CHAPTER 2

Elements of Hyperbolic geometry

1. Upper Half-plane Model

Consider the upper half plane $\mathbb{H}^2 = \{(x, y) \in \mathbb{R}^2 : y > 0\}$. We endow $\mathbb{H}^2$ with the following (Riemannian) metric:

$$ds = \frac{\sqrt{dx^2 + dy^2}}{y}.$$

To be precise, a piecewise differential path $\gamma : [0, 1] \to \mathbb{H}$ has the length defined as follows:

$$\ell(\gamma) = \int_0^1 \frac{\sqrt{x'(t)^2 + y'(t)^2}}{y(t)} dt$$

where $\gamma = (x(t), y(t))$.

For two points $z, w \in \mathbb{H}$, their *hyperbolic* distance is as follows

$$d_{\mathbb{H}}(z, w) = \inf\{\ell(\gamma) : \gamma(0) = z, \gamma(1) = w\}$$

where the infimum is taken over all piecewise differential paths between z and w.

Denote by $\mathrm{Isom}(\mathbb{H}^2)$ the group of all isometries of $\mathbb{H}^2$.

1.1. Orientation-preserving isometries. Consider the general linear groups $GL(2, \mathbb{C})$ of invertible 2×2-matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

where $a, b, c, d \in \mathbb{C}$ such that $ad - bc \neq 0$. The group $\mathcal{M}_2(\mathbb{C})$ of (complex) *linear fractional transformation* (LFT) is a nonconstant function on $\mathbb{C}$ of the form

$$T(z) = \frac{az + b}{cz + d}$$

for $a, b, c, d \in \mathbb{C}$ with $ad - bc \neq 0$. Such LFT is also called *Mobius transformation.* There is a natural map $\Phi : GL(2, \mathbb{C}) \to \mathcal{M}_2(\mathbb{C})$ as follows:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \to \frac{az + b}{cz + d}.$$

Exercise 1.1. *Prove that Φ is homomorphism and the kernel is $\{k \cdot I_{2\times 2} : k \in \mathbb{C} \setminus 0\}$ where $I_{2\times 2}$ is the identity matrix.*

For simplicity, we consider the *special linear* group $SL(2, C)$ consists of the matrices with determinant ± 1 in $GL(2, \mathbb{C})$. The *projective linear* group $PSL(2, \mathbb{C})$ is then the quotient group $SL(2, C)/\{\pm I_{2\times 2}\}$. By the above exercise, $PSL(2, \mathbb{C})$ is isomorphic to $\mathcal{M}_2(\mathbb{C})$.

Lemma 1.2. *Every LFT can be written as a product of the following three elementary transformations:*

(1) $z \to z + c$, *where* $c \in \mathbb{C}$,
(2) $z \to kz$, *where* $k \in \mathbb{C}$,
(3) $z \to \frac{-1}{z}$.

In other words, $\mathcal{M}_2(\mathbb{C})$ *is generated by the set of elementary transformations.*

Every LFT is actually defined on the set $\mathbb{C} \setminus \{\frac{-d}{c}\}$. It will be useful to define LFT over the extended complex plane $\hat{\mathbb{C}} = \mathbb{C} \cup \infty$. Correspondingly, we define

$$T(\frac{-d}{c}) = \infty$$

and

$$T(\infty) = \frac{a}{c},$$

so a LFT T becomes a bijective map on $\hat{\mathbb{C}}$. We equip the topology of $\hat{\mathbb{C}}$ with one-point compactification as follows. The open sets in $\hat{\mathbb{C}}$ are either open sets in $\mathbb{C}$ or the union of ∞ with the complement of a compact set in $\mathbb{C}$.

Exercise 1.3. *Put a metric on* $\hat{\mathbb{C}}$ *such that it induces the one-point compactification* $\hat{\mathbb{C}}$. *(Tips: consider the stereographic projection from the closed upper semi-sphere to* $\hat{\mathbb{C}}$.*)*

Exercise 1.4. *With one-point compactification* $\hat{\mathbb{C}}$, *every LFT is a homeomorphism.*

The above discussion still applies with $\mathbb{C}$ replaced by $\mathbb{R}$. In particular, $\mathcal{M}_2(\mathbb{R})$ denotes the set of LFTs with real coefficients. Then $PSL(2, \mathbb{R})$ is isomorphic to $\mathcal{M}_2(\mathbb{R})$. We now come to the connection of $\mathcal{M}_2(\mathbb{R})$ with $\text{Isom}(\mathbb{H}^2)$.

Lemma 1.5. $\mathcal{M}_2(\mathbb{R}) \subset$ *Isom*$(\mathbb{H}^2)$.

Proof. Note that each type of a real elementary LFT is an isometry so any real LFT belongs to $\text{Isom}(\mathbb{H}^2)$ by Lemma 1.2. □

1.2. Geodesics and reflexions. We now consider the paths $\gamma : I \to \mathbb{H}^2$ where I is an interval in $\mathbb{R}$.

Definition 1.6. A path $\gamma : I \to \mathbb{H}^2$ is called a *geodesic* if it preserves the distance: $|s - t| = d_{\mathbb{H}^2}(\gamma(s), \gamma(t))$ for any $s, t \in I$.

Remark. Sometimes, when I is a finite interval $[a, b]$, the path γ is called a *geodesic segment*. If $I = [0, \infty)$, it is a *geodesic ray*; if $I = \mathbb{R}$, we call it a *geodesic line*.

Theorem 1.7. *The set of geodesic lines in* $\mathbb{H}^2$ *is the set of Euclidean half-lines and half-circles orthogonal to the real axis.*

One may first verify by computations that the positive y-axis is a geodesic line. Then the proof is completed by the following.

Exercise 1.8. $\mathcal{M}_2(\mathbb{R})$ *acts transitively on the set of Euclidean half-lines and half-circles orthogonal to the real axis.*

To obtain the full isometry group of $\mathbb{H}^2$, we need take care of an orientation-reversing isometry. Note that $z \to -\bar{z}$ is such an isometry of $\mathbb{H}^2$, which fixes pointwise the y-axis and exchanges left and right half-planes. So we have the following defintion.

DEFINITION 1.9. A (hyperbolic) *reflexion* in $\mathbb{H}^2$ is a conjugate of $z \to -\bar{z}$ by $\mathcal{M}_2$ so it fixes pointwise a unique geodesic line.

PROPOSITION 1.10. *If an isometry in $\mathbb{H}^2$ fixes pointwise a geodesic line L, then it is either identity or a reflexion about L.*

Before giving a proof, we need make use of the following useful fact about bisectors. Given two points $x, y \in \mathbb{H}^2$, the *bisector* $L_{x,y}$ is the set of points $z \in \mathbb{H}^2$ such that $d_{\mathbb{H}^2}(x,z) = d_{\mathbb{H}^2}(y,z)$.

LEMMA 1.11. *Bisectors $L_{x,y}$ are geodesic lines and the geodesic $[x,y]$ is orthogonal to $L_{x,y}$.*

PROOF. Up to applying LFT (cf. Ex 1.8), we can assume without loss of generality that x, y are symmetric relative to the y-axis. Observe then that the positive y-axis is contained in $L_{x,y}$. Hence, it suffices to prove that any point $z \in L_{x,y}$ has to lie on the y-axis. This can be proved by contradiction; see detailed proof in the Lemma in Stillwell, pp.87. □

Define the distance of a point z to a subset L in $\mathbb{H}^2$:

$$d_{\mathbb{H}^2}(z, L) := \inf\{d_{\mathbb{H}^2}(z, w) : w \in L\}.$$

LEMMA 1.12. *Given a point z outside a geodesic line L, then there exists a unique point $w \in L$ such that $d_{\mathbb{H}^2}(z,w) = d_{\mathbb{H}^2}(z,L)$ and the geodesic through z, w is orthogonal to L.*

PROOF. Note that there exists a geodesic line L_0 passing through z and orthogonal to L. Place L_0 to be the y-axis by a LFT. Then it is clear that the intersection of $L_0 \cap L$ is the shortest point on L to z. □

PROOF OF PROPSOITION 1.10. Suppose the isometry ϕ is not identity so there exists $z \in \mathbb{H}^2 \setminus L$ such that $\phi(z) \neq z$. Consider the bisector $L_{z,\phi(z)}$ which is a geodesic line by Lemma 1.11. Since the geodesic line L is fixed pointwise by ϕ, we have $d(w,z) = d(w,\phi(z))$ for any $w \in L$ so $L \subset L_{z,\phi(z)}$. They are both geodesic lines so they are equal: $L = L_{z,\phi(z)}$.

Up to a translation of LFT, we assume that L is the y-axis. We claim that ϕ coincides the reflexion ρ about the y-axis. That is to say, we need prove that for any $w \in \mathbb{H}^2$, we have $\rho(w) = \phi(w)$. By the same argument for z, we see that the bisector $L_{w,\phi(w)}$ coincides with y-axis. So the geodesic between $w, \phi(w)$ is orthogonal to L, and

$$d_{\mathbb{H}^2}(w, L) = d_{\mathbb{H}^2}(\phi(w), L).$$

By Lemma 1.12, $\phi(w)$ and w is symmetric relative to L. So $\rho(w) = \phi(w)$. □

Let L be a geodesic line. If it is given by half-circles, then the *two endpoints* of L are the intersection points with the real axis. If L is a half line, then the intersection point with the real axis and the infinity point ∞ are the *two endpoints* of L.

