single matrix SDP is as general as a multiple matrix

# Weak duality in SDP

- Just as in LP

$$\langle X, S \rangle = \langle C, X \rangle - b^\top y$$

- Also if both  $X \succeq 0$  and  $S \succeq 0$  then

$$\langle X, S \rangle = \text{Tr}(XS^{1/2}S^{1/2}) = \text{Tr}(S^{1/2}XS^{1/2}) \geq 0$$

because  $S^{1/2}XS^{1/2} \succeq 0$

- Thus

$$\langle X, S \rangle = \langle C, X \rangle - b^\top y \geq 0$$

# Complementarity Slackness Theorem

- $X \succeq 0$  and  $S \succeq 0$  and  $\langle X, S \rangle = 0$  implies

$$XS = 0$$

- Proof:

$$\langle X, S \rangle = \text{Tr}(XS^{1/2}S^{1/2}) = \text{Tr}(S^{1/2}XS^{1/2})$$

Thus  $\text{Tr}(S^{1/2}XS^{1/2}) = 0$ . Since  $S^{1/2}XS^{1/2} \succeq 0$ , then

$$\begin{aligned} S^{1/2}XS^{1/2} = 0 &\implies S^{1/2}X^{1/2}X^{1/2}S^{1/2} = 0 \\ X^{1/2}S^{1/2} = 0 &\implies XS = 0 \end{aligned}$$

# Algebraic properties of SDP

- For reasons to become clear later it is better to write complementary slackness conditions as

$$\frac{XS + SX}{2} = 0$$

- It can be shown that if  $X \succeq 0$  and  $S \succeq 0$ , then  $XS = 0$  iff

$$XS + SX = 0$$

# Algebraic properties of SDP

- Definition:  $X \circ S = \frac{XS+SX}{2}$
- The binary operation  $\circ$  is commutative  $X \circ S = S \circ X$
- $\circ$  is not associative:  $X \circ (Y \circ Z) \neq (X \circ Y) \circ Z$  in general
- But  $X \circ (X \circ X) = (X \circ X) \circ X$ . Thus  $X^{\circ p} = X^p$  is well defined
- In general  $X \circ (X^2 \circ Y) = X^2 \circ (X \circ Y)$
- The identity matrix  $I$  is identity w.r.t  $\circ$
- Define the operator

$$L_X : Y \rightarrow X \circ Y, \text{ thus } X \circ S = L_X(S) = L_X(L_S(I))$$

# Constraint Qualifications

- Unlike LP we need some conditions for the optimal values of Primal and Dual SDP to coincide
- Here are two:
  - If there is primal-feasible  $X \succ 0$  (i.e.  $X$  is positive definite)
  - If there is dual-feasible  $S \succ 0$
- When strong duality holds  $\langle X, S \rangle = 0$

# KKT Condition

- Thus just like LP The system of equations

$$\begin{aligned} \langle A_i, X \rangle &= b_i, & X \succeq 0, & \quad \text{for } i = 1, \dots, m \\ \sum_i y_i A_i + S &= C, & S \succeq 0, & \\ X \circ S &= 0 & & \end{aligned}$$

Gives us a square system



# Second Order Cone Programming (SOCP)

- For simplicity we deal with single variable SOCP:

