Notes on Geometry of Surfaces

Contents

CHAPTER 1

Fundamental groups of Graphs

1. Review of group theory

1.1. Group and generating set.

DEFINITION 1.1. A group $(G, \cdot)$ is a set G endowed with an operation

$$\cdot : G \times G \to G, (a, b) \to a \cdot b$$

such that the following holds.

(1) $\forall a, b, c \in G, a \cdot (b \cdot c) = (a \cdot b) \cdot c,$
(2) $\exists 1 \in G$: $\forall a \in G$, $a \cdot 1 = 1 \cdot a$.
(3) $\forall a \in G$, $\exists a^{-1} \in G$: $a \cdot a^{-1} = a^{-1} \cdot a = 1$.

In the sequel, we usually omit $\cdot$ in $a \cdot b$ if the operation is clear or understood. By the associative law, it makes no ambiguity to write abc instead of $a \cdot (b \cdot c)$ or $(a \cdot b) \cdot c$.

EXAMPLES 1.2. (1) $(\mathbb{Z}^n, +)$ for any integer $n \geq 1$
(2) General Linear groups with matrix multiplication: $GL(n, \mathbb{R})$.
(3) Given a (possibly infinite) set X, the permutation group $Sym(X)$ is the set of all bijections on X, endowed with mapping composition.
(4) Dihedral group $D_{2n} = \langle r, s | s^2 = r^{2n} = 1, srs^{-1} = r^{-1} \rangle$. This group can be visualized as the symmetry group of a regular $(2n)$-polygon: s is the reflection about the axe connecting middle points of the two opposite sides, and r is the rotation about the center of the $2n$-polygon with an angle $\pi/2n$.
(5) Infinite Dihedral group $D_\infty = \langle r, s | s^2 = 1, srs^{-1} = r^{-1} \rangle$. We can think of a regular ∞-polygon as a real line. Consider a group action of D_∞ on the real line.

DEFINITION 1.3. A subset H in a group G is called a *subgroup* if H endowed with the group operation is itself a group. Equivalently, H is a subgroup of G if

(1) $\forall a, b \in H, a \cdot b \in H$
(2) $\forall a \in H$, $\exists a^{-1} \in H$: $a \cdot a^{-1} = a^{-1} \cdot a = 1$.

Note that (1) and (2) imply that the identity 1 lies in H.

Given a subset $X \subset G$, the *subgroup generated by* X, denoted by $\langle X \rangle$, is the minimal subgroup of G containing X. Explicitly, we have

$$\langle X \rangle = \{x_1^{\epsilon_1} x_2^{\epsilon_2} \cdots x_n^{\epsilon_n} : n \in \mathbb{N}, x_i \in X, \epsilon_i \in \{1, -1\}\}.$$

A subset X is called a *generating set* of G if $G = \langle X \rangle$. If X is finite, then G is called *finitely generated*.

Check Examples 1.2 and find out which are finitely generated, and if yes, write a generating set.

EXERCISE 1.4. (1) *Prove that* $(\mathbb{Q}, +)$ *is not finitely generated.*
(2) *Prove that* $\{r, rsr^{-1}\}$ *is a generating set for* D_∞.

EXERCISE 1.5. (1) *Suppose that* G *is a finitely generated group. If* $H \subset G$ *is of finite index in* G*, then* H *is finitely generated.*
(2) *Conversely, suppose that* H *is a finite index subgroup of a group* G*. If* H *is finitely generated, then* G *is also finitely generated.*

EXERCISE 1.6. *Let* N *be a normal subgroup of a group* G*. Suppose that* N *and* G/N *are finitely generated. Then* G *is finitely generated.*

1.2. Group action.

DEFINITION 1.7. Let G be a group and X be set. A *group action of* G *on* X is a function

$$G \times X \to X, (g, x) \to g \cdot x$$

such that the following holds.

(1) $\forall x \in X$, $1 \cdot x = x$.
(2) $\forall g, h \in G, (gh) \cdot x = g \cdot (h \cdot x)$.

Usually we say that G acts on X. Similarly, we often omit $\cdot$ in $g \cdot x$, but keep in mind that $gx \in X$, which is not a group element!

REMARK. A group can act *trivially* on any set X by just setting $g \cdot x = x$. So we are mainly interested in nontrivial group actions.

EXAMPLES 1.8. (1) $\mathbb{Z}$ acts on the real line $\mathbb{R}$: $(n, r) \to n + r$.
(2) $\mathbb{Z}$ acts on the circle $S^1 = \{e^{i\theta} : \theta \in \mathbb{R}\}$: $(n, e^{\theta i}) \to e^{n\theta i}$. Here i is the imaginary unit.
(3) $\mathbb{Z}^n$ acts on $\mathbb{R}^n$.
(4) $GL(n, \mathbb{R})$ acts on $\mathbb{R}^n$.

Recall that $Sym(X)$ is the permutation group of X. We have the following equivalent formulation of a group action.

LEMMA 1.9. *A group* G *acts on a set* X *if and only if there exists a group homomorphism* $G \to Sym(X)$.

PROOF. (=>). Define $\phi : G \to Sym(X)$ as follows. Given $g \in G$, let $\phi(g)(x) = g \cdot x$ for any $x \in X$. Here $g \cdot x$ is given in definition of the group action of G on X.

It is an exercise to verify that $\phi(g)$ is a bijection on X. Moreover, the condition (2) in definition 1.7 is amount to say that ϕ is homomorphism.

(<=). Let $\phi : G \to Sym(X)$ be a group homomorphism. Construct a map $G \times X \to X$: $(g, x) \to \phi(g)(x)$. Then it is easy to see that this map gives a group action of G on X. □

So when we say a group action of G on X, it is same as specifying a group homomorphism from G to $Sym(X)$.

REMARK. (1) A trivial group action is to specify a trivial group homomorphism, sending every element in G to the identity in $Sym(X)$.
(2) In general, the group homomorphism $G \to Sym(X)$ may not injective. If it is injective, we call the group action is *faithful*.

(3) In practice, the set X usually comes with extra nice structures, for example, X is a vector space, a topological space, or a metric space, etc. The homomorphic image of G in $Sym(X)$ may preserve these structures. In this case, we say that G acts on X by linear transformations, by homeomorphisms, or by isometries ...

We now recall Cayley's theorem, which essentially says that we should understand groups via group actions on sets with various good structures.

THEOREM 1.10. *Every group is a subgroup of the permutation group of a set.*

PROOF. Let $X = G$. Clearly the group operation $G \times G \to G$ gives a group action of G on G. Thus, we obtain a homomorphism $G \to Sym(G)$. The injectivity is clear. □

For any $x \in X$, the *orbit* of x under the group action is the set $\{g \cdot x : g \in G\}$. We denote it by $G \cdot x$ or even simply by Gx. The *stabilizer* of x

$$G_x := \{g \in G : g \cdot x = x\}$$

is clearly a group.

LEMMA 1.11. *Suppose that G acts on X. Then for any x, there exists a bijection between $\{gG_x : g \in G\}$ and Gx. In particular, if Gx is finite, then $[G : G_x] = |Gx|$.*

PROOF. We define a map $\phi : gG_x \to gx$. First, we need to show that this map is well-defined: that is to say, if $gG_x = g'G_x$, then $gx = g'x$. This follows from the definition of G_x.

For any $gx \in Gx$, we have $\phi(gG_x) = gx$. So ϕ is surjective.

To see that ϕ is injective, let $gG_x, g'G_x$ such that $gx = g'x$. Then $g^{-1}g'x = x$ and $g^{-1}g' \in G_x$. Hence $gG_x = g'G_x$. This shows that ϕ is injective. □

EXERCISE 1.12. (1) *Let H be a subgroup in G. Then $\cap_{g \in G}(gHg^{-1})$ is a normal subgroup in G.*

(2) *Let H be a finite index subgroup of G. Then there exists a normal subgroup N of G such that $N \subset H$ and $[G : N] < \infty$. (Hint: construct a group action)*

THEOREM 1.13 (M. Hall). *Suppose G is a finitely generated group. Then for any integer $n > 1$, there are only finitely many subgroups H in G such that $[G : H] = n$.*

PROOF. Fix n. Let H be a subgroup of index n. Let $X = \{H, g_1H, \cdots, g_{n-1}H\}$ be the set of all H-cosets. Then G acts on X of by left-multiplication. That is, $(g, g_iH) \to gg_iH$. Clearly, the stabilizer of $H \in X$ is $H \subset G$. Put in other words, the subgroup H can be recovered from the action of G on X.

For any set X with n elements, a finitely generated G has finitely many different actions on X. By Lemma 1.2, a group action is the same as a group homomorphism. A homomorphism is determined by the image of a generating set. As G is finitely generated and $Sym(X)$ is finite, there exist only finitely many group homomorphisms.

Consequently, for any $n > 0$, there exist only finitely many H of finite index n. □

2. Free groups and Ping-Pong Lemma

2.1. Words and their reduced forms. Let $\tilde{X}$ be an alphabet set. A *word* w over $\tilde{X}$ is a finite sequence of letters in $\tilde{X}$. We usually write $w = x_1x_2...x_n$, where $x_i \in \tilde{X}$. The *empty word* is the word with an empty sequence of letters. The *length* of a word w is the length of the sequence of letters.

Two words are *equal* if their sequences of letters are identical. Denote by $\mathcal{W}(\tilde{X})$ the set of all words over $\tilde{X}$. Given two words $w, w' \in \mathcal{W}(\tilde{X})$, the *concatenation* of w and w' is a new word, denoted by ww', which is obtained from w followed by w'.

Given a set X, we take another set X^{-1} such that there exists a bijection $X \to X^{-1} : x \to x^{-1}$. Let $\tilde{X} = X \sqcup X^{-1}$ be the disjoint union of X and X^{-1}. Roughly speaking, the free group $F(X)$ generated by X will be the set of words $\mathcal{W}$ endowed with the operation of word concatenation.

Given a word w, if there exists two consecutive letters of form xx^{-1} or $x^{-1}x$ where $x, x^{-1} \in \tilde{X}$, then we call xx^{-1} or $x^{-1}x$ an *inverse pair* of w. A word w is called *reduced* if w contains no inverse pair. Given a word w, we define an operation on w called a *reduction*, by which we mean deleting an inverse pair xx^{-1} or $x^{-1}x$ to obtain a new word w':

$$w = w_1xx^{-1}w_2 \xrightarrow{reduction} w' = w_1w_2.$$

After a reduction, the length of a word decreases by 2. A finite sequence of reductions

$$w \xrightarrow{reduction\#1} w_1 \xrightarrow{reduction\#2} w_2 ... \xrightarrow{reduction\#n} w_n$$

will be referred to as a reduction process.

Clearly, any word w admits a reduction process to get a reduced word. This reduced word is called a *reduced form* of w. But a word may have different reduction processes to become reduced. For example, $w = xx^{-1}xx^{-1}$. However, we will prove that reduced forms of a word does not depend on the reduction process.

LEMMA 2.1. *Any word w has a unique reduced form.*

PROOF. We prove the lemma by induction on the length $|w|$ of w. The base cases that $|w| = 1, 2$ are trivial. Now assume that the lemma holds for any word of length $|w| \le n$.

Let w be a word of length of n. Let

$$w \xrightarrow{reduction\#1} w_1 \xrightarrow{reduction\#2} w_2 ... \xrightarrow{reduction\#l} w_l$$

and

$$w \xrightarrow{reduction\#1'} w_1' \xrightarrow{reduction\#2'} w_2' ... \xrightarrow{reduction\#m'} w_m'$$

be any two reduction processes of w such that w_l, w_m' are reduced. We will show that $w_l = w_m'$.

We have the following claim.

CLAIM. *Suppose that $w_1 \neq w_1'$. Then there are two reductions*

$$w_1 \xrightarrow{reduction\ \#1} \hat{w}$$

and

$$w_1' \xrightarrow{reduction\ \#1'} \hat{w}'$$

such that $\hat{w} = \hat{w}'$.

PROOF OF CLAIM. Let xx^{-1} be the inverse pair for the reduction #1, and yy^{-1} the inverse pair for the reduction #1′. We have two cases.