EXERCISE 1.13. *Let L_1, L_2 be two geodesic lines such that they have disjoint endpoints. Then there exists a unique geodesic line L orthogonal to both L_1 and L_2.*

We are able to characterize the full isometry group of $\mathrm{Isom}(\mathbb{H}^2)$.

THEOREM 1.14. *The isometry group $Isom(\mathbb{H}^2)$ is generated by $PSL(2,\mathbb{R})$ and the reflexion $z \to -\bar{z}$.*

PROOF. Up to apply LFTs from $PSL(2, \mathbb{R})$,we can assume that an isometry is fixes pointwise the y-axis. Then the proof is completed by Lemma 1.10. □

We now give another description of hyperbolic reflexion without using hyperbolic geometry.

DEFINITION 1.15 (Inversions). Consider the Euclidean plane $\mathbb{E}^2$. If L is a line, an inversion about L is the same as the Euclidean reflexion about L.

If L is a circle of radius $R > 0$ with centers o, an inversion about L sends a point $z \in \mathbb{E}^2$ to $w \in \mathbb{E}^2$ such that

$$|z - o| \cdot |w - o| = R^2,$$

where $| \cdot |$ is the Euclidean distance.

LEMMA 1.16. *Any reflexion in Isom*$(\mathbb{H}^2)$ *is exactly the restriction on* $\mathbb{H}^2$ *of an inversion about lines and circles orthogonal to the* x*-axis.*

PROOF. Observe that the reflexion ρ about y-axis is conjugated to $\phi : z \to \frac{1}{\bar{z}}$ so ϕ is a reflexion. Indeed, there exists a real LFT f such that f maps the y-axis to the unit circle. It suffices to prove that $f\rho f^{-1} = \phi$. Note, $f\rho f^{-1}$ and ϕ keeps y-axis pointwise so by Lemma 1.10 they are either equal or differ by a reflexion. Because f is orientation-preserving, $f\rho f^{-1}$ and ϕ cannot differ by reflexion. Thus, $f\rho f^{-1} = \phi$.

Note also that the hyperbolic isometry $z \to \frac{1}{\bar{z}}$ is an inversion about the unite circle orthogonal to the x-axis. So an reflexion is an inversion.

We prove now that every inversion is a hyperbolic reflexion. If the line L is orthogonal to the x-axis, an inversion about L restricting on $\mathbb{H}^2$ is the same as a hyperbolic reflexion. On ther other hand, any inversion about circles are hyperbolic reflexions, because we can apply LFTs $z \to kz$ and $z \to z + c$ which are isometries to conjugate the inversion to $z \to \frac{1}{\bar{z}}$. The proof is complete. □

1.3. Isometries as products of reflexions.

LEMMA 1.17. *An isometry in Isom*$(\mathbb{H}^2)$ *is determined by three non-linear points: if* $f, g \in$ *Isom*$(\mathbb{H}^2)$ *have same values at* $a, b, c \in \mathbb{H}^2$ *where* a, b, c *are not on the same geodesic line, then* $f = g$.

PROOF. Suppose to the contrary that there exists $z \in \mathbb{H}^2$ such that $f(z) \neq g(z)$. Consider the bisector $L_{f(z),g(z)}$ which contains a, b, c. By Lemma 1.11, $L_{f(z),g(z)}$ is a geodesic line. This contradicts to the hypothesis so we are done. □

LEMMA 1.18. *An isometry in Isom*$(\mathbb{H}^2)$ *can be written as a product of at most three reflexions.*

PROOF. Fix three points $a, b, c \in \mathbb{H}^2$. If the isometry ϕ does not fix a for instance, we compose a reflexion ρ about the bisector $L_{a,\phi(a)}$ such that $\rho(\phi(a)) = a$. In this manner, we can compose at most three reflexions such that the resulted isometry fixes a, b, c simultaneously. The proof is thus completed by Lemma 1.17. □

Another way to study the isometry group of $\mathbb{H}^2$ is to first introduce inversions about Euclidean lines or circles. The group of Mobius transformations is then defined to be the group generated by inversions. By showing that the hyperbolic metric is preserved, ones establishes that the group of Mobius transformations is the full isometry group of some hyperbolic space. This approach applies to higher dimensional hyperbolic spaces, and via Poincare extensions, the group of Mobius transformations in lower dimension naturally embeds into that of higher dimension. We refer the reader to Beardon [**1**] or Ratcliffe [**3**] for this approach.

2. Classification of orientation-reserving isometries

2.1. Ball Model of hyperbolic plane. Consider the unit ball $\mathbb{D}^2 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$. Note that the following complex LFT

$$\Phi(z) = \frac{z - i}{z + i}$$

sends $\mathbb{H}^2$ to $\mathbb{D}^2$. Define the metric on $\mathbb{D}^2$ as follows

$$d_{\mathbb{D}^2}(z, w) = d_{\mathbb{H}^2}(\Phi^{-1}(z), \Phi^{-1}(w))$$

for any $z, w \in \mathbb{D}^2$ so that $\Phi : \mathbb{H}^2 \to \mathbb{D}^2$ is an isometry. Denote by $\mathrm{Isom}^+(\mathbb{D}^2)$ the orientation-preserving isometry group. As a result,

Theorem 2.1.

$$\mathit{Isom}^+(\mathbb{D}^2) = \Phi \cdot \mathit{Isom}^+(\mathbb{H}^2) \cdot \Phi^{-1} = \left\{ \begin{pmatrix} a & \bar{c} \\ c & \bar{a} \end{pmatrix} : a, c \in \mathbb{C}, |a|^2 - |c|^2 = 1. \right\}$$

The full isometry group $\mathit{Isom}(\mathbb{D}^2)$ is generated by the above matrices and $z \to \bar{z}$.

Equivalently, we can consider the following Riemanian metric on $\mathbb{D}^2$:

$$ds = \frac{2\sqrt{dx^2 + dy^2}}{1 - (x^2 + y^2)}.$$

To be precise, a piecewise differential path $\gamma : [0, 1] \to \mathbb{H}$ has the length defined as follows:

$$\ell(\gamma) = \int_0^1 \frac{2\sqrt{x'(t)^2 + y'(t)^2}}{1 - (x(t)^2 + y(t)^2)} dt$$

where $\gamma = (x(t), y(t))$. The distance $d_{\mathbb{D}^2}$ is defined similarly as $d_{\mathbb{H}^2}$.

It is clear that $z \to e^{i\theta} z$ is an isometry of $\mathbb{D}^2$.

Exercise 2.2. *In $\mathbb{D}^2$, let z be a point such that $|z - o| = r < 1$, where o is the origin of $\mathbb{D}^2$ and $|\cdot|$ is the Euclidean distance. Prove that the distance*

$$d_{\mathbb{D}^2}(o, z) = \ln \frac{1 + r}{1 - r}.$$

Conclude that a hyperbolic disk is the same as a Euclidean disk as a set!

Exercise 2.3. *Let $\phi \in \mathcal{M}_2(\mathbb{C})$ be a complex LFT. Then it maps Euclidean circle or lines to Euclidean circle or lines.*

The isometry $\Phi : \mathbb{H}^2 \to \mathbb{D}^2$ transfers geodesic lines from $\mathbb{H}^2$ to $\mathbb{D}^2$ so by the exercise 2.3, we have the following.

Theorem 2.4. *The set of geodesic lines in $\mathbb{D}^2$ is the set of (the intersection with $\mathbb{D}^2$ of) Euclidean lines and circles orthogonal to the unit circle S^1.*

By Exercise 2.3, we also have:

Lemma 2.5. *The topology on $\mathbb{D}^2$ induced by hyperbolic metric $d_{\mathbb{D}^2}$ is the same as the Euclidean topology. The same conclusion for $\mathbb{H}^2$ with induced topology by $d_{\mathbb{H}^2}$.*

Consider the closed disk $\bar{\mathbb{D}}^2 := \mathbb{D}^2 \cup S^1$ with induced Euclidean topology. By Lemma 2.5, the topology on the interior of $\bar{\mathbb{D}}^2$ coincides with the one induced by $d_{\mathbb{D}^2}$. By Theorem 2.1, $\mathrm{Isom}(\mathbb{D}^2)$ acts by homeomorphisms on $\bar{\mathbb{D}}^2$ as they can be seen as LFTs which are homeomorphisms on $\bar{\mathbb{D}}^2$. In this sense, we say that isometries of $\mathbb{D}^2$ *extends* by homeomorphisms to S^1.

Note that the metric topology of $\mathbb{D}^2$ is the same as the Euclidean one. So in view of the hyperbolic geometry, we shall call S_1 the *boundary at infinity* $\partial_\infty \mathbb{D}^2$ of the hyperbolic space $\mathbb{D}^2$. (This boundary is not subset of $\mathbb{D}^2$)

For the upper half space $\mathbb{H}^2$, the *boundary at infinity* $\partial_\infty \mathbb{H}^2$ is the union of $\mathbb{R} \cup \{\infty\}$. Endowing the topology from extended complex numbers, $\bar{\mathbb{H}}^2 = \mathbb{H}^2 \cup \partial_\infty \mathbb{H}^2$ is a compact space with the interior $\mathbb{H}^2$ the Euclidean topology.

