Lecture: Examples of LP, SOCP and SDP

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Problems with absolute values

$$\begin{aligned} \min \quad & \sum_i c_i |x_i|, \qquad \text{assume } c \geq 0 \\ \text{s.t.} \quad & Ax \geq b \end{aligned}$$

- Reformulation 1:

$$\begin{aligned} \min \quad & \sum_i c_i z_i \\ \text{s.t.} \quad & Ax \geq b \\ & |x_i| \leq z_i \end{aligned} \iff \begin{aligned} \min \quad & \sum_i c_i z_i \\ \text{s.t.} \quad & Ax \geq b \\ & -z_i \leq x_i \leq z_i \end{aligned}$$

- Reformulation 2: $x_i = x_i^+ - x_i^-$, $x_i^+, x_i^- \geq 0$. Then $|x_i| = x_i^+ + x_i^-$

$$\begin{aligned} \min \quad & \sum_i c_i (x_i^+ + x_i^-) \\ \text{s.t.} \quad & Ax^+ - Ax^- \geq b, x^+, x^- \geq 0 \end{aligned}$$

Problems with absolute values

- data fitting:

$$\min_x \quad \|Ax - b\|_\infty$$

$$\min_x \quad \|Ax - b\|_1$$

- Compressive sensing

$$\begin{aligned}
&\min \quad \|x\|_1, \text{ s.t. } \quad Ax = b \quad (LP) \\
&\min \quad \mu\|x\|_1 + \tfrac{1}{2}\|Ax + b\|^2 \quad (QP, SOCP) \\
&\min \quad \|Ax - b\|, \text{ s.t. } \quad \|x\|_1 \le 1
\end{aligned}$$

Quadratic Programming (QP)

$$\begin{aligned} \min \quad & q(x) = x^\top Q x + a^\top x + \beta \qquad \text{assume} \quad Q \succ 0, Q = Q^\top \\ \text{s.t.} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

- $q(x) = \|\bar{u}\|^2 + \beta - \frac{1}{4} a^\top Q^{-1} a$, where $\bar{u} = Q^{1/2} x + \frac{1}{2} Q^{-1/2} a$.
- equivalent SOCP

$$\begin{aligned} \min \quad & u_0 \\ \text{s.t.} \quad & \bar{u} = Q^{1/2} x + \frac{1}{2} Q^{-1/2} a \\ & Ax = b \\ & x \geq 0, \quad (u_0, \bar{u}) \succeq_{\mathcal{Q}} 0 \end{aligned}$$

Quadratic constraints

$$q(x) = x^\top B^\top B x + a^\top x + \beta \leq 0$$

is equivalent to

$$(u_0, \bar{u}) \succeq_{\mathcal{Q}} 0,$$

where

$$\bar{u} = \begin{pmatrix} Bx \\ \frac{a^\top x + \beta + 1}{2} \end{pmatrix} \quad \text{and } u_0 = \frac{1 - a^\top x - \beta}{2}$$

Norm minimization problems

Let $\bar{v}_i = A_i x + b_i \in \mathbb{R}^{n_i}$.

- $\min_x \quad \sum_i \|\bar{v}_i\|$ is equivalent to

$$\begin{aligned} \min \quad & \sum_i v_{i0} \\ \text{s.t.} \quad & \bar{v}_i = A_i x + b_i \\ & (v_{i0}, \bar{v}_i) \succeq_{\mathcal{Q}} 0 \end{aligned}$$

- $\min_x \quad \max_{1 \le i \le r} \|\bar{v}_i\|$ is equivalent to

$$\begin{aligned} \min \quad & t \\ \text{s.t.} \quad & \bar{v}_i = A_i x + b_i \\ & (t, \bar{v}_i) \succeq_{\mathcal{Q}} 0 \end{aligned}$$

Norm minimization problems

Let $\bar{v}_i = A_i x + b_i \in \mathbb{R}^{n_i}$.