Case 1. The inverse pairs xx^{-1}, yy^{-1} are disjoint in w. In this case, we let reduction a be reduction #1′, and reduction b be reduction #1. Thus, $\hat{w} = \hat{w}'$.

Case 2. The inverse pairs xx^{-1}, yy^{-1} have overlaps. Then either $x^{-1} = y$ or $y^{-1} = x$. In either cases, we have $w_1 = w_1'$. This contradicts the assumption that $w_1 \neq w_1'$. □

We are now ready to complete the proof of Lemma. First, if $w_1 = w_1'$, then $w_l = w_m'$ by applying the induction assumption to $w_1 = w_1'$ of length $n-2$. Otherwise, by the claim, there are two reductions applying to w_1, w_1' respectively such that the obtained words $\hat{w} = \hat{w}'$ are the same.

Note that $\hat{w}$ is of length $n-4$. Applying induction assumption to $\hat{w}$, we see that any reduction process

$$\hat{w} \xrightarrow{\textit{reduction process}} \bar{w}$$

of $\hat{w}$ gives the same reduced form $\bar{w}$.

By the claim, the reduction a together any reduction process $\hat{w} \xrightarrow{\textit{reduction process}} \bar{w}$ gives a reduction process for w_1 to $\bar{w}$. By induction assumption to w_1, we have $w_l = \bar{w}$. By the same reasoning, we have $w_m' = \bar{w}$. This shows that $w_l = w_m' = \bar{w}$. □

2.2. Construction of free groups by words. Denote by $F(X)$ the set of all reduced words in $\mathcal{W}(\tilde{X})$. By Lemma 2.1, there is a map

$$\mathcal{W}(\tilde{X}) \to F(X)$$

by sending a word to its reduced form.

We now define the group operation on the set $F(X)$. Let w, w' be two words in $F(X)$. The product $w \cdot w'$ is the reduced form of the word ww'.

THEOREM 2.2. *$(F(X), \cdot)$ is a group with a generating set X.*

PROOF. It suffices to prove the associative law for the group operation. Let w_1, w_2, w_3 be words in $F(X)$. We want to show $(w_1 \cdot w_2) \cdot w_3 = w_1 \cdot (w_2 \cdot w_3)$. By Lemma 2.1, the reduced form of a word does not depend on the reduction process. Observe that $(w_1 \cdot w_2) \cdot w_3$ and $w_1 \cdot (w_2 \cdot w_3)$ can be viewed as reduced forms of different reduction processes of the word $w_1w_2w_3$. The proof is thus completed. □

Let $\iota : X \to F(X)$ be the inclusion of X in $F(X)$. Usually we will not distinguish x and $\iota(x)$ below, as ι is injective.

LEMMA 2.3. *For any map of a set X to a group G, there exists a unique homomorphism $\phi : F(X) \to G$ such that*

$$\begin{array}{ccc} X & \to & F(X) \\ & \searrow & \downarrow \\ & & G \end{array}$$

is commutative.

PROOF. Denote by j the map $X \to G$. Define $\phi(x) = j(x)$ for all $x \in X$ and $\phi(x^{-1}) = j(x)^{-1}$ for $x^{-1} \in X^{-1}$. Define ϕ naturally over other elements in $F(X)$.

Let w_1, w_2 be two reduced words in $F(X)$. Without loss of generality, assume that $w_1 = x_1x_2...x_nz_1z_2...z_r$ and $w_2 = z_1^{-1}...z_r^{-1}y_1y_2...y_m$, where $x_i, y_j, z_k \in \tilde{X} = X \sqcup X^{-1}$ and $x_n \neq y_1^{-1}$. Then $w_1 \cdot w_2 = x_1x_2...x_ny_1y_2...y_m$. It is straightforward to verify that $\phi(w_1 \cdot w_2) = \phi(w_1)\phi(w_2)$.

Since a homomorphism of $F(X)$ to G is determined by the value of its restriction over a generating set of $F(X)$, we have that the chosen map $j : X \to G$ determines the uniqueness of ϕ. □

Corollary 2.4. *Every group is a quotient of a free group.*

Proof. Let X be a generating set of G. Let $F(X)$ be the free group generated by X. By Lemma 2.3, we have an epimorphism of $F(X) \to G$. □

Exercise 2.5. *Let X be a set containing only one element. Prove that $F(X) \cong \mathbb{Z}$.*

Analogous to free abelian group, the class of free groups is characterized by the following universal mapping property in GROUP category.

Lemma 2.6. *Let X be a subset, F be a group and $i : X \to F$ be a map. Suppose that for any group G and a map $j : X \to G$, there exists a unique homomorphism $\phi : F \to G$ such that*

$$\begin{array}{ccc} X & \xrightarrow{i} & F \\ & \searrow^{j} & \downarrow{\scriptstyle \phi} \\ & & G \end{array}$$

is commutative. Then $F \cong F(X)$.

Proof. By Lemma 2.3 for free group $F(X)$ and $i : X \to F$, there is a unique homomorphism $\varphi : F(X) \to F$ such that $i = \varphi\iota$, where $\iota : X \to F(X)$ is the inclusion map. ie.

(1)
$$\begin{array}{ccc} X & \xrightarrow{\iota} & F(X) \\ & \searrow^{i} & \downarrow{\scriptstyle \varphi} \\ & & F \end{array}$$

On the other hand, by the assumption to $G = F(X)$ and $\iota : X \to F(X)$, there is a unique homomorphism $\phi : F \to F(X)$ such that we have $\iota = \phi i$.

(2)
$$\begin{array}{ccc} X & \xrightarrow{i} & F \\ & \searrow^{\iota} & \downarrow{\scriptstyle \phi} \\ & & F(X) \end{array}$$

Thus we obtained $\iota = \phi\varphi\iota$, and the following commutative diagram follows from the above (1)(2).

(3)
$$\begin{array}{ccc} X & \xrightarrow{\iota} & F(X) \\ & \searrow^{\iota} & \downarrow{\scriptstyle \phi\varphi} \\ & & F(X) \end{array}$$

Note that the identification $Id_{F(X)}$ between $F(X) \to F(X)$ also makes the above diagram (3) commutative. By the uniqueness statement of Lemma 2.3, $\phi\varphi = Id_{F(X)}$.

It is analogous to prove that $\varphi\phi = Id_F$. Hence ϕ or φ is an isomorphism. □

2.3. (Free) abelian groups. Recall that a group G is called *abelian* if $ab = ba$ for any $a, b \in G$. In this subsection, we study finitely generated abelian group.

Definition 2.7. Let X be a set. The group $A(X) := \bigoplus_{x\in X} \langle x\rangle$ is called the *free abelian group* generated by X. The set X is called a *basis* of $A(X)$.

By definition, we see that there is an injective map $X \to A(X)$ defined by $x \to (0, ...0, x, 0, ...)$ for $x \in X$. Clearly, $A(X)$ is generated by (the image under the injective map) of X.

Let $m \in \mathbb{Z}$ and $a = (n_1 x, ..., n_i x, ...) \in A(X)$. We define the scalar multiplication

$$m \cdot a = (mn_1 x, ..., mn_i x, ...) \in A(X).$$

A *linear combination* of elements $a_i \in A(X), 1 \le i \le n$ is an element in $A(X)$ of the form $\sum_{1\le i\le n} k_i \cdot a_i$ for some $k_i \in \mathbb{Z}, 1 \le i \le n$.

Exercise 2.8.
(1) *Let Y be a subset in a free abelian group G of finite rank. Then Y is basis of G if and only if $G = \langle Y\rangle$ and any element in G can be written as a unique linear combination of elements in Y.*
(2) *Prove that the group of rational numbers $\mathbb{Q}$ is not free abelian.*

Exercise 2.9. *Prove that $\mathbb{Z}^m \cong \mathbb{Z}^n$ if and only if $m = n$.*

If $|X|$ is finite, then $|X|$ is called the *rank* of $A(X)$. In general, a free abelian group may have different basis. The rank of a free abelian group is well-defined, by Exercise 2.9.

Every abelian group is a quotient of a free abelian group.

Lemma 2.10. *Let X be a subset. For any map of X to an abelian group G, there exists a unique homomorphism ϕ such that*

$$\begin{array}{ccc} X & \to & A(X) \\ & \searrow & \downarrow \\ & & G \end{array}$$

is commutative.

Corollary 2.11. *Every abelian group is a quotient of a free abelian group.*

A free abelian group is characterized by the following universal mapping property in the category of abelian groups.

Lemma 2.12. *Let X be a subset, A be an abelian group and $X \to A$ be a map. Suppose that for any abelian group G and a map $X \to G$, there exists a unique homomorphism $\phi : A \to G$ such that*

$$\begin{array}{ccc} X & \to & A \\ & \searrow & \downarrow \\ & & G \end{array}$$

is commutative. Then $A \cong A(X)$.

Recall that the commutator subgroup $[G, G]$ of a group G is the subgroup in G generated by the set of all commutators. That is:

$$[G, G] = \langle \{[f, g] := fgf^{-1}g^{-1} : f, g \in G\} \rangle$$

Use universal mapping property of free groups and free abelian groups to prove the following.

EXERCISE 2.13. *Prove that* $F(X)/[F(X), F(X)] \cong A(X)$*, where* $A(X)$ *is the free abelian group generated by* X*.*

A subset Y is called a *basis* of $F(X)$ if $F(X) \cong F(Y)$. In this case, we often say that $F(X)$ is *freely generated* by X. Use Exercise 2.9 to prove the following.

EXERCISE 2.14. *If* $|X| < \infty$ *and* Y *is a basis of* $F(X)$*, then* $|X| = |Y|$*.*

The *rank* of $F(X)$ is defined to be the cardinality of X. By Exercise 2.14, the rank of a free group is well-defined: does not depend on the choice of basis.

When the rank is finite, we usually write $F_n = F(X)$ for $n = |X|$.

2.4. A criterion of free group by words.

CONVENTION. *Since there is a map* $\mathcal{W}(\tilde{X}) \to F(X) \to G$ *for a generating set* X *of* G*, we write* $w =_G g$ *for a word* $w \in \mathcal{W}(\tilde{X}), g \in G$*, if the image of* w *under the map* $\mathcal{W}(\tilde{X}) \to G$ *is the element* g*.*

THEOREM 2.15. *Let* G *be a group with a generating set* X*. Then* $G \cong F(X)$ *if and only if any non-empty word* $w \in \mathcal{W}(\tilde{X})$ *with* $w =_G 1 \in G$ *contains an inverse pair.*

PROOF. We have first a surjective map $\mathcal{W}(\tilde{X}) \to F(X) \to G$, where $F(X) \to G$ is the epimorphism given by Lemma 2.3.

$=>$. let $w \in \mathcal{W}(\tilde{X})$ be a word such that $w =_G 1$. Since $F(X) \cong G$, we have w is mapped to the empty word in $F(X)$. That is to say, the reduced form of W is the empty word. Thus, w contains an inverse pair.

$<=$. Suppose that $F(X) \to G$ is not injective. Then there exists a non-empty reduced word $w \in F(X)$ such that $w =_G 1$. Then w contains an inverse pair. As w is reduced, this is a contradiction. Hence $F(X) \to G$ is injective. □

COROLLARY 2.16. *A group is freely generated by a set* X *if and only if any non-empty reduced word over* X *is a non-trivial element in* G*.*

EXERCISE 2.17.
(1) *Let* Y *be a set in the free group* $F(X)$ *generated by a set* X *such that* $y^{-1} \notin Y$ *for any* $y \in Y$*. If any reduced word* w *over* $\tilde{Y} = Y \sqcup Y^{-1}$ *is a reduced word over* $\tilde{X} = X \sqcup X^{-1}$*, then* $\langle Y \rangle \cong F(Y)$*.*
(2) *Let* $S = \{b^n a b^{-n} : n \in \mathbb{Z}\}$ *be a set of words in* $F(X)$ *where* $X = \{a, b\}$*. Prove that* $\langle S \rangle \cong F(S)$*.*
(3) *Prove that for any set* X *with* $|X| \geq 2$ *any* $n \geq 1$*,* $F(X)$ *contains a free subgroup of rank* n*.*

2.5. Ping-Pong Lemma and free groups in linear groups. In this subsection, we gives some common practice to construct a free subgroup in concrete groups. We formulate it in Ping-Pong Lemma. Before stating the lemma, we look at the following example.