EXERCISE 2.6. *In $\mathbb{D}^2$, let x_n, y_n be two sequences such that $d_{\mathbb{D}^2}(x_n, y_n)$ are equal and $d_{\mathbb{D}^2}(x_n, o) \to \infty$ for some fixed point $o \in \mathbb{D}^2$. Then their Euclidean distance $|x_n - y_n|$ between x_n and y_n tends to 0 as $n \to \infty$.*

With respect to the compact topology on $\bar{\mathbb{H}}^2$ or $\bar{\mathbb{D}}^2$, the above exercise implies that if one sequence x_n converges to a point $z \in \partial_\infty \mathbb{H}^2$ (resp. $\partial_\infty \mathbb{D}^2$), then any sequence y_n with a uniformly bounded $d_{\mathbb{H}^2}(x_n, y_n)$ (resp. $d_{\mathbb{D}^2}(x_n, y_n)$) tends to the same point z.

2.2. Classification of orientation-preserving isometries. We are interested in classifying the elements in $\mathrm{Isom}^+(\mathbb{D}^2)$ which consists of orientation-preserving isometries (i.e. written as an even products of reflexions). By Theorem 1.14,

$$\mathrm{Isom}^+(\mathbb{D}^2) \cong PSL(2, \mathbb{C}) \cong \mathcal{M}_2(\mathbb{R}).$$

Recall that

THEOREM 2.7 (Brouwer). *Any continuous map of $\bar{\mathbb{D}}^2$ has a fixed point.*

So any $\phi \in \mathrm{Isom}(\mathbb{D}^2)$ has a fixed point in $\bar{\mathbb{D}}^2$. We classify the elements in $\mathrm{Isom}(\mathbb{D}^2)$ according to their action on $\bar{\mathbb{D}}^2$.

DEFINITION 2.8. Let $\phi \in \mathrm{Isom}^+(\mathbb{D}^2)$ be a non-trivial isometry.

(1) It is called *elliptic* element if it has a fixed point in $\mathbb{D}^2$;
(2) It is called *parabolic* element if it has only one fixed point in S^1;
(3) It is called *hyperbolic* element if it has exactly two fixed points in S^1.

REMARK. Since every LFT is determined by three points, every (non-trivial) element in $\mathrm{Isom}^+(\mathbb{D}^2)$ belongs one of these three categories.

The following facts are straightforward:

(1) Every elliptic element is conjugated to $z \to e^{i\theta} z$ in $\mathrm{Isom}(\mathbb{D}^2)$.
(2) Every parabolic element is conjugated to $z \to z + c$ for $c \in \mathbb{R}$ in $\mathrm{Isom}(\mathbb{H}^2)$.
(3) Every hyperbolic element is conjugated to $z \to kz$ for $k > 0$ in $\mathrm{Isom}(\mathbb{H}^2)$.

Since every isometry $\phi(z) = \frac{az+b}{cz+d}$ in $\mathrm{Isom}(\mathbb{H}^2)$ is identified with the collection of matrices

$$A \in \{k \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} : k \neq 0 \in \mathbb{R}, a, b, c, d \in \mathbb{R}; ad - bc \neq 0\}$$

we can define the following function

$$\mathrm{tr}^2(\phi) = \frac{\mathrm{trace}^2(A)}{\det(A)}$$

where trace(A) is the trace of a matrix A.

THEOREM 2.9 (Algebraic chacterization). *Given a non-trivial isometry* $\phi \in Isom^+(\mathbb{D}^2)$*, we have*

(1) ϕ *is elliptic iff* $\mathrm{tr}^2(\phi) \in [0, 4)$.
(2) ϕ *is parabolic iff* $\mathrm{tr}^2(\phi) = 4$.
(3) ϕ *is hyperbolic iff* $\mathrm{tr}^2(\phi) \in (4, \infty)$.

PROOF. We first prove that every LFT ϕ in $PSL(2, \mathbb{R})$ with one fixed point in $\mathbb{R} \cup \{\infty\}$ is conjugated to either $z \to z + 1$ or $z \to kz$ for $k \neq 1 \in \mathbb{R}$. Without loss of generality, we assume that ϕ fixes ∞ so it must be of the form $\phi(z) = az + b$. If $a = 1$, then $f^{-1}\phi f$ is equal to $z \to z + 1$ for the conjugator $f(z) = bz$. If $a \neq 1$, then ϕ has the other fixed point $\frac{b}{1-a}$. Hence, the parabolic element $z \to z + \frac{b}{1-a}$ conjugates ϕ to az.

Since trace(A) is invariant under conjugation, we have $\mathrm{tr}^2(\phi) = \mathrm{tr}^2(h\phi h^{-1})$ for any $h \in \mathcal{M}_2(\mathcal{C})$. So the theorem follows by the above discussion. □

Two geodesic lines L_1, L_2 are called *parallel* if they are disjoint in $\bar{\mathbb{D}}^2$. They are called *asymptotic* if they intersect in only one point in the boundary S^1 of $\bar{\mathbb{D}}^2$. Equivalently, L_1, L_2 are parallel iff their hyperbolic distance is positive; ultra-parallel iff their hyperbolic distance is zero but not realized by any point in $\mathbb{D}^2$.

THEOREM 2.10 (Geometric chacterization). *A non-trivial isometry* $\phi \in Isom^+(\mathbb{D}^2)$ *is a product of two reflexions about lines* L_1, L_2. *Moreover,*

(1) ϕ *is elliptic iff* L_1, L_2 *intersect.*
(2) ϕ *is parabolic iff* L_1, L_2 *are asymtotic.*
(3) ϕ *is hyperbolic iff* L_1, L_2 *are parallel.*

It is worth noting that the there are infinitely many choices of L_i in the above statement. An appropriate choice will be helpful, for instance in the following exercise.

EXERCISE 2.11. *Assume that* g *is a parabolic element and* h *is a hyperbolic element such that they do not have a common fixed point. Give a geometric proof that the commutator* $ghg^{-1}h^{-1}$ *is a hyperbolic element.*

EXERCISE 2.12. *Assume that* g, h *are two elliptic elements without a common fixed point. Give a geometric proof that the commutator* $ghg^{-1}h^{-1}$ *is a hyperbolic element.*

Consider a hyperbolic element $\phi \in \mathrm{Isom}^+(\mathbb{H}^2)$ which can be conjugated to be of the form $z \to kz$. For convenience assume that $k > 1$. It has two fixed points $0, \infty$ in $\bar{\mathbb{H}}^2$. It is clear that given a point $z \in \mathbb{H}^2$, the iterates $\phi^n(z)$ tend to ∞ for $n > 0$; for $n < 0$ they tend to 0. We call ∞ as the attractive fixed point and 0 the repelling fixed point.

In general, one may define a fixed point z of a hyperbolic element to be *attractive* if for some $o \in \mathbb{H}^2$ the iterates $\phi^n(o)$ tend to w for $n > 0$; *repelling* if $\phi^n(o)$ tend to w for $n < 0$. The definition does not depends on the choice of o by Exercise 2.6.

THEOREM 2.13 (North-Sourth Dynamics on $\bar{\mathbb{D}}^2$). *Let* $\phi \in Isom^+(\mathbb{D}^2)$ *be a non-trivial isometry. Then*

(1) *If* ϕ *is parabolic with the fixed point* $z \in S^1$*, then for any open neighborhood* U *of* z *in* S^1*, there exists* $n_0 > 0$ *such that* $\phi^n(S^1 \setminus U) \subset U$ *for any* $n > n_0$.

(2) *If ϕ is hyperbolic with the attractive and repelling points $\phi_+ \neq \phi_- \in S^1$, then for any open neighborhoods U, V of z, w respectively in S^1, there exists $n_0 \in \mathbb{Z}$ such that $\phi^n(S^1 \setminus V) \subset U$ for any $n > n_0$.*

The following lemma is well-known and will be used below.

LEMMA 2.14. *If a continuous $\phi : S^1 \to S^1$ sends a closed arc I of S^1 to be inside the interior $\mathring{I}$ of I, then ϕ contains a fixed point in $\mathring{I}$.*

LEMMA 2.15. *Let g, h be two hyperbolic elements without common fixed points. Then for all sufficiently large $n, m \gg 0$, the element $g^n h^m$ is hyperbolic.*

PROOF. Denote g_-, g_+ the repelling and attractive fixed points respectively of g. Correspondingly, h_-, h_+ for h. By assumption $\{g_-, g_+\} \cap \{h_-, h_+\} = \emptyset$. In order to apply Lemma 2.14, we take a closed arc U of the atractive fixed point g_+ such that $h_-, h_+ \notin U$. By Theorem 2.13 some power h^m for $m > 0$ sends properly U to a small neighborhood V of h_+ which does not contain g_-, g_+ as well. Finally, Theorem 2.13 allows to apply a high power g^n for sending V to the interior $\mathring{U}$ of U. In a word, we have $g^n h^m(U) \subsetneq \mathring{U}$. So Lemma 2.14 implies the exitence of a fixed point in U. A similar argument shows that there exists another fixed point in a closed neighborhood of h_-. There, $g^n h^m$ is a hyperbolic element. □

EXERCISE 2.16. *Under the assumption of Lemma 2.15, prove that the fixed points of $g^n h^m$ are disjoint with those of g, h.*

EXERCISE 2.17. *Let g be parabolic and h be hyperbolic such that they have no common fixed points. Then for all sufficiently large $n, m \gg 0$, the elements $g^n h^m$ and $h^m g^n$ are hyperbolic.*

EXERCISE 2.18. *Let g,h be two parabolic elements without the same fixed point. Then for all sufficiently large $n, m \gg 0$, the element $g^n h^m$ is hyperbolic.*

3. (non-)Elementary Fuchsian groups

We first endow the topology on $SL(2, \mathbb{C})$ from $\mathbb{C}^4$ by understanding each matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ as a 4-tuple of complex numbers (a, b, c, d). Precisely, the topology is generated by the distance $d(A, B) = \parallel A - B \parallel$ where

$$\parallel A \parallel = \sqrt{|a|^2 + |b|^2 + |c|^2 + |d|^2}.$$

EXERCISE 3.1. *For any 2×2 matrix A, we have $\parallel A \parallel^2 \geq 2 \det(A)$.*

Note that the map $(a, b, c, d) \to (-a, -b, -c, -d)$ is a homeomorphism on $SL(2, \mathbb{C})$. In fact, the group $\mathbb{Z}_2$ acts freely on $SL(2, \mathbb{C})$, where the non-trival element in $\mathbb{Z}_2$ sends $(a, b, c, d) \to (-a, -b, -c, -d)$. Thus we know that the orbital map

$$SL(2, \mathbb{C}) \to PSL(2, \mathbb{C})$$

is a 2-sheet covering map, where $PSL(2, \mathbb{C})$ is given by the quotient topology.