**Primal (P)**

$$\min \quad c^\top x$$

$$\text{s.t.} \quad Ax = b$$

$$x \succeq_{\mathcal{Q}} 0$$

**Dual (D)**

$$\max \quad b^\top y$$

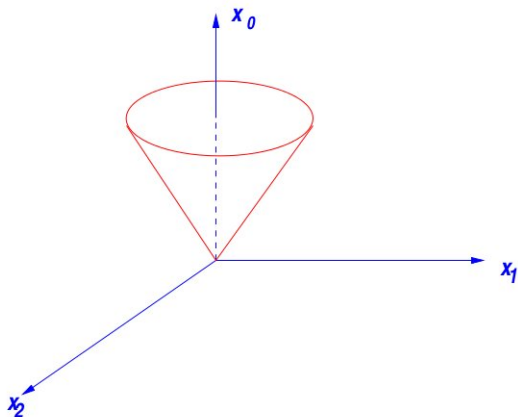
$$\text{s.t.} \quad A^\top y + s = c$$

$$s \succeq_{\mathcal{Q}} 0$$

- the vectors  $x, s, c$  are indexed from zero
- If  $z = (z_0, z_1, \dots, z_n)^\top$  and  $\bar{z} = (z_1, \dots, z_n)^\top$

$$z_{\mathcal{Q}} \geq 0 \iff z_0 \geq \|\bar{z}\|$$

# Illustration of SOC



$$\mathcal{Q} = \{z \mid z_0 \geq \|\bar{z}\|\}$$

# Weak Duality in SOCP

- The single block SOCP is not as trivial as LP but it still can be solved analytically
- weak duality: Again as in LP and SDP

$$x^\top s = c^\top x - b^\top y = \text{duality gap}$$

If  $x, s \succeq_{\mathcal{Q}} 0$ , then

$$\begin{aligned}x^\top s &= x_0 s_0 + \bar{x}^\top \bar{s} \geq \\ &\geq \|\bar{x}\| \cdot \|\bar{s}\| + \bar{x}^\top \bar{s} \quad \text{since } x, s \succeq_{\mathcal{Q}} 0 \\ &\geq |\bar{x}^\top \bar{s}| + \bar{x}^\top \bar{s} \quad \text{Cauchy-Schwartz inequality} \\ &\geq 0\end{aligned}$$

# Complementary Slackness for SOCP

- Given  $x \succeq_{\mathcal{Q}} 0$ ,  $s \succeq_{\mathcal{Q}} 0$  and  $x^\top s = 0$ . Assume  $x_0 > 0$  and  $s_0 > 0$
- We have

$$(*) \quad x_0^2 \geq \sum_{i=1}^n x_i^2$$

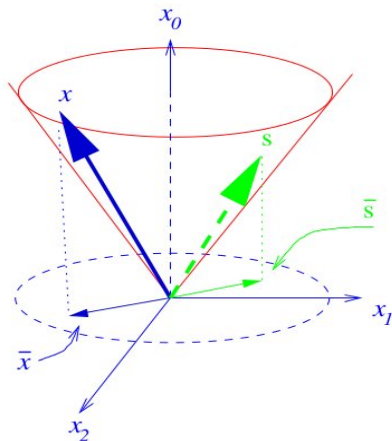
$$(**) \quad s_0^2 \geq \sum_{i=1}^n s_i^2 \iff x_0^2 \geq \sum_{i=1}^n \frac{s_i^2 x_0^2}{s_0^2}$$

$$(***) \quad x^\top s = 0 \iff -x_0 s_0 = \sum_i x_i s_i \iff -2x_0^2 = \sum_{i=1}^n \frac{2x_i s_i x_0}{s_0}$$

- Adding (\*), (\*\*), (\*\*\*), we get  $0 \geq \sum_{i=1}^n \left( x_i + \frac{s_i x_0}{s_0} \right)^2$
- This implies

$$x_i s_0 + x_0 s_i = 0, \text{ for } i = 1, \dots, n$$

# Illustration of SOC



When  $x \succeq_Q 0$ ,  $s \succeq_Q 0$  are orthogonal both must be on the boundary in such a way that their projection on the  $x_1, \dots, x_n$  plane is collinear

# Strong Duality

- at the optimum

$$c^\top x = b^\top y \iff x^\top s = 0$$

- Like SDP constraint qualifications are required
- If there is primal-feasible  $x \succ_Q 0$
- If there is dual-feasible  $s \succ_Q 0$

