

# Lecture: Introduction to LP, SDP and SOCP

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- 1 Linear Programming (LP)
- 2 Semidefinite Programming (SDP)
- 3 Second Order Cone Programming (SOCP)

# Linear Programming (LP)

## Primal

$$\begin{aligned} \min_{x_i} \quad & c_1x_1 + \dots + c_nx_n \\ \text{s.t.} \quad & a_{11}x_1 + \dots + a_{1n}x_n = b_1 \\ & \dots \\ & a_{m1}x_1 + \dots + a_{mn}x_n = b_m \\ & x_i \geq 0 \end{aligned}$$

## Dual

$$\begin{aligned} \max_{y_i} \quad & b_1y_1 + \dots + b_my_m \\ \text{s.t.} \quad & a_{11}y_1 + \dots + a_{m1}y_m \leq c_1 \\ & \dots \\ & a_{1n}y_1 + \dots + a_{mn}y_m \leq c_n \end{aligned}$$

# Linear Programming (LP)

more succinctly

**Primal (P)**

$$\begin{aligned} \min_x \quad & c^\top x \\ \text{s.t.} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

**Dual (D)**

$$\begin{aligned} \max_{y,s} \quad & b^\top y \\ \text{s.t.} \quad & A^\top y + s = c \\ & s \geq 0 \end{aligned}$$

# Problems with absolute values

$$\begin{aligned} \min \quad & \sum_i c_i |x_i|, & \text{assume } c \geq 0 \\ \text{s.t.} \quad & Ax \geq b \end{aligned}$$

- Reformulation 1:

$$\begin{aligned} \min \quad & \sum_i c_i z_i & \min \quad & \sum_i c_i z_i \\ \text{s.t.} \quad & Ax \geq b & \iff & \text{s.t.} \quad Ax \geq b \\ & |x_i| \leq z_i & & -z_i \leq x_i \leq z_i \end{aligned}$$

- Reformulation 2:  $x_i = x_i^+ - x_i^-$ ,  $x_i^+, x_i^- \geq 0$ . Then  $|x_i| = x_i^+ + x_i^-$

$$\begin{aligned} \min \quad & \sum_i c_i (x_i^+ + x_i^-) \\ \text{s.t.} \quad & Ax^+ - Ax^- \geq b, x^+, x^- \geq 0 \end{aligned}$$

# LLM: GPT4

For a given matrix  $A$  and vector  $b$ , consider the basis pursuit problem:  
 $\min \|x\|_1$ , s.t.  $Ax = b$ .

Please convert it to a standard linear programming problem.



## ChatGPT

The Basis Pursuit problem is a fundamental optimization problem that seeks to find the sparsest solution to a linear system. Formally, for a given matrix  $A \in \mathbb{R}^{m \times n}$  and a vector  $b \in \mathbb{R}^m$ , the problem is defined as:

$$\min_{x \in \mathbb{R}^n} \|x\|_1 \quad \text{subject to} \quad Ax = b,$$

where  $\|x\|_1 = \sum_{i=1}^n |x_i|$  denotes the  $l_1$ -norm of  $x$ , which is the sum of the absolute values of the components of  $x$ .

To convert this problem into a standard Linear Programming (LP) form, we leverage the property that the absolute value function can be represented using auxiliary variables and additional constraints. The  $l_1$ -norm objective  $\|x\|_1$  can be linearized as follows:

- Introduce Auxiliary Variables:** For each variable  $x_i$  in  $x$ , introduce two non-negative auxiliary variables  $u_i$  and  $v_i$  where  $x_i = u_i - v_i$  and  $u_i, v_i \geq 0$ . This decomposition allows us to represent both positive and negative values of  $x_i$  using non-negative variables, as required in LP formulations.
- Objective Function Transformation:** The  $l_1$ -norm  $\|x\|_1 = \sum_{i=1}^n |x_i|$  can now be rewritten in terms of  $u_i$  and  $v_i$  as  $\sum_{i=1}^n (u_i + v_i)$ , since  $|x_i| = u_i + v_i$  by construction.
- Constraint Adaptation:** The original constraint  $Ax = b$  must be adapted to reflect the introduction of  $u_i$  and  $v_i$ . Since  $x_i = u_i - v_i$ , we can rewrite  $Ax = A(u - v) = b$ , where  $u$  and  $v$  are vectors consisting of the variables  $u_i$  and  $v_i$ , respectively.

