Lecture: Introduction to LP, SDP and SOCP

Zaiwen Wen

Beijing International Center For Mathematical Research
Peking University

http://bicmr.pku.edu.cn/~wenzw/bigdata2023.html
wenzw@pku.edu.cn

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Outline

1. Linear Programming (LP)
2. Semidefinite Programming (SDP)
3. Second Order Cone Programming (SOCP)

Linear Programming (LP)

Primal

$$\begin{aligned} \min_{x_i} \quad & c_1x_1 + \ldots + c_nx_n \\ \text{s.t.} \quad & a_{11}x_1 + \ldots + a_{1n}x_n = b_1 \\ & \ldots \\ & a_{m1}x_1 + \ldots + a_{mn}x_n = b_m \\ & x_i \geq 0 \end{aligned}$$

Dual

$$\begin{aligned} \max_{y_i} \quad & b_1y_1 + \ldots + b_my_m \\ \text{s.t.} \quad & a_{11}y_1 + \ldots + a_{m1}y_m \leq c_1 \\ & \ldots \\ & a_{1n}y_1 + \ldots + a_{mn}y_m \leq c_n \end{aligned}$$

Linear Programming (LP)

more succinctly

Primal (P)	**Dual (D)**
$\min_{x} \quad c^\top x$	$\max_{y,s} \quad b^\top y$
s.t. $\quad Ax = b$	s.t. $\quad A^\top y + s = c$
$x \geq 0$	$s \geq 0$

Problems with absolute values

$$\begin{aligned} \min \quad & \sum_i c_i |x_i|, \qquad \text{assume } c \geq 0 \\ \text{s.t.} \quad & Ax \geq b \end{aligned}$$

- Reformulation 1:

$$\begin{aligned} \min \quad & \sum_i c_i z_i \\ \text{s.t.} \quad & Ax \geq b \\ & |x_i| \leq z_i \end{aligned} \iff \begin{aligned} \min \quad & \sum_i c_i z_i \\ \text{s.t.} \quad & Ax \geq b \\ & -z_i \leq x_i \leq z_i \end{aligned}$$

- Reformulation 2: $x_i = x_i^+ - x_i^-$, $x_i^+, x_i^- \geq 0$. Then $|x_i| = x_i^+ + x_i^-$

$$\begin{aligned} \min \quad & \sum_i c_i (x_i^+ + x_i^-) \\ \text{s.t.} \quad & Ax^+ - Ax^- \geq b, x^+, x^- \geq 0 \end{aligned}$$

Problems with absolute values

- data fitting:

