Parallel Multi-Block ADMM with $o(1/k)$ Convergence

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Linearly Constrained Separable Problem

$$\begin{aligned} \text{minimize} \quad & f_1(\mathbf{x}_1) + \cdots + f_N(\mathbf{x}_N) \\ \text{subject to} \quad & A_1\mathbf{x}_1 \ + \cdots + A_N\mathbf{x}_N = c, \\ & \mathbf{x}_1 \in \mathcal{X}_1, \ \ldots, \ \mathbf{x}_N \in \mathcal{X}_N. \end{aligned}$$

- $f_i : \mathbb{R}^{n_i} \rightarrow (-\infty, +\infty]$ are convex functions. $N \geq 2$.
- a.k.a. *extended monotropic programming* [Bertsekas, 2008]
- Examples:
  - Linear programming
  - Multi-agent network optimization
  - Exchange problem
  - Regularization model

Parallel and Distributed Algorithms

Motivation:

- Data may be collected and stored in a distributed way
- Often difficult to minimize all the f_i's jointly

Strategy:

- Decompose the problem into N *simpler and smaller* subproblems
- Solve subproblems in parallel
- Coordinate by passing some information

Dual Decomposition

$$\begin{cases} (\mathbf{x}_1^{k+1}, \mathbf{x}_2^{k+1}, \ldots, \mathbf{x}_N^{k+1}) = \arg\min_{\{\mathbf{x}_i\}} \ \mathcal{L}(\mathbf{x}_1, \ldots, \mathbf{x}_N, \lambda^k), \\ \lambda^{k+1} = \lambda^k - \alpha_k \left(\sum_{i=1}^N A_i \mathbf{x}_i^{k+1} - c \right), \quad \alpha_k > 0. \end{cases}$$

- Lagrangian:

$$\mathcal{L}(\mathbf{x}_1, \ldots, \mathbf{x}_N, \lambda) = \sum_{i=1}^N f_i(\mathbf{x}_i) - \lambda^\top \left(\sum_{i=1}^N A_i \mathbf{x}_i - c \right)$$

- $\mathbf{x}$-step has N decoupled $\mathbf{x}_i$-subproblems, *parallelizable*:

$$\mathbf{x}_i^{k+1} = \arg\min_{\mathbf{x}_i} f_i(\mathbf{x}_i) - \langle \lambda^k, A_i \mathbf{x}_i \rangle, \text{ for } i = 1, 2, \ldots, N,$$

- Convergence rate: $O(1/\sqrt{k})$ (for general convex problems)
- Often slow convergence in practice

Distributed ADMM

[Bertsekas and Tsitsiklis, 1997, Boyd et al., 2010, Wang et al., 2013]

- Variable splitting:

$$\begin{aligned}\min_{\{\mathbf{x}_i\},\{\mathbf{z}_i\}} \quad & \sum_{i=1}^{N} f_i(\mathbf{x}_i)\\ \text{s.t.} \quad & A_i\mathbf{x}_i - \mathbf{z}_i = \frac{c}{N},\ i = 1, 2, \ldots, N,\\ & \sum_{i=1}^{N} \mathbf{z}_i = 0.\end{aligned}$$

- Apply ADMM, alternatively update $\{\mathbf{x}_i\}$ and $\{\mathbf{z}_i\}$, then multipliers $\{\lambda_i\}$:

$$\mathbf{z}_i^{k+1} = \left(A_i\mathbf{x}_i^k - \frac{c}{N} - \frac{\lambda_i^k}{\rho}\right) - \frac{1}{N}\sum_{j=1}^{N}\left(A_j\mathbf{x}_j^k - \frac{c}{N} - \frac{\lambda_j^k}{\rho}\right), \forall i;$$

$$\mathbf{x}_i^{k+1} = \arg\min_{\mathbf{x}_i}\ f_i(\mathbf{x}_i) + \frac{\rho}{2}\left\|A_i\mathbf{x}_i - \mathbf{z}_i^{k+1} - \frac{c}{N} - \frac{\lambda_i^k}{\rho}\right\|^2, \forall i;$$

$$\lambda_i^{k+1} = \lambda_i^k - \rho\left(A_i\mathbf{x}_i^{k+1} - \mathbf{z}_i^{k+1} - \frac{c}{N}\right), \forall i.$$