We understand elements $g = \frac{az+b}{cz+d}, ad - bc = 1$ in $PSL(2, \mathbb{C})$ as normalized matrices

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, ad - bc = 1.$$

Then $\min\{\| A-B \|, \| A+B \|\}$ gives a metric on $PSL(2,\mathbb{C})$. Since $\| A-(-A) \|=\| 2A \|> 2\sqrt{2}$, so the map $SL(2,\mathbb{C}) \to PSL(2,\mathbb{C})$ restricting on ball of radius $\sqrt{2}$ is an isometry. This also implies that the quotient topology on $PSL(2,\mathbb{C})$ is the same as the topology induced by the above metric

The *norm* of an element g in $PSL(2,\mathbb{C})$ is defined to be $\| g \|=\| A \|$.

EXERCISE 3.2. *Prove that* $2\cosh d_{\mathbb{H}^2}(i, gi) =\| g \|^2$*, where* $i \in \mathbb{H}^2$ *is the imaginary number.*

EXERCISE 3.3. *With respect to the topology on* $PSL(2,\mathbb{R})$*, construct a sequence of hyperbolic elements* g_n *converging to a parabolic element. Prove that a sequence of elliptic elements cannot converge a hyperbolic element.*

Let G be a subgroup of $PSL(2,\mathbb{R})$. It is called *Fuchsian* if it is discrete in the above-mentioned topology of $PSL(2,\mathbb{C})$.

EXERCISE 3.4. *The group* $G = PSL(2,\mathbb{R})$ *is a* ***topological group*** *endowed with quotient topology: the group multiplication* $(f, g) \in G \times G \to fg \in G$ *is continuous, and the inverse* $g \in G \to g^{-1} \in G$ *is homeomorphism.*

An indirect way to see it is to observe that $SL(2,\mathbb{R})$ covers G so the product $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ covers $G \times G$ as well. The covering map being a local homeomorphism implies that the convergence in $G \times G$ is locally the same as the convergence in $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$.

EXERCISE 3.5. *A group* G *is Fuchsian iff any sequence of elements* $g_n \to 1$ *becomes eventually constant:* $g_n = 1$ *for all but finitely many* n*.*

A Fuchsian group admits a properly discontinuous action on $\mathbb{H}^2$.

THEOREM 3.6. *A subgroup of* $PSL(2,\mathbb{R})$ *is Fuchsian if and only if it acts properly discontinuously on* $\mathbb{H}^2$*.*

PROOF. $\Rightarrow$: Given any compact set K in $\mathbb{H}^2$, let $g \in G$ such that $gK \cap K \neq \emptyset$. Without loss of generality, assume that $i \in K$. Thus, $d_{\mathbb{H}^2}(i, gi) \le 2R$ where R is the diameter of K. By Exercise 3.2, we have $\| g \|=\| A \|$ is uniformly bounded. This implies that only finitely many g satsifies $gK \cap K \neq \emptyset$. If not, there will be a subsequence of g_n such that $A_n \to A$, where A_n are their matrix represenatives. By local homeomorphism of $SL(2,\mathbb{C}) \to PSL(2,\mathbb{C})$, this subsequence converges in G so giving a contradiction to the discreteness of G.

$\Leftarrow$: If G is not discrete, then there exists a seuqence of elements $g_n \in G$ such that $g_n \to 1$ in G. Recall that $SL(2,\mathbb{C}) \to PSL(2,\mathbb{C})$ is a local isometry, so their matrix represenatives A_n converges to the identity in the norm $\| \cdot \|$. This gives a non-dsicrete orbit $g_n x$ for any $x \in \mathbb{H}^2$. This contradicts to the properly discontinuous action. □

A Fuchsian group is called *elementary* if it admits a finite orbit in $\bar{\mathbb{H}}^2$; otherwise it is *non-elementary*: any orbit is infinite.

THEOREM 3.7. *Let* G *be a subgroup of* $PSL(2,\mathbb{R})$ *acting properly discontinuously on* $\mathbb{H}^2$*. Then*

(1) *a parabolic element cannot have a common fixed point with a hyperbolic element.*

(2) *any two hyperbolic element have either disjoint fixed points or the same fixed points.*

Proof. For (1), we can assume that they have a common fixed point at ∞ so we can write $g(z) = z + a$ and $h(z) = kz$ for $a \in \mathbb{R}, k \neq 1$. Up to taking the inverse, we can assume that $k > 1$. By computation we have $h^{-n}gh^n(z) = z + k^{-n}a$. This contradicts the properly discontinuous action of G.

The statement (2) is similar and left to the reader. □

Theorem 3.8. *If all non-trivial element in a subgroup G of $PSL(2,\mathbb{R})$ is elliptic, then G has a global fixed point in $\mathbb{H}^2$.*

Proof. By Exercise 2.12, all elliptic elements fix the same point. □

Theorem 3.9. *Let G be an elementary Fuchsian group of $PSL(2,\mathbb{R})$. Then G belongs to one of the following cases:*

(1) *G is a finite cyclic group generated by an elliptic element,*
(2) *G is an infinite cyclic group generated by either a parabolic element or a hyperbolic element,*
(3) *G is conjugated to a subgroup $\langle z \to kz, z \to -1/z \rangle$ for some $1 \neq k > 0$.*

Proof. If G admits a finite orbit in $\mathbb{H}^2$, then G contains no parabolic and hyperbolic elements; otherwise some power of them would fix pointwise the finite orbit, giving a contradiction. By Theorem 3.8, all elliptic elements fix the same point. Conjugate the fixed point to the orgin so G is conjugated to a subgroup in S^1. Since the group is discrete, we see that G must be a finite cyclic subgroup.

So assume now that G has a finite orbit in $\bar{\mathbb{H}}^2$ and G is infinite. Since G is infinite, it must contain a hyperbolic or parabolic element (by the first paragraph). And the orbit is finite, some power of an infinite order element must fix pointwise this orbit. Thus the orbit consists of at most two points, since every orientation-preserving isometry fixes at most 2 points in the boundary. If it is just one point, then by Theorem 3.7.1, G consists of only parabolic elements. By conjugating the fixed point to ∞, we see that G must be generated by a parabolic element.

If the orbit conatins exactly two points, by Theorem 3.7, G cannot contain a parabolic element so every nontrivial element in G is either hyperbolic or elliptic. We may conjugate these two points to $0, \infty$ in $\bar{\mathbb{H}}^2$. Note that G must contain hyperbolic elements. If it consists of only hyperbolic elements, then we see that G is cyclic generated by a hyperbolic element.

If G does contain an elliptic element e, then e must switch the two fixed points $0, \infty$ so e can be conjugated to $z \to -1/z$.

Let H be the subgroup of G fixing 0 and ∞. As above, we have that H is generated by a hyperbolic element $z \to kz$ for some $k > 0$. We claim now that $G = \langle z \to kz, e \rangle$. Indeed, it suffices to consider $g \in G \setminus H$ so it switchs 0 and ∞. Then $e \cdot g$ fixes 0 and ∞ and thus belongs to H. Therefore, G is conjugated to $\langle z \to kz, z \to -1/z \rangle$. □

Exercise 3.10. *Prove that if an element in $PSL(2,\mathbb{R})$ switches two points $z, w \in \partial_\infty \mathbb{H}^2$ then it is conjugated to $z \to -1/z$.*

Theorem 3.11. *A non-elementary Fuchsian group contains infinitely many hyperbolic elements, none two of which has the same fixed points.*

PROOF. By Theorem 3.9, there exist at least two hyperbolic elements g, h such that $Fix(g) \cap Fix(h) = \emptyset$. By Lemma 2.15, $g^n h^m$ is hyperbolic for any sufficiently large $n, m > 0$. By Exercise 2.16, the fixed points of $g^n h^m$ are disjoint with those of g, h, but it could arbitrarily close to those of g! Consequently, we could produce infinitely many hyperbolic elements without sharing the same fixed points. □

3.1. Limit sets of Fuchsian groups. Since a Fuchsian group G acts properly discontinuously so any orbit is discrete in $\mathbb{D}^2$, it will be useful to look at their asymptotics at the infinity, $\partial_\infty \mathbb{D}^2$, of $\mathbb{D}^2$. In what follows, we usually consider the ball model, since its compactification by $\partial_\infty \mathbb{D}^2 = S^1$ is obvious and easy to visualize then in upper half space model.

