Notes on Geometry of Surfaces

Contents

CHAPTER 1

Fundamental groups of Graphs

1. Review of group theory

1.1. Group and generating set.

DEFINITION 1.1. A group $(G, \cdot)$ is a set G endowed with an operation

$$\cdot : G \times G \to G, (a, b) \to a \cdot b$$

such that the following holds.

(1) $\forall a, b, c \in G, a \cdot (b \cdot c) = (a \cdot b) \cdot c$,
(2) $\exists 1 \in G$: $\forall a \in G$, $a \cdot 1 = 1 \cdot a$.
(3) $\forall a \in G$, $\exists a^{-1} \in G$: $a \cdot a^{-1} = a^{-1} \cdot a = 1$.

In the sequel, we usually omit $\cdot$ in $a \cdot b$ if the operation is clear or understood. By the associative law, it makes no ambiguity to write abc instead of $a \cdot (b \cdot c)$ or $(a \cdot b) \cdot c$.

EXAMPLES 1.2. (1) $(\mathbb{Z}^n, +)$ for any integer $n \geq 1$
(2) General Linear groups with matrix multiplication: $GL(n, \mathbb{R})$.
(3) Given a (possibly infinite) set X, the permutation group $Sym(X)$ is the set of all bijections on X, endowed with mapping composition.
(4) Dihedral group $D_{2n} = \langle r, s | s^2 = r^{2n} = 1, srs^{-1} = r^{-1} \rangle$. This group can be visualized as the symmetry group of a regular $(2n)$-polygon: s is the reflection about the axe connecting middle points of the two opposite sides, and r is the rotation about the center of the $2n$-polygon with an angle $\pi/2n$.
(5) Infinite Dihedral group $D_\infty = \langle r, s | s^2 = 1, srs^{-1} = r^{-1} \rangle$. We can think of a regular ∞-polygon as a real line. Consider a group action of D_∞ on the real line.

DEFINITION 1.3. A subset H in a group G is called a *subgroup* if H endowed with the group operation is itself a group. Equivalently, H is a subgroup of G if

(1) $\forall a, b \in H, a \cdot b \in H$
(2) $\forall a \in H$, $\exists a^{-1} \in H$: $a \cdot a^{-1} = a^{-1} \cdot a = 1$.

Note that (1) and (2) imply that the identity 1 lies in H.

Given a subset $X \subset G$, the *subgroup generated by* X, denoted by $\langle X \rangle$, is the minimal subgroup of G containing X. Explicitly, we have

$$\langle X \rangle = \{x_1^{\epsilon_1} x_2^{\epsilon_2} \cdots x_n^{\epsilon_n} : n \in \mathbb{N}, x_i \in X, \epsilon_i \in \{1, -1\}\}.$$

A subset X is called a *generating set* of G if $G = \langle X \rangle$. If X is finite, then G is called *finitely generated*.

Check Examples 1.2 and find out which are finitely generated, and if yes, write a generating set.

EXERCISE 1.4. (1) *Prove that* $(\mathbb{Q}, +)$ *is not finitely generated.*
(2) *Prove that* $\{r, rsr^{-1}\}$ *is a generating set for* D_∞.

EXERCISE 1.5. (1) *Suppose that* G *is a finitely generated group. If* $H \subset G$ *is of finite index in* G*, then* H *is finitely generated.*
(2) *Conversely, suppose that* H *is a finite index subgroup of a group* G*. If* H *is finitely generated, then* G *is also finitely generated.*

EXERCISE 1.6. *Let* N *be a normal subgroup of a group* G*. Suppose that* N *and* G/N *are finitely generated. Then* G *is finitely generated.*

1.2. Group action.

DEFINITION 1.7. Let G be a group and X be set. A *group action of* G *on* X is a function

$$G \times X \to X, (g, x) \to g \cdot x$$

such that the following holds.
(1) $\forall x \in X$, $1 \cdot x = x$.
(2) $\forall g, h \in G, (gh) \cdot x = g \cdot (h \cdot x)$.

Usually we say that G acts on X. Similarly, we often omit $\cdot$ in $g \cdot x$, but keep in mind that $gx \in X$, which is not a group element!

REMARK. A group can act *trivially* on any set X by just setting $g \cdot x = x$. So we are mainly interested in nontrivial group actions.

EXAMPLES 1.8. (1) $\mathbb{Z}$ acts on the real line $\mathbb{R}$: $(n, r) \to n + r$.
(2) $\mathbb{Z}$ acts on the circle $S^1 = \{e^{i\theta} : \theta \in \mathbb{R}\}$: $(n, e^{\theta i}) \to e^{n\theta i}$. Here i is the imaginary unit.
(3) $\mathbb{Z}^n$ acts on $\mathbb{R}^n$.
(4) $GL(n, \mathbb{R})$ acts on $\mathbb{R}^n$.

Recall that $Sym(X)$ is the permutation group of X. We have the following equivalent formulation of a group action.

LEMMA 1.9. *A group* G *acts on a set* X *if and only if there exists a group homomorphism* $G \to Sym(X)$.

PROOF. (=>). Define $\phi : G \to Sym(X)$ as follows. Given $g \in G$, let $\phi(g)(x) = g \cdot x$ for any $x \in X$. Here $g \cdot x$ is given in definition of the group action of G on X.

It is an exercise to verify that $\phi(g)$ is a bijection on X. Moreover, the condition (2) in definition 1.7 is amount to say that ϕ is homomorphism.

(<=). Let $\phi : G \to Sym(X)$ be a group homomorphism. Construct a map $G \times X \to X$: $(g, x) \to \phi(g)(x)$. Then it is easy to see that this map gives a group action of G on X. □

So when we say a group action of G on X, it is same as specifying a group homomorphism from G to $Sym(X)$.

REMARK. (1) A trivial group action is to specify a trivial group homomorphism, sending every element in G to the identity in $Sym(X)$.
(2) In general, the group homomorphism $G \to Sym(X)$ may not injective. If it is injective, we call the group action is *faithful*.

(3) In practice, the set X usually comes with extra nice structures, for example, X is a vector space, a topological space, or a metric space, etc. The homomorphic image of G in $Sym(X)$ may preserve these structures. In this case, we say that G acts on X by linear transformations, by homeomorphisms, or by isometries ...

We now recall Cayley's theorem, which essentially says that we should understand groups via group actions on sets with various good structures.

THEOREM 1.10. *Every group is a subgroup of the permutation group of a set.*

PROOF. Let $X = G$. Clearly the group operation $G \times G \to G$ gives a group action of G on G. Thus, we obtain a homomorphism $G \to Sym(G)$. The injectivity is clear. □

For any $x \in X$, the *orbit* of x under the group action is the set $\{g \cdot x : g \in G\}$. We denote it by $G \cdot x$ or even simply by Gx. The *stabilizer* of x

$$G_x := \{g \in G : g \cdot x = x\}$$

is clearly a group.

LEMMA 1.11. *Suppose that G acts on X. Then for any x, there exists a bijection between $\{gG_x : g \in G\}$ and Gx. In particular, if Gx is finite, then $[G : G_x] = |Gx|$.*

PROOF. We define a map $\phi : gG_x \to gx$. First, we need to show that this map is well-defined: that is to say, if $gG_x = g'G_x$, then $gx = g'x$. This follows from the definition of G_x.

For any $gx \in Gx$, we have $\phi(gG_x) = gx$. So ϕ is surjective.

To see that ϕ is injective, let $gG_x, g'G_x$ such that $gx = g'x$. Then $g^{-1}g'x = x$ and $g^{-1}g' \in G_x$. Hence $gG_x = g'G_x$. This shows that ϕ is injective. □

EXERCISE 1.12. (1) *Let H be a subgroup in G. Then $\cap_{g \in G}(gHg^{-1})$ is a normal subgroup in G.*
(2) *Let H be a finite index subgroup of G. Then there exists a normal subgroup N of G such that $N \subset H$ and $[G : N] < \infty$. (Hint: construct a group action)*

THEOREM 1.13 (M. Hall). *Suppose G is a finitely generated group. Then for any integer $n > 1$, there are only finitely many subgroups H in G such that $[G : H] = n$.*

PROOF. Fix n. Let H be a subgroup of index n. Let $X = \{H, g_1H, \cdots, g_{n-1}H\}$ be the set of all H-cosets. Then G acts on X of by left-multiplication. That is, $(g, g_iH) \to gg_iH$. Clearly, the stabilizer of $H \in X$ is $H \subset G$. Put in other words, the subgroup H can be recovered from the action of G on X.