Therefore, the Basis Pursuit problem can be reformulated as a standard LP problem:

$$\begin{aligned} \min_{u, v \in \mathbb{R}^n} \quad & \sum_{i=1}^n (u_i + v_i) \\ \text{subject to} \quad & A(u - v) = b, \\ & u_i \geq 0, \quad \forall i = 1, \dots, n, \\ & v_i \geq 0, \quad \forall i = 1, \dots, n. \end{aligned}$$

# LLM: GPT4

- Great! Please provide alternative formulation by introducing a new variable  $z_i$ , and enforcing  $|x_i| \leq z_i$



## ChatGPT

To reformulate the Basis Pursuit problem by introducing a new variable  $z$  and enforcing  $|x_i| \leq z_i$  for each component  $i$  of the vector  $x$ , we can take advantage of the relationship between the  $l_1$ -norm objective and these new variables. This approach aims to directly model the  $l_1$ -norm minimization within a linear programming framework by explicitly capturing the absolute value constraints.

The original Basis Pursuit problem is given by:

$$\min_{x \in \mathbb{R}^n} \|x\|_1 \quad \text{subject to} \quad Ax = b,$$

where  $\|x\|_1 = \sum_{i=1}^n |x_i|$ .

### Reformulation Steps:

- Introduce New Variables:** For each  $x_i$  in the vector  $x$ , introduce a corresponding variable  $z_i$  intended to capture the absolute value of  $x_i$ , i.e.,  $|x_i| \leq z_i$ . The variable  $z_i$  is non-negative for all  $i$ , since it represents the magnitude of  $x_i$ .
- Objective Function:** The objective becomes the minimization of the sum of the new variables  $z$ , which represent the absolute values of the components of  $x$ . Hence, the objective function is  $\min \sum_{i=1}^n z_i$ .
- Constraints:** In addition to the original equality constraint  $Ax = b$ , we add constraints to ensure  $|x_i| \leq z_i$  for all  $i$ . This is achieved by adding the following inequalities for each  $i$ :
  - $x_i \leq z_i$ , to capture the case when  $x_i$  is positive or zero.
  - $-x_i \leq z_i$ , to capture the case when  $x_i$  is negative.

Together, these inequalities ensure that  $z_i$  is at least as large as the absolute value of  $x_i$ , effectively modeling  $|x_i| \leq z_i$ .

### Linear Programming Formulation:

The Basis Pursuit problem can now be reformulated as:

$$\begin{aligned} \min_{x, z \in \mathbb{R}^n} \quad & \sum_{i=1}^n z_i \\ \text{subject to} \quad & Ax = b, \\ & x_i \leq z_i, \quad \forall i = 1, \dots, n, \\ & -x_i \leq z_i, \quad \forall i = 1, \dots, n, \\ & z_i \geq 0, \quad \forall i = 1, \dots, n. \end{aligned}$$

# Problems with absolute values

- data fitting:

$$\min_x \|Ax - b\|_\infty$$

$$\min_x \|Ax - b\|_1$$

- Compressive sensing

$$\min \|x\|_1, \text{ s.t. } Ax = b \quad (LP)$$

$$\min \mu \|x\|_1 + \frac{1}{2} \|Ax - b\|^2 \quad (QP, SOCP)$$

$$\min \|Ax - b\|, \text{ s.t. } \|x\|_1 \leq 1$$

# An example of linear programming: 菜鸟

2018 杭州·云栖大会 Alibaba Group

## 智能履行决策

商家

杭州-上海 杭州-广州 杭州-北京 杭州-武汉 ...

YTO ZTO YUNDA

商家

菜鸟智能发货引擎

时效	服务	成本	单量平衡	...
线路容量	网点容量	局部优化	全局优化	...