DEFINITION 3.12. Let G be a Fuchsian group. The *limit set* denoted by $\Lambda(G)$ is the set of accumulation points of an orbit Go where $o \in \mathbb{D}^2$ is a preferred basepoint. Each point in $\Lambda(G)$ will be called a *limit point*

By Exercise 2.6, the limit set does not depend on the choice of basepoints.

EXERCISE 3.13. *The limit set of G is a G-invariant, closed subset in the topology of $\bar{\mathbb{H}}^2$.*

The following result is a consequence of Theorem 3.8.

LEMMA 3.14. *A Fuchsian group is elementary iff its limit set consists of at most two points (it may be 0, 1, 2). A non-elementay Fuchsian group must have infinitely many limit points.*

The limit set can be characterized by the following property.

THEOREM 3.15. *Let G be a non-elementary Fuchsian group. Then the limit set $\Lambda(G)$ is the minimal G-invariant closed set in $\partial_\infty \mathbb{D}^2$. And there is no isolated point in $\Lambda(G)$.*

By definition, a *perfect* set is a subset of a topological space that is closed and has no isolated points. It is known that a perfect set has uncountablely many points. So the limit set of a non-elementay Fuchsian group is a prefect set so contains uncountably many points.

PROOF. Let L be a G-invariant closed set in $\partial_\infty \mathbb{D}^2$. We shall prove that $\Lambda(G) \subset L$. Recall that G contains infinitely many hyperbolic elements g_n without same fixed points. Since L is closed and G-invaraint so $g_n L = L$, by dynamics of hyperbolic elements in Theorem 2.13, the set L contains at least three points.

By definition, $\Lambda(G)$ is the set of accumulation points of Gz. So for any $x \in \Lambda(G)$, there exists a sequence of elements $h_n \in G$ such that $h_n o \to x$ for some $o \in \bar{\mathbb{D}}^2$. Let $z \neq w \in L \setminus \{x\}$ be two points, which exist by the first paragraph. We connect z and w by a geodesic γ. We claim that up to passage of subsequences, one of the two sequences $\{h_n z\}$ and $\{h_n w\}$ converges to x.

Indeed, we choose the basepoint o on the geodesic γ for convenience. Passing to a subsequence, we assume that the endpoints $h_n z$ and $h_n w$ of geodesics $h_n \gamma$ converge to a, b respectively. It is possible that $a = b$.

Since L is closed and $z, w \in L$, we thus obtain $a, b \in L$. Note that $h_n o$ belongs to the geodesics $h_n \gamma$ so it must converge to a point in $\{a, b\}$ (cf. Exercise 3.16). Hence, the claim follows. As a consequence, x belongs to $\{a, b\}$ so $\Lambda(G) \subset L$ is proved.

Now it remains to show that x is not isolated in $\Lambda(G)$. Indeed, since $\Lambda(G)$ contains infinitely many points, we then choose three distinct points $z_1, w_1, w_2 \in \Lambda(G) \setminus \{x\}$. We apply the claim above to there pairs (z_1, w_1), (z_1, w_2) and (w_1, w_2) separately: there must be a pair of points, denoted by (z, w), from z_1, w_1, w_2 such that $g_n z \to x$ and $g_n w \to x$. Since $g_n z \neq g_n w$, we thus obtain a sequence of distinct points tending to x, thereby completing the proof that x is not isolated. □

EXERCISE 3.16. *Give a proof of the above fact that if a sequence of points z_n on geodesics γ_n converges to a point $z \in \partial_\infty \mathbb{D}^2$, then z must lie in the set of accumulation points of endpoints of γ_n.*

One way to prove this exercise is to use the following fact:

Let γ be a geodesic in $\mathbb{D}^2$ outside the ball $B(0, r)$ of Euclidean radius $r < 1$ centered at the origin. Then the Euclidean diameter of γ tends to 0 as $r \to 1$.

EXERCISE 3.17. *Consider a Fuchsian group G with a subgroup H.*

(1) *If H is of finite index in G, then $\Lambda(H) = \Lambda(G)$.*
(2) *If H is an infinite normal subgroup in G, then $\Lambda(H) = \Lambda(G)$. In particular, if G is non-elementary, then H is also non-elementary. (Tips: use Theorem 3.15.)*

COROLLARY 3.18. *Let G be a non-elementay Fuchsian group. Then the following holds:*

(1) *Any orbit is dense in the limit set $\Lambda(G)$.*
(2) *The closure of fixed points of parabolic elements coincides with $\Lambda(G)$, provided that parabolic elements exist.*
(3) *The closure of fixed points of hyperbolic elements coincides with $\Lambda(G)$.*

CHAPTER 3

Geometry of Fuchsian groups

In this chapter, we will always consider a Fuchsian group G acting on $\mathbb{H}^2$ or $\mathbb{D}^2$ if no explicit mention. We shall begin with some examples of non-elementary Fuchisan groups.

1. Schottky groups

Fix a basepoint $o \in \mathbb{H}^2$. If g is a non-elliptic element of $\mathbb{D}^2$, then the set X_g represents the open half-plane in $\mathbb{D}^2$ bounded by the bisector $L_{o,go}$, containing $g(o)$. The sets X_g and $X_{g^{-1}}$ are disjoint (resp. tangent) if and only if g is hyperbolic (resp. parabolic).

Exercise 1.1. *Prove that the sets X_g and $X_{g^{-1}}$ are disjoint (resp. tangent) in $\bar{\mathbb{H}}^2$ if and only if g is hyperbolic (resp. parabolic).*

We have

$$gX_{g^{-1}} = \mathbb{H}^2 \setminus \overline{X_g}.$$

Definition 1.2. Let $g_1, g_2, \cdots, g_n$ be a set of non-elliptic elements such that

$$\overline{(X_{g_i} \cup X_{g_i^{-1}})} \cap \overline{(X_{g_j} \cup X_{g_j^{-1}})} = \emptyset$$

for any $i \neq j$. The group generated by $\{g_1, g_2, \cdots, g_n\}$ is called *Schottky group.*

Lets repeat the Ping-Pong Lemma 2.20 here.

Lemma 1.3 (Ping-Pong Lemma). *Suppose that G is generated by a set S, and G acts on a set X. Assume, in addition, that for each $s \in \tilde{S} = S \sqcup S^{-1}$, there exists a set $X_s \subset X$ with the following properties.*

(1) $\forall s \in \tilde{S}$, $s \cdot X_t \subset X_s$, *where* $t \in \tilde{S} \setminus \{s^{-1}\}$.
(2) $\exists o \in X \setminus \cup_{s \in \tilde{S}} X_s$, *such that* $s \cdot o \in X_S$ *for any* $s \in \tilde{S}$.

Then $G \cong F(S)$.

Corollary 1.4. *A Schottky group is free.*

1.1. Fundamental domain. We give a general introduction to the notion of a fundamental domain. More details can be found in [**3**, Ch. 6.6] or [**1**, Ch. 9].

Definition 1.5. A closed subset F is called a *fundamental domain* for the action of G on $\mathbb{H}^2$ if the following two conditions hold:

(1) $\cup_{g \in G} gF = \mathbb{H}^2$,
(2) $g\mathring{F} \cap \mathring{F} = \emptyset$ for any $g \neq 1 \in G$.

Exercise 1.6. *If there exists a point $o \in \mathbb{H}^2$ such that Go is discrete and the point-stabilizer G_o is finite, then G acts properly and discontinuously on $\mathbb{H}^2$.*

LEMMA 1.7. *If a group action of G on $\mathbb{H}^2$ admits a fundamental domain then G is a Fuchsian group.*

PROOF. Let F be a fundamental domain for the action of G on $\mathbb{H}^2$. For any interior point $o \in \mathring{F}$, we see that Go is discrete, and G_o is trivial. Hence, G acts properly discontinuously on $\mathbb{H}^2$ so it is a Fuchsian group. □

A fundamental domain F is called *locally finite* if any compact set intersects only finitely many translates gF for $g \in G$. The importance of a locally finite fundamental domain lies in the following fact.

THEOREM 1.8. [**3**, Theorem 6.6.7][**1**, Theorem 9.2.4] *Let F be a locally finite fundamental domain for the action of G on $\mathbb{H}^2$. Then $\mathbb{H}^2/G$ is homeomorphic to the quotient space of F by the restriction of the map $\mathbb{H}^2 \to \mathbb{H}^2/G$.*

Assume that G acts properly discontinuously on $\mathbb{H}^2$. We define a metric on $\mathbb{H}^2/G$ as follows:

$$\bar{d}(Gx, Gy) = \inf\{d(x, Gy)\}$$

for $x, y \in \mathbb{H}^2$.