LEMMA 2.18. *The subgroup of* $\mathbb{SL}(2,\mathbb{Z})$ *generated by the following matrices*

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$$

is isomorphic to F_2.

PROOF. See Proposition 3.7, on page 59 in our reference [**2**]. □

EXERCISE 2.19. *The subgroup of* $\mathbb{SL}(2,\mathbb{C})$ *generated by the following matrices*

$$\begin{pmatrix} 1 & a_1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ a_2 & 1 \end{pmatrix}, |a_1| \ge 2, |a_2| \ge 2;$$

is isomorphic to F_2.

LEMMA 2.20 (Ping-Pong Lemma). *Suppose that* G *is generated by a set* S, *and* G *acts on a set* X. *Assume, in addition, that for each* $s \in \tilde{S} = S \sqcup S^{-1}$, *there exists a set* $X_s \subset X$ *with the following properties.*

(1) $\forall s \in \tilde{S}$, $s \cdot X_t \subset X_s$, *where* $t \in \tilde{S} \setminus \{s^{-1}\}$.
(2) $\exists o \in X \setminus \cup_{s\in\tilde{S}} X_s$, *such that* $s \cdot o \in X_S$ *for any* $s \in \tilde{S}$.

Then $G \cong F(S)$.

PROOF. By Lemma 2.3 and Lemma 1.9, we have the following homomorphism:

$$\iota : F(S) \to G \to Sym(X).$$

Let w be a reduced non-empty word in $F(S)$. Write $w = s_1 s_2 ... s_n$ for $s_i \in \tilde{S}$. By Theorem 2.15, it suffices to show that $g = \iota(s_1)\iota(s_2)...\iota(s_n)$ is not an identity in $Sym(X)$.

We now apply the permutation g to $o \in X$ to get

$$g \cdot o = \iota(s_1)\iota(s_2)...\iota(s_{n-1})\iota(s_n) \cdot o \subset \iota(s_1)\iota(s_2)...\iota(s_{n-1})X_{s_n} \subset ... \subset X_{s_1}.$$

However, as $o \in X_{s_1}$, we have $g \ne 1 \in Sym(X)$. This shows that $F(S) \cong G$. □

Ping-Pong Lemma has a variety of forms, for instance:

EXERCISE 2.21. *Let* G *be a group generated by two elements* a, b *of infinite order. Assume that* G *acts on a set* X *with the following properties.*

(1) *There exists non-empty subsets* $A, B \subset X$ *such that* A *is not included in* B.
(2) $a^n(B) \subset A$ *and* $b^n(A) \subset B$ *for all* $n \in \mathbb{Z} \setminus \{0\}$.

Prove that G *is freely generated by* $\{a, b\}$.

We now prove that $\mathbb{SL}(2,\mathbb{R})$ contains many free subgroups.

PROPOSITION 2.22. *Let* $A \in \mathbb{SL}(2,\mathbb{R})$ *with two eigenvalues* λ, λ^{-1} *for* $\lambda > 1$, *and corresponding eigenvectors* $v_\lambda, v_{\lambda^{-1}}$. *Choose* $B \in \mathbb{SL}(2,\mathbb{R})$ *such that* $B\langle v_\lambda\rangle \ne \langle v_\lambda\rangle$, $B\langle v_\lambda\rangle \ne \langle v_{\lambda^{-1}}\rangle$ *and* $B\langle v_{\lambda^{-1}}\rangle \ne \langle v_\lambda\rangle$, $B\langle v_{\lambda^{-1}}\rangle \ne \langle v_{\lambda^{-1}}\rangle$.

Then there exist $N, M > 0$ *depending only on* A, B *such that*

$$F(a,b) = \langle a, b\rangle$$

where $a = A^n, b = BA^mB^{-1}$ *for* $n, > N, m > M$.

PROOF. Observe that BAB^{-1} has the same eigenvalues λ, λ^{-1}, but eigenvectors $Bv_\lambda, Bv_{\lambda^{-1}}$ respectively.

Let $\theta \in (0, 2\pi)$ be a (very small) angle. Denote by $X_{v,\theta} \subset \mathbb{R}^2$ the open sector around the line $\langle v_\lambda \rangle$ with angle θ.

We claim the following fact about the dynamics of A on vectors.

CLAIM. $\forall \theta \in (0, 2\pi), \exists N > 0$ *such that the following holds.*

$$\textit{For } \forall n > N, v \in \mathbb{R}^2 \setminus \langle v_\lambda^{-1} \rangle, \textit{ we have } A^n v \in X_{v_\lambda,\theta}.$$

and

$$\textit{For } \forall n > N, v \in \mathbb{R}^2 \setminus \langle v_\lambda \rangle, \textit{ we have } A^{-n} v \in X_{v_\lambda^{-1},\theta}.$$

PROOF OF CLAIM. Since $\{v_\lambda, v_\lambda^{-1}\}$ is a basis of $\mathbb{R}^2$, the conclusion follows by a simple calculation. $\square$

By the same reasoning, we also have

CLAIM. $\forall \theta \in (0, 2\pi), \exists M > 0$ *such that the following holds.*

$$\textit{For } \forall m > M, v \in \mathbb{R}^2 \setminus \langle Bv_\lambda^{-1} \rangle, \textit{ we have } BA^m B^{-1} v \in X_{Bv_\lambda,\theta}.$$

and

$$\textit{For } \forall m > M, v \in \mathbb{R}^2 \setminus \langle Bv_\lambda \rangle, \textit{ we have } BA^{-m} B^{-1} v \in X_{Bv_{\lambda^{-1}},\theta}.$$

Denote $a = A^n, b = BA^m B^{-1}, X_a = X_{v_\lambda,\theta}, X_a^{-1} = X_{v_\lambda^{-1},\theta}, X_b = X_{Bv_\lambda,\theta}, X_b^{-1} = X_{Bv_\lambda,\theta}$. Let $S = \{a, b\}$. By the above claims, we obtain the following.

$$\forall s \in \tilde{S},\ s \cdot X_t \subset X_s, \text{ where } t \in \tilde{S} \setminus \{s^{-1}\}.$$

Choose θ small enough such that $X_a \cup X_a^{-1} \cup X_b \cup X_b^{-1} \neq \mathbb{R}^2$. Choose any $o \in \mathbb{R}^2 \setminus \cup_{s \in \tilde{S}} X_s$. By the claims, $s \cdot o \in X_s$. Hence, all conditions of Ping-Pong Lemma are satisfied. We obtain that $F(\{a, b\}) = \langle a, b \rangle$. $\square$

In fact, Jacques Tits proved the following celebrated result in 1972, which is usually called Tits alternative.

THEOREM 2.23. *Let G be a finitely generated linear group. Then either G is virtually solvable or contains a free subgroup of rank at least 2.*

REMARK. Note that a virtually solvable group does not contain any free group of rank at least 2. This explains the name of Tits alternative.

3. Subgroups of free groups

We shall give two proofs of the following theorem of Nielsen.

THEOREM 3.1. *Any subgroup of a free group is free.*

3.1. Group action on graphs. The first proof is to consider a group action on trees, and to use Ping-Pong Lemma. We first introduce the notion of a metric graph.

DEFINITION 3.2. A *metric graph* $\mathcal{G}$ consists of a set V of vertices and a set E of *undirected* edges which are copies of intervals $[0, 1]$ (with length 1). Each edge $e \in E$ are associated with two endpoints in V.

We can endow the graph with the following metric. The distance of two points $v, w \in \mathcal{G}$ is the length of shortest path between v, w.

REMARK. We do allow two edges with the same endpoints, and the two endpoints of an edge can be the same.

A *graph morphism* $\phi : \mathcal{G} \to \mathcal{G}'$ between two metric graphs $\mathcal{G}, \mathcal{G}'$ is a map sending edges to edges isometrically. It is called a *graph isomorphism* if ϕ is bijective. So, a graph isomorphism is an isometry of the metric graph.

An graph isomorphism is called an *inversion* if it switch two endponts of some edge. By inserting a vertex at each fixed point, an inversion of a graph can induce a non-inversion isomorphism of a new graph, which captures essential information of the original one.

Suppose now that a group G acts on a graph $\mathcal{G}$ by *isomorphisms without inversion*: we have a homomorphism

$$G \to \mathrm{Aut}(\mathcal{G})$$

where $\mathrm{Aut}(\mathcal{G})$ is the group of all isomorphisms of $\mathcal{G}$ such that the images do not contain inversions.

Given a metric graph, we consider the topology induced by the metric so the notion of connectedness, closed subset etc can be talked about. It is obvious that a connected graph is amount to saying that any two vertices are connected by a path.

LEMMA 3.3. *If a group G acts on a connected graph $\mathcal{G}$ without inversions, then there exists a subset F in $\mathcal{G}$ such that*

(1) *F is a closed subset,*
(2) *the set $\{gF : g \in G\}$ covers the graph,*
(3) *no subset of F satisfies properties (1) and (2).*

A set F satisfying the above properties is called *fundamental domain* for the action of G on the graph. In what follows, it is usually connected.

PROOF. We first construct the *core* C of the desired fundamental domain F. The core C will be a connected subset and contain exactly one point from each orbit or vertices.

Fix a vertex $o \in \mathcal{G}$ as the basepoint. We need use the form of Axiom of Choice - Zorn's lemma to conclude the construction of X: every non-empty partially ordered set in which every chain (i.e., totally ordered subset) has an upper bound contains at least one maximal element. We consider the collection $\mathbb{X}$ of connected subgraphs X with the property:

(1) $o \in X$,
(2) If $v, w \in X$ are two different vertices, then no $g \in G$ satisfies $gv = w$.

Note that every chain $X_0 \subset X_1 \subset \cdots \subset X_n \cdots$ has upper bound $\cup X_i$. By Zorn's Lemma, we have the collection of sets $\mathbb{X}$ contains a maximal element C.

We claim that C is indeed the core, i.e. containing exactly one point from each orbit of vertices. In other words, the vertex set is contained in $G \cdot C$. Suppose to the contrary that there exists $v \in V(\mathcal{G})$ such that $v \notin G \cdot C$. Without loss of generality, assume that there exists an edge e with one endpoint v and the other endpoint in $G \cdot C$. Then we add this edge e to C for getting a bigger set which belongs to $\mathbb{X}$. This is a contraction, as C is maximal by Zorn's Lemma.

To get the desired fundamental domain, it is important to note that $G \cdot C$ may not contain all edges in $\mathcal{G}$. We have to enlarge C by adding additional edges. Let e be such edge not contained in $G \cdot C$. We add half of the edge, the subinterval $[0, 1/2]$, to C. In a similar way by using Zorn's lemma, we get a fundamental domain F as wanted in the hypothesis. □

By the third condition of minimality, two distinct translates of a fundamental domain intersect only in their boundary.

Corollary 3.4. *The interior of the fundamental domain F contains exactly one vertex from each orbit Gv for $v \in V(\mathcal{G})$: for any vertex $w \in F$ and $1 \neq g \in G$, we have either $gw \notin F$ or $gw = w$.*

A connected fundamental domain determines a system of generating set.

Theorem 3.5. *Let G act on a connected graph $\mathcal{G}$ with a connected fundamental domain F. Then the set of elements $S = \{gG : g \neq 1, gF \cap F \neq \emptyset\}$ is a generating set for G.*

Proof. We fix a basepoint o in F. For any element $g \in G$, we connect o and go by a path γ. The aim of the proof is to cover the path γ by finitely many hF where $h \in G$.

Note that the following two facts.

(1) gF cannot intersect entirely in the interior of an edge: $gF \cap e \subsetneq \mathring{e}$, for $gF \cap e$ is connected.
(2) If $\mathring{e} \cap gF \cap F \neq \emptyset$ and one endpoint e_+ of e belongs to $gF \cap F$. Then $gF \cap e = F \cap e$.