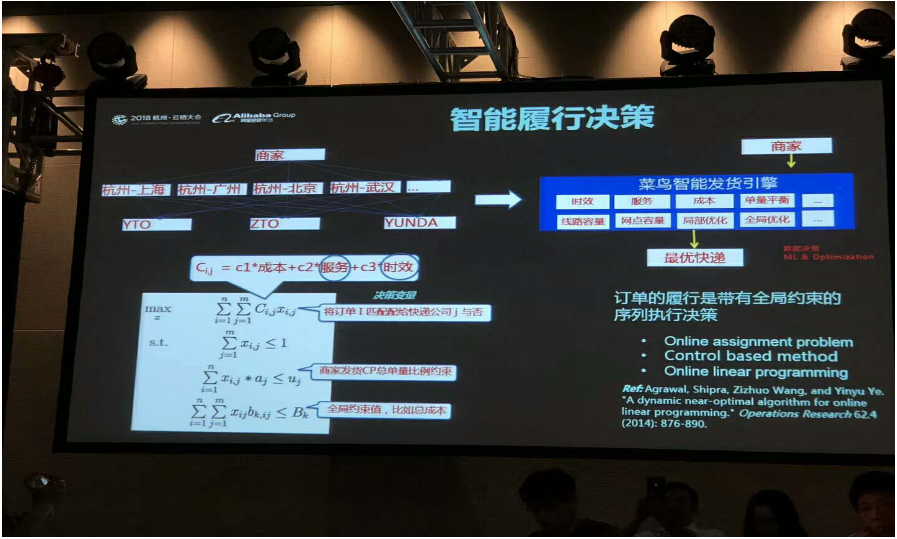
$$\min_x \quad \|Ax - b\|_\infty$$

$$\min_x \quad \|Ax - b\|_1$$

- Compressive sensing

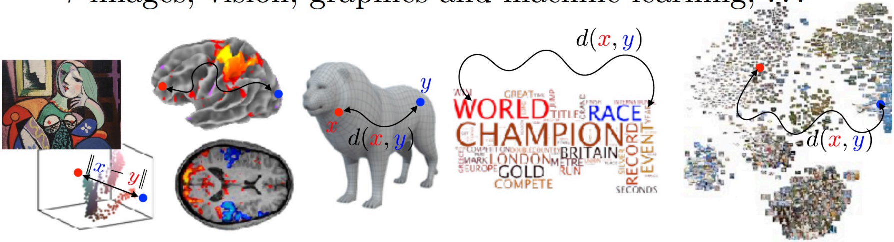
$$\begin{aligned}
&\min && \|x\|_1, \text{ s.t. } \quad Ax = b \quad (LP)\\
&\min && \mu\|x\|_1 + \tfrac{1}{2}\|Ax - b\|^2 \quad (QP, SOCP)\\
&\min && \|Ax - b\|, \text{ s.t. } \quad \|x\|_1 \le 1
\end{aligned}$$

An example of linear programming: 菜鸟



Optimal transport

→ images, vision, graphics and machine learning, ...



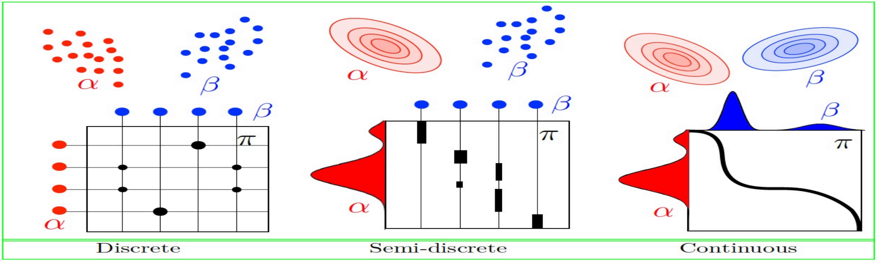
Optimal transport: LP

$$\min_{\pi \in \mathbb{R}^{m\times n}} \quad \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}\pi_{ij}$$

$$\text{s.t.} \quad \sum_{j=1}^{n} \pi_{ij} = \mu_i, \quad \forall i = 1,\dots,m,$$

$$\sum_{i=1}^{m} \pi_{ij} = \nu_i, \quad \forall j = 1,\dots,n$$

$$\pi \geq 0$$



Weak duality

Suppose

- x is feasible to (P)
- (y, s) is feasible to (D)

Then

$$\begin{aligned} 0 &\leq x^\top s \quad \text{because } x_i s_i \geq 0 \\ &= x^\top (c - A^\top y) \\ &= c^\top x - (Ax)^\top y \\ &= c^\top x - b^\top y \\ &= \text{duality gap} \end{aligned}$$

Key Properties of LP

- Strong duality: If both Primal and Dual are feasible then at the optimum

$$c^\top x = b^\top y \Longleftrightarrow x^\top s = 0$$

- complementary slackness: This implies

$$\begin{aligned} x^\top s &= x_1 s_1 + \ldots + x_n s_n = 0 \quad \text{and therefore} \\ x_i s_i &= 0 \end{aligned}$$

complementarity

- Putting together primal feasibility, dual feasibility and complementarity together we get a square system of equations

$$\begin{array}{rcll} Ax & = & b, & x \geq 0, \\ A^\top y + s & = & c, & s \geq 0, \\ x_i s_i & = & 0 & \text{for } i = 1, \ldots, n \end{array}$$

- At least in principle this system determines the primal and dual optimal values

Outline

1. Linear Programming (LP)
2. Semidefinite Programming (SDP)
3. Second Order Cone Programming (SOCP)

Semidefinite Programming (SDP)

