13. Douglas-Rachford method and ADMM

- Douglas-Rachford splitting method
- examples
- alternating direction method of multipliers
- image deblurring example
- convergence

Douglas-Rachford splitting algorithm

$$\text{minimize} \quad f(x) = g(x) + h(x)$$

g and h are closed convex functions

Douglas-Rachford iteration: starting at any $z^{(0)}$, repeat

$$\begin{aligned} x^{(k)} &= \mathrm{prox}_{th}(z^{(k-1)}) \\ y^{(k)} &= \mathrm{prox}_{tg}(2x^{(k)} - z^{(k-1)}) \\ z^{(k)} &= z^{(k-1)} + y^{(k)} - x^{(k)} \end{aligned}$$

- t is a positive constant (simply scales the objective)
- useful when g and h have inexpensive prox-operators
- under weak conditions (existence of a minimizer), $x^{(k)}$ converges

Equivalent form

- start iteration at y-update

$$y^+ = \mathrm{prox}_{tg}(2x - z), \qquad z^+ = z + y^+ - x, \qquad x^+ = \mathrm{prox}_{th}(z^+)$$

- switch z- and x-updates

$$y^+ = \mathrm{prox}_{tg}(2x - z), \qquad x^+ = \mathrm{prox}_{th}(z + y^+ - x), \qquad z^+ = z + y^+ - x$$

- make change of variables $w = z - x$

alternate form of DR iteration: start at $x^{(0)} \in \mathbf{dom}\, h$, $w^{(0)} \in t\partial h(x^{(0)})$

$$\begin{aligned} y^+ &= \mathrm{prox}_{tg}(x - w) \\ x^+ &= \mathrm{prox}_{th}(y^+ + w) \\ w^+ &= w + y^+ - x^+ \end{aligned}$$

Interpretation as fixed-point iteration

Douglas-Rachford iteration (p. 13-2) can be written as

$$z^{(k)} = F(z^{(k-1)})$$

where $F(z) = z + \mathrm{prox}_{tg}(2\mathrm{prox}_{th}(z) - z) - \mathrm{prox}_{th}(z)$

fixed points of F and minimizers of $g + h$

- if z is a fixed point, then $x = \mathrm{prox}_{th}(z)$ is a minimizer:

$$\begin{aligned} z = F(z), \quad x = \mathrm{prox}_{th}(z) \quad &\Longrightarrow \quad \mathrm{prox}_{tg}(2x - z) = x = \mathrm{prox}_{th}(z) \\ &\Longrightarrow \quad -x + z \in t\partial g(x), \quad z - x \in t\partial h(x) \\ &\Longrightarrow \quad 0 \in t\partial g(x) + t\partial h(x) \end{aligned}$$

- if x is a minimizer and $u \in t\partial g(x) \cap -t\partial h(x)$, then $x + u = F(x + u)$

Douglas-Rachford iteration with relaxation

fixed-point iteration with relaxation

$$z^+ = z + \rho(F(z) - z)$$

$1 < \rho < 2$ is overrelaxation, $0 < \rho < 1$ is underrelaxation

first version of DR method

$$\begin{aligned} x^+ &= \operatorname{prox}_{th}(z) \\ y^+ &= \operatorname{prox}_{tg}(2x^+ - z) \\ z^+ &= z + \rho(y^+ - x^+) \end{aligned}$$

alternate version

$$\begin{aligned} y^+ &= \operatorname{prox}_{tg}(x - w) \\ x^+ &= \operatorname{prox}_{th}((1-\rho)x + \rho y^+ + w) \\ w^+ &= w + \rho y^+ + (1-\rho)x - x^+ \end{aligned}$$

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Sparse inverse covariance selection

$$\text{minimize} \quad \mathbf{tr}(CX) - \log\det X + \rho \sum_{i>j} |X_{ij}|$$

variable is $X \in \mathbf{S}^n$; parameters $C \in \mathbf{S}^n_{++}$ and $\rho > 0$ are given

Douglas-Rachford splitting

$$g(X) = \mathbf{tr}(CX) - \log\det X, \qquad h(X) = \rho \sum_{i>j} |X_{ij}|$$

- $X = \operatorname{prox}_{tg}(\hat{X})$ is positive solution of $C - X^{-1} + (1/t)(X - \hat{X}) = 0$

  easily solved via eigenvalue decomposition of $\hat{X} - tC$ (see homework)