- $\|\bar{v}_{[1]}\|, \ldots, \|\bar{v}_{[r]}\|$ are the norms $\|\bar{v}_1\|, \ldots, \|\bar{v}_r\|$ sorted in nonincreasing order
- $\min_x \quad \sum_{i=1}^{k} \|\bar{v}_{[i]}\|$ is equivalent to

$$
\begin{aligned}
\min \quad & \sum_{i=1}^{m} u_i + kt \\
\text{s.t.} \quad & \bar{v}_i = A_i x + b_i, \quad i = 1, \ldots, m \\
& \|\bar{v}_i\| \le u_i + t, \quad i = 1, \ldots, m \\
& u_i \ge 0, \quad i = 1, \ldots, m
\end{aligned}
$$

Rotated Quadratic Cone

- rotated cone $w^\top w \le xy$, where $x, y \ge 0$, is equivalent to

$$\left\| \begin{pmatrix} 2w \\ x - y \end{pmatrix} \right\| \le x + y$$

- Minimize the harmonic mean of positive affine functions

$$\min \quad \sum_i 1/(a_i^\top x + \beta_i), \text{ s.t.} \quad a_i^\top x + \beta_i > 0$$

is equivalent to

$$\begin{aligned} \min \quad & \sum_i u_i \\ \text{s.t.} \quad & \bar{v}_i = a_i^\top x + \beta_i \\ & 1 \le u_i v_i \\ & u_i \ge 0 \end{aligned}$$

- Logarithmic Tchebychev approximation

$$\min_{x} \quad \max_{1 \le i \le r} \quad |\ln(a_i^\top x) - \ln b_i|$$

Since $|\ln(a_i^\top x) - \ln b_i| = \ln \max(a_i^\top x/b_i, b_i/a_i^\top x)$, the problem is equivalent to

$$\begin{aligned} \min \quad & t \\ \text{s.t.} \quad & 1 \le (a_i^\top x/b_i)t \\ & a_i^\top x/b_i \le t \\ & t \ge 0 \end{aligned}$$

- Inequalities involving geometric means

$$\left(\prod_{i=1}^{n} (a_i^\top x + b_i) \right)^{1/n} \ge t$$

- n=4

$$\max \prod_{i=1}^{4}(a_i^\top x - b_i) \iff \begin{array}{ll} \max & w_3 \\ \text{s.t.} & a_i^\top x - b_i \geq 0 \\ & (a_1^\top x - b_1)(a_2^\top x - b_2) \geq w_1^2 \\ & (a_3^\top x - b_3)(a_4^\top x - b_4) \geq w_2^2 \\ & w_1 w_2 \geq w_3^2 \\ & w_i \geq 0 \end{array}$$

- This can be extended to products of rational powers of affine functions

Robust linear programming

the parameters in LP are often uncertain

$$\begin{aligned} \min \quad & c^\top x \\ \text{s.t.} \quad & a_i^\top x \le b_i \end{aligned}$$

There can be uncertainty in c, a_i, b.

two common approaches to handling uncertainty (in a_i, for simplicity)

- deterministic model: constraints must hold for all $a_i \in \mathcal{E}_i$

$$\begin{aligned} \min \quad & c^\top x \\ \text{s.t.} \quad & a_i^\top x \le b_i, \ \text{for all } a_i \in \mathcal{E}_i \end{aligned}$$

- stochastic model: a_i is random variable; constraints must hold with probability η

$$\begin{aligned} \min \quad & c^\top x \\ \text{s.t.} \quad & \text{prob}(a_i^\top x \le b_i) \ge \eta \end{aligned}$$

deterministic approach via SOCP

- Choose an ellipsoid as $\mathcal{E}_i$:

$$\mathcal{E}_i = \{\bar{a}_i + P_i u \mid \|u\|_2 \leq 1\}, \quad \bar{a}_i \in \mathbb{R}^n, P_i \in \mathbb{R}^{n\times n}$$

- Robust LP

$$\begin{aligned} \min \quad & c^\top x \\ \text{s.t.} \quad & a_i^\top x \leq b_i, \text{ for all } a_i \in \mathcal{E}_i \end{aligned}$$

is equivalent to the SOCP

$$\begin{aligned} \min \quad & c^\top x \\ \text{s.t.} \quad & \bar{a}_i^\top x + \|P_i^\top x\|_2 \leq b_i \end{aligned}$$

since

$$\sup_{\|u\|_2 \leq 1} (\bar{a}_i + P_i u)^\top x = \bar{a}_i^\top x + \|P_i^\top x\|_2$$