For the second fact, let $x \in \mathring{e} \cap gF \cap F$. Then $gx, x \in \mathring{F}$. By the minimality of F, we have $gx = x$. Since g is an isometry but not inversion, we have $g|_e = id$. If there exists $y \in gF \cap e \setminus (F \cap e)$, then $yg^{-1}y \in F \cap e$, so we get a contradiciton. Hence $gF \cap e = F \cap e$.

We choose these hF in the following way. Note that the two endpoints have been already covered by F and gF. Set $g_0 = 1$ and so $o \in g_0F$. Let x be the intersection point of an edge e in the path γ with $X_i := \cup_{j \le i} g_j F$.

If the point x lies in the interior of the edge e, then we denote by y the other endpoint of e not lying in X_i. Since $e \subset GF$, there exists $g_{i+1} \in G$ such that $y \in g_{i+1}F$. We claim that $e \subset \cup_{j \le i+1} g_j F$. If not, then there exists a subinterval K of e outside $\cup_{j \le i+1} g_j F$. On the other hand, there exists a translate of F intersecting K. By the second fact, this is impossible. So the claim is proved.

If the point x is the endpoint of an edge e outside X_i, denote by y the middle point of e. Then there exists $g_{i+1}F$ such that $y \in g_{i+1}F$. By the first fact, we have

$g_{i+1}F$ contains at least one of endpoints of e. We then consider the intersection point of $g_{i+1} \cap e$. Repeating these two cases whether it is an interior point or endpoint, we are able to choose a sequence of g_iF $(0 \leq i \leq n)$ such that $g_iF \cap g_{i+1}F \neq \emptyset$, where $g_n = g$. Then we can write explicitely g as a product of elements h such that $F \cap hF \neq \emptyset$. The proof is then complete. □

The following exercise is a corollary for the previous theorem.

EXERCISE 3.6. *Suppose G acts by graph isomorphisms without inversions on a connected graph X such that there exists a finite subgraph K with $G \cdot K = X$. Assume that the edge stabilizers and the vertex stabilizers are finitely generated. Then G is finitely generated.*

There is a straightforward connection between fundamental domains of subgroups and groups.

EXERCISE 3.7. *Let G act freely on a connected graph $\mathcal{G}$ by isomorphisms without inversions with a connected fundamental domain F. Then for a subgroup H of G, there exists a set of elements $R \subset G$ such that $\cup_{r \in R} r \cdot F$ is a connected fundamental domain for the action of H on $\mathcal{G}$.*

3.2. Groups acting on trees. By definition, a *tree* is a graph where every reduced circuit is a point. Equivalently, there exists a unique reduced path between two points.

Now lets consider the free group $F(S)$ over a set S. We define a tree $\mathcal{G}$ for which the vertex set V is all elements in $F(S)$. Two reduced words $W, W' \in F(S)$ are connected by an edge if there exists $s \in \tilde{S}$ such that $W' = Ws$. Formally, the edge set E is defined to be $F(S) \times \tilde{S}$. The map $^-$ sends $(W, s) \in F(S) \times \tilde{S}$ to $(Ws, s^{-1}) \in F(S) \times \tilde{S}$. Such a graph $\mathcal{G}$ is indeed a tree, and $F(S)$ acts on $\mathcal{G}$ by graph isomorphisms.

We shall use Ping-Pong Lemma to prove the following theorem, which implies Theorem 3.1.

THEOREM 3.8. *Suppose that G acts on a tree T such that the stabilizer of each vertex is trivial. In other words, G acts on a tree T freely. Then G is a free group.*

PROOF. We divide the proof into three steps.

Step 1. Find a fundamental domain. We consider the core C of a fundamental set defined for the action of G on T. Note that C will be a connected subset such that it contains exactly one vertex from each orbit Gv for $v \in T$.

Since $G \cdot C$ may not contain all edges in T, in order that $G \cdot X = T$, we have to include some half edges to C to get the fundmental domain F.

We denote by E_0 the set of edges e of F such that C contains exactly one endpoint of e. We also denote by e_- the endpoint of e in X, and e_+ the other endpoint of e outside X. Define $\bar{X} = C \cup E_0$. Then $\bar{X}$ is still connected and $G \times \bar{X} = T$.

REMARK. The set $\bar{X}$ is not a fundamentail domain, as it contains FULL edges and but F only does half edges.

Step 2. Find free basis of G. For each $e \in E_0$, we know that $e_- \in X$ and $e_+ \notin X$. Recall that X contains (exactly) one vertex from each G-orbit in T. Thus, there exist an element $g_e \in G \setminus 1$ and a unique vertex $v \in X$ such that

$g_e v = e_+$. The element g_e is unique, otherwise the stabilizer of v is nontrivial. This is a contradiction, since G acts on T freely.

Observe that $g_e^{-1}(e_-) \in T \setminus X$ is connected by the edge $g_e^{-1}(e)$ to $v \in X$. Denote $e' = g_e^{-1}(e)$. Thus, $e \neq e'$ and $e' \in E_0$. By the uniqueness of $g_{e'}$, we also see that $g_{e'} = g_e^{-1}$.

In conclusion, for each $e \in E_0$, there exists a unique $e \neq e' \in E_0$ and a unique $g_e \in G \setminus 1$ such that $g_e^{-1}(e) = e'$. Moreover, $g_{e'} = g_e^{-1}$. We call $g_e, g_{e'}$ the **edge paring transformation** of the pair of edges (e, e').

Denote $\tilde{S} = \{g_e : e \in E_0\}$. Note that edges e, e' in E_0 are paired. From each such pair, we choose exactly one edge and denote them by $E_1 \subset E_0$. Define $S = \{g_e : e \in E_1\}$. Obviously, $\tilde{S} = S \cup S^{-1}$.

Step 3. Verify Ping-Pong Lemma. We now prove that $G = F(S)$ by using Ping-Pong Lemma.

For each $e \in E_0$, we define X_e to be the subgraph of T such that for each vertex z in X_v, there exists a (unique) reduced path from o to z containing the edge e. We note that X_e is connected, since it contains the endpoint e_+ of e. Moreover, $X_{e_1} \cap X_{e_2} = \emptyset$ for $e_1 \neq e_2 \in E_0$, and any path between two points in X_{e_1} and X_{e_2} respectively have to intersect X. These two properties follow from the fact that T is a tree: if not, we would be able find a nontrivial circuit.

As a result, if a path γ intersects X_e but $\gamma \cap X = \emptyset$, then γ lies in X_e.

We first verify that $g_e(o) \in X_e$, where $e \in E_0$. By definition, we need prove that the reduced path between o and $g_e(o)$ contains the edge e. For this purpose, we connect o and $g_e^{-1} e_+ \in X$ by a unique reduced path γ in X. Since X is the core of the fundamental domain, we have that $g_e \gamma \cap X = \emptyset$. Since $g_e \gamma$ contains the endpoint e_+ of e and $e_+ \in X_e$, we obtain that $g_e o \in g_e \gamma \subset X_e$ by the above discussion.

Secondly, we prove that $g_e X_t \subset X_e$ for $t \neq e' \in E_0$. Indeed, for any $z \in X_t$, we connect $g_{e'} o$ and z by a shortest geodesic γ. Since $g_{e'} o \in X_{e'}$ and $X_{e'} \cap X_t = \emptyset$, the path γ must intersect C and contain e'. So the path $g_e \gamma$ contains e and its endpoint are $\{o, g_e z\}$. By definition of X_e, we have that $g_e z \subset X_e$ and so $g_e X_t \subset X_e$.

Therefore, we have verified the conditions of Ping-Pong Lemma 2.20. So $G = F(S)$. □

In the above proof, we see that the rank of the free group G is the number of paired edges of the fundamental domain. From this fact and Exercise 3.7, we can deduce the following.

EXERCISE 3.9. *Let H be a subgroup of index n in a free group F_r of rank r ($r > 1$). Then the rank of H is $rn - (n-1)$. In particular, for each $n > 1$, F_2 contains a finite index subgroup of rank n.*

4. Fundamental groups of graphs

The second proof of Theorem 3.1 is to use a combinatorial notion of fundamental groups of a graph.

DEFINITION 4.1. A *graph* $\mathcal{G}$ consists of a set V of vertices and a set E of directed edges. For each directed edge $e \in E$, we associate to e the *initial point* $e_- \in V$ and *terminal point* $e_+ \in V$. There is an orientation-reversing map

$$\bar{\ }: E \to E, e \to \bar{e}$$

such that $e \neq \bar{e}$, $e = \bar{\bar{e}}$ and $e_- = (\bar{e})_+, e_+ = (\bar{e})_-$.

An *orientation* of $\mathcal{G}$ picks up exactly one directed edge in $\{e, \bar{e}\}$ for all $e \in E$. Formally, an orientation is a subset in E such that it contains exactly one element in $\{e, \bar{e}\}$ for all $e \in E$

REMARK. Clearly, such a map $^-$ has to be bijective. Moreover, $e_+ = (\bar{e})_-$ can be deduced from other conditions: $e_+ = \bar{\bar{e}}_+ = \bar{e}_-$.

REMARK. Every combinatorial graph can be geometrically realized by a common graph in the sense of CW-complex. We take the set of points V, and for each pair $(e, \bar{e})$, we take an interval $[0, 1]$ and attach its endpoints to $e_-, e_+ \in V$ respectively. Then we get a 1-dimensional CW-complex.

Combinatorially, we define a *path* to be a concatenation of directed edges:

$$\gamma = e_1 e_2 ... e_n, e_i \in E$$

where $(e_i)_+ = (e_{i+1})_-$ for $1 \leq i < n$. The initial point γ_- and terminal point γ_+ of γ are defined as follows:

$$\gamma_- = (e_1)_-, \gamma_+ = (e_n)_+.$$

If $(e_n)_+ = (e_1)_-$, the path γ is called a *circuit* at $(e_1)_-$. By convention, we think of a vertex in $\mathcal{G}$ as a path (or circuit), where there are no edges.

A *backtracking* in γ is a subpath of form $e_i e_{i+1}$ such that $e_i = \bar{e}_{i+1}$. A path without backtracking is called *reduced*. If a path γ contains a backtracking, we can obtain a new path after deleting the backtracking. So any path can be converted to a reduced path by a reduction process. Similarly as Lemma 2.1, we can prove the following.

LEMMA 4.2. *The reduced path is independent of the reduction process, and thus is unique.*

A *graph morphism* $\phi : \mathcal{G} \to \mathcal{G}'$ between two graphs $\mathcal{G}, \mathcal{G}'$ is a vertex-to-vertex, edge-to-edge map such that $\phi(e_-) = \phi(e)_-, \phi(e_+) = \phi(e)_+$ and $\phi(\bar{e}) = \overline{\phi(e)}$. It is called a *graph isomorphism* if ϕ is bijective.

The *concatenation* $\gamma\gamma'$ of two paths γ, γ' is defined in the obvious way, if $\gamma_+ = \gamma'_-$.

DEFINITION 4.3. Let $\mathcal{G}$ be a graph and $o \in \mathcal{G}$ be a basepoint. Then the *fundamental group* $\pi_1(\mathcal{G}, o)$ of $\mathcal{G}$ consists of all reduced circuits based at o, where the group multiplication is defined by sending two reduced circuits to the reduced form of their concatenation.

The group identity in $\pi_1(\mathcal{G}, o)$ is the just the base point $o \in \mathcal{G}$, the constant circuit.

REMARK. We can consider an equivalence relation over the set of all circuits based at o: two circuits are *equivalent* if they have the same reduced form. By Lemma 4.2, this is indeed an equivalence relation. Then the fundamental group $\pi_1(\mathcal{G}, o)$ can be also defined as the set of equivalent classes $[\gamma]$ of all circuits based at o, endowed with the multiplication:

$$[\gamma] \cdot [\gamma'] \to [\gamma\gamma'].$$

It is easy to see that these two definitions give the isomorphic fundamental groups.

A particularly important graph is the *graph of a rose* which consists of one vertex o with all other edges $e \in E$ such that $e_- = e_+ = o$. Topologically, the rose is obtained by attaching a collection of circles to one point.

Here we list a few properties about the fundamental group of a graph. Taking into account Lemma 4.2, the following is just an interpretation of definitions .