- $X = \operatorname{prox}_{th}(\hat{X})$ is soft-thresholding

Spingarn's method of partial inverses

equality constrained convex problem

$$\begin{array}{ll} \text{minimize} & h(x) \\ \text{subject to} & x \in V \end{array}$$

h a closed convex function; V a subspace

Douglas-Rachford splitting: take $g = I_V$ (indicator of V)

$$\begin{aligned} x^+ &= \operatorname{prox}_{th}(z) \\ y^+ &= P_V(2x^+ - z) \\ z^+ &= z + y^+ - x^+ \end{aligned}$$

Application to composite optimization problem

$$\text{minimize} \quad f_1(x) + f_2(Ax)$$

f_1 and f_2 have simple prox-operators

- equivalent to minimizing $h(x_1, x_2)$ over subspace V where

$$h(x_1, x_2) = f_1(x_1) + f_2(x_2), \qquad V = \{(x_1, x_2) \mid x_2 = Ax_1\}$$

- prox_{th} is separable: $\mathrm{prox}_{th}(x_1, x_2) = (\mathrm{prox}_{tf_1}(x_1), \mathrm{prox}_{tf_2}(x_2))$

- projection of (x_1, x_2) on V reduces to linear equation:

$$\begin{aligned} P_V(x_1, x_2) &= \begin{bmatrix} I \\ A \end{bmatrix} (I + A^T A)^{-1}(x_1 + A^T x_2) \\ &= \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} A^T \\ -I \end{bmatrix} (I + AA^T)^{-1}(x_2 - Ax_1) \end{aligned}$$

Decomposition of separable problems

$$\text{minimize} \quad \sum_{j=1}^{n} f_j(x_j) + \sum_{i=1}^{m} g_i(A_{i1}x_1 + \cdots + A_{in}x_n)$$

- same problem as p.12-10, but without strong convexity assumption
- we assume the functions f_j and g_i have inexpensive prox-operators

equivalent formulation

$$\begin{array}{ll} \text{minimize} & \sum_{j=1}^{n} f_j(x_j) + \sum_{i=1}^{m} g_i(y_{i1} + \cdots + y_{in}) \\ \text{subject to} & y_{ij} = A_{ij}x_j, \quad i = 1, \ldots, m, \quad j = 1, \ldots, n \end{array}$$

- prox-operator of cost involves uncoupled prox-evaluations for f_j, g_i
- projection on constraint set reduces to n independent linear equations

Decomposition of separable problems

second equivalent formulation with extra splitting variables x_{ij}:

$$
\begin{array}{ll}
\text{minimize} & \sum_{j=1}^{n} f_j(x_j) + \sum_{i=1}^{m} g_i(y_{i1} + \cdots + y_{in}) \\
\text{subject to} & x_{ij} = x_j, \qquad i = 1, \ldots, m, \quad j = 1, \ldots, n \\
& y_{ij} = A_{ij} x_{ij}, \quad i = 1, \ldots, m, \quad j = 1, \ldots, n
\end{array}
$$

- make first set of constraints part of domain of f_j:

$$
\tilde{f}_j(x_j, x_{1j}, \ldots, x_{mj}) = \begin{cases} f_j(x_j) & x_{ij} = x_j, \quad i = 1, \ldots, m \\ +\infty & \text{otherwise} \end{cases}
$$

  prox-operator of $\tilde{f}_j$ reduces to prox-operator of f_j

- projection on other constraints involves mn independent linear equations

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Dual application of Douglas-Rachford method

separable convex problem

$$\begin{array}{ll} \text{minimize} & f_1(x_1) + f_2(x_2) \\ \text{subject to} & A_1 x_1 + A_2 x_2 = b \end{array}$$

dual problem

$$\text{maximize} \quad -b^T z - f_1^*(-A_1^T z) - f_2^*(-A_2^T z)$$

we apply the Douglas-Rachford method (page 13-3) to minimize

$$\underbrace{b^T z + f_1^*(-A_1^T z)}_{g(z)} + \underbrace{f_2^*(-A_2^T z)}_{h(z)}$$