EXERCISE 1.9.
(1) *Prove that $\bar{d}$ is indeed a metric on the set $\mathbb{H}^2/G$ of orbits.*
(2) *The map $\pi : \mathbb{H}^2 \to \mathbb{H}^2/G$ sends $B(x, r)$ onto $B(\pi(x), r)$ for each $r > 0$. In particular, π is an open map.*
(3) *The quotient topology on $\mathbb{H}^2/G$ coincides with the metric topology by $\bar{d}$.*

THEOREM 1.10 (Covering is local isometry). *Assume that G acts freely and properly discontinuously on $\mathbb{H}^2$. Then the covering map $\pi : \mathbb{H}^2 \to \mathbb{H}^2/G$ is a local isometry: for each point $x \in \mathbb{H}^2$, there exists $r > 0$ (depending on x) such that $\pi : B(x, r) \to B(\pi(x), r)$ is an isometry.*

PROOF. First note that for each $x \in \mathbb{H}^2$ there exists $r > 0$ such that $B(x, r) \cap B(gx, r) = \emptyset$ for all $1 \neq g \in G$. The constant r is thus the desired one. □

EXERCISE 1.11. *Prove that the quotient spaces $\mathbb{H}^2/\langle h\rangle$ and $\mathbb{H}^2/\langle p\rangle$ endowed with the above metrics are not isometric, where h is a hyperbolic element and p is a parabolic element. (Tips: find metric-invariants to distinguish them: for instance, whether they contain closed loops which are locally shortest (i.e.: closed geodesics), or the maximal radius of embeded disks in spaces (i.e. injective radius)...)*

1.2. Dirichlet domain. In this subsection, we are going to construct a fundamental domain for any Fuchsian group. This in particular implies that the converse of Lemma 1.7 is also true.

LEMMA 1.12. *Suppose that G acts properly discontinuously on $\mathbb{H}^2$. Then there exists a point o such that it is not fixed by any non-trivial element $g \in G$.*

PROOF. Fix arbitrary point $z \in \mathbb{H}^2$, and consider the discrete orbit Gz. Then there exists $r > 0$ such that $B(z, r) \cap gB(z, r) = \emptyset$ if $go \neq o$. Thus, any point o in $B(z, r)$ satisfies the conclusion, since the point z is the only fixed point of the stabilizer of G_z. □

A special kind of fundamental domain called *Dirichlet domain* can be constructed as follows. Let o be a point not fixed by any nontrivial element in G.

Denote by $H_o(g)$ be the closed half-plane containing o bounded by the bisector $L_{o,go}$. The *Dirichlet domain* is defined as follows:

$$D_o(G) := \cap_{g\in G} H_o(g).$$

Equivalently, it contains exactly the shortest points from each orbit Gz. This is formulated in the following.

LEMMA 1.13. $D_o(G) = \{z \in \mathbb{H}^2 : d(o,z) = d(Go,z) = d(o,Gz)\}$.

PROOF. Let $z \in D_o(G)$ so $d(o,z) \le d(go,z)$ for any $g \in G$. Hence, $d(z,o) = d(z,Go)$. For the other direction, take $z \in \mathbb{H}^2$ such that $d(o,z) = d(Go,z)$. Since Go is discrete, for any $g \in G$, we have $d(o,z) \le d(go,z)$ so $z \in H_o(g)$. This implies that $z \in D_o(G)$ completing the proof. □

LEMMA 1.14. *For any point $o \in \mathbb{H}^2$ fixed only by the trivial element in G, the Dirichlet domain $D_o(G)$ is a connected convex fundamental domain.*

PROOF. The set $D_o(G)$ is path connected, and convex as the intersection of convex half-planes. Since it consists of points $z \in \mathbb{H}^2$ such that $d(o,z) = d(o,Gz)$, the condition (1) for a fundamental domain holds. So it remains to prove (2).

Suppose not, there exist $z, w \in \mathring{D}_o(G)$ such that they are in the same G-orbit: there exists $1 \ne g \in G$ such that $w = gz$. Hence, we have $d(o,z) = d(o,Gz) = d(o,w)$ thus $d(o,z) = d(g^{-1}o,z)$: $z \in L_{o,g^{-1}o}$ lies in the boundary of $D_o(G)$. This is a contradiction. □

COROLLARY 1.15. *For any $z \in \mathbb{H}^2$, the intersection $Gz \cap D_o(G)$ is a finite nonempty set.*

PROOF. By the proof of Lemma 1.14, any two points w_1, w_2 has the same distance to o. By the properly discontinuous action, there are only finitely many such points in $Gz \cap D_o(G)$. □

In what follows, the set $Gz \cap D_o(G)$ shall be referred to as a *cycle*.

LEMMA 1.16 (Local finiteness). *The Dirichlet domain is locally finite: any compact set K intersects only finitely many tranlsates of $D_o(G)$.*

PROOF. Without loss of generality, assume that K is a closed ball of radius R centered at o. Given $gD_o(G) \cap K \ne \emptyset$, we are going to prove that $d(o,go) \le 2R$ so the conclusion follows by proper actions.

Let $z \in gD_o(G) \cap K$. Then $d(o,z) \le R$ and $g^{-1}z \in D_o(G)$. Since $D_o(G)$ contains closet points in each orbit, we see that $d(g^{-1}z,o) \le d(z,o) \le R$. Hence, $d(o,go) \le 2R$. □

Let F be a convex set in $\mathbb{D}^2$. The *sides* of F correspond to the collection of maximal non-empty convex subsets of the boundary of F in $\mathbb{D}^2$, and two sides intersect at a *vertex*.

LEMMA 1.17 (Sides paired). *For each side S of $D_o(G)$, there exists a unique element $g \in G$ such that the following holds:*

(1) *S is contained in a bisector $L_{o,go}$.*
(2) *$S = D_o(G) \cap gD_o(G)$.*
(3) *$g^{-1}S$ is also a side of $D_o(G)$.*

Proof. Observe that the collection of bisectors $\{L_{o,go} : g \in G\}$ is locally finite: any compact set K intersects finitely many of them. Indeed, we can assume that K is a closed ball of radius R centered at o. If $K \cap L_{o,go} \neq \emptyset$, then $d(o, go) \leq 2R$. The properly discontinuously action thus implies the local finitenes of bisectors.

As a consequence of local finiteness, each side S contains at least two points so has positive length. Moreover, S must belong to a bisector $L_{o,go}$ for some $g \in G$.

We first prove that $D_o(G) \cap gD_o(G) = S$. If not, then $D_o(G) \cap gD_o(G)$ is a proper subset of S, and there exists $g \neq h \in G$ such that $S \cap hD_o(G)$ contains at least two points so has positive length. Let $z \in D_o(G) \cap gD_o(G)$ so $z, g^{-1}z \in D_o(G)$. Thus, $d(z, o) = d(go, z)$ by Lemma 1.13. This implies that o, go are symmetric with respect to $L_{o,go}$. By the same reasoning, we see that o, ho are symmetric about the same line $L_{o,go}$. Thus, we must have $go = ho$. By the choice of the basepoint o, we have $g = h$. This is a contradiciton, so $T = D_o(G) \cap gD_o(G)$.

By the maximality of sides by definition, we see that $g^{-1}S$ is also an edge of $D_o(G)$.

Let us prove the uniqueness of the above g. If there exists $g \neq h$ such that $S = D_o(G) \cap hD_o(G)$, then S lies on $L_{o,ho}$ so $L_{o,go} = L_{o,ho}$. Hence, we would obtain $go = ho$ and then $g = h$, a contradiction. □

Remark. When a side of a convex fundamental domain is preserved by an elliptic element, the middle point is fixed by the elliptic element. In this case, we shall divide this side into two sides with a new vertex at the middle point. It is clear that the above statements still hold for these new sides.

Note that the pair (g, g^{-1}) corresponds to the pair of sides $(S, g^{-1}S)$. It is possible that $S = gS$. If this happens, then g must have fixed point inside S and $g^2 = 1$.

The set Φ of elements g determined by sides S shall be called *side pairings* of the Dirichlet domain.

Corollary 1.18 (Generating sets). *The set of side pairings Φ generates the group G.*

Sketch of proof. By the same argument of Theorem 3.5, the set of elements $\{g \in G : gF \cap F\}$ generates G. Thus it remains to show that the elements in vertex stabilizers can be written as products over Φ. □

Exercise 1.19. *Give a proof of the above corollary.*

1.3. Schottky groups are Fuchsian.

Lemma 1.20 (Fundamental domain). *The set $F = \mathbb{D}^2 \setminus \cup_{1 \leq i \leq n}(X_{g_i} \cup X_{g_i^{-1}})$ coincides with the Dirichlet domain $D_o(G)$ based at o.*

Proof. By definition of $D_o(G)$, we know that $D_o(G)$ is a subset of F. For the other direction, suppose that there exists $x \in F \setminus D_o(G)$. Then there exists $1 \neq g \in G$ such that $gx \in D_o(G)$. Since G is a free group on the generators $S = \{g_1, g_2, \cdots, g_n\}$, we write $g = s_1 s_2 \cdots s_m$ as a reduced word where $s_i \in S$. It thus follows that $gx \in X_{s_1}$. However, $X_{s_1} \cap F = \emptyset$ so this gives a contradiction that $x \in F$. Hence, it is proved that $F = D_o(G)$. □

Theorem 1.21. *A Schottky group is a free Fuchsian group.*

1.4. Modular groups. The *modular group* $PSL(2,\mathbb{Z})$ is clearly a Fuchsian group, since the entries in matrices are integers so the group is discrete in $PSL(2,\mathbb{R})$.

LEMMA 1.22. *The Dirichlet fundamental domain at $o = ki$ for $k > 1$ is*

$$D_o(G) = \{z \in \mathbb{H}^2 : |z| > 1, |Re(z)| \le 1/2\}.$$

PROOF. It is clear that $D_o(G) = H_o(g) \cap H_o(h)$ for $g(z) = z+1$ and $h(z) = -1/z$. So it remains to show that for any $\phi = \frac{az+b}{cz+d}$, $\phi\mathring{F} \cap \mathring{F} = \emptyset$.