最优快递

智能决策  
ML & Optimization

$C_{i,j} = c1 * \text{成本} + c2 * \text{服务} + c3 * \text{时效}$

决策变量

$\max_x \sum_{i=1}^n \sum_{j=1}^m C_{i,j} x_{i,j}$  将订单 i 匹配合适快递公司 j 与否

s.t.  $\sum_{j=1}^m x_{i,j} \leq 1$

$\sum_{i=1}^n x_{i,j} * a_j \leq u_j$  商家发货 CP 总单量比例约束

$\sum_{i=1}^n \sum_{j=1}^m x_{i,j} b_{k,i,j} \leq B_k$  全局约束值, 比如总成本

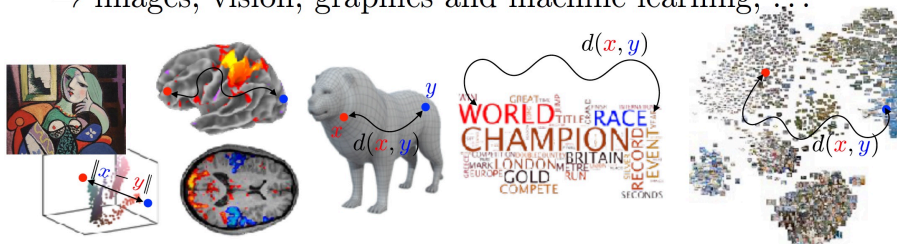
订单的履行是带有全局约束的序列执行决策

- Online assignment problem
- Control based method
- Online linear programming

Ref: Agrawal, Shipra, Zizhuo Wang, and Yinyu Ye. "A dynamic near-optimal algorithm for online linear programming." *Operations Research* 62.4 (2014): 876-890.

# Optimal transport

→ images, vision, graphics and machine learning, ...



Monge



Kantorovich



Koopmans



Dantzig



Brenier



Otto



McCann



Villani



Figalli

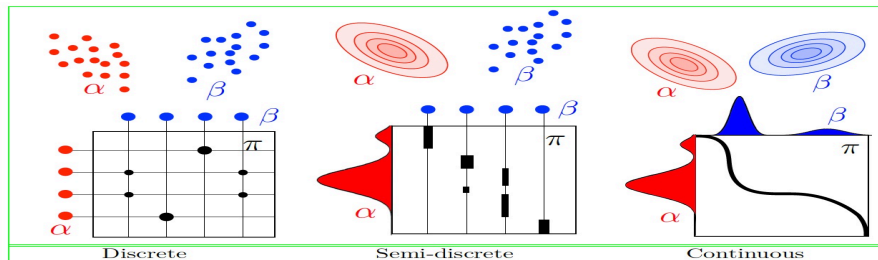
Nobel '75

Fields '10

Fields'18

# Optimal transport: LP

$$\begin{aligned} \min_{\pi \in \mathbb{R}^{m \times n}} \quad & \sum_{i=1}^m \sum_{j=1}^n c_{ij} \pi_{ij} \\ \text{s.t.} \quad & \sum_{j=1}^n \pi_{ij} = \mu_i, \quad \forall i = 1, \dots, m, \\ & \sum_{i=1}^m \pi_{ij} = \nu_j, \quad \forall j = 1, \dots, n \\ & \pi \geq 0 \end{aligned}$$



# Weak duality

Suppose

- $x$  is feasible to (P)
- $(y, s)$  is feasible to (D)

Then

$$\begin{aligned} 0 &\leq x^\top s \quad \text{because } x_i s_i \geq 0 \\ &= x^\top (c - A^\top y) \\ &= c^\top x - (Ax)^\top y \\ &= c^\top x - b^\top y \\ &= \text{duality gap} \end{aligned}$$

# Key Properties of LP

- Strong duality: If both Primal and Dual are feasible then at the optimum

$$c^T x = b^T y \iff x^T s = 0$$

- complementary slackness: This implies

$$\begin{aligned}x^T s &= x_1 s_1 + \dots + x_n s_n = 0 \quad \text{and therefore} \\x_i s_i &= 0\end{aligned}$$

# complementarity

- Putting together primal feasibility, dual feasibility and complementarity together we get a square system of equations

$$\begin{aligned}Ax &= b, & x &\geq 0, \\A^\top y + s &= c, & s &\geq 0, \\x_i s_i &= 0 & \text{for } i &= 1, \dots, n\end{aligned}$$

- At least in principle this system determines the primal and dual optimal values

- 1 Linear Programming (LP)
- 2 Semidefinite Programming (SDP)**
- 3 Second Order Cone Programming (SOCP)