LEMMA 4.4. *We fix an orientation on a rose. Then the fundamental group of a rose is isomorphic to the free group generated by the alphabet set as the orientation.*

Any graph contains a *spanning* tree which is a tree with the vertex set of the graph. We can collapse a spanning tree to get a rose, called the *spin* of the graph.

EXERCISE 4.5. *The fundamental group of a graph is isomorphic to that of its spin.*

A graph morphism $\phi : \mathcal{G} \to \mathcal{G}'$ naturally defines a homomorphism between the fundamental group as follows:

$$\phi_* : \pi_1(\mathcal{G}, o) \to \pi_1(\mathcal{G}', \phi(o))$$

by sending a reduced circuit γ in $\pi_1(\mathcal{G}, o)$ to the reduced path of $\phi(\gamma)$ in $\pi_1(\mathcal{G}', \phi(o))$.

Given a vertex v in $\mathcal{G}$, consider the star

$$Star_{\mathcal{G}}(v) = \{e \in E(\mathcal{G}) : e_- = v\}.$$

A graph morphism $\phi : \mathcal{G} \to \mathcal{G}'$ naturally induces a graph morphism between the stars of v and $\phi(v)$.

DEFINITION 4.6. A graph morphism $\phi : \mathcal{G} \to \mathcal{G}'$ is called an *immersion* if for every vertex v, the induced graph morphism between the stars of v and $\phi(v)$

$$Star_{\mathcal{G}}(v) \to Star_{\mathcal{G}'}(\phi(v))$$

is injective. That is, ϕ is locally injective. If, in addition, ϕ is surjective, then it is called a *covering.*

The following lemma is a consequence of the defition of an immersion.

LEMMA 4.7 (Unique lifting). *Let $\phi : (\mathcal{G}, o) \to (\mathcal{G}', o')$ be an immersion where $o' := \phi(o)$. Then for any path γ and $x \in \mathcal{G}'$ satisfying $\phi(x) = o'$, if the lift $\hat{\gamma}$ of the path γ exists, then it is unique.*

If ϕ is a covering, then the lift of γ always exists and is thus unique.

REMARK. The difference between an immersion and a covering leads that the lift of a path may not exist!

Here is a corollary of Lemma 4.7.

LEMMA 4.8. *Let $\phi : (\mathcal{G}, o) \to (\mathcal{G}', \phi(o))$ be an immersion, and γ be a circuit in $\mathcal{G}'$ based at $\phi(o)$. If γ is not in $\phi_*(\pi_1(\mathcal{G}, o))$, then any lift of γ is not a circuit.*

It is clear that lifting preserves backtracking in an immersion so a reduction process is lifted from the downstair to the upstair.

LEMMA 4.9 (Backtracking). *Let $\phi : (\mathcal{G}, o) \to (\mathcal{G}', o')$ be an immersion where $o' := \phi(o)$. If a path γ has backtracking, then so does the lift $\hat{\gamma}$.*

An important consequence of an immersion is the following result.

LEMMA 4.10. *An immersion induces an imbedding of fundamental groups. That is, ϕ_* is injective.*

PROOF. Suppose not. There exists a non-empty reduced circuit γ based at o in $\mathcal{G}$ such that $\phi(\gamma)$ has the reduced circuit as the constant circuit c'_o. However, backtracking is preserved under lifting. During the reduction process from $\phi(\gamma)$ to c'_o, each backtracking is lifted to $\mathcal{G}$ and so a reduction process is inducted between γ and o. This contradicts to the choice of the non-empty reduced circuit γ. The lemma is thus proved. □

5. J. Stalling's Folding and separability of subgroups

5.1. J. Stalling's Folding. Let $\phi : \mathcal{G} \to \mathcal{G}'$ be a graph morphism. We shall make use of an operation called *folding* to convert the graph morphism ϕ to an immersion on a **new graph** $\mathcal{G}'$.

A pair of edges e, e' in $\mathcal{G}$ is called *foldable* if $e_- = (e')_-$, $\bar{e} \neq e'$, and $\phi(e) = \phi(e')$. Given a foldable pair of edges e, e', we can define a graph morphism ϕ_e to a new graph $\bar{\mathcal{G}}$ called *folding* as follows

$$\phi_e : \mathcal{G} \to \bar{\mathcal{G}} := \mathcal{G}/\{e = e', \bar{e} = \bar{e}'\}$$

by identifying the edges $e = e'$ and $\bar{e} = \bar{e}'$ respectively.

Observe that such an operation strictly decreases the number of edges and vertices. It is also possible that two loops can be identified. In this case, the fundamental group of the new graph $\bar{\mathcal{G}}$ changes.

Moreover, given a foldable pair of edges e, e', we can naturally define a new graph morphism $\bar{\phi} : \bar{\mathcal{G}} \to \mathcal{G}'$ such that the following diagram

$$\begin{array}{ccc} \mathcal{G} & \xrightarrow{\phi_e} & \bar{\mathcal{G}} \\ & \searrow^{\phi} & \downarrow{\scriptstyle \bar{\phi}} \\ & & \mathcal{G}' \end{array}$$

is commutative.

We do the above *folding process* for each foldable pair of edges, and finally obtain an immersion from a new graph $\bar{\mathcal{G}}$ to $\mathcal{G}'$. Precisely, we have the following.

LEMMA 5.1. *Let $\phi : \mathcal{G} \to \mathcal{G}'$ be a graph morphism. Then there exists a sequence of foldings $\phi_i : \mathcal{G}_i \to \mathcal{G}_{i+1}$ for $0 \leq i < n$ and an immersion $\bar{\phi} : \bar{\mathcal{G}} \to \mathcal{G}'$ such that $\phi = \bar{\phi}\phi_n \cdots \phi_0$, where $\mathcal{G}_0 = \mathcal{G}, \mathcal{G}_n = \bar{\mathcal{G}}$.*

A direct corollary is as follows.

COROLLARY 5.2. *Let $\phi : \Gamma \to \mathcal{G}$ be a graph immersion between two finite graphs. Then there exists a finite covering π of $\hat{\Gamma} \to \mathcal{G}$ such that Γ is a subgraph of $\hat{\Gamma}$ and $\pi(\iota) = \phi$, where ι is the natural embeding of Γ into $\hat{\Gamma}$.*

This corrollary implies that a wraped/immersed object, for instance the image $\phi(\Gamma)$, can be unwrapped to be embeded in a finite covering. The key notion making this possible is the separability of the subgroup $\phi_*(\pi_1(\Gamma))$ in $\pi_1(\mathcal{G})$.

An important consequence of the above folding process is that ϕ_* and $\bar{\phi}_*$ have the same image in the fundamental group of $\mathcal{G}'$. We apply the above theory to subgroups of a free group and to prove Theorem 3.1.

THEOREM 5.3 (Nielsen basis). *Let H be a subgroup of a free group $F(S)$. Then H is a free group. Moreover, given any generating set T of H, there exists an algorithm to find a free basis for H.*

PROOF. Let H be a subgroup of a free group $F(S)$. Suppose that H is generated by a set $T \subset F(S)$. By the above discussion, there exists a rose $\mathcal{G}'$ with one vertex and $2|S|$ edges whose fundamental group is $F(S)$. Here in fact, we choose an orientation on $\mathcal{G}'$ and then identify $\pi_1(\mathcal{G}')$ as $F(S)$.

Note that T are a set of reduced words. For each word $W \in T$, we associate to W a circuit graph $\mathcal{C}_W$ of $2|T|$ edges with a basepoint o and an orientation such that the clock-wise "label" of $\mathcal{C}_W$ is the word W. It is obvious that there exists a graph morphism $\mathcal{C}_W \to \mathcal{G}'$.

We attach all $\mathcal{C}_W$ at o for $W \in T$ to get a graph $\mathcal{G}$. Then we have a graph morphism $\phi : \mathcal{G} \to \mathcal{G}'$. It is also clear that the image $\phi_*(\pi_1(\mathcal{G}))$ is the subgroup H in $F(S)$. Hence, a consequence of Lemma 5.1 is that any subgroup of a free group is free. Moreover, since the immersion given by Lemma 5.1 induces an injective homomorphism, we can easily obtain a free basis of H by writing down the generating elements of the fundamental group of $\bar{\mathcal{G}}$. □

5.2. Separability of subgroups. In this subsection, we present the proof of J. Stallings of a theorem of M. Hall.

THEOREM 5.4 (M. Hall). *Let H be a finitely generated subgroup in a free group F of finite rank. For any element $g \in F \setminus H$, there exists a finite index subgroup Γ of F such that $H \subset \Gamma$ and $g \notin \Gamma$.*

REMARK. A subgroup with the above property is called *separable*. In other words, a subgroup H is *separable* in G if it is the intersection of all finite index subgroups of G containing H.

LEMMA 5.5. *Let $\phi : (\mathcal{G}, o) \to (\mathcal{G}', \phi(o))$ be a covering for two finite graphs $\mathcal{G}$ and $\mathcal{G}'$. Then $\phi_*(\pi_1(\mathcal{G}, o))$ is of finite index in $\pi_1(\mathcal{G}', o)$.*

PROOF. Denote by H the subgroup $\phi_*(\pi_1(\mathcal{G}, o))$. We count the right coset Hg where $g \in \pi_1(\mathcal{G}', o)$. Then any lift of the circuit in Hg based at o has the same terminal endpoint. Moreover, if $Hg \neq Hg'$, then the endpoints of corresponding lifts are different. Indeed, if not, we get a circuit and by Lemma 4.7 we see that $g'g^{-1} \in H$.

Since $\mathcal{G}$ is finite, we see that there are only finitely many different right H-cosets. □

We are now in a position to give the Stalling's proof of Theorem 5.4.

PROOF OF THEOREM 5.4. Let $\mathcal{G}'$ be a rose. We have put an orientation on $\mathcal{G}'$, a subset E_0 of edges, such that $\pi_1(\mathcal{G}')$ is identical to $F(E_0)$.

Let H be a finitely generated subgroup in F with a finite generating set T. Given $g \notin H$, we write g as a reduced word W_g over S, and similarly for each $t \in T$ a word W_t. As in the proof of Theorem 5.3, we construct a graph by gluing circuits labeled by W_t for $t \in T$, and use the folding to get an immersion $\phi : (\mathcal{G}, o) \to (\mathcal{G}', \phi(o))$, where $\mathcal{G}$ has the fundamental group H. This naturally induces an orientation E_1 on $\mathcal{G}$. Now we attach a path labeled by W_g at o by following the orientation $\mathcal{G}$. Since $g \notin H$, the endpoint of the path must be different from o, i.e.: the path is not closed. The new graph is still denoted by $\mathcal{G}$ for simplicity. And $\phi : (\mathcal{G}, o) \to (\mathcal{G}', \phi(o))$ is still an immersion.

Denote by V the vertex set of $\mathcal{G}$. For each $e \in E_0$, we have a set of directed edges $\phi_e^{-1}(e)$ in E_1. Since ϕ is an immersion, each edge in $\phi_e^{-1}(e)$ defines an ordered pair of endpoints in V. Thus, each $e \in E_0$ defines a bijective map ι_e on a subset of the vertex set V of $\mathcal{G}$. Similarly, we can define $\iota_{\bar{e}}$ for $e \in E_0$.

Since V is finite, ι_e can be extended to a bijective map of V. (We actually have many choices). Let's denote again by ι_e one such bijective map of V.

It is easy to use these maps $\iota_e, \iota_{\bar{e}}$ for $e \in E_0$ to complete the immersion ϕ : $(\mathcal{G}, o) \to (\mathcal{G}', \phi(o))$ to a covering $\tilde{\phi} : (\tilde{\mathcal{G}}, o) \to (\mathcal{G}', \phi(o))$. Precisely,

For each $e \in E_0$, we use ι_e to connect v and $\iota_e(v)$ by a directed edge e, if such an edge was not in $\phi_e^{-1}(e)$. We do similarly for each $\bar{e}$ where $e \in E_0$. It is clear that the such obtained graph $\tilde{\mathcal{G}}$ is a finite covering. By Lemma 5.5, the fundamental group $\Gamma = \phi_\star(\pi_1(\tilde{\mathcal{G}}, o))$ of $\tilde{\mathcal{G}}$ is of finite index in G.