For any set X with n elements, a finitely generated G has finitely many different actions on X. By Lemma 1.2, a group action is the same as a group homomorphism. A homomorphism is determined by the image of a generating set. As G is finitely generated and $Sym(X)$ is finite, there exist only finitely many group homomorphisms.

Consequently, for any $n > 0$, there exist only finitely many H of finite index n. □

2. Free groups and Ping-Pong Lemma

2.1. Words and their reduced forms. Let $\tilde{X}$ be an alphabet set. A *word* w over $\tilde{X}$ is a finite sequence of letters in $\tilde{X}$. We usually write $w = x_1 x_2 ... x_n$, where $x_i \in \tilde{X}$. The *empty word* is the word with an empty sequence of letters. The *length* of a word w is the length of the sequence of letters.

Two words are *equal* if their sequences of letters are identical. Denote by $\mathcal{W}(\tilde{X})$ the set of all words over $\tilde{X}$. Given two words $w, w' \in \mathcal{W}(\tilde{X})$, the *concatenation* of w and w' is a new word, denoted by ww', which is obtained from w followed by w'.

Given a set X, we take another set X^{-1} such that there exists a bijection $X \to X^{-1} : x \to x^{-1}$. Let $\tilde{X} = X \sqcup X^{-1}$ be the disjoint union of X and X^{-1}. Roughly speaking, the free group $F(X)$ generated by X will be the set of words $\mathcal{W}$ endowed with the operation of word concatenation.

Given a word w, if there exists two consecutive letters of form xx^{-1} or $x^{-1}x$ where $x, x^{-1} \in \tilde{X}$, then we call xx^{-1} or $x^{-1}x$ an *inverse pair* of w. A word w is called *reduced* if w contains no inverse pair. Given a word w, we define an operation on w called a *reduction*, by which we mean deleting an inverse pair xx^{-1} or $x^{-1}x$ to obtain a new word w':

$$w = w_1 x x^{-1} w_2 \xrightarrow{reduction} w' = w_1 w_2.$$

After a reduction, the length of a word decreases by 2. A finite sequence of reductions

$$w \xrightarrow{reduction\#1} w_1 \xrightarrow{reduction\#2} w_2 ... \xrightarrow{reduction\#n} w_n$$

will be referred to as a reduction process.

Clearly, any word w admits a reduction process to get a reduced word. This reduced word is called a *reduced form* of w. But a word may have different reduction processes to become reduced. For example, $w = xx^{-1}xx^{-1}$. However, we will prove that reduced forms of a word does not depend on the reduction process.

LEMMA 2.1. *Any word w has a unique reduced form.*

PROOF. We prove the lemma by induction on the length $|w|$ of w. The base cases that $|w| = 1, 2$ are trivial. Now assume that the lemma holds for any word of length $|w| \leq n$.

Let w be a word of length of n. Let

$$w \xrightarrow{reduction\#1} w_1 \xrightarrow{reduction\#2} w_2 ... \xrightarrow{reduction\#l} w_l$$

and

$$w \xrightarrow{reduction\#1'} w_1' \xrightarrow{reduction\#2'} w_2' ... \xrightarrow{reduction\#m'} w_m'$$

be any two reduction processes of w such that w_l, w_m' are reduced. We will show that $w_l = w_m'$.

We have the following claim.

CLAIM. *Suppose that $w_1 \neq w_1'$. Then there are two reductions*

$$w_1 \xrightarrow{reduction\ \#1} \hat{w}$$

and

$$w_1' \xrightarrow{reduction\ \#1'} \hat{w}'$$

such that $\hat{w} = \hat{w}'$.

Proof of Claim. Let xx^{-1} be the inverse pair for the reduction #1, and yy^{-1} the inverse pair for the reduction #1′. We have two cases.

Case 1. The inverse pairs xx^{-1}, yy^{-1} are disjoint in w. In this case, we let reduction a be reduction #1′, and reduction b be reduction #1. Thus, $\hat{w} = \hat{w}'$.

Case 2. The inverse pairs xx^{-1}, yy^{-1} have overlaps. Then either $x^{-1} = y$ or $y^{-1} = x$. In either cases, we have $w_1 = w_1'$. This contradicts the assumption that $w_1 \neq w_1'$. □

We are now ready to complete the proof of Lemma. First, if $w_1 = w_1'$, then $w_l = w_m'$ by applying the induction assumption to $w_1 = w_1'$ of length $n-2$. Otherwise, by the claim, there are two reductions applying to w_1, w_1' respectively such that the obtained words $\hat{w} = \hat{w}'$ are the same.

Note that $\hat{w}$ is of length $n-4$. Applying induction assumption to $\hat{w}$, we see that any reduction process

$$\hat{w} \xrightarrow{\textit{reduction process}} \bar{w}$$

of $\hat{w}$ gives the same reduced form $\bar{w}$.

By the claim, the reduction a together any reduction process $\hat{w} \xrightarrow{\textit{reduction process}} \bar{w}$ gives a reduction process for w_1 to $\bar{w}$. By induction assumption to w_1, we have $w_l = \bar{w}$. By the same reasoning, we have $w_m' = \bar{w}$. This shows that $w_l = w_m' = \bar{w}$. □

2.2. Construction of free groups by words. Denote by $F(X)$ the set of all reduced words in $\mathcal{W}(\tilde{X})$. By Lemma 2.1, there is a map

$$\mathcal{W}(\tilde{X}) \to F(X)$$

by sending a word to its reduced form.

We now define the group operation on the set $F(X)$. Let w, w' be two words in $F(X)$. The product $w \cdot w'$ is the reduced form of the word ww'.

Theorem 2.2. *$(F(X), \cdot)$ is a group with a generating set X.*

Proof. It suffices to prove the associative law for the group operation. Let w_1, w_2, w_3 be words in $F(X)$. We want to show $(w_1 \cdot w_2) \cdot w_3 = w_1 \cdot (w_2 \cdot w_3)$. By Lemma 2.1, the reduced form of a word does not depend on the reduction process. Observe that $(w_1 \cdot w_2) \cdot w_3$ and $w_1 \cdot (w_2 \cdot w_3)$ can be viewed as reduced forms of different reduction processes of the word $w_1 w_2 w_3$. The proof is thus completed. □

Let $\iota : X \to F(X)$ be the inclusion of X in $F(X)$. Usually we will not distinguish x and $\iota(x)$ below, as ι is injective.

Lemma 2.3. *For any map of a set X to a group G, there exists a unique homomorphism $\phi : F(X) \to G$ such that*

$$\begin{array}{ccc} X & \to & F(X) \\ & \searrow & \downarrow \\ & & G \end{array}$$

is commutative.

Proof. Denote by j the map $X \to G$. Define $\phi(x) = j(x)$ for all $x \in X$ and $\phi(x^{-1}) = j(x)^{-1}$ for $x^{-1} \in X^{-1}$. Define ϕ naturally over other elements in $F(X)$.

Let w_1, w_2 be two reduced words in $F(X)$. Without loss of generality, assume that $w_1 = x_1 x_2 ... x_n z_1 z_2 ... z_r$ and $w_2 = z_1^{-1} ... z_r^{-1} y_1 y_2 ... y_m$, where $x_i, y_j, z_k \in \tilde{X} = X \sqcup X^{-1}$ and $x_n \neq y_1^{-1}$. Then $w_1 \cdot w_2 = x_1 x_2 ... x_n y_1 y_2 ... y_m$. It is straightforward to verify that $\phi(w_1 \cdot w_2) = \phi(w_1)\phi(w_2)$.

Since a homomorphism of $F(X)$ to G is determined by the value of its restriction over a generating set of $F(X)$, we have that the chosen map $j : X \to G$ determines the uniqueness of ϕ. □

COROLLARY 2.4. *Every group is a quotient of a free group.*

PROOF. Let X be a generating set of G. Let $F(X)$ be the free group generated by X. By Lemma 2.3, we have an epimorphism of $F(X) \to G$. □

EXERCISE 2.5. *Let X be a set containing only one element. Prove that $F(X) \cong \mathbb{Z}$.*

Analogous to free abelian group, the class of free groups is characterized by the following universal mapping property in GROUP category.