# Semidefinite Programming (SDP)

- $X \succeq Y$  means that the the symmetric matrix  $X - Y$  is positive semidefinite
- $X$  is positive semidefinite

$$a^\top X a \geq 0 \text{ for all vector } a \iff X = B^\top B \iff$$

all eigenvalues of  $X$  is nonnegative

- For simplicity we deal with single variable SDP:

**Primal (P)**

$$\begin{aligned} \min_X \quad & \langle C, X \rangle \\ \text{s.t.} \quad & \langle A_1, X \rangle = b_1 \\ & \dots \\ & \langle A_m, X \rangle = b_m \\ & X \succeq 0 \end{aligned}$$

**Dual (D)**

$$\begin{aligned} \max_{y, S} \quad & b^\top y \\ \text{s.t.} \quad & \sum_i y_i A_i + S = C \\ & S \succeq 0 \end{aligned}$$

- A single variable LP is trivial
- But a single matrix SDP is as general as a multiple matrix

# Facts on matrix calculation

- If  $A, B \in \mathbb{R}^{m \times n}$ , then  $\text{Tr}(AB^\top) = \text{Tr}(B^\top A)$
- If  $U, V \in \mathcal{S}^n$  and  $Q$  is orthogonal, then  $\langle U, V \rangle = \langle Q^\top U Q, Q^\top V Q \rangle$
- If  $X \in \mathcal{S}^n$ , then  $U = Q^\top \Lambda Q$ , where  $Q^\top Q = I$  and  $\Lambda$  is diagonal.
- Matrix norms:  $\|X\|_F = \|\lambda(X)\|_2$ ,  $\|X\|_2 = \|\lambda(X)\|_\infty$ ,  $\lambda(X) = \text{diag}(\Lambda)$
- $X \succeq 0 \iff v^\top X v \geq 0$  for all  $v \in \mathbb{R}^n \iff \lambda(X) \geq 0 \iff X = B^\top B$
- The dual cone of  $\mathcal{S}_+^n$  is  $\mathcal{S}_+^n$
- If  $X \succeq 0$ , then  $X_{ii} \geq 0$ . If  $X_{ii} = 0$ , then  $X_{ik} = X_{ki} = 0$  for all  $k$ .
- If  $X \succeq 0$ , then  $PXP^\top \succeq 0$  for any  $P$  of appropriate dimensions
- If  $X = \begin{pmatrix} X_{11} & X_{12} \\ X_{12}^\top & X_{22} \end{pmatrix} \succeq 0$ , then  $X_{11} \succeq 0$ .
- $X \succeq 0$  iff every principal submatrix is positive semidefinite (psd).

## Facts on matrix calculation

- Let  $U = \begin{pmatrix} A & B \\ B^\top & C \end{pmatrix}$  with  $A$  and  $C$  symmetric and  $A \succ 0$ . Then

$$U \succeq 0 \text{ ( or } \succ 0) \iff C - B^\top A^{-1} B \succeq 0 \text{ ( or } \succ 0).$$

The matrix  $C - B^\top A^{-1} B$  is the **Schur complement** of  $A$  in  $U$ :

$$\begin{pmatrix} A & B \\ B^\top & C \end{pmatrix} = \begin{pmatrix} I & 0 \\ B^\top A^{-1} & I \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & C - B^\top A^{-1} B \end{pmatrix} \begin{pmatrix} I & A^{-1} B \\ 0 & I \end{pmatrix}$$

- If  $A \in S^n$ , then  $x^\top A x = \langle A, x x^\top \rangle$
- If  $A \succ 0$ , then  $\langle A, B \rangle > 0$  for every nonzero  $B \succeq 0$  and  $\{B \succeq 0 \mid \langle A, B \rangle \leq \beta\}$  is bounded for  $\beta > 0$
- If  $A, B \succeq 0$ , then  $\langle A, B \rangle = 0$  iff  $AB = 0$
- $A, B \in S^n$ , then  $A$  and  $B$  commute iff  $AB$  is symmetric, iff  $A$  and  $B$  can be simultaneously diagonalized