Douglas Rachford on the dual

$$y^+ = \mathrm{prox}_{tg}(z - w), \qquad z^+ = \mathrm{prox}_{th}(y^+ + w), \qquad w^+ = w + y^+ - z^+$$

first line: use result on page 10-10 to compute $y^+ = \mathrm{prox}_{tg}(z - w)$

$$\begin{aligned} \hat{x}_1 &= \operatorname*{argmin}_{x_1} \left(f_1(x_1) + z^T(A_1x_1 - b) + \frac{t}{2}\|A_1x_1 - b - w/t\|_2^2\right) \\ y^+ &= z - w + t(A_1\hat{x}_1 - b) \end{aligned}$$

second line: similarly, compute $z^+ = \mathrm{prox}_{th}(z + t(A_1\hat{x}_1 - b))$

$$\begin{aligned} \hat{x}_2 &= \operatorname*{argmin}_{x_2} \left(f_2(x_2) + z^TA_2x_2 + \frac{t}{2}\|A_1\hat{x}_1 + A_2x_2 - b\|_2^2\right. \\ z^+ &= z + t(A_1\hat{x}_1 + A_2\hat{x}_2 - b)) \end{aligned}$$

third line reduces to $w^+ = -tA_2\hat{x}_2$

Alternating direction method of multipliers

1. minimize augmented Lagrangian over x_1

$$x_1^{(k)} = \underset{x_1}{\operatorname{argmin}} \left(f_1(x_1) + (z^{(k-1)})^T A_1 x_1 + \frac{t}{2} \|A_1 x_1 + A_2 x_2^{(k-1)} - b\|_2^2 \right)$$

2. minimize augmented Lagrangian over x_2

$$x_2^{(k)} = \underset{x_2}{\operatorname{argmin}} \left(f_2(x_2) + (z^{(k-1)})^T A_2 x_2 + \frac{t}{2} \|A_1 x_1^{(k)} + A_2 x_2 - b\|_2^2 \right)$$

3. dual update

$$z^{(k)} = z^{(k-1)} + t(A_1 x_1^{(k)} + A_2 x_2^{(k)} - b)$$

also known as split Bregman method

Comparison with other multiplier methods

alternating minimization method (page 12-13) with $g(y) = I_{\{b\}}(y)$

- same dual update, same update for x_2
- x_1-update in alternating minimization method is simpler:

$$x_1^{(k)} = \operatorname*{argmin}_{x_1} \left(f_1(x_1) + (z^{(k-1)})^T A_1 x_1 \right)$$

- ADMM does not require strong convexity of f_1

augmented Lagrangian method (page 12-14) with $g(y) = I_{\{b\}}(y)$

- dual update is the same
- AL method requires joint minimization of the augmented Lagrangian

$$f_1(x_1) + f_2(x_2) + (z^{(k-1)})^T (A_1 x_1 + A_2 x_2) + \frac{t}{2} \|A_1 x_1 + A_2 x_2 - b\|_2^2$$

Application to composite optimization (method 1)

$$\text{minimize} \quad f_1(x) + f_2(Ax)$$

apply ADMM to

$$\begin{array}{ll} \text{minimize} & f_1(x_1) + f_2(x_2) \\ \text{subject to} & Ax_1 = x_2 \end{array}$$

- augmented Lagrangian is

$$f_1(x_1) + f_2(x_2) + \frac{t}{2}\|Ax_1 - x_2 + z/t\|_2^2$$

- x_1-update requires minimization of $f_1(x_1) + (t/2)\|Ax_1 - x_2 + z/t\|_2^2$
- x_2-update is evaluation of $\mathrm{prox}_{t^{-1}f_2}$

Application to composite optimization (method 2)

introduce extra 'splitting' or 'dummy' variable x_3

$$\begin{array}{ll} \text{minimize} & f_1(x_3) + f_2(x_2) \\ \text{subject to} & \begin{bmatrix} A \\ I \end{bmatrix} x_1 = \begin{bmatrix} x_2 \\ x_3 \end{bmatrix} \end{array}$$

- alternate minimization of augmented Lagrangian over x_1 and (x_2, x_3)

$$f_1(x_3) + f_2(x_2) + \frac{t}{2}\left(\|Ax_1 - x_2 + z_1/t\|_2^2 + \|x_1 - x_3 + z_2/t\|_2^2\right)$$

- x_1-update: linear equation with coefficient $I + A^T A$
- (x_2, x_3)-update: decoupled evaluations of $\operatorname{prox}_{t^{-1}f_1}$ and $\operatorname{prox}_{t^{-1}f_2}$