- $X \succeq Y$ means that the the symmetric matrix $X - Y$ is positive semidefinite
- X is positive semidefinite

$$a^\top X a \geq 0 \text{ for all vector } a \iff X = B^\top B \iff$$

$$\text{all eigenvalues of } X \text{ is nonnegative}$$

SDP

- For simplicity we deal with single variable SDP:

Primal (P)

$$\begin{aligned} \min_{X} \quad & \langle C, X \rangle \\ \text{s.t.} \quad & \langle A_1, X \rangle = b_1 \\ & \dots \\ & \langle A_m, X \rangle = b_m \\ & X \succeq 0 \end{aligned}$$

Dual (D)

$$\begin{aligned} \max_{y,S} \quad & b^\top y \\ \text{s.t.} \quad & \sum_i y_i A_i + S = C \\ & S \succeq 0 \end{aligned}$$

- A single variable LP is trivial
- But a single matrix SDP is as general as a multiple matrix

Facts on matrix calcuation

- If $A, B \in \mathbb{R}^{m\times n}$, then $\mathrm{Tr}(AB^\top) = \mathrm{Tr}(B^\top A)$
- If $U, V \in \mathcal{S}^n$ and Q is orthogonal, then $\langle U, V\rangle = \langle Q^\top UQ, Q^\top UQ\rangle$
- If $X \in \mathcal{S}^n$, then $U = Q^\top \Lambda Q$, where $Q^\top Q = I$ and Λ is diagonal.
- Matrix norms: $\|X\|_F = \|\lambda(X)\|_2$, $\|X\|_2 = \|\lambda(X)\|_\infty$, $\lambda(X) = \mathrm{diag}(\Lambda)$
- $X \succeq 0 \iff v^\top Xv \geq$ for all $v \in \mathbb{R}^n \iff \lambda(X) \geq 0 \iff X = B^\top B$
- The dual cone of $\mathcal{S}_+^n$ is $\mathcal{S}_+^n$
- If $X \succeq 0$, then $X_{ii} \geq 0$. If $X_{ii} = 0$, then $X_{ik} = X_{ki} = 0$ for all k.
- If $X \succeq 0$, then $PXP^\top \succeq 0$ for any P of approriate dimensions
- If $X = \begin{pmatrix} X_{11} & X_{12} \\ X_{12}^\top & X_{22} \end{pmatrix} \succeq 0$, then $X_{11} \succeq 0$.
- $X \succeq 0$ iff every principal submatrix is positive semidefinite (psd).

Facts on matrix calcuation

- Let $U = \begin{pmatrix} A & B \\ B^\top & C \end{pmatrix}$ with A and C symmetric and $A \succ 0$. Then

$$U \succeq 0 \text{ (or } \succ 0) \quad \Longleftrightarrow \quad C - B^\top A^{-1} B \succeq 0 \text{ (or } \succ 0).$$

  The matrix $C - B^\top A^{-1} B$ is the **Schur complement** of A in U:

$$\begin{pmatrix} A & B \\ B^\top & C \end{pmatrix} = \begin{pmatrix} I & 0 \\ B^\top A^{-1} & I \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & C - B^\top A^{-1} B \end{pmatrix} \begin{pmatrix} I & A^{-1}B \\ 0 & I \end{pmatrix}$$

- If $A \in \mathcal{S}^n$, then $x^\top A x = \langle A, xx^\top \rangle$
- If $A \succ 0$, then $\langle A, B \rangle > 0$ for every nonzero $B \succeq 0$ and $\{B \succeq 0 \mid \langle A, B \rangle \leq \beta\}$ is bounded for $\beta > 0$
- If $A, B \succeq 0$, then $\langle A, B \rangle = 0$ iff $AB = 0$
- $A, B \in \mathcal{S}^n$, then A and B are commute iff AB is symmetric, iff A and B can be simultaneously diagonalized

Eigenvalue optimization

- minimizing the largest eigenvalue $\lambda_{\max}(A_0 + \sum_i x_i A_i)$:

$$\min \quad \lambda_{\max}(A_0 + \sum_i x_i A_i)$$

can be expressed as an SDP

$$\begin{aligned} \min \quad & z \\ \text{s.t.} \quad & zI - \sum_i x_i A_i \succeq A_0 \end{aligned}$$

and its dual is

$$\begin{aligned} \max \quad & \langle A_0, Y \rangle \\ \text{s.t.} \quad & \langle A_i, Y \rangle = k \\ & \langle I, Y \rangle = 1 \\ & Y \succeq 0 \end{aligned}$$