# Eigenvalue optimization

- minimizing the largest eigenvalue  $\lambda_{\max}(A_0 + \sum_i x_i A_i)$ :

$$\min \lambda_{\max}(A_0 + \sum_i x_i A_i)$$

can be expressed as an SDP

$$\begin{aligned} \min \quad & z \\ \text{s.t.} \quad & zI - \sum_i x_i A_i \succeq A_0 \end{aligned}$$

and its dual is

$$\begin{aligned} \max \quad & \langle A_0, Y \rangle \\ \text{s.t.} \quad & \langle A_i, Y \rangle = k \\ & \langle I, Y \rangle = 1 \\ & Y \succeq 0 \end{aligned}$$

- follows from

$$\lambda_{\max}(A) \leq t \iff A \preceq tI$$

# Eigenvalue optimization

- Let  $A_i \in \mathbb{R}^{m \times n}$ . Minimizing the 2-norm of  $A(x) = A_0 + \sum_i x_i A_i$ :

$$\min_x \|A(x)\|_2$$

can be expressed as an SDP

$$\begin{aligned} \min_{x,t} \quad & t \\ \text{s.t.} \quad & \begin{pmatrix} tI & A(x) \\ A(x)^\top & tI \end{pmatrix} \succeq 0 \end{aligned}$$

- Constraint follows from

$$\begin{aligned} \|A\|_2 \leq t & \iff A^\top A \preceq t^2 I, \quad t \geq 0 \\ & \iff \begin{pmatrix} tI & A(x) \\ A(x)^\top & tI \end{pmatrix} \succeq 0 \end{aligned}$$

# Quadratically Constrained Quadratic Programming

Consider QCQP

$$\begin{aligned} \min \quad & x^\top A_0 x + 2b_0^\top x + c_0 \quad \text{assume } A_i \in \mathcal{S}^n \\ \text{s.t.} \quad & x^\top A_i x + 2b_i^\top x + c_i \leq 0, \quad i = 1, \dots, m \end{aligned}$$

- If  $A_0 \succ 0$  and  $A_i = B_i^\top B_i$ ,  $i = 1, \dots, m$ , then it is a SOCP
- If  $A_i \in \mathcal{S}^n$  but may be indefinite

$$x^\top A_i x + 2b_i^\top x + c_i = \langle A_i, xx^\top \rangle + 2b_i^\top x + c_i$$

- The original problem is equivalent to

$$\begin{aligned} \min \quad & \text{Tr}A_0 X + 2b_0^\top x + c_0 \\ \text{s.t.} \quad & \text{Tr}A_i X + 2b_i^\top x + c_i \leq 0, \quad i = 1, \dots, m \\ & X = xx^\top \end{aligned}$$

- If  $A_i \in \mathcal{S}^n$  but may be indefinite

$$x^\top A_i x + 2b_i^\top x + c_i = \left\langle \begin{pmatrix} A_i & b_i \\ b_i^\top & c_i \end{pmatrix}, \begin{pmatrix} X & x \\ x^\top & 1 \end{pmatrix} \right\rangle := \langle \bar{A}_i, \bar{X} \rangle$$

$\bar{X} \succeq 0$  is equivalent to  $X \succeq xx^\top$

- The SDP relaxation is

$$\begin{aligned} \min \quad & \text{Tr} \bar{A}_0 \bar{X} \\ \text{s.t.} \quad & \text{Tr} \bar{A}_i \bar{X} \leq 0, \quad i = 1, \dots, m \\ & \bar{X} \succeq 0 \end{aligned}$$

- Maxcut:  $\max x^\top Wx, \quad \text{s.t.} \quad x_i^2 = 1$
- Phase retrieval:  $|a_i^\top x| = b_i$ , the value of  $a_i^\top x$  is complex

# Max cut

- For graph  $(V, E)$  and weights  $w_{ij} = w_{ji} \geq 0$ , the maxcut problem is

$$(Q) \quad \max_x \frac{1}{2} \sum_{i < j} w_{ij} (1 - x_i x_j), \quad \text{s.t. } x_i \in \{-1, 1\}$$

- Relaxation:

$$(P) \quad \max_{v_i \in \mathbb{R}^n} \frac{1}{2} \sum_{i < j} w_{ij} (1 - v_i^\top v_j), \quad \text{s.t. } \|v_i\|_2 = 1$$