Moreover, by Lemma 4.7, the subgroup Γ contains H but not g, since the path labeled by W_g is not closed in $\tilde{\mathcal{G}}$. The proof is complete. □

The following two exercises are consequences of Theorem 5.4.

EXERCISE 5.6. *A free group F is residually finite: for any $g \neq 1 \in F$, there exists a homorphism $\phi : F \to G$ to a finite group G such that $\phi(g) \neq 1$.*

In fact, another way is to note that a linear group is residually finite, and free groups are linear.

EXERCISE 5.7. *Free groups are Hopfian: any endomorphism is an isomorphism.*

6. More about covering spaces of graphs

In this section, we list a few theorems in theory of covering spaces of graphs. They will serve a template for the corresponding ones in general topological spaces.

LEMMA 6.1. *Let $\phi : (\Gamma, x) \to (\mathcal{G}, o)$ be a covering for two graphs Γ and $\mathcal{G}$. Then there exists a bijection between the fiber $\phi^{-1}(o)$ and the collection of right cosets of $\phi_*(\pi_1(\Gamma, o))$.*

PROOF. Denote $H := \phi_*(\pi_1(\mathcal{G}, o))$. We count the right coset Hg where $g \in \pi_1(\mathcal{G}, o)$. By Lemma 4.9, any lift of a circuit in Hg based at x has the same terminal endpoint. This establishes that the corresponding map Φ from $\{Hg : g \in \pi_1(\mathcal{G}, o)\}$ to the fiber $\phi^{-1}(o)$ is well-defined.

Moreover, if $Hg \neq Hg'$, then the endpoints of the corresponding lifts are distinct. Indeed, if not, let $g \in Hg$ and $g' \in Hg'$ such that their lifts $\hat{g}$ and $\hat{g}'$ at x have the same other endpoint y. Then we get a circuit $\hat{g} \cdot \hat{g}'^{-1}$ at x. By Lemma 4.7 we see that $gg'^{-1} \in H$, contradicting to the assumption of $Hg \neq Hg'$. So this implies that the aboved defined map is injective.

To see the surjectivity, let y be a point in $\phi^{-1}(o)$ and connect x, y by a path γ. The image of γ is then a loop $g \in \pi_1(\mathcal{G}, o)$. By definition of the map, we see that $\Phi(g) = y$. So it is proved that the map Φ is a bijection. □

Universal covering. A connected graph is *simply connected* if its fundamental group is trivial. A covering $\Gamma \to \mathcal{G}$ is called *universal* if Γ is simply connected.

It is straightforwad to construct the universal covering of a rose, so of any graph by blowing up each vertex by a spanning tree of the graph in that of the rose.

THEOREM 6.2 (Lift graph morphisms). *Let $\phi : (\mathcal{G}, o) \to (\mathcal{G}', \phi(o))$ be a graph morphism. Suppose we have two coverings $\pi : (\hat{\mathcal{G}}, x) \to (\mathcal{G}, o)$ and $\pi' : (\hat{\mathcal{G}}', y) \to (\mathcal{G}', \phi(o))$. Then there exists a unique lift $\hat{\phi} : \hat{\mathcal{G}} \to \hat{\mathcal{G}}'$ such that $\hat{\phi}(x) = y$ if and only if $\phi_*(\pi_*(\pi_1(\hat{\mathcal{G}}, x))) \subset \pi'_*(\pi_1(\hat{\mathcal{G}}', y))$.*

$$\begin{array}{ccc} \hat{\mathcal{G}} & \xrightarrow{\hat{\phi}} & \hat{\mathcal{G}}' \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi'} \\ \mathcal{G} & \xrightarrow{\phi} & \mathcal{G}' \end{array}$$

As a corollary, we produce the following.

THEOREM 6.3 (Uniqueness of universal covering). *Let $\mathcal{G}$ be a graph. Let $\pi_1 : \Gamma_1 \to \mathcal{G}$ and $\pi_2 : \Gamma_2 \to \mathcal{G}$ be two universal coverings. Then there exists a graph isomorphism $\phi : \Gamma_1 \to \Gamma_2$ such that the diagram is commutative.*

$$\begin{array}{ccc} \Gamma_1 & \xrightarrow{\phi} & \Gamma_2 \\ & {\scriptstyle \pi_1} \searrow \quad \swarrow {\scriptstyle \pi_2} & \\ & \mathcal{G} & \end{array}$$

THEOREM 6.4 (Correspondence of subgroups $\leftrightarrow$ covering spaces). *Let $(\mathcal{G}, o)$ be a graph. Then for any subgroup H in $\pi_1(\mathcal{G}, o)$, there exists a covering $\phi : (\Gamma, x) \to (\mathcal{G}, o)$ such that H is the image of $\phi_*(\pi_1(\Gamma, x))$.*

PROOF. We only explain the case that H is finitely generated by a set S, where each $s \in S$ is a word with respect to the free generators of $\pi_1(\mathcal{G}, o)$.

We first draw down explitely a graph Γ according to this set S. Using Folding process, we can assume that $\Gamma \to \mathcal{G}$ is an immersion. Thus, the fundamental group of Γ is just isomorphic to H.

To get a covering with fundamental group H, it suffices to add infinite trees to the vertices in Γ which have incomplete stars. It is obvious such a completion is always possible and does not change the fundamental group of Γ. $\square$

THEOREM 6.5 (Covering transformations $\leftrightarrow$ Normalizer). *Let $\phi : (\Gamma, x) \to (\mathcal{G}, o)$ be a covering. Then the group of covering transformations is isomorphic to the normalizer of the image $\phi_*(\pi_1(\Gamma, x))$ in $\pi_1(\mathcal{G}, o)$.*

A covering $\phi : (\Gamma, x) \to (\mathcal{G}, o)$ is called *normal* if the image $\phi_*(\pi_1(\Gamma, x))$ is a normal subgroup of $\pi_1(\mathcal{G}, o)$.

THEOREM 6.6 (Free actions on trees). *Let Γ be a tree on which a group G of automorphisms acts freely. Then the fundamental group of the quotient graph is isomorphic to the group G.*

CHAPTER 2

Elements of Hyperbolic geometry

1. Upper Half-plane Model

Consider the upper half plane $\mathbb{H}^2 = \{(x, y) \in \mathbb{R}^2 : y > 0\}$. We endow $\mathbb{H}^2$ with the following (Riemannian) metric:

$$ds = \frac{\sqrt{dx^2 + dy^2}}{y}.$$

To be precise, a piecewise differential path $\gamma : [0, 1] \to \mathbb{H}$ has the length defined as follows:

$$\ell(\gamma) = \int_0^1 \frac{\sqrt{x'(t)^2 + y'(t)^2}}{y(t)} dt$$

where $\gamma = (x(t), y(t))$.

For two points $z, w \in \mathbb{H}$, their *hyperbolic* distance is as follows

$$d_{\mathbb{H}}(z, w) = \inf\{\ell(\gamma) : \gamma(0) = z, \gamma(1) = w\}$$

where the infimum is taken over all piecewise differential paths between z and w.

Denote by $\mathrm{Isom}(\mathbb{H}^2)$ the group of all isometries of $\mathbb{H}^2$.

1.1. Orientation-preserving isometries. Consider the general linear groups $GL(2, \mathbb{C})$ of invertible 2×2-matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

where $a, b, c, d \in \mathbb{C}$ such that $ad - bc \neq 0$. The group $\mathcal{M}_2(\mathbb{C})$ of (complex) *linear fractional transformation* (LFT) is a nonconstant function on $\mathbb{C}$ of the form

$$T(z) = \frac{az + b}{cz + d}$$

for $a, b, c, d \in \mathbb{C}$ with $ad - bc \neq 0$. Such LFT is also called *Mobius transformation.* There is a natural map $\Phi : GL(2, \mathbb{C}) \to \mathcal{M}_2(\mathbb{C})$ as follows:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \to \frac{az + b}{cz + d}.$$

EXERCISE 1.1. *Prove that Φ is homomorphism and the kernel is $\{k \cdot I_{2\times 2} : k \in \mathbb{C} \setminus 0\}$ where $I_{2\times 2}$ is the identity matrix.*

For simplicity, we consider the *special linear* group $SL(2, C)$ consists of the matrices with determinant ± 1 in $GL(2, \mathbb{C})$. The *projective linear* group $PSL(2, \mathbb{C})$ is then the quotient group $SL(2, C)/\{\pm I_{2\times 2}\}$. By the above exercise, $PSL(2, \mathbb{C})$ is isomorphic to $\mathcal{M}_2(\mathbb{C})$.

LEMMA 1.2. *Every LFT can be written as a product of the following three elementary transformations:*

(1) $z \to z + c$, *where* $c \in \mathbb{C}$,
(2) $z \to kz$, *where* $k \in \mathbb{C}$,
(3) $z \to \frac{-1}{z}$.

In other words, $\mathcal{M}_2(\mathbb{C})$ *is generated by the set of elementary transformations.*

Every LFT is actually defined on the set $\mathbb{C} \setminus \{\frac{-d}{c}\}$. It will be useful to define LFT over the extended complex plane $\hat{\mathbb{C}} = \mathbb{C} \cup \infty$. Correspondingly, we define

$$T(\frac{-d}{c}) = \infty$$

and

$$T(\infty) = \frac{a}{c},$$

so a LFT T becomes a bijective map on $\hat{\mathbb{C}}$. We equip the topology of $\hat{\mathbb{C}}$ with one-point compactification as follows. The open sets in $\hat{\mathbb{C}}$ are either open sets in $\mathbb{C}$ or the union of ∞ with the complement of a compact set in $\mathbb{C}$.

Exercise 1.3. *Put a metric on* $\hat{\mathbb{C}}$ *such that it induces the one-point compactification* $\hat{\mathbb{C}}$. *(Tips: consider the stereographic projection from the closed upper semi-sphere to* $\hat{\mathbb{C}}$.*)*

Exercise 1.4. *With one-point compactification* $\hat{\mathbb{C}}$, *every LFT is a homeomorphism.*

The above discussion still applies with $\mathbb{C}$ replaced by $\mathbb{R}$. In particular, $\mathcal{M}_2(\mathbb{R})$ denotes the set of LFTs with real coefficients. Then $PSL(2, \mathbb{R})$ is isomorphic to $\mathcal{M}_2(\mathbb{R})$. We now come to the connection of $\mathcal{M}_2(\mathbb{R})$ with $\mathrm{Isom}(\mathbb{H}^2)$.

Lemma 1.5. $\mathcal{M}_2(\mathbb{R}) \subset$ *Isom*$(\mathbb{H}^2)$.

Proof. Note that each type of a real elementary LFT is an isometry so any real LFT belongs to $\mathrm{Isom}(\mathbb{H}^2)$ by Lemma 1.2. □

1.2. Geodesics and reflexions. We now consider the paths $\gamma : I \to \mathbb{H}^2$ where I is an interval in $\mathbb{R}$.

Definition 1.6. A path $\gamma : I \to \mathbb{H}^2$ is called a *geodesic* if it preserves the distance: $|s - t| = d_{\mathbb{H}^2}(\gamma(s), \gamma(t))$ for any $s, t \in I$.

Remark. Sometimes, when I is a finite interval $[a, b]$, the path γ is called a *geodesic segment*. If $I = [0, \infty)$, it is a *geodesic ray*; if $I = \mathbb{R}$, we call it a *geodesic line*.

Theorem 1.7. *The set of geodesic lines in* $\mathbb{H}^2$ *is the set of Euclidean half-lines and half-circles orthogonal to the real axis.*

One may first verify by computations that the positive y-axis is a geodesic line. Then the proof is completed by the following.

Exercise 1.8. $\mathcal{M}_2(\mathbb{R})$ *acts transitively on the set of Euclidean half-lines and half-circles orthogonal to the real axis.*

To obtain the full isometry group of $\mathbb{H}^2$, we need take care of an orientation-reversing isometry. Note that $z \to -\bar{z}$ is such an isometry of $\mathbb{H}^2$, which fixes pointwise the y-axis and exchanges left and right half-planes. So we have the following defintion.