LEMMA 2.6. *Let X be a subset, F be a group and $i : X \to F$ be a map. Suppose that for any group G and a map $j : X \to G$, there exists a unique homomorphism $\phi : F \to G$ such that*

$$\begin{array}{ccc} X & \xrightarrow{i} & F \\ & \searrow^{j} & \downarrow{\scriptstyle \phi} \\ & & G \end{array}$$

is commutative. Then $F \cong F(X)$.

PROOF. By Lemma 2.3 for free group $F(X)$ and $i : X \to F$, there is a unique homomorphism $\varphi : F(X) \to F$ such that $i = \varphi\iota$, where $\iota : X \to F(X)$ is the inclusion map. ie.

$$\begin{array}{ccc} X & \xrightarrow{\iota} & F(X) \\ & \searrow^{i} & \downarrow{\scriptstyle \varphi} \\ & & F \end{array} \tag{1}$$

On the other hand, by the assumption to $G = F(X)$ and $\iota : X \to F(X)$, there is a unique homomorphism $\phi : F \to F(X)$ such that we have $\iota = \phi i$.

$$\begin{array}{ccc} X & \xrightarrow{i} & F \\ & \searrow^{\iota} & \downarrow{\scriptstyle \phi} \\ & & F(X) \end{array} \tag{2}$$

Thus we obtained $\iota = \phi\varphi\iota$, and the following commutative diagram follows from the above (1)(2).

$$\begin{array}{ccc} X & \xrightarrow{\iota} & F(X) \\ & \searrow^{\iota} & \downarrow{\scriptstyle \phi\varphi} \\ & & F(X) \end{array} \tag{3}$$

Note that the identification $Id_{F(X)}$ between $F(X) \to F(X)$ also makes the above diagram (3) commutative. By the uniqueness statement of Lemma 2.3, $\phi\varphi = Id_{F(X)}$.

It is analogous to prove that $\varphi\phi = Id_F$. Hence ϕ or φ is an isomorphism. □

2.3. (Free) abelian groups. Recall that a group G is called *abelian* if $ab = ba$ for any $a, b \in G$. In this subsection, we study finitely generated abelian group.

DEFINITION 2.7. Let X be a set. The group $A(X) := \bigoplus_{x\in X} \langle x \rangle$ is called the *free abelian group* generated by X. The set X is called a *basis* of $A(X)$.

By definition, we see that there is an injective map $X \to A(X)$ defined by $x \to (0, ...0, x, 0, ...)$ for $x \in X$. Clearly, $A(X)$ is generated by (the image under the injective map) of X.

Let $m \in \mathbb{Z}$ and $a = (n_1 x, ..., n_i x, ...) \in A(X)$. We define the scalar multiplication

$$m \cdot a = (mn_1 x, ..., mn_i x, ...) \in A(X).$$

A *linear combination* of elements $a_i \in A(X), 1 \le i \le n$ is an element in $A(X)$ of the form $\sum_{1\le i\le n} k_i \cdot a_i$ for some $k_i \in \mathbb{Z}, 1 \le i \le n$.

EXERCISE 2.8. (1) *Let Y be a subset in a free abelian group G of finite rank. Then Y is basis of G if and only if $G = \langle Y \rangle$ and any element in G can be written as a unique linear combination of elements in Y.*

(2) *Prove that the group of rational numbers $\mathbb{Q}$ is not free abelian.*

EXERCISE 2.9. *Prove that $\mathbb{Z}^m \cong \mathbb{Z}^n$ if and only if $m = n$.*

If $|X|$ is finite, then $|X|$ is called the *rank* of $A(X)$. In general, a free abelian group may have different basis. The rank of a free abelian group is well-defined, by Exercise 2.9.

Every abelian group is a quotient of a free abelian group.

LEMMA 2.10. *Let X be a subset. For any map of X to an abelian group G, there exists a unique homomorphism ϕ such that*

$$\begin{array}{ccc} X & \to & A(X) \\ & \searrow & \downarrow \\ & & G \end{array}$$

is commutative.

COROLLARY 2.11. *Every abelian group is a quotient of a free abelian group.*

A free abelian group is characterized by the following universal mapping property in the category of abelian groups.

LEMMA 2.12. *Let X be a subset, A be an abelian group and $X \to A$ be a map. Suppose that for any abelian group G and a map $X \to G$, there exists a unique homomorphism $\phi : A \to G$ such that*

$$\begin{array}{ccc} X & \to & A \\ & \searrow & \downarrow \\ & & G \end{array}$$

is commutative. Then $A \cong A(X)$.

Recall that the commutator subgroup $[G, G]$ of a group G is the subgroup in G generated by the set of all commutators. That is:

$$[G, G] = \langle \{[f, g] := fgf^{-1}g^{-1} : f, g \in G\} \rangle$$

Use universal mapping property of free groups and free abelian groups to prove the following.

EXERCISE 2.13. *Prove that $F(X)/[F(X), F(X)] \cong A(X)$, where $A(X)$ is the free abelian group generated by X.*

A subset Y is called a *basis* of $F(X)$ if $F(X) \cong F(Y)$. In this case, we often say that $F(X)$ is *freely generated* by X. Use Exercise 2.9 to prove the following.

EXERCISE 2.14. *If $|X| < \infty$ and Y is a basis of $F(X)$, then $|X| = |Y|$.*

The *rank* of $F(X)$ is defined to be the cardinality of X. By Exercise 2.14, the rank of a free group is well-defined: does not depend on the choice of basis.

When the rank is finite, we usually write $F_n = F(X)$ for $n = |X|$.

2.4. A criterion of free group by words.

CONVENTION. *Since there is a map $\mathcal{W}(\tilde{X}) \to F(X) \to G$ for a generating set X of G, we write $w =_G g$ for a word $w \in \mathcal{W}(\tilde{X}), g \in G$, if the image of w under the map $\mathcal{W}(\tilde{X}) \to G$ is the element g.*

THEOREM 2.15. *Let G be a group with a generating set X. Then $G \cong F(X)$ if and only if any non-empty word $w \in \mathcal{W}(\tilde{X})$ with $w =_G 1 \in G$ contains an inverse pair.*

PROOF. We have first a surjective map $\mathcal{W}(\tilde{X}) \to F(X) \to G$, where $F(X) \to G$ is the epimorphism given by Lemma 2.3.

=>. let $w \in \mathcal{W}(\tilde{X})$ be a word such that $w =_G 1$. Since $F(X) \cong G$, we have w is mapped to the empty word in $F(X)$. That is to say, the reduced form of W is the empty word. Thus, w contains an inverse pair.

<=. Suppose that $F(X) \to G$ is not injective. Then there exists a non-empty reduced word $w \in F(X)$ such that $w =_G 1$. Then w contains an inverse pair. As w is reduced, this is a contradiction. Hence $F(X) \to G$ is injective. □

COROLLARY 2.16. *A group is freely generated by a set X if and only if any non-empty reduced word over X is a non-trivial element in G.*

EXERCISE 2.17.
(1) *Let Y be a set in the free group $F(X)$ generated by a set X such that $y^{-1} \notin Y$ for any $y \in Y$. If any reduced word w over $\tilde{Y} = Y \sqcup Y^{-1}$ is a reduced word over $\tilde{X} = X \sqcup X^{-1}$, then $\langle Y \rangle \cong F(Y)$.*
(2) *Let $S = \{b^n a b^{-n} : n \in \mathbb{Z}\}$ be a set of words in $F(X)$ where $X = \{a, b\}$. Prove that $\langle S \rangle \cong F(S)$.*
(3) *Prove that for any set X with $|X| \geq 2$ any $n \geq 1$, $F(X)$ contains a free subgroup of rank n.*

2.5. Ping-Pong Lemma and free groups in linear groups. In this subsection, we gives some common practice to construct a free subgroup in concrete groups. We formulate it in Ping-Pong Lemma. Before stating the lemma, we look at the following example.

LEMMA 2.18. *The subgroup of* $\mathbb{SL}(2,\mathbb{Z})$ *generated by the following matrices*

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$$

is isomorphic to F_2.

PROOF. See Proposition 3.7, on page 59 in our reference [**1**]. □

EXERCISE 2.19. *The subgroup of* $\mathbb{SL}(2,\mathbb{C})$ *generated by the following matrices*

$$\begin{pmatrix} 1 & a_1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ a_2 & 1 \end{pmatrix}, |a_1| \geq 2, |a_2| \geq 2;$$

is isomorphic to F_2.