- follows from

$$\lambda_{\max}(A) \le t \iff A \preceq tI$$

Eigenvalue optimization

- Let $A_i \in \mathbb{R}^{m \times n}$. Minimizing the 2-norm of $A(x) = A_0 + \sum_i x_i A_i$:

$$\min_{x} \quad \|A(x)\|_2$$

can be expressed as an SDP

$$\begin{aligned} \min_{x,t} \quad & t \\ \text{s.t.} \quad & \begin{pmatrix} tI & A(x) \\ A(x)^\top & tI \end{pmatrix} \succeq 0 \end{aligned}$$

- Constraint follows from

$$\begin{aligned} \|A\|_2 \le t \quad & \iff \quad A^\top A \preceq t^2 I, \quad t \ge 0 \\ & \iff \quad \begin{pmatrix} tI & A(x) \\ A(x)^\top & tI \end{pmatrix} \succeq 0 \end{aligned}$$

Quadratically Constrained Quadratic Programming

Consider QCQP

$$\begin{array}{ll} \min & x^\top A_0 x + 2b_0^\top x + c_0 \qquad \text{assume } A_i \in \mathcal{S}^n \\ \text{s.t.} & x^\top A_i x + 2b_i^\top x + c_i \le 0, \quad i = 1, \dots, m \end{array}$$

- If $A_0 \succ 0$ and $A_i = B_i^\top B_i$, $i = 1, \dots, m$, then it is a SOCP
- If $A_i \in \mathcal{S}^n$ but may be indefinite

$$x^\top A_i x + 2b_i^\top x + c_i = \left\langle A_i, xx^\top \right\rangle + 2b_i^\top x + c_i$$

- The original problem is equivalent to

$$\begin{array}{ll} \min & \mathrm{Tr} A_0 X + 2b_0^\top x + c_0 \\ \text{s.t.} & \mathrm{Tr} A_i X + 2b_i^\top x + c_i \le 0, \quad i = 1, \dots, m \\ & X = xx^\top \end{array}$$

QCQP

- If $A_i \in \mathcal{S}^n$ but may be indefinite

$$x^\top A_i x + 2b_i^\top x + c_i = \left\langle \begin{pmatrix} A_i & b_i \\ b_i^\top & c_i \end{pmatrix}, \begin{pmatrix} X & x \\ x^\top & 1 \end{pmatrix} \right\rangle := \langle \bar{A}_i, \bar{X} \rangle$$

$\bar{X} \succeq 0$ is equivalent to $X \succeq xx^\top$

- The SDP relaxation is

$$\begin{aligned} \min \quad & \mathrm{Tr} \bar{A}_0 \bar{X} \\ \text{s.t.} \quad & \mathrm{Tr} \bar{A}_i \bar{X} \leq 0, \quad i = 1, \ldots, m \\ & \bar{X} \succeq 0 \end{aligned}$$

- Maxcut: $\max \ x^\top W x, \quad$ s.t. $\quad x_i^2 = 1$
- Phase retrieval: $|a_i^\top x| = b_i$, the value of $a_i^\top x$ is complex

Max cut

- For graph (V, E) and weights $w_{ij} = w_{ji} \geq 0$, the maxcut problem is

$$(Q) \quad \max_{x} \ \frac{1}{2} \sum_{i<j} w_{ij}(1 - x_i x_j), \quad \text{s.t. } \ x_i \in \{-1, 1\}$$

- Relaxation:

$$(P) \quad \max_{v_i \in \mathbb{R}^n} \ \frac{1}{2} \sum_{i<j} w_{ij}(1 - v_i^\top v_j), \quad \text{s.t. } \ \|v_i\|_2 = 1$$

- Equivalent SDP of (P):

$$(SDP) \quad \max_{X \in S^n} \ \frac{1}{2} \sum_{i<j} w_{ij}(1 - X_{ij}), \quad \text{s.t. } \ X_{ii} = 1, X \succeq 0$$