- Equivalent SDP of (P):

$$(SDP) \quad \max_{X \in \mathcal{S}^n} \frac{1}{2} \sum_{i < j} w_{ij} (1 - X_{ij}), \quad \text{s.t. } X_{ii} = 1, X \succeq 0$$

# Max cut: rounding procedure

Goemans and Williamson's randomized approach

- Solve (SDP) to obtain an optimal solution  $X$ . Compute the decomposition  $X = V^T V$ , where

$$V = [v_1, v_2, \dots, v_n]$$

- Generate a vector  $r$  uniformly distributed on the unit sphere, i.e.,  $\|r\|_2 = 1$
- Set

$$x_i = \begin{cases} 1 & v_i^T r \geq 0 \\ -1 & \text{otherwise} \end{cases}$$

## Max cut: theoretical results

- Let  $W$  be the objective function value of  $x$  and  $E(W)$  be the expected value. Then

$$E(W) = \frac{1}{\pi} \sum_{i < j} w_{ij} \arccos(v_i^\top v_j)$$

- Goemans and Williamson showed:

$$E(W) \geq \alpha \frac{1}{2} \sum_{i < j} w_{ij} (1 - v_i^\top v_j)$$

where

$$\alpha = \min_{0 \leq \theta < \pi} \frac{2}{\pi} \frac{\theta}{1 - \cos \theta} > 0.878$$

- Let  $Z_{(SDP)}^*$  and  $Z_{(Q)}^*$  be the optimal values of (SDP) and (Q)

$$E(W) \geq \alpha Z_{(SDP)}^* \geq \alpha Z_{(Q)}^*$$

# Weak duality in SDP

- Just as in LP

$$\langle X, S \rangle = \langle C, X \rangle - b^\top y$$

- Also if both  $X \succeq 0$  and  $S \succeq 0$  then

$$\langle X, S \rangle = \text{Tr}(XS^{1/2}S^{1/2}) = \text{Tr}(S^{1/2}XS^{1/2}) \geq 0$$

because  $S^{1/2}XS^{1/2} \succeq 0$

- Thus

$$\langle X, S \rangle = \langle C, X \rangle - b^\top y \geq 0$$

# Complementarity Slackness Theorem

- $X \succeq 0$  and  $S \succeq 0$  and  $\langle X, S \rangle = 0$  implies

$$XS = 0$$

- Proof:

$$\langle X, S \rangle = \text{Tr}(XS^{1/2}S^{1/2}) = \text{Tr}(S^{1/2}XS^{1/2})$$

Thus  $\text{Tr}(S^{1/2}XS^{1/2}) = 0$ . Since  $S^{1/2}XS^{1/2} \succeq 0$ , then

$$\begin{aligned} S^{1/2}XS^{1/2} = 0 &\implies S^{1/2}X^{1/2}X^{1/2}S^{1/2} = 0 \\ X^{1/2}S^{1/2} = 0 &\implies XS = 0 \end{aligned}$$

# Equivalent complementarity slackness

- For reasons to become clear later it is better to write complementary slackness conditions as

$$\frac{XS + SX}{2} = 0$$

- It can be shown that if  $X \succeq 0$  and  $S \succeq 0$ , then  $XS = 0$  iff

$$XS + SX = 0$$

# Constraint Qualifications

- Unlike LP we need some conditions for the optimal values of Primal and Dual SDP to coincide
- Here are two:
  - If there is primal-feasible  $X \succ 0$  (i.e.  $X$  is positive definite)
  - If there is dual-feasible  $S \succ 0$
- When strong duality holds  $\langle X, S \rangle = 0$

# KKT Condition

- Thus just like LP, the system of equations are

$$\begin{aligned}\langle A_i, X \rangle &= b_i, & X \succeq 0, \\ \sum_i y_i A_i + S &= C, & S \succeq 0, \\ X \circ S &= 0.\end{aligned}$$

It gives us a square system.