Jacobi ADMM

- Augmented Lagrangian:

$$\mathcal{L}_\rho(\mathbf{x}_1, \ldots, \mathbf{x}_N, \lambda) = \sum_{i=1}^N f_i(\mathbf{x}_i) - \lambda^\top \left(\sum_{i=1}^N A_i \mathbf{x}_i - c \right) + \frac{\rho}{2} \left\| \sum_{i=1}^N A_i \mathbf{x}_i - c \right\|^2$$

- Do not introduce $\{\mathbf{z}_i\}$, directly apply Jacobi-type block minimization:

$$\begin{aligned} \mathbf{x}_i^{k+1} &= \arg\min_{\mathbf{x}_i} \; \mathcal{L}_\rho(\mathbf{x}_1^k, \ldots, \mathbf{x}_{i-1}^k, \mathbf{x}_i, \mathbf{x}_{i+1}^k, \ldots, \mathbf{x}_N^k, \lambda^k) \\ &= \arg\min_{\mathbf{x}_i} \; f_i(\mathbf{x}_i) + \frac{\rho}{2} \left\| A_i \mathbf{x}_i + \sum_{j \neq i} A_j \mathbf{x}_j^k - c - \frac{\lambda^k}{\rho} \right\|^2 \end{aligned}$$

for $i = 1, \ldots, N$ **in parallel**;

$$\lambda^{k+1} = \lambda^k - \rho \left(\sum_{i=1}^N A_i \mathbf{x}_i^{k+1} - c \right).$$

- Not necessarily convergent (even if $N = 2$)
- Need *either* conditions *or* modifications to converge

A Sufficient Condition for Convergence

Theorem

Suppose that there exists $\delta > 0$ such that

$$\|A_i^\top A_j\| \le \delta,\ \forall\ i \ne j, \quad \textit{and} \quad \lambda_{\min}(A_i^\top A_i) > 3(N-1)\delta,\ \forall\ i,$$

Then Jacobi ADMM converges to a solution.

The assumption basically says:

- $\{A_i,\ i = 1, 2, \ldots, N\}$ are mutually "near-orthogonal"
- every A_i has full column rank and is sufficiently strong.

Proximal Jacobi ADMM

1. for $i = 1, \ldots, N$ in **parallel**,

$$\mathbf{x}_i^{k+1} = \arg\min_{\mathbf{x}_i} f_i(\mathbf{x}_i) + \frac{\rho}{2} \left\| A_i \mathbf{x}_i + \sum_{j \neq i} A_j \mathbf{x}_j^k - b - \frac{\lambda^k}{\rho} \right\|^2 + \frac{1}{2} \left\| \mathbf{x}_i - \mathbf{x}_i^k \right\|_{P_i}^2;$$

2. $\lambda^{k+1} = \lambda^k - \gamma\rho \left(\sum_{i=1}^N A_i \mathbf{x}_i^{k+1} - b \right), \ \gamma > 0.$

- The added proximal term is critical to convergence.
- Some forms of $P_i \succeq 0$ make subproblems *easier to solve* and *more stable*.
- Global $o(1/k)$ convergence if P_i and γ are properly chosen.
- Suitable for parallel and distributed computing.

Little-o convergence

Lemma

If a sequence $\{a_k\} \subset \mathbb{R}$ *obeys:*

$$a_k \geq 0 \quad \textit{and} \quad \sum_{t=1}^{\infty} a_t < \infty,$$

then we have:

1. *(convergence)* $\lim_{k\to\infty} a_k = 0$;
2. *(ergodic convergence)* $\frac{1}{k}\sum_{t=1}^{k} a_t = O\left(\frac{1}{k}\right)$;
3. *(running best)* $\min_{t\leq k}\{a_t\} = o\left(\frac{1}{k}\right)$;
4. *(non-ergodic convergence) if* a_k *is monotonically nonincreasing, then* $a_k = o\left(\frac{1}{k}\right)$.