Max cut: rounding procedure

Goemans and Williamson's randomized approach

- Solve (SDP) to obtain an optimal solution X. Compute the decomposition $X = V^\top V$, where

$$V = [v_1, v_2, \ldots, v_n]$$

- Generate a vector r uniformly distributed on the unit sphere, i.e., $\|r\|_2 = 1$

- Set

$$x_i = \begin{cases} 1 & v_i^\top r \geq 0 \\ -1 & \text{otherwise} \end{cases}$$

Max cut: theoretical results

- Let W be the objective function value of x and $E(W)$ be the expected value. Then

$$E(W) = \frac{1}{\pi} \sum_{i<j} w_{ij} \arccos(v_i^\top v_j)$$

- Goemans and Williamson showed:

$$E(W) \geq \alpha \frac{1}{2} \sum_{i<j} w_{ij} (1 - v_i^\top v_j)$$

where

$$\alpha = \min_{0 \leq \theta \pi} \frac{2}{\pi} \frac{\theta}{1 - \cos \theta} > 0.878$$

- Let $Z^*_{(SDP)}$ and $Z^*_{(Q)}$ be the optimal values of (SDP) and (Q)

$$E(W) \geq \alpha Z^*_{(SDP)} \geq \alpha Z^*_{(Q)}$$

Weak duality in SDP

- Just as in LP
$$\langle X, S\rangle = \langle C, X\rangle - b^\top y$$
- Also if both $X \succeq 0$ and $S \succeq 0$ then
$$\langle X, S\rangle = \mathrm{Tr}(XS^{1/2}S^{1/2}) = \mathrm{Tr}(S^{1/2}XS^{1/2}) \geq 0$$
because $S^{1/2}XS^{1/2} \succeq 0$
- Thus
$$\langle X, S\rangle = \langle C, X\rangle - b^\top y \geq 0$$

Complementarity Slackness Theorem

- $X \succeq 0$ and $S \succeq 0$ and $\langle X, S \rangle = 0$ implies

$$XS = 0$$

- Proof:

$$\langle X, S \rangle = \mathrm{Tr}(XS^{1/2}S^{1/2}) = \mathrm{Tr}(S^{1/2}XS^{1/2})$$

Thus $\mathrm{Tr}(S^{1/2}XS^{1/2}) = 0$. Since $S^{1/2}XS^{1/2} \succeq 0$, then

$$S^{1/2}XS^{1/2} = 0 \Longrightarrow S^{1/2}X^{1/2}X^{1/2}S^{1/2} = 0$$
$$X^{1/2}S^{1/2} = 0 \Longrightarrow XS = 0$$

Equivalent complementarity slackness

- For reasons to become clear later it is better to write complementary slackness conditions as

$$\frac{XS + SX}{2} = 0$$

- It can be shown that if $X \succeq 0$ and $S \succeq 0$, then $XS = 0$ iff

$$XS + SX = 0$$

Constraint Qualifications

- Unlike LP we need some conditions for the optimal values of Primal and Dual SDP to coincide
- Here are two:
  - If there is primal-feasible $X \succ 0$ (i.e. X is positive definite)
  - If there is dual-feasible $S \succ 0$
- When strong duality holds $\langle X, S \rangle = 0$

KKT Condition

- Thus just like LP, the system of equations are

$$\begin{aligned} \langle A_i, X \rangle &= b_i, \quad X \succeq 0, \\ \sum_i y_i A_i + S &= C, \quad S \succeq 0, \\ X \circ S &= 0. \end{aligned}$$

It gives us a square system.