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Image blurring model

$$b = Kx_{\mathrm{t}} + w$$

- x_{t} is unknown image
- b is observed (blurred and noisy) image; w is noise
- $N \times N$-images are stored in column-major order as vectors of length N^2

blurring matrix K

- represents 2D convolution with space-invariant point spread function
- with periodic boundary conditions, block-circulant with circulant blocks
- can be diagonalized by multiplication with unitary 2D DFT matrix W:

$$K = W^H \mathbf{diag}(\lambda) W$$

equations with coefficient $I + K^T K$ can be solved in $O(N^2 \log N)$ time

Total variation deblurring with 1-norm

$$\begin{array}{ll} \text{minimize} & \|Kx - b\|_1 + \gamma \|Dx\|_{\text{tv}} \\ \text{subject to} & 0 \preceq x \preceq \mathbf{1} \end{array}$$

second term in objective is **total variation penalty**

- Dx is discretized first derivative in vertical and horizontal direction

$$D = \begin{bmatrix} I \otimes D_1 \\ D_1 \otimes I \end{bmatrix}, \qquad D_1 = \begin{bmatrix} -1 & 0 & 0 & \cdots & 0 & 0 & 1 \\ 1 & -1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & -1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 1 & -1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & -1 \end{bmatrix}$$

- $\|\cdot\|_{\text{tv}}$ is a sum of Euclidean norms: $\|(u,v)\|_{\text{tv}} = \sum_{i=1}^{n} \sqrt{u_i^2 + v_i^2}$

Solution via Douglas-Rachford method

an example of a composite optimization problem

$$\text{minimize} \quad f_1(x) + f_2(Ax)$$

with f_1 the indicator of $[0,1]^n$ and

$$A = \begin{bmatrix} K \\ D \end{bmatrix}, \qquad f_2(u,v) = \|u\|_1 + \gamma\|v\|_{\mathrm{tv}}$$

primal DR method (page 13-8) and **ADMM** (page 13-16) require:

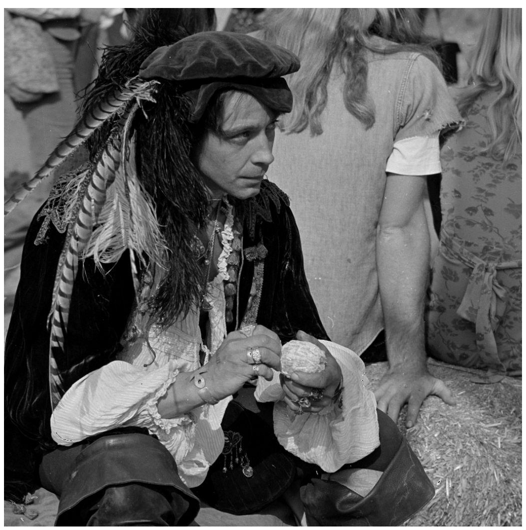
- decoupled prox-evaluations of $\|u\|_1$ and $\gamma\|v\|_{\mathrm{tv}}$, and projections on C
- solution of linear equations with coefficient matrix

$$I + K^T K + D^T D$$

  solvable in $O(N^2 \log N)$ time

Example

- 1024×1024 image, periodic boundary conditions
- Gaussian blur
- salt-and-pepper noise (50% pixels randomly changed to $0/1$)