Convergence of Proximal Jacobi ADMM

Sufficient condition: there exist $\epsilon_i > 0,\ i = 1, 2, \ldots, N$ such that

$$\textbf{(C1)} \quad \begin{cases} P_i \succ \rho(\frac{1}{\epsilon_i} - 1)A_i^\top A_i,\ i = 1, 2, \ldots, N \\ \sum_{i=1}^N \epsilon_i < 2 - \gamma. \end{cases}$$

Simplification to (C1): set $\epsilon_i < \frac{2-\gamma}{N}$:

$$P_i \succ \rho \left(\frac{N}{2-\gamma} - 1 \right) A_i^\top A_i,\ i = 1, 2, \ldots, N$$

- $P_i = \tau_i \mathbf{I}$ (standard proximal method): $\tau_i > \rho \left(\frac{N}{2-\gamma} - 1 \right) \|A_i\|^2$
- $P_i = \tau_i \mathbf{I} - \rho A_i^\top A_i$ (prox-linear method): $\tau_i > \frac{\rho N}{2-\gamma} \|A_i\|^2$

$o(1/k)$ **Convergence Rate**

Notation:

$$G_x := \begin{pmatrix} P_1 + \rho A_1^\top A_1 & & \\ & \ddots & \\ & & P_N + \rho A_N^\top A_N \end{pmatrix}, \; G_x' := G_x - \rho A^\top A$$

Theorem

If $G_x' \succeq 0$ and the condition (C1) holds, then

$$\|\mathbf{x}^k - \mathbf{x}^{k+1}\|_{G_x'}^2 = o(1/k) \quad \textit{and} \quad \|\lambda^k - \lambda^{k+1}\|^2 = o(1/k).$$

Note: $(\mathbf{x}^{k+1}, \lambda^{k+1})$ is optimal if $\|\mathbf{x}^k - \mathbf{x}^{k+1}\|_{G_x'}^2 = 0,\ \|\lambda^k - \lambda^{k+1}\|^2 = 0$. The quantity $\|\mathbf{u}^k - \mathbf{u}^{k+1}\|_{G'}^2$ as a measure of the convergence rate. Proof is similar to He and Yuan [2012], He et al. [2013].

Adaptive Parameter Tuning

- Condition (C1) may be rather conservative.
- Adaptively adjusting the matrices $\{P_i\}$ with guaranteed convergence.

Initialize with small $P_i^0 \succeq 0$ $(i = 1, 2, \ldots, N)$ and a small $\eta > 0$;

for $k = 1, 2, \ldots$ **do**

  if $h(\mathbf{u}^{k-1}, \mathbf{u}^k) > \eta \cdot \|\mathbf{u}^{k-1} - \mathbf{u}^k\|_G^2$ **then**

    $P_i^{k+1} \leftarrow P_i^k, \ \forall i$;

  else

    Increase P_i: $P_i^{k+1} \leftarrow \alpha_i P_i^k + \beta_i Q_i \ (\alpha_i > 1, \ \beta_i \geq 0, \ Q_i \succ 0), \forall i$;

    Restart: $\mathbf{u}^k \leftarrow \mathbf{u}^{k-1}$;

Note: $h(\mathbf{u}^k, \mathbf{u}^{k+1})$ can be computed at little extra cost in the algorithm.

- Often yields much smaller paramters $\{P_i\}$ than those required by condition (C1), leading to substantially faster convergence in practice.

Numerical Experiments

Compare several parallel splitting algorithms:

- **Prox-JADMM**: Proximal Jacobi ADMM [this work]
- **VSADMM**: distributed ADMM, **variable splitting** [Bertsekas, 2008]
- **Corr-JADMM**: Jacobian ADMM with correction steps [He et al., 2013]

They have roughly the same per-iteration cost (in terms of both computation and communication).

Exchange Problem

Consider a network of N agents that exchange n commodities.

$$\min_{\{\mathbf{x}_i\}} \sum_{i=1}^{N} f_i(\mathbf{x}_i) \qquad \text{s.t.} \quad \sum_{i=1}^{N} \mathbf{x}_i = 0.$$

- $\mathbf{x}_i \in \mathbb{R}^n$ $(i = 1, 2, \ldots, N)$: quantities of commodities that are exchanged by agents i.
- $f_i : \mathbb{R}^n \to \mathbb{R}$: cost function for agent i.