DEFINITION 1.9. A (hyperbolic) *reflexion* in $\mathbb{H}^2$ is a conjugate of $z \to -\bar{z}$ by $\mathcal{M}_2$ so it fixes pointwise a unique geodesic line.

PROPOSITION 1.10. *If an isometry in $\mathbb{H}^2$ fixes pointwise a geodesic line L, then it is either identity or a reflexion about L.*

Before giving a proof, we need make use of the following useful fact about bisectors. Given two points $x, y \in \mathbb{H}^2$, the *bisector* $L_{x,y}$ is the set of points $z \in \mathbb{H}^2$ such that $d_{\mathbb{H}^2}(x, z) = d_{\mathbb{H}^2}(y, z)$.

LEMMA 1.11. *Bisectors $L_{x,y}$ are geodesic lines and the geodesic $[x, y]$ is orthogonal to $L_{x,y}$.*

PROOF. Up to applying LFT (cf. Ex 1.8), we can assume without loss of generality that x, y are symmetric relative to the y-axis. Observe then that the positive y-axis is contained in $L_{x,y}$. Hence, it suffices to prove that any point $z \in L_{x,y}$ has to lie on the y-axis. This can be proved by contradiction; see detailed proof in the Lemma in Stillwell, pp.87. □

Define the distance of a point z to a subset L in $\mathbb{H}^2$:

$$d_{\mathbb{H}^2}(z, L) := \inf\{d_{\mathbb{H}^2}(z, w) : w \in L\}.$$

LEMMA 1.12. *Given a point z outside a geodesic line L, then there exists a unique point $w \in L$ such that $d_{\mathbb{H}^2}(z, w) = d_{\mathbb{H}^2}(z, L)$ and the geodesic through z, w is orthogonal to L.*

PROOF. Note that there exists a geodesic line L_0 passing through z and orthogonal to L. Place L_0 to be the y-axis by a LFT. Then it is clear that the intersection of $L_0 \cap L$ is the shortest point on L to z. □

PROOF OF PROPSOITION 1.10. Suppose the isometry ϕ is not identity so there exists $z \in \mathbb{H}^2 \setminus L$ such that $\phi(z) \neq z$. Consider the bisector $L_{z,\phi(z)}$ which is a geodesic line by Lemma 1.11. Since the geodesic line L is fixed pointwise by ϕ, we have $d(w, z) = d(w, \phi(z))$ for any $w \in L$ so $L \subset L_{z,\phi(z)}$. They are both geodesic lines so they are equal: $L = L_{z,\phi(z)}$.

Up to a translation of LFT, we assume that L is the y-axis. We claim that ϕ coincides the reflexion ρ about the y-axis. That is to say, we need prove that for any $w \in \mathbb{H}^2$, we have $\rho(w) = \phi(w)$. By the same argument for z, we see that the bisector $L_{w,\phi(w)}$ coincides with y-axis. So the geodesic between $w, \phi(w)$ is orthogonal to L, and

$$d_{\mathbb{H}^2}(w, L) = d_{\mathbb{H}^2}(\phi(w), L).$$

By Lemma 1.12, $\phi(w)$ and w is symmetric relative to L. So $\rho(w) = \phi(w)$. □

Let L be a geodesic line. If it is given by half-circles, then the *two endpoints* of L are the intersection points with the real axis. If L is a half line, then the intersection point with the real axis and the infinity point ∞ are the *two endpoints* of L.

EXERCISE 1.13. *Let L_1, L_2 be two geodesic lines such that they have disjoint endpoints. Then there exists a unique geodesic line L orthogonal to both L_1 and L_2.*

We are able to characterize the full isometry group of $\mathrm{Isom}(\mathbb{H}^2)$.

THEOREM 1.14. *The isometry group $Isom(\mathbb{H}^2)$ is generated by $PSL(2, \mathbb{R})$ and the reflexion $z \to -\bar{z}$.*

PROOF. Up to apply LFTs from $PSL(2, \mathbb{R})$,we can assume that an isometry is fixes pointwise the y-axis. Then the proof is completed by Lemma 1.10. □

We now give another description of hyperbolic reflexion without using hyperbolic geometry.

DEFINITION 1.15 (Inversions). Consider the Euclidean plane $\mathbb{E}^2$. If L is a line, an inversion about L is the same as the Euclidean reflexion about L.

If L is a circle of radius $R > 0$ with centers o, an inversion about L sends a point $z \in \mathbb{E}^2$ to $w \in \mathbb{E}^2$ such that

$$|z - o| \cdot |w - o| = R^2,$$

where $| \cdot |$ is the Euclidean distance.

LEMMA 1.16. *Any reflexion in Isom*($\mathbb{H}^2$) *is exactly the restriction on* $\mathbb{H}^2$ *of an inversion about lines and circles orthogonal to the* x*-axis.*

PROOF. Observe that the reflexion ρ about y-axis is conjugated to $\phi : z \to \frac{1}{\bar{z}}$ so ϕ is a reflexion. Indeed, there exists a real LFT f such that f maps the y-axis to the unit circle. It suffices to prove that $f\rho f^{-1} = \phi$. Note, $f\rho f^{-1}$ and ϕ keeps y-axis pointwise so by Lemma 1.10 they are either equal or differ by a reflexion. Because f is orientation-preserving, $f\rho f^{-1}$ and ϕ cannot differ by reflexion. Thus, $f\rho f^{-1} = \phi$.

Note also that the hyperbolic isometry $z \to \frac{1}{\bar{z}}$ is an inversion about the unite circle orthogonal to the x-axis. So an reflexion is an inversion.

We prove now that every inversion is a hyperbolic reflexion. If the line L is orthogonal to the x-axis, an inversion about L restricting on $\mathbb{H}^2$ is the same as a hyperbolic reflexion. On ther other hand, any inversion about circles are hyperbolic reflexions, because we can apply LFTs $z \to kz$ and $z \to z + c$ which are isometries to conjugate the inversion to $z \to \frac{1}{\bar{z}}$. The proof is complete. □

1.3. Isometries as products of reflexions.

LEMMA 1.17. *An isometry in Isom*($\mathbb{H}^2$) *is determined by three non-linear points: if* $f, g \in$ *Isom*($\mathbb{H}^2$) *have same values at* $a, b, c \in \mathbb{H}^2$ *where* a, b, c *are not on the same geodesic line, then* $f = g$.

PROOF. Suppose to the contrary that there exists $z \in \mathbb{H}^2$ such that $f(z) \neq g(z)$. Consider the bisector $L_{f(z),g(z)}$ which contains a, b, c. By Lemma 1.11, $L_{f(z),g(z)}$ is a geodesic line. This contradicts to the hypothesis so we are done. □

LEMMA 1.18. *An isometry in Isom*($\mathbb{H}^2$) *can be written as a product of at most three reflexions.*

PROOF. Fix three points $a, b, c \in \mathbb{H}^2$. If the isometry ϕ does not fix a for instance, we compose a reflexion ρ about the bisector $L_{a,\phi(a)}$ such that $\rho(\phi(a)) = a$. In this manner, we can compose at most three reflexions such that the resulted isometry fixes a, b, c simultaneously. The proof is thus completed by Lemma 1.17. □

Another way to study the isometry group of $\mathbb{H}^2$ is to first introduce inversions about Euclidean lines or circles. The group of Mobius transformations is then defined to be the group generated by inversions. By showing that the hyperbolic metric is preserved, ones establishes that the group of Mobius transformations is the full isometry group of some hyperbolic space. This approach applies to higher dimensional hyperbolic spaces, and via Poincare extensions, the group of Mobius transformations in lower dimension naturally embeds into that of higher dimension. We refer the reader to Beardon [**1**] or Ratcliffe [**4**] for this approach.

2. Classification of orientation-reserving isometries

2.1. Ball Model of hyperbolic plane. Consider the unit ball $\mathbb{D}^2 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$. Note that the following complex LFT

$$\Phi(z) = \frac{z - i}{z + i}$$

sends $\mathbb{H}^2$ to $\mathbb{D}^2$. Define the metric on $\mathbb{D}^2$ as follows

$$d_{\mathbb{D}^2}(z, w) = d_{\mathbb{H}^2}(\Phi^{-1}(z), \Phi^{-1}(w))$$

for any $z, w \in \mathbb{D}^2$ so that $\Phi : \mathbb{H}^2 \to \mathbb{D}^2$ is an isometry. Denote by $\mathrm{Isom}^+(\mathbb{D}^2)$ the orientation-preserving isometry group. As a result,

THEOREM 2.1.

$$Isom^+(\mathbb{D}^2) = \Phi \cdot Isom^+(\mathbb{H}^2) \cdot \Phi^{-1} = \left\{ \begin{pmatrix} a & \bar{c} \\ c & \bar{a} \end{pmatrix} : a, c \in \mathbb{C}, |a|^2 - |c|^2 = 1. \right\}$$

The full isometry group $Isom(\mathbb{D}^2)$ is generated by the above matrices and $z \to \bar{z}$.

Equivalently, we can consider the following Riemanian metric on $\mathbb{D}^2$:

$$ds = \frac{2\sqrt{dx^2 + dy^2}}{1 - (x^2 + y^2)}.$$

To be precise, a piecewise differential path $\gamma : [0, 1] \to \mathbb{H}$ has the length defined as follows:

$$\ell(\gamma) = \int_0^1 \frac{2\sqrt{x'(t)^2 + y'(t)^2}}{1 - (x(t)^2 + y(t)^2)} dt$$

where $\gamma = (x(t), y(t))$. The distance $d_{\mathbb{D}^2}$ is defined similarly as $d_{\mathbb{H}^2}$.

It is clear that $z \to e^{i\theta} z$ is an isometry of $\mathbb{D}^2$.

EXERCISE 2.2. *In $\mathbb{D}^2$, let z be a point such that $|z - o| = r < 1$, where o is the origin of $\mathbb{D}^2$ and $|\cdot|$ is the Euclidean distance. Prove that the distance*

$$d_{\mathbb{D}^2}(o, z) = \ln \frac{1 + r}{1 - r}.$$

Conclude that a hyperbolic disk is the same as a Euclidean disk as a set!

EXERCISE 2.3. *Let $\phi \in \mathcal{M}_2(\mathbb{C})$ be a complex LFT. Then it maps Euclidean circle or lines to Euclidean circle or lines.*

The isometry $\Phi : \mathbb{H}^2 \to \mathbb{D}^2$ transfers geodesic lines from $\mathbb{H}^2$ to $\mathbb{D}^2$ so by the exercise 2.3, we have the following.

THEOREM 2.4. *The set of geodesic lines in $\mathbb{D}^2$ is the set of (the intersection with $\mathbb{D}^2$ of) Euclidean lines and circles orthogonal to the unit circle S^1.*

By Exercise 2.3, we also have:

LEMMA 2.5. *The topology on $\mathbb{D}^2$ induced by hyperbolic metric $d_{\mathbb{D}^2}$ is the same as the Euclidean topology. The same conclusion for $\mathbb{H}^2$ with induced topology by $d_{\mathbb{H}^2}$.*

Consider the closed disk $\bar{\mathbb{D}}^2 := \mathbb{D}^2 \cup S^1$ with induced Euclidean topology. By Lemma 2.5, the topology on the interior of $\bar{\mathbb{D}}^2$ coincides with the one induced by $d_{\mathbb{D}^2}$. By Theorem 2.1, $\mathrm{Isom}(\mathbb{D}^2)$ acts by homeomorphisms on $\bar{\mathbb{D}}^2$ as they can be seen as LFTs which are homeomorphisms on $\bar{\mathbb{D}}^2$. In this sense, we say that isometries of $\mathbb{D}^2$ *extends* by homeomorphisms to S^1.

Note that the metric topology of $\mathbb{D}^2$ is the same as the Euclidean one. So in view of the hyperbolic geometry, we shall call S_1 the *boundary at infinity* $\partial_\infty \mathbb{D}^2$ of the hyperbolic space $\mathbb{D}^2$. (This boundary is not subset of $\mathbb{D}^2$)

For the upper half space $\mathbb{H}^2$, the *boundary at infinity* $\partial_\infty \mathbb{H}^2$ is the union of $\mathbb{R} \cup \{\infty\}$. Endowing the topology from extended complex numbers, $\bar{\mathbb{H}}^2 = \mathbb{H}^2 \cup \partial_\infty \mathbb{H}^2$ is a compact space with the interior $\mathbb{H}^2$ the Euclidean topology.