LEMMA 2.20 (Ping-Pong Lemma). *Suppose that* G *is generated by a set* S*, and* G *acts on a set* X*. Assume, in addition, that for each* $s \in \tilde{S} = S \sqcup S^{-1}$*, there exists a set* $X_s \subset X$ *with the following properties.*

(1) $\forall s \in \tilde{S}$, $s \cdot X_t \subset X_s$, *where* $t \in \tilde{S} \setminus \{s^{-1}\}$.
(2) $\exists o \in X \setminus \cup_{s \in \tilde{S}} X_s$, *such that* $s \cdot o \in X_S$ *for any* $s \in \tilde{S}$.

Then $G \cong F(S)$.

PROOF. By Lemma 2.3 and Lemma 1.9, we have the following homomorphism:

$$\iota : F(S) \to G \to Sym(X).$$

Let w be a reduced non-empty word in $F(S)$. Write $w = s_1 s_2 ... s_n$ for $s_i \in \tilde{S}$. By Theorem 2.15, it suffices to show that $g = \iota(s_1)\iota(s_2)...\iota(s_n)$ is not an identity in $Sym(X)$.

We now apply the permutation g to $o \in X$ to get

$$g \cdot o = \iota(s_1)\iota(s_2)...\iota(s_{n-1})\iota(s_n) \cdot o \subset \iota(s_1)\iota(s_2)...\iota(s_{n-1})X_{s_n} \subset ... \subset X_{s_1}.$$

However, as $o \in X_{s_1}$, we have $g \neq 1 \in Sym(X)$. This shows that $F(S) \cong G$. □

Ping-Pong Lemma has a variety of forms, for instance:

EXERCISE 2.21. *Let* G *be a group generated by two elements* a, b *of infinite order. Assume that* G *acts on a set* X *with the following properties.*

(1) *There exists non-empty subsets* $A, B \subset X$ *such that* A *is not included in* B.
(2) $a^n(B) \subset A$ *and* $b^n(A) \subset B$ *for all* $n \in \mathbb{Z} \setminus \{0\}$.

Prove that G *is freely generated by* $\{a, b\}$.

We now prove that $\mathbb{SL}(2,\mathbb{R})$ contains many free subgroups.

PROPOSITION 2.22. *Let* $A \in \mathbb{SL}(2,\mathbb{R})$ *with two eigenvalues* λ, λ^{-1} *for* $\lambda > 1$*, and corresponding eigenvectors* $v_\lambda, v_{\lambda^{-1}}$*. Choose* $B \in \mathbb{SL}(2,\mathbb{R})$ *such that* $B\langle v_\lambda \rangle \neq \langle v_\lambda \rangle$, $B\langle v_\lambda \rangle \neq \langle v_{\lambda^{-1}} \rangle$ *and* $B\langle v_{\lambda^{-1}} \rangle \neq \langle v_\lambda \rangle$, $B\langle v_{\lambda^{-1}} \rangle \neq \langle v_{\lambda^{-1}} \rangle$.

Then there exist $N, M > 0$ *depending only on* A, B *such that*

$$F(a, b) = \langle a, b \rangle$$

where $a = A^n, b = BA^mB^{-1}$ *for* $n, > N, m > M$.

PROOF. Observe that BAB^{-1} has the same eigenvalues λ, λ^{-1}, but eigenvectors $Bv_\lambda, Bv_{\lambda^{-1}}$ respectively.

Let $\theta \in (0, 2\pi)$ be a (very small) angle. Denote by $X_{v,\theta} \subset \mathbb{R}^2$ the open sector around the line $\langle v_\lambda \rangle$ with angle θ.

We claim the following fact about the dynamics of A on vectors.

CLAIM. $\forall \theta \in (0, 2\pi), \exists N > 0$ *such that the following holds.*

$$\text{For } \forall n > N, v \in \mathbb{R}^2 \setminus \langle v_\lambda^{-1} \rangle, \text{ we have } A^n v \in X_{v_\lambda, \theta}.$$

and

$$\text{For } \forall n > N, v \in \mathbb{R}^2 \setminus \langle v_\lambda \rangle, \text{ we have } A^{-n} v \in X_{v_\lambda^{-1}, \theta}.$$

PROOF OF CLAIM. Since $\{v_\lambda, v_\lambda^{-1}\}$ is a basis of $\mathbb{R}^2$, the conclusion follows by a simple calculation. □

By the same reasoning, we also have

CLAIM. $\forall \theta \in (0, 2\pi), \exists M > 0$ *such that the following holds.*

$$\text{For } \forall m > M, v \in \mathbb{R}^2 \setminus \langle Bv_\lambda^{-1} \rangle, \text{ we have } BA^m B^{-1} v \in X_{Bv_\lambda, \theta}.$$

and

$$\text{For } \forall m > M, v \in \mathbb{R}^2 \setminus \langle Bv_\lambda \rangle, \text{ we have } BA^{-m} B^{-1} v \in X_{Bv_{\lambda^{-1}}, \theta}.$$

Denote $a = A^n, b = BA^m B^{-1}, X_a = X_{v_\lambda, \theta}, X_a^{-1} = X_{v_\lambda^{-1}, \theta}, X_b = X_{Bv_\lambda, \theta}, X_b^{-1} = X_{Bv_\lambda, \theta}$. Let $S = \{a, b\}$. By the above claims, we obtain the following.

$$\forall s \in \tilde{S},\ s \cdot X_t \subset X_s, \text{ where } t \in \tilde{S} \setminus \{s^{-1}\}.$$

Choose θ small enough such that $X_a \cup X_a^{-1} \cup X_b \cup X_b^{-1} \neq \mathbb{R}^2$. Choose any $o \in \mathbb{R}^2 \setminus \cup_{s \in \tilde{S}} X_s$. By the claims, $s \cdot o \in X_s$. Hence, all conditions of Ping-Pong Lemma are satisfied. We obtain that $F(\{a, b\}) = \langle a, b \rangle$. □

In fact, Jacques Tits proved the following celebrated result in 1972, which is usually called Tits alternative.

THEOREM 2.23. *Let G be a finitely generated linear group. Then either G is virtually solvable or contains a free subgroup of rank at least 2.*

REMARK. Note that a virtually solvable group does not contain any free group of rank at least 2. This explains the name of Tits alternative.

3. Subgroups of free groups

We shall give two proofs of the following theorem of Nielsen.

THEOREM 3.1. *Any subgroup of a free group is free.*

3.1. Group action on graphs. The first proof is to consider a group action on trees, and to use Ping-Pong Lemma. We first introduce the notion of a metric graph.

DEFINITION 3.2. A *metric graph* $\mathcal{G}$ consists of a set V of vertices and a set E of *undirected* edges which are copies of intervals $[0, 1]$ (with length 1). Each edge $e \in E$ are associated with two endpoints in V.

We can endow the graph with the following metric. The distance of two points $v, w \in \mathcal{G}$ is the length of shortest path between v, w.

REMARK. We do allow two edges with the same endpoints, and the two endpoints of an edge can be the same.

A *graph morphism* $\phi : \mathcal{G} \to \mathcal{G}'$ between two metric graphs $\mathcal{G}, \mathcal{G}'$ is a map sending edges to edges isometrically. It is called a *graph isomorphism* if ϕ is bijective. So, a graph isomorphism is an isometry of the metric graph.

An graph isomorphism is called an *inversion* if it switch two endponts of some edge. By inserting a vertex at each fixed point, an inversion of a graph can induce a non-inversion isomorphism of a new graph, which captures essential information of the original one.

Suppose now that a group G acts on a graph $\mathcal{G}$ by *isomorphisms without inversion*: we have a homomorphism

$$G \to \mathrm{Aut}(\mathcal{G})$$

where $\mathrm{Aut}(\mathcal{G})$ is the group of all isomorphisms of $\mathcal{G}$ such that the images do not contain inversions.

Given a metric graph, we consider the topology induced by the metric so the notion of connectedness, closed subset etc can be talked about. It is obvious that a connected graph is amount to saying that any two vertices are connected by a path.

LEMMA 3.3. *If a group G acts on a connected graph $\mathcal{G}$ without inversions, then there exists a subset F in $\mathcal{G}$ such that*

(1) *F is a closed subset,*
(2) *the set $\{gF : g \in G\}$ covers the graph,*
(3) *no subset of F satisfies properties (1) and (2).*

A set F satisfying the above properties is called *fundamental domain* for the action of G on the graph. In what follows, it is usually connected.