Outline

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Second Order Cone Programming (SOCP)

- For simplicity we deal with single variable SOCP:

Primal (P)

$$\begin{aligned} \min \quad & c^\top x \\ \text{s.t.} \quad & Ax = b \\ & x_{\mathcal{Q}} \succeq 0 \end{aligned}$$

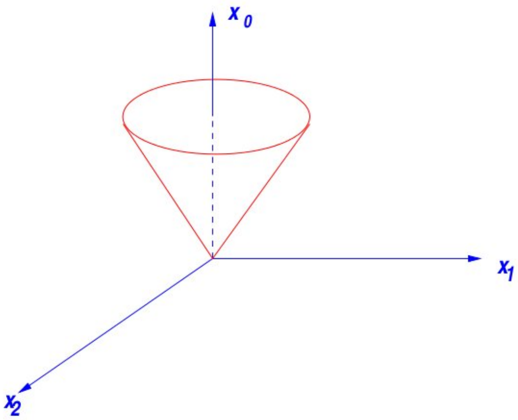
Dual (D)

$$\begin{aligned} \max \quad & b^\top y \\ \text{s.t.} \quad & A^\top y + s = c \\ & s_{\mathcal{Q}} \succeq 0 \end{aligned}$$

- the vectors x, s, c are indexed from zero
- If $z = (z_0, z_1, \ldots, z_n)^\top$ and $\bar{z} = (z_1, \ldots, z_n)^\top$

$$z_{\mathcal{Q}} \geq 0 \iff z_0 \geq \|\bar{z}\|$$

Illustration of SOC



$$\mathcal{Q} = \{z \mid z_0 \geq \|\bar{z}\|\}$$

Quadratic Programming (QP)

$$\begin{aligned} \min \quad & q(x) = x^\top Q x + a^\top x + \beta \qquad \text{assume} \quad Q \succ 0, Q = Q^\top \\ \text{s.t.} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

- $q(x) = \|\bar{u}\|^2 + \beta - \frac{1}{4} a^\top Q^{-1} a$, where $\bar{u} = Q^{1/2} x + \frac{1}{2} Q^{-1/2} a$.
- equivalent SOCP

$$\begin{aligned} \min \quad & u_0 \\ \text{s.t.} \quad & \bar{u} = Q^{1/2} x + \frac{1}{2} Q^{-1/2} a \\ & Ax = b \\ & x \geq 0, \quad (u_0, \bar{u}) \succeq_{\mathcal{Q}} 0 \end{aligned}$$

Robust linear programming

the parameters in LP are often uncertain

$$\begin{aligned} \min \quad & c^\top x \\ \text{s.t.} \quad & a_i^\top x \le b_i \end{aligned}$$

There can be uncertainty in c, a_i, b.

two common approaches to handling uncertainty (in a_i, for simplicity)

- deterministic model: constraints must hold for all $a_i \in \mathcal{E}_i$

$$\begin{aligned} \min \quad & c^\top x \\ \text{s.t.} \quad & a_i^\top x \le b_i, \text{ for all } a_i \in \mathcal{E}_i \end{aligned}$$

- stochastic model: a_i is random variable; constraints must hold with probability η

$$\begin{aligned} \min \quad & c^\top x \\ \text{s.t.} \quad & \text{prob}(a_i^\top x \le b_i) \ge \eta \end{aligned}$$

deterministic approach via SOCP

- Choose an ellipsoid as $\mathcal{E}_i$:

$$\mathcal{E}_i = \{\bar{a}_i + P_i u \mid \|u\|_2 \le 1\}, \quad \bar{a}_i \in \mathbb{R}^n, P_i \in \mathbb{R}^{n\times n}$$

- Robust LP

$$\begin{aligned} \min \quad & c^\top x \\ \text{s.t.} \quad & a_i^\top x \le b_i, \text{ for all } a_i \in \mathcal{E}_i \end{aligned}$$

is equivalent to the SOCP

$$\begin{aligned} \min \quad & c^\top x \\ \text{s.t.} \quad & \bar{a}_i^\top x + \|P_i^\top x\|_2 \le b_i \end{aligned}$$

since

$$\sup_{\|u\|_2 \le 1} (\bar{a}_i + P_i u)^\top x = \bar{a}_i^\top x + \|P_i^\top x\|_2$$

stochastic approach via SOCP

- a_i is Gaussian with mean $\bar{a}_i$, covariance Σ_i ($a_i \sim \mathcal{N}(\bar{a}_i, \Sigma_i)$
- $a_i^\top x$ is Gaussian r.v. with mean $\bar{a}_i^\top x$, variance $x^\top \Sigma_i x$; hence