For any $z \in \mathring{F}$, we see that $|cz+d|^2 > 1$ so $Im(\phi(z)) = \frac{Im(z)}{|cz+d|^2} < Im(z)$. The conclusion thus follows. □

2. Geometry of Dirichlet domains

2.1. Ford domains. The reference to this subsection is [**1**, Section 9.5], where the notion of a generalized Dirichlet domain is introduced.

We first give an alternative way to interprete the Dirichlet domain. This is best illustrated in the upper plane model $\mathbb{H}^2$. Consider a LFT

$$\phi(z) = \frac{az+b}{cz+d}$$

where $a, b, c, d \in \mathbb{R}$ and $ad - bc = 1$. By computation, we see that

$$\phi'(z) = \frac{1}{(cz+d)^2}.$$

Hence, the Euclidean length $|dz|$ is sent under ϕ to the Euclidean length $|d\phi(z)|$ by a ratio $\frac{1}{|cz+d|^2}$. If $c \ne 0$, then ϕ is a Euclidean isomtery restricting on the points satsifying $|cz+d| = 1$. Since $c, d \in \mathbb{R}$, the set $|cz+d| = 1$ is a circle centered at $z = -d/c \in \mathbb{R}$ with radius $|1/c|$, which is orthgonal to the x-axis.

Equivalently, $c \ne 0$ is amount to saying that ϕ does not fix ∞.

DEFINITION 2.1. If $c \ne 0$, then $|z + d/c| = |1/c|$ is called the *isometric circle* of $\phi(z) = \frac{az+b}{cz+d}$.

Recall that an orientation-preserving isometry is a product of two reflexions about two geodesics whose configuration determines the isometry type (cf. Theorem 2.10). An isometric circle is clearly a geodesic, so giving rise to the following decompostion of an element as a product of an inversion about isometric circle and a Euclidean reflexion.

LEMMA 2.2. *If $g \in PSL(2,\mathbb{R})$ does not fix ∞ in $\mathbb{H}^2$, then $g = \rho_{L_1}\rho_{L_2}$, where L_2 is its isometric circle and L_1 is orthogonal to the real axis so ρ_{L_1} is a Euclidean reflexion. Moreover, $\rho_{L_1}(L_2)$ is the isometric circle of g^{-1}.*

By Theorem 2.10, we see that the isometric circles of g and g^{-1} are parallel (resp. asymptotic / intersecting) iff g is hyperbolic (resp. parabolic / elliptic).

The following theorem is proved in [**1**, Theorem 9.5.2].

THEOREM 2.3. *The intersection of exteriors of the isometric circles of all elements in G is a fundamental domain. In particular, when o is the origin in $\mathbb{D}^2$, it coincides with the Dirichlet domain based at o.*

2.2. Classification of limit points. We shall introduce a class of limit points called *conical points* which generalize the fixed points of hyperbolic elements. They constitute the most frequently occurring points in limit sets.

DEFINITION 2.4 (Conical points). Let G be a Fuchsian group. A limit point $z \in \Lambda(G)$ is called a *conical point* if there exists a sequence of elements $g_n \in G$ such that $g_n o \to z$, and for some basepoint o and some geodesic ray γ ending at z, the points $g_n o$ stay within a finite neighborhood of γ.

The definition is independent of the choice of the basepoints and geodesic rays:

EXERCISE 2.5. *If z is a conical point given by the above definition, then the last statement holds for any basepoint o and any geodesic ray γ ending at z.*

EXERCISE 2.6. *In a Fuchsian group, the fixed points of a hyperbolic element are conical points.*

Via the above exercise, the following result generalizes the first statement of Theorem 3.7.

LEMMA 2.7. *In a Fuchsian group, a conical point cannot be fixed by a parabolic element.*

PROOF. Assume that the conical point is at ∞ and is fixed by a parabolic element p which has the form $p(z) = z + c$ for $c \in \mathbb{R}$. By Exercise 2.5, we fix the basepoint at i, and the geodesic ray γ is put on the y-axis, for instance. By definition, there exists a sequence of elements $g_n \in G$ such that $g_n i \in N_M(\gamma)$ converges to ∞ for a uniform constant $M > 0$. The idea of the proof is similar to that of Theorem 3.7: we shall examine the values of a sequence of parabolic elements $g_n^{-1} p g_n$ at i.

First, after passage to subsequence, we see that $p(g_n(i)) = g_n(i) + c$ has a uniform bounded hyperbolic distance to $g_n(i)$. Indeed, since $g_n i \in N_M(\gamma) \to \infty$, the y-cooridnate of $g_n i$ tends to ∞. By definition of hyperbolic distance $\frac{|dz|}{y}$, there exists a constant K depending on c such that $d(p(g_n(i)), g_n(i)) \le K$. Hence, we see that $d(g_n^{-1} p g_n(i), i) \le K$ for all n. Since G acts properly on $\mathbb{H}^2$, we obtain that the set of elements $g_n^{-1} p g_n$ is finite.

As a consequence, there exist infinitely many distinct n_i such that $g_{n_i}^{-1} p g_{n_i}$ equal to the same element so $g_{n_0} g_{n_i}^{-1} p = p g_{n_0} g_{n_i}^{-1}$. Thus, $g_{n_0} g_{n_i}^{-1}$ is a parabolic element fixing ∞ as well, sending $g_{n_0} g_{n_i}^{-1}$ to $g_{n_i} i$ to $g_0 i$. However, the y-coordinate of $g_{n_i} i$ differs from that of $g_0 i$ as $g_n i \to \infty$. This is a contradiction, because a parabolic element fixing ∞ preserves the y-coordinate. Therefore, the proof is complete. □

In $\mathbb{D}^2$, a *horocycle* based at $q \in S^1$ is a Euclidean circle in $\mathbb{D}^2$ tangent at q with S^1. The Euclidean disk bounded by a horocycle is called *horodisk*.

EXERCISE 2.8. *In a Fuchsian group G, let $q \in \partial_\infty \mathbb{H}^2$ be a point fixed by a parabolic element p. Denote by G_q the stabilizer of q in G. Prove that there exists a horodisk H based at q such that $gH \cap H = \emptyset$ for any $g \in G \setminus G_q$.*

[Tips: use Lemma 2.7 prove that for any point $o \in \mathbb{H}^2$, there exists a finite number $M > 0$ such that y-coordinates of $go \in Go$ are bounded by M.]

Let H be a subgroup of a Fuchsian group G. A subset K in $\mathbb{H}^2$ is called *strictly H-invariant* if $hK = K$ for any $h \in H$, and $gK \cap K = \emptyset$ for any $g \in G \setminus H$.

Then Exercise 2.8 implies that every maximal parabolic subgroup P has a strictly invariant horodisk H. By the following exercise, we see that the corresponding quotient space H/P is embedded into $\mathbb{H}^2/G$, which shall be referred to as a *cusp* of $\mathbb{H}^2/G$.

EXERCISE 2.9. *Let K be a strictly H-invariant open subset in $\mathbb{H}^2$. Prove that the quotient space K/H is homeomorphic to $\pi(K)$ in $\mathbb{H}^2/G$ where $\pi : \mathbb{H}^2 \to \mathbb{H}^2/G$.*

Let F be a convex set in $\mathbb{D}^2$. It will be useful to consider the *infinity boundary* of F, denoted by F^∞, which is the intersection with S^1 the closure of F in the compactification $\bar{\mathbb{D}}^2$. A *free side* is a connected component of F^∞ of positive length in S^1.

LEMMA 2.10. *The interior of a free side of the Dirichlet domain is not a limit point.*

PROOF. This is straightforward by definition of a limit point. □

2.3. Parabolic fixed points and proper vertex. Recall that a *vertex* of a convex set F is the intersection of two sides. When considering the infinity boundary of F, it is useful to define vertices there as follows. A *proper vertex* of F is a point on S^1 which is the intersection of two sides; otherwise it is called an *improper vertex* if one of the two sides is a free side.

LEMMA 2.11. *Every parabolic fixed point is sent by an element $g \in G$ into the infinity boundary $D_o^\infty(G)$ of $D_o(G)$. Moreover, it is sent to a proper vertex.*

PROOF. Let q be a point fixed by a parabolic element p. We fix a geodesic ray γ ending at q. For convenience, we consider the upper plane model $\mathbb{H}^2$ and assume $q = \infty$, so γ belongs to the y-axis. Write $F = D_o(G)$ in the proof.

Since interior points of the infinity boundary of F are not limit points, it suffices to prove that γ will eventually stay in a translate gF for some $g \in G$. Equivalently, we need to show there are only finitely many gF intersecting γ.

We argue by contradiction. Assume that there exists infinitely many g_nF such that $g_nF \cap \gamma \neq \emptyset$. Choose $z_n \in g_nF \cap \gamma$. Since the Dirichlet domain is locally finite, we conclude that $z_n \to \infty$. We claim now that $d(g_no, \gamma) < M$ for a uniform constant M.

Indeed, since F is exacty the set of shortest points to the basepoint o in each orbit Gz, it follows that the set g_nF consists of shortest points in orbits to g_no. Since $z_n \in g_nF$, we see that $d(z_n, g_no) \le d(\langle p\rangle z_n, g_no)$ for each fixed n. Since p is of the form $z \to z + c$, it preseves the horocycle H through z_n. Note that the shortest path from g_no to H is orthogonal to H, so we see that the x-coordinate of $g_n(o)$ differs that of z_n at most $c/2$. This implies that there exists a uniform constant M such that $d(g_no, \gamma) < M$ where M depends on c. The claim thus follows.