PROOF. We first construct the *core* C of the desired fundamental domain F. The core C will be a connected subset and contain exactly one point from each orbit or vertices.

Fix a vertex $o \in \mathcal{G}$ as the basepoint. We need use the form of Axiom of Choice - Zorn's lemma to conclude the construction of X: every non-empty partially ordered set in which every chain (i.e., totally ordered subset) has an upper bound contains at least one maximal element. We consider the collection $\mathbb{X}$ of connected subgraphs X with the property:

(1) $o \in X$,
(2) If $v, w \in X$ are two different vertices, then no $g \in G$ satisfies $gv = w$.

Note that every chain $X_0 \subset X_1 \subset \cdots \subset X_n \cdots$ has upper bound $\cup X_i$. By Zorn's Lemma, we have the collection of sets $\mathbb{X}$ contains a maximal element C.

We claim that C is indeed the core, i.e. containing exactly one point from each orbit of vertices. In other words, the vertex set is contained in $G \cdot C$. Suppose to the contrary that there exists $v \in V(\mathcal{G})$ such that $v \notin G \cdot C$. Without loss of generality, assume that there exists an edge e with one endpoint v and the other endpoint in $G \cdot C$. Then we add this edge e to C for getting a bigger set which belongs to $\mathbb{X}$. This is a contraction, as C is maximal by Zorn's Lemma.

To get the desired fundamental domain, it is important to note that $G \cdot C$ may not contain all edges in $\mathcal{G}$. We have to enlarge C by adding additional edges. Let e be such edge not contained in $G \cdot C$. We add half of the edge, the subinterval $[0, 1/2]$, to C. In a similar way by using Zorn's lemma, we get a fundamental domain F as wanted in the hypothesis. $\square$

By the third condition of minimality, two distinct translates of a fundamental domain intersect only in their boundary.

Corollary 3.4. *The interior of the fundamental domain F contains exactly one vertex from each orbit Gv for $v \in V(\mathcal{G})$: for any vertex $w \in F$ and $1 \neq g \in G$, we have either $gw \notin F$ or $gw = w$.*

A connected fundamental domain determines a system of generating set.

Theorem 3.5. *Let G act on a connected graph $\mathcal{G}$ with a connected fundamental domain F. Then the set of elements $S = \{gG : g \neq 1, gF \cap F \neq \emptyset\}$ is a generating set for G.*

Proof. We fix a basepoint o in F. For any element $g \in G$, we connect o and go by a path γ. The aim of the proof is to cover the path γ by finitely many hF where $h \in G$.

Note that the following two facts.

(1) gF cannot intersect entirely in the interior of an edge: $gF \cap e \subsetneq e^o$, for $gF \cap e$ is connected.
(2) If $e^o \cap gF \cap F \neq \emptyset$ and one endpoint e_+ of e belongs to $gF \cap F$. Then $gF \cap e = F \cap e$.

For the second fact, let $x \in e^o \cap gF \cap F$. Then $gx, x \in F^o$. By the minimality of F, we have $gx = x$. Since g is an isometry but not inversion, we have $g|_e = id$. If there exists $y \in gF \cap e \setminus (F \cap e)$, then $yg^{-1}y \in F \cap e$, so we get a contradiciton. Hence $gF \cap e = F \cap e$.

We choose these hF in the following way. Note that the two endpoints have been already covered by F and gF. Set $g_0 = 1$ and so $o \in g_0 F$. Let x be the intersection point of an edge e in the path γ with $X_i := \cup_{j \leq i} g_j F$.

If the point x lies in the interior of the edge e, then we denote by y the other endpoint of e not lying in X_i. Since $e \subset GF$, there exists $g_{i+1} \in G$ such that $y \in g_{i+1} F$. We claim that $e \subset \cup_{j \leq i+1} g_j F$. If not, then there exists a subinterval K of e outside $\cup_{j \leq i+1} g_j F$. On the other hand, there exists a translate of F intersecting K. By the second fact, this is impossible. So the claim is proved.

If the point x is the endpoint of an edge e outside X_i, denote by y the middle point of e. Then there exists $g_{i+1} F$ such that $y \in g_{i+1} F$. By the first fact, we have

$g_{i+1}F$ contains at least one of endpoints of e. We then consider the intersection point of $g_{i+1} \cap e$. Repeating these two cases whether it is an interior point or endpoint, we are able to choose a sequence of g_iF $(0 \leq i \leq n)$ such that $g_iF \cap g_{i+1}F \neq \emptyset$, where $g_n = g$. Then we can write explicitely g as a product of elements h such that $F \cap hF \neq \emptyset$. The proof is then complete. □

The following exercise is a corollary for the previous theorem.

EXERCISE 3.6. *Suppose G acts by graph isomorphisms without inversions on a connected graph X such that there exists a finite subgraph K with $G \cdot K = X$. Assume that the edge stabilizers and the vertex stabilizers are finitely generated. Then G is finitely generated.*

There is a straightforward connection between fundamental domains of subgroups and groups.

EXERCISE 3.7. *Let G act on a connected graph $\mathcal{G}$ by isomorphisms without inversions with a connected fundamental domain F. Then for a subgroup H of G, there exists a set of elements $R \subset G$ such that $\cup_{r \in R} r \cdot F$ is a connected fundamental domain for the action of H on $\mathcal{G}$.*

3.2. Groups acting on trees. By definition, a *tree* is a graph where every reduced circuit is a point. Equivalently, there exists a unique reduced path between two points.

Now lets consider the free group $F(S)$ over a set S. We define a tree $\mathcal{G}$ for which the vertex set V is all elements in $F(S)$. Two reduced words $W, W' \in F(S)$ are connected by an edge if there exists $s \in \tilde{S}$ such that $W' = Ws$. Formally, the edge set E is defined to be $F(S) \times \tilde{S}$. The map $^-$ sends $(W, s) \in F(S) \times \tilde{S}$ to $(Ws, s^{-1}) \in F(S) \times \tilde{S}$. Such a graph $\mathcal{G}$ is indeed a tree, and $F(S)$ acts on $\mathcal{G}$ by graph isomorphisms.

We shall use Ping-Pong Lemma to prove the following theorem, which implies Theorem 3.1.

THEOREM 3.8. *Suppose that G acts on a tree T such that the stabilizer of each vertex is trivial. In other words, G acts on a tree T freely. Then G is a free group.*

PROOF. We divide the proof into three steps.

Step 1. Find a fundamental domain. We consider the core C of a fundamental set defined for the action of G on T. Note that C will be a connected subset such that it contains exactly one vertex from each orbit Gv for $v \in T$.

Since $G \cdot C$ may not contain all edges in T, in order that $G \cdot X = T$, we have to include some half edges to C to get the fundmental domain F.

We denote by E_0 the set of edges e of F such that C contains exactly one endpoint of e. We also denote by e_- the endpoint of e in X, and e_+ the other endpoint of e outside X. Define $\bar{X} = C \cup E_0$. Then $\bar{X}$ is still connected and $G \times \bar{X} = T$.

REMARK. The set $\bar{X}$ is not a fundamentail domain, as it contains FULL edges and but F only does half edges.

Step 2. Find free basis of G. For each $e \in E_0$, we know that $e_- \in X$ and $e_+ \notin X$. Recall that X contains (exactly) one vertex from each G-orbit in T. Thus, there exist an element $g_e \in G \setminus 1$ and a unique vertex $v \in X$ such that

$g_e v = e_+$. The element g_e is unique, otherwise the stabilizer of v is nontrivial. This is a contradiction, since G acts on T freely.

Observe that $g_e^{-1}(e_-) \in T \setminus X$ is connected by the edge $g_e^{-1}(e)$ to $v \in X$. Denote $e' = g_e^{-1}(e)$. Thus, $e \neq e'$ and $e' \in E_0$. By the uniqueness of $g_{e'}$, we also see that $g_{e'} = g_e^{-1}$.

In conclusion, for each $e \in E_0$, there exists a unique $e \neq e' \in E_0$ and a unique $g_e \in G \setminus 1$ such that $g_e^{-1}(e) = e'$. Moreover, $g_{e'} = g_e^{-1}$. We call $g_e, g_{e'}$ the **edge paring transformation** of the pair of edges (e, e').