$$\text{prob}(a_i^\top x \leq b_i) = \Phi\left(\frac{b_i - \bar{a}_i^\top x}{\|\Sigma^{1/2} x\|_2}\right)$$

  where $\Phi(x) = (1/\sqrt{2\pi}) \int_{-\infty}^{x} e^{-t^2/2} dt$ is CDF of $\mathcal{N}(0,1)$
- robust LP

$$\begin{array}{ll} \min & c^\top x \\ \text{s.t.} & \text{prob}(a_i^\top x \leq b_i) \geq \eta \end{array}$$

  is equivalent to the SOCP

$$\begin{array}{ll} \min & c^\top x \\ \text{s.t.} & \bar{a}_i^\top x + \Phi^{-1}(\eta) \|\Sigma^{1/2} x\|_2 \leq b_i \end{array}$$

Weak Duality in SOCP

- The single block SOCP is not as trivial as LP but it still can be solved analytically
- weak duality: Again as in LP and SDP

$$x^\top s = c^\top x - b^\top y = \text{ duality gap}$$

If $x, s \succeq_{\mathcal{Q}} 0$, then

$$\begin{aligned} x^\top s &= x_0 s_0 + \bar{x}^\top \bar{s} \\ &\geq \|\bar{x}\| \cdot \|\bar{s}\| + \bar{x}^\top \bar{s} \quad \text{since } x, s \succeq_{\mathcal{Q}} 0 \\ &\geq |\bar{x}^\top \bar{s}| + \bar{x}^\top \bar{s} \quad \text{Cauchy-Schwartz inequality} \\ &\geq 0 \end{aligned}$$

Complementary Slackness for SOCP

- Given $x \succeq_{\mathcal{Q}} 0$, $s \succeq_{\mathcal{Q}} 0$ and $x^\top s = 0$. Assume $x_0 > 0$ and $s_0 > 0$
- We have

$$(*) \quad x_0^2 \geq \sum_{i=1}^{n} x_i^2$$

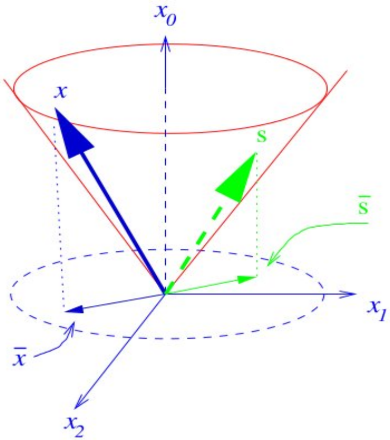
$$(**) \quad s_0^2 \geq \sum_{i=1}^{n} s_i^2 \Longleftrightarrow x_0^2 \geq \sum_{i=1}^{n} \frac{s_i^2 x_0^2}{s_0^2}$$

$$(***) \quad x^\top s = 0 \Longleftrightarrow -x_0 s_0 = \sum_{i} x_i s_i \Longleftrightarrow -2x_0^2 = \sum_{i=1}^{n} \frac{2 x_i s_i x_0}{s_0}$$

- Adding (*), (**), (***), we get $0 \geq \sum_{i=1}^{n} \left(x_i + \frac{s_i x_0}{s_0} \right)^2$
- This implies

$$x_i s_0 + x_0 s_i = 0, \text{ for } i = 1, \ldots, n$$

Illustration of SOC



When $x \succeq_{\mathcal{Q}} 0$, $s \succeq_{\mathcal{Q}} 0$ are orthogonal both must be on the boundary in such a way that their projection on the $x_1, \ldots, x_n$ plane is collinear

Strong Duality

- at the optimum

$$c^\top x = b^\top y \iff x^\top s = 0$$

- Like SDP constraint qualifications are required
- If there is primal-feasible $x \succ_{\mathcal{Q}} 0$
- If there is dual-feasible $s \succ_{\mathcal{Q}} 0$

Complementary Slackness for SOCP

- Thus again we have a square system

$$
\begin{aligned}
Ax &= b, \quad x \succeq_{\mathcal{Q}} 0, \\
A^\top y + s &= c, \quad s \succeq_{\mathcal{Q}} 0, \\
x^\top s &= 0 \\
x_0 s_i + s_0 x_i &= 0
\end{aligned}
$$