# Complementary Slackness for SOCP

- Thus again we have a square system

$$\begin{aligned}Ax &= b, & x &\succeq_{\mathcal{Q}} 0, \\A^{\top}y + s &= c, & s &\succeq_{\mathcal{Q}} 0, \\x^{\top}s &= 0, \\x_0s_i + s_0x_i &= 0\end{aligned}$$

# Algebraic properties of SOCP

- Let us define a binary operation for vectors  $x$  and  $s$  both indexed from zero

$$\begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_n \end{pmatrix} \circ \begin{pmatrix} s_0 \\ s_1 \\ \vdots \\ s_n \end{pmatrix} = \begin{pmatrix} x^\top s \\ x_0 s_1 + s_0 x_1 \\ \vdots \\ x_0 s_n + s_0 x_n \end{pmatrix}$$



# Algebraic properties of SOCP

- The binary operation  $\circ$  is commutative  $x \circ s = s \circ x$
- $\circ$  is not associative:  $x \circ (y \circ z) \neq (x \circ y) \circ z$  in general
- But  $x \circ (x \circ x) = (x \circ x) \circ x$ . Thus  $x^{\circ p} = x^p$  is well defined
- In general  $x \circ (x^2 \circ y) = x^2 \circ (x \circ y)$
- The identity matrix  $I$  is identity w.r.t  $\circ$
- $e = (1, 0, \dots, 0)^\top$  is the identity:  $x \circ e = x$

# Algebraic properties of SOCP

- Define the operator

$$\mathbb{L}_x : y \rightarrow x \circ y$$

$$\mathbb{L}_x = \text{Arw}(x) = \begin{pmatrix} x_0 & \bar{x}^\top \\ \bar{x} & x_0 I \end{pmatrix}$$

$$x \circ s = \text{Arw}(x)s = \text{Arw}(x)\text{Arw}(s)e$$

# Summary

## Properties

	LP	SDP	SOCP
binary operator	$x \circ s = (x_i s_i)$	$X \circ S = \frac{XS+SX}{2}$	$x \circ s = \begin{pmatrix} x^\top s \\ x_0 \bar{s} + s_0 \bar{x} \end{pmatrix}$
identity	1	I	$e = (1, 0, \dots, 0)^\top$
associative	yes	no	no
$L_X$	$y \rightarrow \text{Diag}(x)y$	$Y \rightarrow \frac{XY+YX}{2}$	$y \rightarrow \text{Arw}(x)y$
Primal feasibility	$Ax = b$	$\langle A_i, X \rangle = b_i$	$Ax = b$
dual feasibility	$A^\top y + s = c$	$\sum_i y_i A_i + S = C$	$A^\top y + s = c$
complementarity	$L_x L_s 1 = 0$	$L_X(L_S(I)) = 0$	$L_x L_s e = 0$

A set  $K \subseteq \mathbb{R}^n$  is a proper cone if

- It is a cone:  $x \in K \implies \alpha x \in K$  for all  $\alpha \geq 0$
- It is convex:  $x, y \in K \implies \alpha x + (1 - \alpha)y \in K$  for  $\alpha \in [0, 1]$
- It is pointed:  $K \cap (-K) = \{0\}$
- It is closed
- It has non-empty interior in  $\mathbb{R}^n$
- dual cone:

$$K^* = \{x \mid \text{for all } z \in K, \langle x, z \rangle \geq 0\}$$

Conic-LP is defined as the following optimization problem:

**Primal (P)**

$$\begin{aligned} \min \quad & c^\top x \\ \text{s.t.} \quad & Ax = b \\ & x \in K \end{aligned}$$

**Dual (D)**

$$\begin{aligned} \max \quad & b^\top y \\ \text{s.t.} \quad & A^\top y + s = c \\ & s \in K^* \end{aligned}$$

- For LP  $K$  is the nonnegative orthant
- For SDP  $K$  is the cone of positive semidefinite matrices
- For SOCP  $K$  is the circular or Lorentz cone
- In all three cases the cones are self-dual  $K = K^*$