Denote $\tilde{S} = \{g_e : e \in E_0\}$. Note that edges e, e' in E_0 are paired. From each such pair, we choose exactly one edge and denote them by $E_1 \subset E_0$. Define $S = \{g_e : e \in E_1\}$. Obviously, $\tilde{S} = S \cup S^{-1}$.

Step 3. Verify Ping-Pong Lemma. We now prove that $G = F(S)$ by using Ping-Pong Lemma.

For each $e \in E_0$, we define X_e to be the subgraph of T such that for each vertex z in X_v, there exists a (unique) reduced path from o to z containing the edge e. We note that X_e is connected, since it contains the endpoint e_+ of e. Moreover, $X_{e_1} \cap X_{e_2} = \emptyset$ for $e_1 \neq e_2 \in E_0$, and any path between two points in X_{e_1} and X_{e_2} respectively have to intersect X. These two properties follow from the fact that T is a tree: if not, we would be able find a nontrivial circuit.

As a result, if a path γ intersects X_e but $\gamma \cap X = \emptyset$, then γ lies in X_e.

We first verify that $g_e(o) \in X_e$, where $e \in E_0$. By definition, we need prove that the reduced path between o and $g_e(o)$ contains the edge e. For this purpose, we connect o and $g_e^{-1}e_+ \in X$ by a unique reduced path γ in X. Since X is the core of the fundamental domain, we have that $g_e\gamma \cap X = \emptyset$. Since $g_e\gamma$ contains the endpoint e_+ of e and $e_+ \in X_e$, we obtain that $g_e o \in g_e\gamma \subset X_e$ by the above discussion.

Secondly, we prove that $g_e X_t \subset X_e$ for $t \neq e' \in E_0$. Indeed, for any $z \in X_t$, we connect $g_{e'}o$ and z by a shortest geodesic γ. Since $g_{e'}o \in X_{e'}$ and $X_{e'} \cap X_t = \emptyset$, the path γ must intersect C and contain e'. So the path $g_e\gamma$ contains e and its endpoint are $\{o, g_e z\}$. By definition of X_e, we have that $g_e z \subset X_e$ and so $g_e X_t \subset X_e$.

Therefore, we have verified the conditions of Ping-Pong Lemma 2.20. So $G = F(S)$. □

In the above proof, we see that the rank of the free group G is the number of paired edges of the fundamental domain. From this fact and Exercise 3.7, we can deduce the following.

Exercise 3.9. *Let H be a subgroup of index n in a free group F_r of rank r ($r > 1$). Then the rank of H is $rn - (n-1)$. In particular, for each $n > 1$, F_2 contains a finite index subgroup of rank n.*

4. Fundamental groups of graphs

The second proof of Theorem 3.1 is to use a combinatorial notion of fundamental groups of a graph.

Definition 4.1. A *graph* $\mathcal{G}$ consists of a set V of vertices and a set E of directed edges. For each directed edge $e \in E$, we associate to e the *initial point* $e_- \in V$ and *terminal point* $e_+ \in V$. There is an orientation-reversing map

$$\bar{}: E \to E, e \to \bar{e}$$

such that $e \neq \bar{e}$, $e = \bar{\bar{e}}$ and $e_- = (\bar{e})_+, e_+ = (\bar{e})_-$.

An *orientation* of $\mathcal{G}$ picks up exactly one directed edge in $\{e, \bar{e}\}$ for all $e \in E$. Formally, an orientation is a subset in E such that it contains exactly one element in $\{e, \bar{e}\}$ for all $e \in E$

REMARK. Clearly, such a map $^-$ has to be bijective. Moreover, $e_+ = (\bar{e})_-$ can be deduced from other conditions: $e_+ = \bar{\bar{e}}_+ = \bar{e}_-$.

REMARK. Every combinatorial graph can be geometrically realized by a common graph in the sense of CW-complex. We take the set of points V, and for each pair $(e, \bar{e})$, we take an interval $[0, 1]$ and attach its endpoints to $e_-, e_+ \in V$ respectively. Then we get a 1-dimensional CW-complex.

Combinatorially, we define a *path* to be a concatenation of directed edges:

$$\gamma = e_1 e_2 ... e_n, e_i \in E$$

where $(e_i)_+ = (e_{i+1})_-$ for $1 \leq i < n$. The initial point γ_- and terminal point γ_+ of γ are defined as follows:

$$\gamma_- = (e_1)_-, \gamma_+ = (e_n)_+.$$

If $(e_n)_+ = (e_1)_-$, the path γ is called a *circuit* at $(e_1)_-$. By convention, we think of a vertex in $\mathcal{G}$ as a path (or circuit), where there are no edges.

A *backtracking* in γ is a subpath of form $e_i e_{i+1}$ such that $e_i = \bar{e}_{i+1}$. A path without backtracking is called *reduced.* If a path γ contains a backtracking, we can obtain a new path after deleting the backtracking. So any path can be converted to a reduced path by a reduction process. Similarly as Lemma 2.1, we can prove the following.

LEMMA 4.2. *The reduced path is independent of the reduction process, and thus is unique.*

A *graph morphism* $\phi : \mathcal{G} \to \mathcal{G}'$ between two graphs $\mathcal{G}, \mathcal{G}'$ is a vertex-to-vertex, edge-to-edge map such that $\phi(e_-) = \phi(e)_-, \phi(e_+) = \phi(e)_+$ and $\phi(\bar{e}) = \overline{\phi(e)}$. It is called a *graph isomorphism* if ϕ is bijective.

The *concatenation* $\gamma\gamma'$ of two paths γ, γ' is defined in the obvious way, if $\gamma_+ = \gamma'_-$.

DEFINITION 4.3. Let $\mathcal{G}$ be a graph and $o \in \mathcal{G}$ be a basepoint. Then the *fundamental group* $\pi_1(\mathcal{G}, o)$ of $\mathcal{G}$ consists of all reduced circuits based at o, where the group multiplication is defined by sending two reduced circuits to the reduced form of their concatenation.

The group identity in $\pi_1(\mathcal{G}, o)$ is the just the base point $o \in \mathcal{G}$, the constant circuit.

REMARK. We can consider an equivalence relation over the set of all circuits based at o: two circuits are *equivalent* if they have the same reduced form. By Lemma 4.2, this is indeed an equivalence relation. Then the fundamental group $\pi_1(\mathcal{G}, o)$ can be also defined as the set of equivalent classes $[\gamma]$ of all circuits based at o, endowed with the multiplication:

$$[\gamma] \cdot [\gamma'] \to [\gamma\gamma'].$$

It is easy to see that these two definitions give the isomorphic fundamental groups.

A particularly important graph is the *graph of a rose* which consists of one vertex o with all other edges $e \in E$ such that $e_- = e_+ = o$. Topologically, the rose is obtained by attaching a collection of circles to one point.

Here we list a few properties about the fundamental group of a graph. Taking into account Lemma 4.2, the following is just an interpretation of definitions .

LEMMA 4.4. *We fix an orientation on a rose. Then the fundamental group of a rose is isomorphic to the free group generated by the alphabet set as the orientation.*

Any graph contains a *spanning* tree which is a tree with the vertex set of the graph. We can collapse a spanning tree to get a rose, called the *spin* of the graph.

EXERCISE 4.5. *The fundamental group of a graph is isomorphic to that of its spin.*

A graph morphism $\phi : \mathcal{G} \to \mathcal{G}'$ naturally defines a homomorphism between the fundamental group as follows:

$$\phi_* : \pi_1(\mathcal{G}, o) \to \pi_1(\mathcal{G}', \phi(o))$$

by sending a reduced circuit γ in $\pi_1(\mathcal{G}, o)$ to the reduced path of $\phi(\gamma)$ in $\pi_1(\mathcal{G}', \phi(o))$.

Given a vertex v in $\mathcal{G}$, consider the star

$$Star_{\mathcal{G}}(v) = \{e \in E(\mathcal{G}) : e_- = v\}.$$

A graph morphism $\phi : \mathcal{G} \to \mathcal{G}'$ naturally induces a graph morphism between the stars of v and $\phi(v)$.

DEFINITION 4.6. A graph morphism $\phi : \mathcal{G} \to \mathcal{G}'$ is called an *immersion* if for every vertex v, the induced graph morphism between the stars of v and $\phi(v)$

$$Star_{\mathcal{G}}(v) \to Star_{\mathcal{G}'}(\phi(v))$$

is injective. That is, ϕ is locally injective. If, in addition, ϕ is surjective, then it is called a *covering*.