# Outline

- 1 Linear Programming (LP)
- 2 Semidefinite Programming (SDP)
- 3 Second Order Cone Programming (SOCP)**

# Second Order Cone Programming (SOCP)

- For simplicity we deal with single variable SOCP:

**Primal (P)**

$$\min \quad c^\top x$$

$$\text{s.t.} \quad Ax = b$$

$$x_Q \succeq 0$$

**Dual (D)**

$$\max \quad b^\top y$$

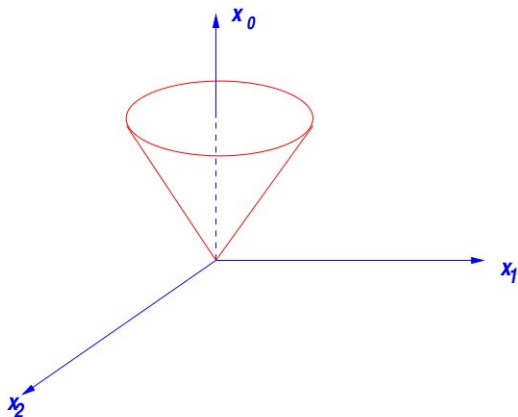
$$\text{s.t.} \quad A^\top y + s = c$$

$$s_Q \succeq 0$$

- the vectors  $x, s, c$  are indexed from zero
- If  $z = (z_0, z_1, \dots, z_n)^\top$  and  $\bar{z} = (z_1, \dots, z_n)^\top$

$$z_Q \succeq 0 \iff z_0 \geq \|\bar{z}\|$$

# Illustration of SOC



$$\mathcal{Q} = \{z \mid z_0 \geq \|\bar{z}\|\}$$

# Quadratic Programming (QP)

$$\begin{array}{ll} \min & q(x) = x^\top Qx + a^\top x + \beta \quad \text{assume } Q \succ 0, Q = Q^\top \\ \text{s.t.} & Ax = b \\ & x \geq 0 \end{array}$$

- $q(x) = \|\bar{u}\|^2 + \beta - \frac{1}{4}a^\top Q^{-1}a$ , where  $\bar{u} = Q^{1/2}x + \frac{1}{2}Q^{-1/2}a$ .
- equivalent SOCP

$$\begin{array}{ll} \min & u_0 \\ \text{s.t.} & \bar{u} = Q^{1/2}x + \frac{1}{2}Q^{-1/2}a \\ & Ax = b \\ & x \geq 0, \quad (u_0, \bar{u}) \succeq_Q 0 \end{array}$$

# Robust linear programming

the parameters in LP are often uncertain

$$\begin{array}{ll} \min & c^\top x \\ \text{s.t.} & a_i^\top x \leq b_i \end{array}$$

There can be uncertainty in  $c, a_i, b$ .

two common approaches to handling uncertainty (in  $a_i$ , for simplicity)

- deterministic model: constraints must hold for all  $a_i \in \mathcal{E}_i$

$$\begin{array}{ll} \min & c^\top x \\ \text{s.t.} & a_i^\top x \leq b_i, \text{ for all } a_i \in \mathcal{E}_i \end{array}$$

- stochastic model:  $a_i$  is random variable; constraints must hold with probability  $\eta$

$$\begin{array}{ll} \min & c^\top x \\ \text{s.t.} & \text{prob}(a_i^\top x \leq b_i) \geq \eta \end{array}$$

# deterministic approach via SOCP

- Choose an ellipsoid as  $\mathcal{E}_i$ :

$$\mathcal{E}_i = \{\bar{a}_i + P_i u \mid \|u\|_2 \leq 1\}, \quad \bar{a}_i \in \mathbb{R}^n, P_i \in \mathbb{R}^{n \times n}$$

- Robust LP

$$\begin{aligned} \min \quad & c^\top x \\ \text{s.t.} \quad & a_i^\top x \leq b_i, \text{ for all } a_i \in \mathcal{E}_i \end{aligned}$$

is equivalent to the SOCP

$$\begin{aligned} \min \quad & c^\top x \\ \text{s.t.} \quad & \bar{a}_i^\top x + \|P_i^\top x\|_2 \leq b_i \end{aligned}$$

since

$$\sup_{\|u\|_2 \leq 1} (\bar{a}_i + P_i u)^\top x = \bar{a}_i^\top x + \|P_i^\top x\|_2$$