A consequence of the claim shows that $g_no \to \infty$ and $g_n \in N_M(\gamma)$. This contradicts to Lemma 2.7. The proof is thus complete. □

The claim of the above proof proves the following fact. See [**1**, Thm 9.2.8] for a general statement with ANY LOCALLY FINITE fundamental domain.

COROLLARY 2.12. *Let p be a parabolic element with the fixed point at q. Then any geodesic ray ending at q intersects in only finitely many translates of Dirichlet domains.*

LEMMA 2.13. [**1**, Thm 9.3.8] *Let $q \in S^1$ be any point of $D_o^\infty(G)$ fixed by a nontrivial element p. Then p must be a parabolic element. Moreover, the cycle of q consists of a finite number of proper vertices.*

PROOF. Assume to the contrary that p is hyperbolic. Let γ be the axis of p with one endpoint at q. Let $z_n \in [o, q]$ tending to q where $[o, q] \subset D_o(G)$ by the convexity. Clearly, there exists $w_n \in \gamma$ such that $d(z_n, w_n) \to 0$ as $n \to \infty$. Since $\langle p \rangle$ acts cocompactly on γ, there exists a sequence of distinct elements $h_n \in \langle p \rangle$ sending z_n to a compact set K of γ: $h_n z_n \in K$. Noting that $d(z_n, w_n) \to 0$, there exists a compact set $K \subset K'$ such that $h_n D_o(G)$ intersects K' for infinitely many h_n. This is a contradiciton to the local finiteness of $D_o(G)$. Thus, p must be parabolic.

It remains to show that the cycle of q is finite. If not, there exist infinitely many $q_n \in D_o^\infty(G)$ and $g_n q_n = q$ for $g_n \in G$. As a consequence, each $g_n D_o(G)$ intersects a fixed geodesic ray ending at q so it is impossible by the proof of Lemma 2.11. Thus g_n must be a finite set, contradicting that $q_n \in D_o^\infty(G)$ are distinct. So the proof is finished. □

Recall that an improper vertex is the intersection of a side with a free side.

EXERCISE 2.14. *Every improper vertex in a Dirichlet domain is not a limit point.*

2.4. Conjugacy classes of elliptic and parabolic elements. A *cycle* is a maximal subset of vertices in F if they belong to the same G-orbit. If one of point in a cycle is fixed by an elliptic element, then the cycle is called an *elliptic cycle*. A cycle of proper vertices is called a *parabolic cycle*.

LEMMA 2.15 (Elliptic cycle). *Let C be a cycle of vertices in a Dirichlet domain F, and Θ be the sum of the angles at vertices in C. Then there exists some integer $m \ge 1$ such that $\Theta = 2\pi/m$. If $m > 1$, then every vertex in C is fixed by an elliptic element order m, otherwise its stabilizer is trivial.*

PROOF. By Corollary 1.15, C is a finite set. We list $C = \{x_0, x_1, \cdots, x_n\}$ such that $h_i x_i = x_{i-1}$ for some h_i where $1 \le i \mod (n+1)$. Note that $h_0 x_0 = x_n$. Thus the product $h_1 h_2 \cdots h_n h_0$ fixes x_0.

Since the sides of F is paired by Lemma 1.17, the point x_i is the common endpoint of two sides e_i and e_i' such that $h_i e_i = e_{i-1}'$ is the intersection $F \cap h_i F$ and the other side of $h_i F$ is $h_i e_i'$. Note that $h_0 e_0 = e_n'$ and $F \cap h_0 F = e_n'$. Let θ_i be the angle between e_i and e_i'.

Note that $h_1 F \cap F = e_0'$, then $h_1 h_2 F \cap h_1 F = h_1 e_1'$, continuously we get

$$h_1 h_2 \cdots h_i F \cap h_1 h_2 \cdots h_{i-1} F = h_1 h_2 \cdots h_{i-1} e_{i-1}' = h_1 h_2 \cdots h_i e_i$$

for $i \le n$. The other side of $h_1 h_2 \cdots h_n F$ is $h_1 h_2 \cdots h_n e_n'$. Noting that $e_n' = h_0 e_0$, the sides $h_1 h_2 \cdots h_n h_0 e_0$ and e_0 extends a total angle $\theta_0 + \theta_1 + \cdots + \theta_n$.

Since $h_1 h_2 \cdots h_n h_0$ fixes x_0 and sends e_0 to $h_1 h_2 \cdots h_n h_0 e_0$ with angle Θ, it must be an elliptic element of order $2\pi/\Theta$. □

LEMMA 2.16. *If a Dirichlet domain has finitely many sides, then each proper vertex is fixed by a parabolic element.*

PROOF. Let v be a proper vertex so it is the intersection of two sides. Then there exists infinitely many translates of Dirichlet domains $g_n D_o(G)$ in which v is a proper vertex. Hence, $g_n^{-1}v$ are proper vertices in $D_o(G)$. By hypothesis, the cycle of proper vertices is finite. As a result, there are infinitely many distinct elements $g_{n_i}^{-1}$ such that $g_{n_i}^{-1}v$ are the same. Thus, the proper vertex v is fixed by a non-trivial element which must be parabolic by Lemma 2.13. □

By defintion, a subgroup is called a *parabolic (resp. elliptic) subgroup* if every nontrivial element is parabolic (resp. elliptic). It is called *maximal* if it is maximal with respect to the inclusion.

By Lemma 2.18, a parabolic (resp. elliptic) subgroup fixes a unique point v so it is included in a *unique* maximal parabolic (resp. elliptic) subgroup which is the stabilizer of the point v.

THEOREM 2.17. *In a Dirichlet domain, there exists a one-one correspondence between elliptic cycles and conjugacy classes of maximal elliptic subgroups. If the Dirichlet domain has finitely many sides, then parabolic cycles correspond to conjugacy classes of maximal parabolic subgroups.*

PROOF. The correspondence for elliptic cycles and conjugacy classes of maximal elliptic subgroups is straightforward. We prove the correspondence for parabolic cycles.

Let C be a parabolic cycle which consists of proper vertices in the same G-orbit. Then each $v \in$ is fixed by a parabolic element by Lemma 2.16 so the stabilizer G_v of v is a maximal parabolic subgroup. Hence, C corresponds to the conjugacy class of G_v.

Conversely, a maximal parabolic subgroup fixes a unqiue point $v \in S^1$ so its conjugacy class corresponds to the orbit Gv. By Lemma 2.11, v is sent by an element g to a proper vertex. This clearly establishes the correspondence between parabolic cycles and conjugacy classes of maximal parabolic subgroups. □

3. Geometrically finite Fuchsian groups

3.1. Convex hull and Nieslen kernel. Let K be a closed subset in S^1. The *convex hull* $C(K)$ of K is the minimal convex subset of $\mathbb{D}^2$ such that the infinity boundary of $C(K)$ contains K. Equivalently, $C(K)$ is the intersection of half planes H whose infinity boundary contains K.

EXERCISE 3.1. *The infinity boundary of $C(K)$ coincides with K.*

Let G be a non-elementary Fuchsian group with limit set $\Lambda(G)$. The *Nieslen kernel* $N(\Lambda(G))$ is defined to be the convex hull of $\Lambda(G)$. Thus, $N(\Lambda(G))$ is G-invariant.

LEMMA 3.2. *There exists a G-equivariant retraction map $r : \bar{\mathbb{D}}^2 \setminus \Lambda(G) \to N(\Lambda(G))$.*

COROLLARY 3.3. *A Fuchsian group G acts properly discontinuously on $\bar{\mathbb{D}}^2 \setminus \Lambda(G)$.*

The set $S^1 \setminus \Lambda(G)$ is called the *discontinuity domain* of the action. It is the maximal open set in S^1 on which G acts properly discontinuously.

THEOREM 3.4. *If G is torsion-free, then the quotient space $N(\Lambda(G))/G$ is the minimal convex submanifold which is homotopic to $\mathbb{D}^2/G$.*

3.2. Geometrically finite groups.

Definition 3.5. A non-elementary Fuchsian group is called *geometrically finite* if $N(\Lambda(G))/G$ has finite area.

By convention, any elementary Fuchsian group is geometrically finite.

Lemma 3.6. *If G admits a Dirichlet domain with finitely many sides, then G is geometrically finite.*

If a Dirichlet domain has finitely many sides, then it has finitely many parabolic cycles. For each parabolic cycle C, we choose a horodisk H_v at each v such that $H_{gv} = gH_v$ for $gv \in C$.

Lemma 3.7 (Cusp decomposition). *If G admits a Dirichlet domain with finitely many sides, then there exists finitely many horodisk H_i centered at proper vertices for each parabolic cycle and a compact subset $K \subset D_o(G)$ such that*

$$C(\Lambda(G)) \setminus GH_i = GK.$$

Theorem 3.8. *A group is geometrically finite iff one of the following statements holds:*

(1) *G is finitely generated;*
(2) *G has a Dirichlet domain with finitely many sides;*
(3) *The limit set consists of conical points and parabolic fixed points.*

4. Hyperbolic surfaces

Definition 4.1. A metric space Σ is called a hyperbolic surface if every point $p \in \Sigma$ has an open neighborhood which is isometric to an open disk in $\mathbb{H}^2$.

4.1. Glueing polygons.

Theorem 4.2. *Every closed orientable surfaces of genus ≥ 2 admits a hyperbolic structure.*

4.2. Developping hyperbolic surfaces.

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