The following lemma is a consequence of the defition of an immersion.

LEMMA 4.7 (Unique lifting). *Let $\phi : (\mathcal{G}, o) \to (\mathcal{G}', o')$ be an immersion where $o' := \phi(o)$. Then for any path γ and $x \in \mathcal{G}'$ satisfying $\phi(x) = o'$, if the lift $\hat{\gamma}$ of the path γ exists, then it is unique.*

If ϕ is a covering, then the lift of γ always exists and is thus unique.

REMARK. The difference between an immersion and a covering leads that the lift of a path may not exist!

Here is a corollary of Lemma 4.7.

LEMMA 4.8. *Let $\phi : (\mathcal{G}, o) \to (\mathcal{G}', \phi(o))$ be an immersion, and γ be a circuit in $\mathcal{G}'$ based at $\phi(o)$. If γ is not in $\phi_*(\pi_1(\mathcal{G}, o))$, then any lift of γ is not a circuit.*

It is clear that lifting preserves backtracking in an immersion so a reduction process is lifted from the downstair to the upstair.

LEMMA 4.9 (Backtracking). *Let $\phi : (\mathcal{G}, o) \to (\mathcal{G}', o')$ be an immersion where $o' := \phi(o)$. If a path γ has backtracking, then so does the lift $\hat{\gamma}$.*

An important consequence of an immersion is the following result.

LEMMA 4.10. *An immersion induces an imbedding of fundamental groups. That is, ϕ_* is injective.*

PROOF. Suppose not. There exists a non-empty reduced circuit γ based at o in $\mathcal{G}$ such that $\phi(\gamma)$ has the reduced circuit as the constant circuit c'_o. However, backtracking is preserved under lifting. During the reduction process from $\phi(\gamma)$ to c'_o, each backtracking is lifted to $\mathcal{G}$ and so a reduction process is inducted between γ and o. This contradicts to the choice of the non-empty reduced circuit γ. The lemma is thus proved. □

5. J. Stalling's Folding and separability of subgroups

5.1. J. Stalling's Folding. Let $\phi : \mathcal{G} \to \mathcal{G}'$ be a graph morphism. We shall make use of an operation called *folding* to convert the graph morphism ϕ to an immersion on a **new graph** $\mathcal{G}'$.

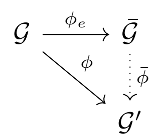
A pair of edges e, e' in $\mathcal{G}$ is called *foldable* if $e_- = (e')_-$, $\bar{e} \neq e'$, and $\phi(e) = \phi(e')$. Given a foldable pair of edges e, e', we can define a graph morphism ϕ_e to a new graph $\bar{\mathcal{G}}$ called *folding* as follows

$$\phi_e : \mathcal{G} \to \bar{\mathcal{G}} := \mathcal{G}/\{e = e', \bar{e} = \bar{e}'\}$$

by identifying the edges $e = e'$ and $\bar{e} = \bar{e}'$ respectively.

Observe that such an operation strictly decreases the number of edges and vertices. It is also possible that two loops can be identified. In this case, the fundamental group of the new graph $\bar{\mathcal{G}}$ changes.

Moreover, given a foldable pair of edges e, e', we can naturally define a new graph morphism $\bar{\phi} : \bar{\mathcal{G}} \to \mathcal{G}'$ such that the following diagram



is commutative.

We do the above *folding process* for each foldable pair of edges, and finally obtain an immersion from a new graph $\bar{\mathcal{G}}$ to $\mathcal{G}'$. Precisely, we have the following.

LEMMA 5.1. *Let $\phi : \mathcal{G} \to \mathcal{G}'$ be a graph morphism. Then there exists a sequence of foldings $\phi_i : \mathcal{G}_i \to \mathcal{G}_{i+1}$ for $0 \leq i < n$ and an immersion $\bar{\phi} : \bar{\mathcal{G}} \to \mathcal{G}'$ such that $\phi = \bar{\phi}\phi_n \cdots \phi_0$, where $\mathcal{G}_0 = \mathcal{G}, \mathcal{G}_n = \bar{\mathcal{G}}$.*

A direct corollary is as follows.

COROLLARY 5.2. *Let $\phi : \Gamma \to \mathcal{G}$ be a graph immersion between two finite graphs. Then there exists a finite covering π of $\hat{\Gamma} \to \mathcal{G}$ such that Γ is a subgraph of $\hat{\Gamma}$ and $\pi(\iota) = \phi$, where ι is the natural embeding of Γ into $\hat{\Gamma}$.*

This corollary implies that a wraped/immersed object, for instance the image $\phi(\Gamma)$, can be unwrapped to be embeded in a finite covering. The key notion making this possible is the separability of the subgroup $\phi_*(\pi_1(\Gamma))$ in $\pi_1(\mathcal{G})$.

An important consequence of the above folding process is that ϕ_* and $\bar{\phi}_*$ have the same image in the fundamental group of $\mathcal{G}'$. We apply the above theory to subgroups of a free group and to prove Theorem 3.1.

THEOREM 5.3 (Nielsen basis). *Let H be a subgroup of a free group $F(S)$. Then H is a free group. Moreover, given any generating set T of H, there exists an algorithm to find a free basis for H.*

PROOF. Let H be a subgroup of a free group $F(S)$. Suppose that H is generated by a set $T \subset F(S)$. By the above discussion, there exists a rose $\mathcal{G}'$ with one vertex and $2|S|$ edges whose fundamental group is $F(S)$. Here in fact, we choose an orientation on $\mathcal{G}'$ and then identify $\pi_1(\mathcal{G}')$ as $F(S)$.

Note that T are a set of reduced words. For each word $W \in T$, we associate to W a circuit graph $\mathcal{C}_W$ of $2|T|$ edges with a basepoint o and an orientation such that the clock-wise "label" of $\mathcal{C}_W$ is the word W. It is obvious that there exists a graph morphism $\mathcal{C}_W \to \mathcal{G}'$.

We attach all $\mathcal{C}_W$ at o for $W \in T$ to get a graph $\mathcal{G}$. Then we have a graph morphism $\phi : \mathcal{G} \to \mathcal{G}'$. It is also clear that the image $\phi_*(\pi_1(\mathcal{G}))$ is the subgroup H in $F(S)$. Hence, a consequence of Lemma 5.1 is that any subgroup of a free group is free. Moreover, since the immersion given by Lemma 5.1 induces an injective homomorphism, we can easily obtain a free basis of H by writing down the generating elements of the fundamental group of $\bar{\mathcal{G}}$. □

5.2. Separability of subgroups. In this subsection, we present the proof of J. Stallings of a theorem of M. Hall.

THEOREM 5.4 (M. Hall). *Let H be a finitely generated subgroup in a free group F of finite rank. For any element $g \in F \setminus H$, there exists a finite index subgroup Γ of F such that $H \subset \Gamma$ and $g \notin \Gamma$.*

REMARK. A subgroup with the above property is called *separable*. In other words, a subgroup H is *separable* in G if it is the intersection of all finite index subgroups of G containing H.

LEMMA 5.5. *Let $\phi : (\mathcal{G}, o) \to (\mathcal{G}', \phi(o))$ be a covering for two finite graphs $\mathcal{G}$ and $\mathcal{G}'$. Then $\phi_*(\pi_1(\mathcal{G}, o))$ is of finite index in $\pi_1(\mathcal{G}', o)$.*

PROOF. Denote by H the subgroup $\phi_*(\pi_1(\mathcal{G}, o))$. We count the right coset Hg where $g \in \pi_1(\mathcal{G}', o)$. Then any lift of the circuit in Hg based at o has the same terminal endpoint. Moreover, if $Hg \neq Hg'$, then the endpoints of corresponding lifts are different. Indeed, if not, we get a circuit and by Lemma 4.7 we see that $g'g^{-1} \in H$.

Since $\mathcal{G}$ is finite, we see that there are only finitely many different right H-cosets. □

We are now in a position to give the Stalling's proof of Theorem 5.4.

PROOF OF THEOREM 5.4. Let $\mathcal{G}'$ be a rose. We have put an orientation on $\mathcal{G}'$, a subset E_0 of edges, such that $\pi_1(\mathcal{G}')$ is identical to $F(E_0)$.