# stochastic approach via SOCP

- $a_i$  is Gaussian with mean  $\bar{a}_i$ , covariance  $\Sigma_i$  ( $a_i \sim \mathcal{N}(\bar{a}_i, \Sigma_i)$ )
- $a_i^\top x$  is Gaussian r.v. with mean  $\bar{a}_i^\top x$ , variance  $x^\top \Sigma_i x$ ; hence

$$\text{prob}(a_i^\top x \leq b_i) = \Phi \left( \frac{b_i - \bar{a}_i^\top x}{\|\Sigma_i^{1/2} x\|_2} \right)$$

where  $\Phi(x) = (1/\sqrt{2\pi}) \int_{-\infty}^x e^{-t^2/2} dt$  is CDF of  $\mathcal{N}(0, 1)$

- robust LP

$$\begin{aligned} \min \quad & c^\top x \\ \text{s.t.} \quad & \text{prob}(a_i^\top x \leq b_i) \geq \eta \end{aligned}$$

is equivalent to the SOCP

$$\begin{aligned} \min \quad & c^\top x \\ \text{s.t.} \quad & \bar{a}_i^\top x + \Phi^{-1}(\eta) \|\Sigma_i^{1/2} x\|_2 \leq b_i \end{aligned}$$

# Weak Duality in SOCP

- The single block SOCP is not as trivial as LP but it still can be solved analytically
- weak duality: Again as in LP and SDP

$$x^\top s = c^\top x - b^\top y = \text{duality gap}$$

If  $x, s \succeq_{\mathcal{Q}} 0$ , then

$$\begin{aligned}x^\top s &= x_0 s_0 + \bar{x}^\top \bar{s} \\ &\geq \|\bar{x}\| \cdot \|\bar{s}\| + \bar{x}^\top \bar{s} \quad \text{since } x, s \succeq_{\mathcal{Q}} 0 \\ &\geq |\bar{x}^\top \bar{s}| + \bar{x}^\top \bar{s} \quad \text{Cauchy-Schwartz inequality} \\ &\geq 0\end{aligned}$$

# Complementary Slackness for SOCP

- Given  $x \succeq_{\mathcal{Q}} 0$ ,  $s \succeq_{\mathcal{Q}} 0$  and  $x^\top s = 0$ . Assume  $x_0 > 0$  and  $s_0 > 0$
- We have

$$(*) \quad x_0^2 \geq \sum_{i=1}^n x_i^2$$

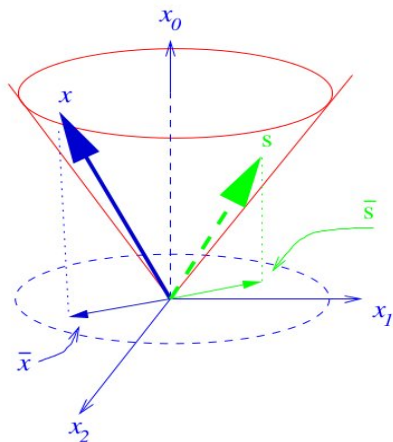
$$(**) \quad s_0^2 \geq \sum_{i=1}^n s_i^2 \iff x_0^2 \geq \sum_{i=1}^n \frac{s_i^2 x_0^2}{s_0^2}$$

$$(***) \quad x^\top s = 0 \iff -x_0 s_0 = \sum_i x_i s_i \iff -2x_0^2 = \sum_{i=1}^n \frac{2x_i s_i x_0}{s_0}$$

- Adding (\*), (\*\*), (\*\*\*), we get  $0 \geq \sum_{i=1}^n \left( x_i + \frac{s_i x_0}{s_0} \right)^2$
- This implies

$$x_i s_0 + x_0 s_i = 0, \text{ for } i = 1, \dots, n$$

# Illustration of SOC



When  $x \succeq_Q 0$ ,  $s \succeq_Q 0$  are orthogonal both must be on the boundary in such a way that their projection on the  $x_1, \dots, x_n$  plane is collinear

# Strong Duality

- at the optimum

$$c^T x = b^T y \iff x^T s = 0$$

- Like SDP constraint qualifications are required
- If there is primal-feasible  $x \succ_Q 0$
- If there is dual-feasible  $s \succ_Q 0$

# Complementary Slackness for SOCP

- Thus again we have a square system

$$\begin{aligned}Ax &= b, & x &\succeq_{\mathcal{Q}} 0, \\A^{\top}y + s &= c, & s &\succeq_{\mathcal{Q}} 0, \\x^{\top}s &= 0 \\x_0s_i + s_0x_i &= 0\end{aligned}$$