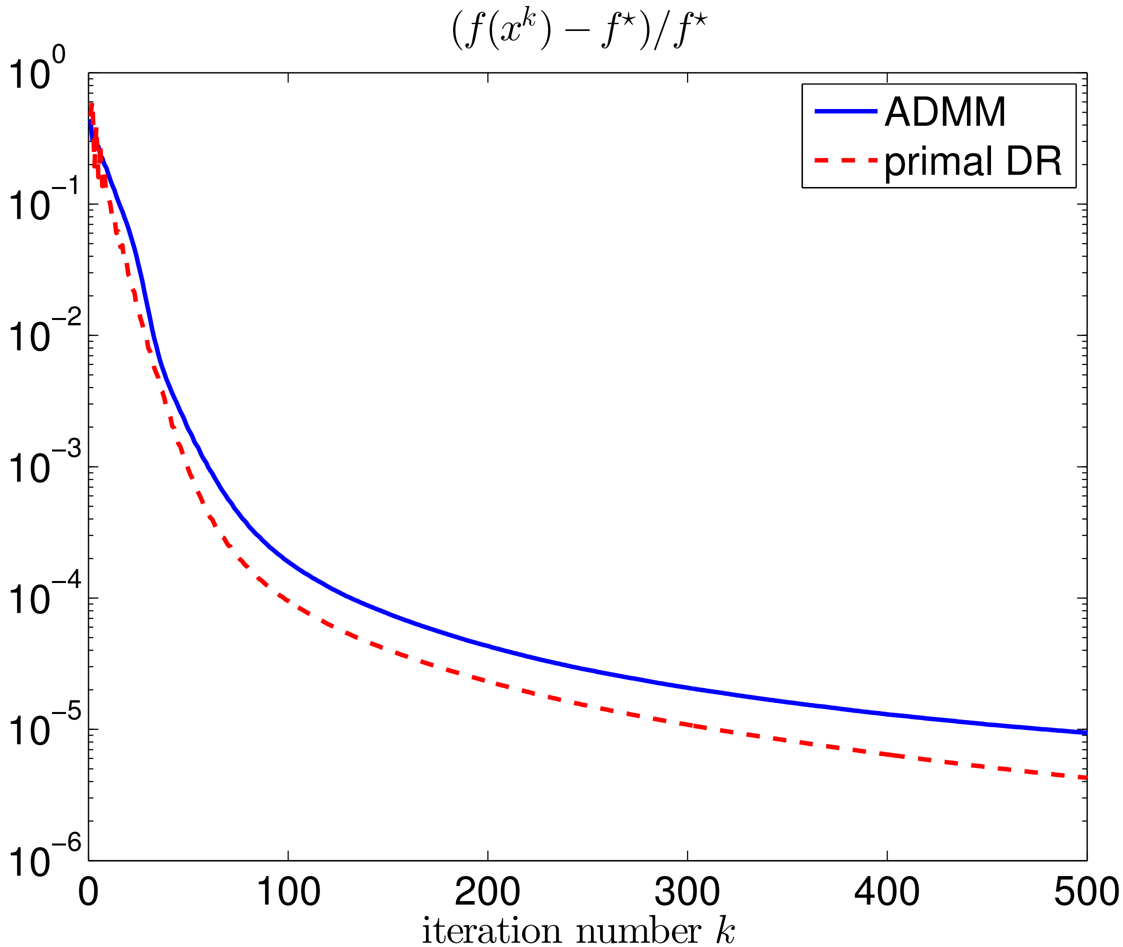


original

noisy/blurred

restored

Convergence



cost per iteration is dominated by 2D FFTs

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Douglas-Rachford iteration mappings

define iteration map F and negative step G (in notation of p. 13-5)

$$\begin{aligned} F(z) &= z + \mathrm{prox}_{tg}(2\mathrm{prox}_{th}(z) - z) - \mathrm{prox}_{th}(z) \\ G(z) &= z - F(z) \\ &= \mathrm{prox}_{th}(z) - \mathrm{prox}_{tg}(2\mathrm{prox}_{th}(z) - z) \end{aligned}$$

- F is firmly nonexpansive (co-coercive with parameter 1)

$$(F(z) - F(\hat{z}))^T(z - \hat{z}) \geq \|F(z) - F(\hat{z})\|_2^2 \qquad \forall z, \hat{z}$$

- implies that G is firmly nonexpansive:

$$\begin{aligned} &(G(z) - G(\hat{z}))^T(z - \hat{z}) \\ &\quad = \|G(z) - G(\hat{z})\|_2^2 + (F(z) - F(\hat{z}))^T(z - \hat{z}) - \|F(z) - F(\hat{z})\|_2^2 \\ &\quad \geq \|G(z) - G(\hat{z})\|_2^2 \end{aligned}$$

proof (firm nonexpansiveness of F)

- define $x = \mathrm{prox}_{th}(z)$, $\hat{x} = \mathrm{prox}_{th}(\hat{z})$, and

$$y = \mathrm{prox}_{tg}(2x - z), \qquad \hat{y} = \mathrm{prox}_{g}(2\hat{x} - \hat{z})$$