Let H be a finitely generated subgroup in F with a finite generating set T. Given $g \notin H$, we write g as a reduced word W_g over S, and similarly for each $t \in T$ a word W_t. As in the proof of Theorem 5.3, we construct a graph by gluing circuits labeled by W_t for $t \in T$, and use the folding to get an immersion $\phi : (\mathcal{G}, o) \to (\mathcal{G}', \phi(o))$, where $\mathcal{G}$ has the fundamental group H. This naturally induces an orientation E_1 on $\mathcal{G}$. Now we attach a path labeled by W_g at o by following the orientation $\mathcal{G}$. Since $g \notin H$, the endpoint of the path must be different from o, i.e.: the path is not closed. The new graph is still denoted by $\mathcal{G}$ for simplicity. And $\phi : (\mathcal{G}, o) \to (\mathcal{G}', \phi(o))$ is still an immersion.

Denote by V the vertex set of $\mathcal{G}$. For each $e \in E_0$, we have a set of directed edges $\phi_e^{-1}(e)$ in E_1. Since ϕ is an immersion, each edge in $\phi_e^{-1}(e)$ defines an ordered pair of endpoints in V. Thus, each $e \in E_0$ defines a bijective map ι_e on a subset of the vertex set V of $\mathcal{G}$. Similarly, we can define $\iota_{\bar{e}}$ for $e \in E_0$.

Since V is finite, ι_e can be extended to a bijective map of V. (We actually have many choices). Let's denote again by ι_e one such bijective map of V.

It is easy to use these maps $\iota_e, \iota_{\bar{e}}$ for $e \in E_0$ to complete the immersion $\phi : (\mathcal{G}, o) \to (\mathcal{G}', \phi(o))$ to a covering $\tilde{\phi} : (\tilde{\mathcal{G}}, o) \to (\mathcal{G}', \phi(o))$. Precisely,

For each $e \in E_0$, we use ι_e to connect v and $\iota_e(v)$ by a directed edge e, if such an edge was not in $\phi_e^{-1}(e)$. We do similarly for each $\bar{e}$ where $e \in E_0$. It is clear that the such obtained graph $\tilde{\mathcal{G}}$ is a finite covering. By Lemma 5.5, the fundamental group $\Gamma = \phi_\star(\pi_1(\tilde{\mathcal{G}}, o))$ of $\tilde{\mathcal{G}}$ is of finite index in G.

Moreover, by Lemma 4.7, the subgroup Γ contains H but not g, since the path labeled by W_g is not closed in $\tilde{\mathcal{G}}$. The proof is complete. □

The following two exercises are consequences of Theorem 5.4.

EXERCISE 5.6. *A free group F is residually finite: for any $g \neq 1 \in F$, there exists a homorphism $\phi : F \to G$ to a finite group G such that $\phi(g) \neq 1$.*

In fact, another way is to note that a linear group is residually finite, and free groups are linear.

EXERCISE 5.7. *Free groups are Hopfian: any endomorphism is an isomorphism.*

6. More about covering spaces of graphs

In this section, we list a few theorems in theory of covering spaces of graphs. They will serve a template for the corresponding ones in general topological spaces.

LEMMA 6.1. *Let $\phi : (\Gamma, x) \to (\mathcal{G}, o)$ be a covering for two graphs Γ and $\mathcal{G}$. Then there exists a bijection between the fiber $\phi^{-1}(o)$ and the collection of right cosets of $\phi_*(\pi_1(\Gamma, o))$.*

PROOF. Denote $H := \phi_*(\pi_1(\mathcal{G}, o))$. We count the right coset Hg where $g \in \pi_1(\mathcal{G}, o)$. By Lemma 4.9, any lift of a circuit in Hg based at x has the same terminal endpoint. This establishes that the corresponding map Φ from $\{Hg : g \in \pi_1(\mathcal{G}, o)\}$ to the fiber $\phi^{-1}(o)$ is well-defined.

Moreover, if $Hg \neq Hg'$, then the endpoints of the corresponding lifts are distinct. Indeed, if not, let $g \in Hg$ and $g' \in Hg'$ such that their lifts $\hat{g}$ and $\hat{g}'$ at x have the same other endpoint y. Then we get a circuit $\hat{g} \cdot \hat{g}'^{-1}$ at x. By Lemma 4.7 we see that $gg'^{-1} \in H$, contradicting to the assumption of $Hg \neq Hg'$. So this implies that the aboved defined map is injective.

To see the surjectivity, let y be a point in $\phi^{-1}(o)$ and connect x, y by a path γ. The image of γ is then a loop $g \in \pi_1(\mathcal{G}, o)$. By definition of the map, we see that $\Phi(g) = y$. So it is proved that the map Φ is a bijection. □

Universal covering. A connected graph is *simply connected* if its fundamental group is trivial. A covering $\Gamma \to \mathcal{G}$ is called *universal* if Γ is simply connected.

It is straightforwad to construct the universal covering of a rose, so of any graph by blowing up each vertex by a spanning tree of the graph in that of the rose.

THEOREM 6.2 (Lift graph morphisms). *Let* $\phi : (\mathcal{G}, o) \to (\mathcal{G}', \phi(o))$ *be a graph morphism. Suppose we have two coverings* $\pi : (\hat{\mathcal{G}}, x) \to (\mathcal{G}, o)$ *and* $\pi' : (\hat{\mathcal{G}}', y) \to (\mathcal{G}', \phi(o))$. *Then there exists a unique lift* $\hat{\phi} : \hat{\mathcal{G}} \to \hat{\mathcal{G}}'$ *such that* $\hat{\phi}(x) = y$ *if and only if* $\phi_*(\pi_*(\pi_1(\hat{\mathcal{G}}, x))) \subset \pi'_*(\pi_1(\hat{\mathcal{G}}', y))$.

$$\begin{array}{ccc} \hat{\mathcal{G}} & \xrightarrow{\hat{\phi}} & \hat{\mathcal{G}}' \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi'} \\ \mathcal{G} & \xrightarrow{\phi} & \mathcal{G}' \end{array}$$

As a corollary, we produce the following.

THEOREM 6.3 (Uniqueness of universal covering). *Let* $\mathcal{G}$ *be a graph. Let* $\pi_1 : \Gamma_1 \to \mathcal{G}$ *and* $\pi_2 : \Gamma_2 \to \mathcal{G}$ *be two universal coverings. Then there exists a graph isomorphism* $\phi : \Gamma_1 \to \Gamma_2$ *such that the diagram is commutative.*

$$\begin{array}{ccc} \Gamma_1 & \xrightarrow{\quad\phi\quad} & \Gamma_2 \\ & {\scriptstyle \pi_1}\searrow \quad \swarrow{\scriptstyle \pi_2} & \\ & \mathcal{G} & \end{array}$$

THEOREM 6.4 (Correspondence of subgroups $\leftrightarrow$ covering spaces). *Let* $(\mathcal{G}, o)$ *be a graph. Then for any subgroup* H *in* $\pi_1(\mathcal{G}, o)$, *there exists a covering* $\phi : (\Gamma, x) \to (\mathcal{G}, o)$ *such that* H *is the image of* $\phi_*(\pi_1(\Gamma, x))$.

PROOF. We only explain the case that H is finitely generated by a set S, where each $s \in S$ is a word with respect to the free generators of $\pi_1(\mathcal{G}, o)$.

We first draw down explitely a graph Γ according to this set S. Using Folding process, we can assume that $\Gamma \to \mathcal{G}$ is an immersion. Thus, the fundamental group of Γ is just isomorphic to H.

To get a covering with fundamental group H, it suffices to add infinite trees to the vertices in Γ which have incomplete stars. It is obvious such a completion is always possible and does not change the fundamental group of Γ. □

THEOREM 6.5 (Covering transformations $\leftrightarrow$ Normalizer). *Let* $\phi : (\Gamma, x) \to (\mathcal{G}, o)$ *be a covering. Then the group of covering transformations is isomorphic to the normalizer of the image* $\phi_*(\pi_1(\Gamma, x))$ *in* $\pi_1(\mathcal{G}, o)$.

A covering $\phi : (\Gamma, x) \to (\mathcal{G}, o)$ is called *normal* if the image $\phi_*(\pi_1(\Gamma, x))$ is a normal subgroup of $\pi_1(\mathcal{G}, o)$.

THEOREM 6.6 (Free actions on trees). *Let* Γ *be a tree on which a group* G *of automorphisms acts freely. Then the fundamental group of the quotient graph is isomorphic to the group* G.

Bibliography

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