stochastic approach via SOCP

- a_i is Gaussian with mean $\bar{a}_i$, covariance Σ_i ($a_i \sim \mathcal{N}(\bar{a}_i, \Sigma_i)$
- $a_i^\top x$ is Gaussian r.v. with mean $\bar{a}_i^\top x$, variance $x^\top \Sigma_i x$; hence

$$\text{prob}(a_i^\top x \le b_i) = \Phi\left(\frac{b_i - \bar{a}_i^\top x}{\|\Sigma^{1/2} x\|_2}\right)$$

  where $\Phi(x) = (1/\sqrt{2\pi}) \int_{-\infty}^{x} e^{-t^2/2} dt$ is CDF of $\mathcal{N}(0, 1)$
- robust LP

$$\begin{aligned} \min \quad & c^\top x \\ \text{s.t.} \quad & \text{prob}(a_i^\top x \le b_i) \ge \eta \end{aligned}$$

  is equivalent to the SOCP

$$\begin{aligned} \min \quad & c^\top x \\ \text{s.t.} \quad & \bar{a}_i^\top x + \Phi^{-1}(\eta) \|\Sigma^{1/2} x\|_2 \le b_i \end{aligned}$$

SDP Standard Form

- $\mathcal{S}^n = \{X \in \mathbb{R}^{n\times n} \mid X^\top = X\}$, $\mathcal{S}^n_+ = \{X \in \mathcal{S}^n \mid X \succeq 0\}$, $\mathcal{S}^n_{++} = \{X \in \mathcal{S}^n \mid X \succ 0\}$
- Define linear operator $\mathcal{A} : \mathcal{S}^n \to \mathbb{R}^m$:

$$\mathcal{A}(X) = \left(\langle A_1, X\rangle, \ldots, \langle A_m, X\rangle\right)^\top, \quad X \in \mathcal{S}^n.$$

Since $\mathcal{A}(X)^\top y = \sum_{i=1}^m y_i \langle A_i, X\rangle = \langle \sum_{i=1}^m y_i A_i, X\rangle$, the adjoint of $\mathcal{A}$:

$$\mathcal{A}^*(y) = \sum_{i=1}^m y_i A_i$$

- The SDP standard form:

$$\textbf{(P)} \quad \begin{aligned} \min \quad & \langle C, X\rangle \\ \text{s.t.} \quad & \mathcal{A}(X) = b \\ & X \succeq 0 \end{aligned} \qquad \textbf{(D)} \quad \begin{aligned} \max \quad & b^\top y \\ \text{s.t.} \quad & \mathcal{A}^*(y) + S = C \\ & S \succeq 0 \end{aligned}$$

Facts on matrix calcuation

- If $A, B \in \mathbb{R}^{m\times n}$, then $\mathrm{Tr}(AB^\top) = \mathrm{Tr}(B^\top A)$
- If $U, V \in \mathcal{S}^n$ and Q is orthogonal, then $\langle U, V\rangle = \langle Q^\top UQ, Q^\top UQ\rangle$
- If $X \in \mathcal{S}^n$, then $U = Q^\top \Lambda Q$, where $Q^\top Q = I$ and Λ is diagonal.
- Matrix norms: $\|X\|_F = \|\lambda(X)\|_2$, $\|X\|_2 = \|\lambda(X)\|_\infty$, $\lambda(X) = \mathrm{diag}(\Lambda)$
- $X \succeq 0 \iff v^\top Xv \geq$ for all $v \in \mathbb{R}^n \iff \lambda(X) \geq 0 \iff X = B^\top B$
- The dual cone of $\mathcal{S}^n_+$ is $\mathcal{S}^n_+$
- If $X \succeq 0$, then $X_{ii} \geq 0$. If $X_{ii} = 0$, then $X_{ik} = X_{ki} = 0$ for all k.
- If $X \succeq 0$, then $PXP^\top \succeq 0$ for any P of approriate dimensions
- If $X = \begin{pmatrix} X_{11} & X_{12} \\ X_{12}^\top & X_{22} \end{pmatrix} \succeq 0$, then $X_{11} \succeq 0$.
- $X \succeq 0$ iff every principal submatrix is positive semidefinite (psd).