Numerical Result

Let $f_i(\mathbf{x}_i) := \frac{1}{2}\|C_i\mathbf{x}_i - d_i\|^2$, $C_i \in \mathbb{R}^{p\times n}$ and $d_i \in \mathbb{R}^p$.

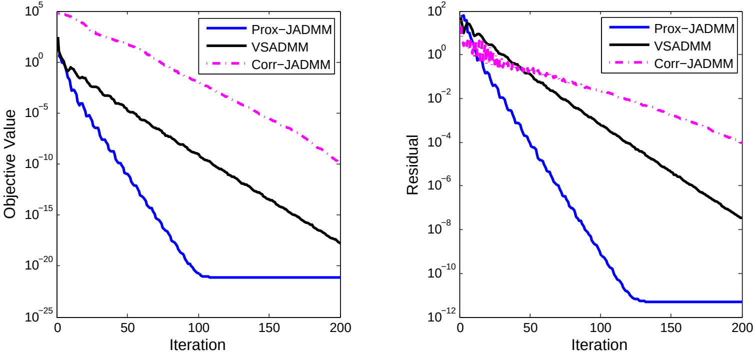


Figure: Exchange problem $(n = 100,\ N = 100,\ p = 80)$.

Basis Pursuit

Finding sparse solutions of an under-determined linear system:

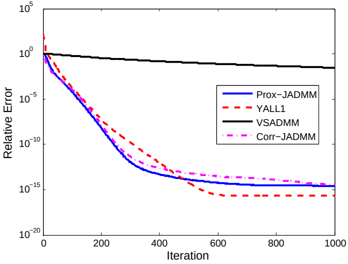
$$\min_{\mathbf{x}} \ \|\mathbf{x}\|_1 \quad \text{s.t.} \ \ A\mathbf{x} = c$$

- $\mathbf{x} \in \mathbb{R}^n$, $A \in \mathbb{R}^{m \times n}$ $(m < n)$
- Partition data into N blocks:

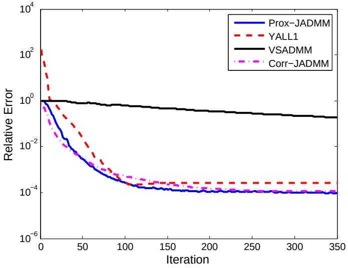
$$\mathbf{x} = [\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_N], \ A = [A_1, A_2, \dots, A_N], \ f_i(\mathbf{x}_i) = \|\mathbf{x}_i\|_1$$

- **YALL1**: a dual-ADMM solver for the basis pursuit problem.

Numerical Result



(a) Noise-free $(\sigma = 0)$



(b) Noise added $(\sigma = 10^{-3})$

Figure: ℓ_1-problem $(n = 1000,\ m = 300,\ k = 60)$.

Amazon EC2

Tested solve two basis pursuit problems.

	m	n	k	Size
dataset 1	1.0×10^5	2.0×10^5	2.0×10^3	150GB
dataset 2	1.5×10^5	3.0×10^5	3.0×10^3	337GB

Environment:

- C code uses GSL and MPI, about 300 lines
- 10 instances from Amazon, each with 8 cores and 68GB RAM
- price: $17 each hour

	150GB Test			337GB Test		
	Itr	Time(s)	Cost($)	Itr	Time(s)	Cost($)
Data generation	–	44.4	0.21	–	99.5	0.5
CPU per iteration	–	1.32	–	–	2.85	–
Comm. per iteration	–	0.07	–	–	0.15	–
Reach 10^{-1}	23	30.4	0.14	27	79.08	0.37
Reach 10^{-2}	30	39.4	0.18	39	113.68	0.53
Reach 10^{-3}	86	112.7	0.53	84	244.49	1.15
Reach 10^{-4}	234	307.9	1.45	89	259.24	1.22

Summary

- It is feasible to extend ADMM from 2 blocks to **3 or more blocks**
- Jacobi ADMM is good for problems with large and distributed data
- Gauss-Seidel ADMM is good for 3 or a few more blocks,
  Jacobi ADMM is good for many blocks
- Asynchronous subproblems become a real need (talk to Dong Qian)

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