- substitute expressions $F(z) = z + y - x$ and $F(\hat{z}) = \hat{z} + \hat{y} - \hat{x}$:

$$\begin{aligned}
&(F(z) - F(\hat{z}))^T(z - \hat{z}) \\
&\quad \geq \quad (z + y - x - \hat{z} - \hat{y} + \hat{x})^T(z - \hat{z}) - (x - \hat{x})^T(z - \hat{z}) + \|x - \hat{x}\|_2^2 \\
&\quad = \quad (y - \hat{y})^T(z - \hat{z}) + \|z - x - \hat{z} + \hat{x}\|_2^2 \\
&\quad = \quad (y - \hat{y})^T(2x - z - 2\hat{x} + \hat{z}) - \|y - \hat{y}\|_2^2 + \|F(z) - F(\hat{z})\|_2^2 \\
&\quad \geq \quad \|F(z) - F(\hat{z})\|_2^2
\end{aligned}$$

inequalities use firm nonexpansiveness of prox_{th} and prox_{tg} (p. 6-9):

$$(x - \hat{x})^T(z - \hat{z}) \geq \|x - \hat{x}\|_2^2, \qquad (2x - z - 2\hat{x} + \hat{z})^T(y - \hat{y}) \geq \|y - \hat{y}\|_2^2$$

Convergence result

$$
\begin{aligned}
z^{(k)} &= (1-\rho_k)z^{(k-1)} + \rho_k F(z^{(k-1)}) \\
&= z^{(k-1)} - \rho_k G(z^{(k-1)})
\end{aligned}
$$

assumptions

- optimal value $f^\star = \inf_x (g(x) + h(x))$ is finite and attained
- $\rho_k \in [\rho_{\min}, \rho_{\max}]$ with $0 < \rho_{\min} < \rho_{\max} < 2$

result

- $z^{(k)}$ converges to a fixed point $z^\star$ of F
- $x^{(k)} = \mathrm{prox}_{th}(z^{(k-1)})$ converges to a minimizer $x^\star = \mathrm{prox}_{th}(z^\star)$

  (follows from continuity of prox_{th})

proof: let $z^\star$ be any fixed point of $F(z)$ (zero of $G(z)$)

consider iteration k (with $z = z^{(k-1)}$, $\rho = \rho_k$, $z^+ = z^{(k)}$):

$$\begin{aligned}
\|z^+ - z^\star\|_2^2 - \|z - z^\star\|_2^2 &= 2(z^+ - z)^T(z - z^\star) + \|z^+ - z\|_2^2 \\
&= -2\rho G(z)^T(z - z^\star) + \rho^2\|G(z)\|_2^2 \\
&\leq -\rho(2-\rho)\|G(z))\|_2^2 \\
&\leq -M\|G(z))\|_2^2 \qquad (1)
\end{aligned}$$

where $M = \rho_{\min}(2 - \rho_{\max})$ (line 3 is firm nonexpansiveness of G)

- (1) implies that
$$M\sum_{k=0}^{\infty}\|G(z^{(k)})\|_2^2 \leq \|z^{(0)} - z^\star\|_2^2, \qquad \|G(z^{(k)})\|_2 \to 0$$
- (1) implies that $\|z^{(k)} - z^\star\|_2$ is nonincreasing; hence $z^{(k)}$ is bounded
- since $\|z^{(k)} - z^\star\|_2$ is nonincreasing, the limit $\lim_{k\to\infty}\|z^{(k)} - z^\star\|_2$ exists

proof (continued)

- since the sequence $z^{(k)}$ is bounded, it has a convergent subsequence
- let $\bar{z}_k$ be a convergent subsequence with limit $\bar{z}$; by continuity of G,

$$0 = \lim_{k\to\infty} G(\bar{z}_k) = G(\bar{z})$$

hence, $\bar{z}$ is a zero of G and the limit $\lim_{k\to\infty} \|z^{(k)} - \bar{z}\|_2$ exists

- let $\bar{z}_1$ and $\bar{z}_2$ be two limit points; the limits

$$\lim_{k\to\infty} \|z^{(k)} - \bar{z}_1\|_2, \qquad \lim_{k\to\infty} \|z^{(k)} - \bar{z}_2\|_2$$

exist, and subsequences of $z^{(k)}$ converge to $\bar{z}_1$, resp. $\bar{z}_2$; therefore

$$\|\bar{z}_2 - \bar{z}_1\|_2 = \lim_{k\to\infty} \|z^{(k)} - \bar{z}_1\|_2 = \lim_{k\to\infty} \|z^{(k)} - \bar{z}_2\|_2 = 0$$

References

Douglas-Rachford method, ADMM, Spingarn's method

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image deblurring: the example is taken from

D. O'Connor and L. Vandenberghe, *Primal-dual decomposition by operator splitting and applications to image deblurring* (2014)