Facts on matrix calcuation

- Let $U = \begin{pmatrix} A & B \\ B^\top & C \end{pmatrix}$ with A and C symmetric and $A \succ 0$. Then

$$U \succeq 0 \text{ (or } \succ 0) \quad \Longleftrightarrow \quad C - B^\top A^{-1} B \succeq 0 \text{ (or } \succ 0).$$

  The matrix $C - B^\top A^{-1} B$ is the **Schur complement** of A in U:

$$\begin{pmatrix} A & B \\ B^\top & C \end{pmatrix} = \begin{pmatrix} I & 0 \\ B^\top A^{-1} & I \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & C - B^\top A^{-1} B \end{pmatrix} \begin{pmatrix} I & A^{-1} B \\ 0 & I \end{pmatrix}$$

- If $A \in \mathcal{S}^n$, then $x^\top A x = \langle A, xx^\top \rangle$
- If $A \succ 0$, then $\langle A, B \rangle > 0$ for every nonzero $B \succeq 0$ and $\{B \succeq 0 \mid \langle A, B \rangle \leq \beta\}$ is bounded for $\beta > 0$
- If $A, B \succeq 0$, then $\langle A, B \rangle = 0$ iff $AB = 0$
- $A, B \in \mathcal{S}^n$, then A and B are commute iff AB is symmetric, iff A and B can be simultaneously diagonalized

Eigenvalue optimization

- minimizing the largest eigenvalue $\lambda_{\max}(A_0 + \sum_i x_i A_i)$:

$$\min \quad \lambda_{\max}(A_0 + \sum_i x_i A_i)$$

can be expressed as an SDP

$$\begin{array}{ll} \min & z \\ \text{s.t.} & zI - \sum_i x_i A_i \succeq A_0 \end{array}$$

and its dual is

$$\begin{array}{ll} \max & \langle A_0, Y \rangle \\ \text{s.t.} & \langle A_i, Y \rangle = k \\ & \langle I, Y \rangle = 1 \\ & Y \succeq 0 \end{array}$$

- follows from

$$\lambda_{\max}(A) \le t \iff A \preceq tI$$

Eigenvalue optimization

- Let $A_i \in \mathbb{R}^{m \times n}$. Minimizing the 2-norm of $A(x) = A_0 + \sum_i x_i A_i$:

$$\min_x \quad \|A(x)\|_2$$

  can be expressed as an SDP

$$\begin{array}{ll} \min\limits_{x,t} & t \\ \text{s.t.} & \begin{pmatrix} tI & A(x) \\ A(x)^\top & tI \end{pmatrix} \succeq 0 \end{array}$$

- Constraint follows from

$$\begin{aligned} \|A\|_2 \leq t \quad & \Longleftrightarrow \quad A^\top A \preceq t^2 I, \quad t \geq 0 \\ & \Longleftrightarrow \quad \begin{pmatrix} tI & A(x) \\ A(x)^\top & tI \end{pmatrix} \succeq 0 \end{aligned}$$

Eigenvalue optimization

- Let $\Lambda_k(A)$ indicate sum of the k largest eigenvalues of A. Then minimizing $\Lambda_k(A_0 + \sum_i x_i A_i)$:

$$\min \quad \Lambda_k(A_0 + \sum_i x_i A_i)$$

can be expressed as an SDP

$$\begin{array}{ll} \min & kz + \mathrm{Tr}(X) \\ \text{s.t.} & zI + X - \sum_i x_i A_i \succeq A_0 \\ & X \succeq 0 \end{array}$$

since

$$\Lambda_k(A) \le t \iff t - kz - \mathrm{Tr}(X) \ge 0, zI + X \succeq A, X \succeq 0$$

The following problems can be expressed as SDP

- maximizing sum of the k smallest eigenvalues of $A_0 + \sum_i x_i A_i$
- minimizing sum of the k absolute-value-wise largest eigenvalues
- minimizing sum of the k largest singular values of $A_0 + \sum_i x_i A_i$

Quadratically Constrained Quadratic Programming

Consider QCQP

$$\begin{aligned} \min \quad & x^\top A_0 x + 2b_0^\top x + c_0 \qquad \text{assume } A_i \in \mathcal{S}^n \\ \text{s.t.} \quad & x^\top A_i x + 2b_i^\top x + c_i \le 0, \quad i = 1, \ldots, m \end{aligned}$$

- If $A_0 \succ 0$ and $A_i = B_i^\top B_i$, $i = 1, \ldots, m$, then it is a SOCP
- If $A_i \in \mathcal{S}^n$ but may be indefinite

$$x^\top A_i x + 2b_i^\top x + c_i \quad = \quad \left\langle A_i, xx^\top \right\rangle + 2b_i^\top x + c_i$$

- The original problem is equivalent to

$$\begin{aligned} \min \quad & \mathrm{Tr} A_0 X + 2b_0^\top x + c_0 \\ \text{s.t.} \quad & \mathrm{Tr} A_i X + 2b_i^\top x + c_i \le 0, \quad i = 1, \ldots, m \\ & X = xx^\top \end{aligned}$$

QCQP

- If $A_i \in \mathcal{S}^n$ but may be indefinite

$$x^\top A_i x + 2b_i^\top x + c_i \quad = \quad \left\langle \begin{pmatrix} A_i & b_i \\ b_i^\top & c_i \end{pmatrix}, \begin{pmatrix} X & x \\ x^\top & 1 \end{pmatrix} \right\rangle := \langle \bar{A}_i, \bar{X} \rangle$$

$\bar{X} \succeq 0$ is equivalent to $X \succeq xx^\top$

- The SDP relaxation is

$$\begin{aligned} \min \quad & \mathrm{Tr} A_0 X + 2b_0^\top x + c_0 \\ \text{s.t.} \quad & \mathrm{Tr} A_i X + 2b_i^\top x + c_i \le 0, \quad i = 1, \ldots, m \\ & X \succeq xx^\top \end{aligned}$$

- Maxcut: $\max\ x^\top W x$, s.t. $x_i^2 = 1$
- Phase retrieval: $|a_i^\top x| = b_i$, the value of $a_i^\top x$ is complex

Max cut

- For graph (V, E) and weights $w_{ij} = w_{ji} \geq 0$, the maxcut problem is

$$(Q) \quad \max_{x} \frac{1}{2} \sum_{i<j} w_{ij}(1 - x_i x_j), \quad \text{s.t.} \ x_i \in \{-1, 1\}$$

- Relaxation:

$$(P) \quad \max_{v_i \in \mathbb{R}^n} \frac{1}{2} \sum_{i<j} w_{ij}(1 - v_i^\top v_j), \quad \text{s.t.} \ \|v_i\|_2 = 1$$

- Equivalent SDP of (P):

$$(SDP) \quad \max_{X \in S^n} \frac{1}{2} \sum_{i<j} w_{ij}(1 - X_{ij}), \quad \text{s.t.} \ X_{ii} = 1, X \succeq 0$$

Max cut: rounding procedure

Goemans and Williamson's randomized approach

- Solve (SDP) to obtain an optimal solution X. Compute the decomposition $X = V^\top V$, where

$$V = [v_1, v_2, \ldots, v_n]$$

- Generate a vector r uniformly distributed on the unit sphere, i.e., $\|r\|_2 = 1$

- Set

$$x_i = \begin{cases} 1 & v_i^\top r \geq 0 \\ -1 & \text{otherwise} \end{cases}$$

Max cut: theoretical results

- Let W be the objective function value of x and $E(W)$ be the expected value. Then

$$E(W) = \frac{1}{\pi} \sum_{i<j} w_{ij} \arccos(v_i^\top v_j)$$

- Goemans and Williamson showed:

$$E(W) \geq \alpha \frac{1}{2} \sum_{i<j} w_{ij} (1 - v_i^\top v_j)$$

where

$$\alpha = \min_{0 \leq \theta \pi} \frac{2}{\pi} \frac{\theta}{1 - \cos\theta} > 0.878$$

- Let $Z^*_{(SDP)}$ and $Z^*_{(Q)}$ be the optimal values of (SDP) and (Q)

$$E(W) \geq \alpha Z^*_{(SDP)} \geq \alpha Z^*_{(Q)}$$

SDP-Representablity

What kind of problems can be expressed by SDP and SOCP?

- Definition: A set X $\subseteq R^n$ is SDP-representable (or SDP-Rep for short) if it can be expressed linearly as the feasible region of an SDP

$$\mathsf{X} = \Big\{x \mid \text{there exist } u \in \mathbb{R}^k \text{ such that for some} \\ A_i, B_j, C \in \mathbb{R}^{m\times m} : \sum_i x_i A_i + \sum_j u_j B_j + C \succeq 0 \Big\}$$

SDP-Representablity

- Definition: A function $f(x)$ is SDP-Rep if its epigraph

$$\text{epi}(f) = \{(x_0, x) \mid f(x) \leq x_0\}$$

  is SDP-representable

- If X is SDP-Rep, then $\min_{x \in X} c^\top x$ is an SDP
- If $f(x)$ is SDP-Rep, then $\min_x \quad f(x)$ is an SDP

A "calculus" of SDP-Rep sets and functions

SDP-Rep sets and functions remain so under finitely many applications of most convex-preserving operations.
If X and Y are SDP-Rep then so are

- Minkowski sum $X + Y$
- intersection $X \cap Y$
- Affine pre-image $A^{-1}(X)$ if A is affine
- Affine map $A(X)$ if A is affine
- Cartesian Product: $X \times Y = \{(x, y) \mid x \in X, y \in Y\}$

SDP-Rep Functions

If functions f_i, $i = 1, \ldots, m$ and g are SDP-Rep. Then the following are SDP-Rep

- nonnegative sum $\sum_i \alpha_i f_i$ for $\alpha_i \geq 0$
- maximum $\max_i f_i$
- composition: $g(f_1(x), \ldots, f_m(x))$ if $f_i : \mathbb{R}^n \to R$ and $g : \mathbb{R}^m \to \mathbb{R}$
- Legendre transform

$$f^*(y) = \max_x \quad y^\top x - f(x)$$

Positive Polynomials

- The set of nonnegative polynomials of a given degree forms a proper cone

$$\mathcal{P}_n = \{(p_0, \ldots, p_n) \mid p_0 + p_1 t + \ldots + p_n t^t > 0 \text{ for all } t \in I\}$$

  where I is any of $[a, b]$, $[a, \infty)$ or $(-\infty, \infty)$

- Important fact: The cone of positive polynomials is SDP-Rep
- To see this we need to introduce another problem

The Moment cone

The Moment space and Moment cone

- Let $(c_0, c_1, \ldots, c_n)^\top$ be such that there is a probability measure F where
$$c_i = \int_I t^i dF, \text{ for } i = 0, \ldots, n.$$
The set of such vectors is the Moment space

- The Moment cone

$$\mathcal{M}_n = \{ac \mid \text{There is a distribution } F : c_i = \int_I t^i dF \text{ and } a \geq 0\}$$

The Moment cone

The moment cone is also SDP-Rep:

- The discrete Hamburger moment problem:

$$I = \mathbb{R}, \quad c \in \mathcal{M}_{2n+1} \Longleftrightarrow$$

$$\begin{pmatrix} c_0 & c_1 & \dots & c_n \\ c_1 & c_2 & \dots & c_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ c_n & c_{n+1} & \dots & c_{2n} \end{pmatrix} \succeq 0$$

- This is the Hankel matrix

The Moment cone

- The discrete Stieltjes moment problem

$$I = [0, \infty), \quad c \in \mathcal{M}_m \iff$$

$$\begin{pmatrix} c_0 & c_1 & \dots & c_n \\ c_1 & c_2 & \dots & c_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ c_n & c_{n+1} & \dots & c_{2n} \end{pmatrix} \succeq 0, \text{ and } \begin{pmatrix} c_1 & c_2 & \dots & c_{n-1} \\ c_2 & c_3 & \dots & c_n \\ \vdots & \vdots & \ddots & \vdots \\ c_{n-1} & c_n & \dots & c_{2n-1} \end{pmatrix} \succeq 0$$

where $m = \lfloor \frac{n}{2} \rfloor$

- The Hausdorff moment problem where $I = [0, 1]$ is similarly SDP-Rep

Moment and positive polynomial cones

- $\mathcal{P}_n^* = \mathcal{M}_n$, i.e., i.e. moment cones and nonnegative polynomials are dual of each other
- If $\{u_0(x), \dots, u_n(x)\}$, $x \in I$ are linearly independent functions (possibly of several variables)
- The cone of polynomials that can be expressed as sum of squares is SDP-Rep.
- if I is a one dimensional set then positive polynomials and sum of square polynomials coincide
- In general except for I one-dimensional positive polynomials properly include sum of square polynomials