Exercise 2.6. *In $\mathbb{D}^2$, let x_n, y_n be two sequences such that $d_{\mathbb{D}^2}(x_n, y_n)$ are equal and $d_{\mathbb{D}^2}(x_n, o) \to \infty$ for some fixed point $o \in \mathbb{D}^2$. Then their Euclidean distance $|x_n - y_n|$ between x_n and y_n tends to 0 as $n \to \infty$.*

With respect to the compact topology on $\bar{\mathbb{H}}^2$ or $\bar{\mathbb{D}}^2$, the above exercise implies that if one sequence x_n converges to a point $z \in \partial_\infty \mathbb{H}^2$ (resp. $\partial_\infty \mathbb{D}^2$), then any sequence y_n with a uniformly bounded $d_{\mathbb{H}^2}(x_n, y_n)$ (resp. $d_{\mathbb{D}^2}(x_n, y_n)$) tends to the same point z.

2.2. Classification of orientation-preserving isometries. We are interested in classifying the elements in $\mathrm{Isom}^+(\mathbb{D}^2)$ which consists of orientation-preserving isometries (i.e. written as an even products of reflexions). By Theorem 1.14,

$$\mathrm{Isom}^+(\mathbb{D}^2) \cong PSL(2, \mathbb{C}) \cong \mathcal{M}_2(\mathbb{R}).$$

Recall that

Theorem 2.7 (Brouwer). *Any continuous map of $\bar{\mathbb{D}}^2$ has a fixed point.*

So any $\phi \in \mathrm{Isom}(\mathbb{D}^2)$ has a fixed point in $\bar{\mathbb{D}}^2$. We classify the elements in $\mathrm{Isom}(\mathbb{D}^2)$ according to their action on $\bar{\mathbb{D}}^2$.

Definition 2.8. Let $\phi \in \mathrm{Isom}^+(\mathbb{D}^2)$ be a non-trivial isometry.

(1) It is called *elliptic* element if it has a fixed point in $\mathbb{D}^2$;
(2) It is called *parabolic* element if it has only one fixed point in S^1;
(3) It is called *hyperbolic* element if it has exactly two fixed points in S^1.

Remark. Since every LFT is determined by three points, every (non-trivial) element in $\mathrm{Isom}^+(\mathbb{D}^2)$ belongs one of these three categories.

The following facts are straightforward:

(1) Every elliptic element is conjugated to $z \to e^{i\theta} z$ in $\mathrm{Isom}(\mathbb{D}^2)$.
(2) Every parabolic element is conjugated to $z \to z + c$ for $c \in \mathbb{R}$ in $\mathrm{Isom}(\mathbb{H}^2)$.
(3) Every hyperbolic element is conjugated to $z \to kz$ for $k > 0$ in $\mathrm{Isom}(\mathbb{H}^2)$.

Since every isometry $\phi(z) = \frac{az+b}{cz+d}$ in $\mathrm{Isom}(\mathbb{H}^2)$ is identified with the collection of matrices

$$A \in \{k \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} : k \neq 0 \in \mathbb{R}, a, b, c, d \in \mathbb{R}; ad - bc \neq 0\}$$

we can define the following function

$$\mathrm{tr}^2(\phi) = \frac{\mathrm{trace}^2(A)}{\det(A)}$$

where trace(A) is the trace of a matrix A.

THEOREM 2.9 (Algebraic chacterization). *Given a non-trivial isometry $\phi \in Isom^+(\mathbb{D}^2)$, we have*

(1) *ϕ is elliptic iff* $\mathrm{tr}^2(\phi) \in [0, 4)$.
(2) *ϕ is parabolic iff* $\mathrm{tr}^2(\phi) = 4$.
(3) *ϕ is hyperbolic iff* $\mathrm{tr}^2(\phi) \in (4, \infty)$.

PROOF. We first prove that every LFT ϕ in $PSL(2, \mathbb{R})$ with one fixed point in $\mathbb{R} \cup \{\infty\}$ is conjugated to either $z \to z + 1$ or $z \to kz$ for $k \neq 1 \in \mathbb{R}$. Without loss of generality, we assume that ϕ fixes ∞ so it must be of the form $\phi(z) = az + b$. If $a = 1$, then $f^{-1}\phi f$ is equal to $z \to z + 1$ for the conjugator $f(z) = bz$. If $a \neq 1$, then ϕ has the other fixed point $\frac{b}{1-a}$. Hence, the parabolic element $z \to z + \frac{b}{1-a}$ conjugates ϕ to az.

Since trace(A) is invariant under conjugation, we have $\mathrm{tr}^2(\phi) = \mathrm{tr}^2(h\phi h^{-1})$ for any $h \in \mathcal{M}_2(\mathcal{C})$. So the theorem follows by the above discussion. □

Two geodesic lines L_1, L_2 are called *parallel* if they are disjoint in $\bar{\mathbb{D}}^2$. They are called *asymptotic* if they intersect in only one point in the boundary S^1 of $\bar{\mathbb{D}}^2$. Equivalently, L_1, L_2 are parallel iff their hyperbolic distance is positive; ultra-parallel iff their hyperbolic distance is zero but not realized by any point in $\mathbb{D}^2$.

THEOREM 2.10 (Geometric chacterization). *A non-trivial isometry $\phi \in Isom^+(\mathbb{D}^2)$ is a product of two reflexions about lines L_1, L_2. Moreover,*

(1) *ϕ is elliptic iff L_1, L_2 intersect.*
(2) *ϕ is parabolic iff L_1, L_2 are asymtotic.*
(3) *ϕ is hyperbolic iff L_1, L_2 are parallel.*

It is worth noting that the there are infinitely many choices of L_i in the above statement. An appropriate choice will be helpful, for instance in the following exercise.

EXERCISE 2.11. *Assume that g is a parabolic element and h is a hyperbolic element such that they do not have a common fixed point. Give a geometric proof that the commutator $ghg^{-1}h^{-1}$ is a hyperbolic element.*

EXERCISE 2.12. *Assume that g, h are two elliptic elements without a common fixed point. Give a geometric proof that the commutator $ghg^{-1}h^{-1}$ is a hyperbolic element.*

Consider a hyperbolic element $\phi \in \mathrm{Isom}^+(\mathbb{H}^2)$ which can be conjugated to be of the form $z \to kz$. For convenience assume that $k > 1$. It has two fixed points $0, \infty$ in $\bar{\mathbb{H}}^2$. It is clear that given a point $z \in \mathbb{H}^2$, the iterates $\phi^n(z)$ tend to ∞ for $n > 0$; for $n < 0$ they tend to 0. We call ∞ as the attractive fixed point and 0 the repelling fixed point.

In general, one may define a fixed point z of a hyperbolic element to be *attractive* if for some $o \in \mathbb{H}^2$ the iterates $\phi^n(o)$ tend to w for $n > 0$; *repelling* if $\phi^n(o)$ tend to w for $n < 0$. The definition does not depends on the choice of o by Exercise 2.6.

THEOREM 2.13 (North-Sourth Dynamics on $\bar{\mathbb{D}}^2$). *Let $\phi \in Isom^+(\mathbb{D}^2)$ be a non-trivial isometry. Then*

(1) *If ϕ is parabolic with the fixed point $z \in S^1$, then for any open neighborhood U of z in S^1, there exists $n_0 > 0$ such that $\phi^n(S^1 \setminus U) \subset U$ for any $n > n_0$.*

(2) *If ϕ is hyperbolic with the attractive and repelling points $\phi_+ \neq \phi_- \in S^1$, then for any open neighborhoods U, V of z, w respectively in S^1, there exists $n_0 \in \mathbb{Z}$ such that $\phi^n(S^1 \setminus V) \subset U$ for any $n > n_0$.*

The following lemma is well-known and will be used below.

LEMMA 2.14. *If a continuous $\phi : S^1 \to S^1$ sends a closed arc I of S^1 to be inside the interior $\mathring{I}$ of I, then ϕ contains a fixed point in $\mathring{I}$.*

LEMMA 2.15. *Let g, h be two hyperbolic elements without common fixed points. Then for all sufficiently large $n, m \gg 0$, the element $g^n h^m$ is hyperbolic.*

PROOF. Denote g_-, g_+ the repelling and attractive fixed points respectively of g. Correspondingly, h_-, h_+ for h. By assumption $\{g_-, g_+\} \cap \{h_-, h_+\} = \emptyset$. In order to apply Lemma 2.14, we take a closed arc U of the atractive fixed point g_+ such that $h_-, h_+ \notin U$. By Theorem 2.13 some power h^m for $m > 0$ sends properly U to a small neighborhood V of h_+ which does not contain g_-, g_+ as well. Finally, Theorem 2.13 allows to apply a high power g^n for sending V to the interior $\mathring{U}$ of U. In a word, we have $g^n h^m(U) \subsetneq \mathring{U}$. So Lemma 2.14 implies the exitence of a fixed point in U. A similar argument shows that there exists another fixed point in a closed neighborhood of h_-. There, $g^n h^m$ is a hyperbolic element. □

EXERCISE 2.16. *Under the assumption of Lemma 2.15, prove that the fixed points of $g^n h^m$ are disjoint with those of g, h.*

EXERCISE 2.17. *Let g be parabolic and h be hyperbolic such that they have no common fixed points. Then for all sufficiently large $n, m \gg 0$, the elements $g^n h^m$ and $h^m g^n$ are hyperbolic.*

EXERCISE 2.18. *Let g,h be two parabolic elements without the same fixed point. Then for all sufficiently large $n, m \gg 0$, the element $g^n h^m$ is hyperbolic.*

3. (non-)Elementary Fuchsian groups

We first endow the topology on $SL(2, \mathbb{C})$ from $\mathbb{C}^4$ by understanding each matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ as a 4-tuple of complex numbers (a, b, c, d). Precisely, the topology is generated by the distance $d(A, B) = \parallel A - B \parallel$ where

$$\parallel A \parallel = \sqrt{|a|^2 + |b|^2 + |c|^2 + |d|^2}.$$

EXERCISE 3.1. *For any 2×2 matrix A, we have $\parallel A \parallel^2 \geq 2 \det(A)$.*

Note that the map $(a, b, c, d) \to (-a, -b, -c, -d)$ is a homeomorphism on $SL(2, \mathbb{C})$. In fact, the group $\mathbb{Z}_2$ acts freely on $SL(2, \mathbb{C})$, where the non-trival element in $\mathbb{Z}_2$ sends $(a, b, c, d) \to (-a, -b, -c, -d)$. Thus we know that the orbital map

$$SL(2, \mathbb{C}) \to PSL(2, \mathbb{C})$$

is a 2-sheet covering map, where $PSL(2, \mathbb{C})$ is given by the quotient topology.

We understand elements $g = \frac{az+b}{cz+d}, ad - bc = 1$ in $PSL(2, \mathbb{C})$ as normalized matrices

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, ad - bc = 1.$$

Then $\min\{\| A-B \|, \| A+B \|\}$ gives a metric on $PSL(2,\mathbb{C})$. Since $\| A-(-A) \| = \| 2A \| > 2\sqrt{2}$, so the map $SL(2,\mathbb{C}) \to PSL(2,\mathbb{C})$ restricting on ball of radius $\sqrt{2}$ is an isometry. This also implies that the quotient topology on $PSL(2,\mathbb{C})$ is the same as the topology induced by the above metric

The *norm* of an element g in $PSL(2,\mathbb{C})$ is defined to be $\| g \| = \| A \|$.

EXERCISE 3.2. *Prove that* $2\cosh d_{\mathbb{H}^2}(i, gi) = \| g \|^2$*, where* $i \in \mathbb{H}^2$ *is the imaginary number.*

EXERCISE 3.3. *With respect to the topology on* $PSL(2,\mathbb{R})$*, construct a sequence of hyperbolic elements* g_n *converging to a parabolic element. Prove that a sequence of elliptic elements cannot converge a hyperbolic element.*

Let G be a subgroup of $PSL(2,\mathbb{R})$. It is called *Fuchsian* if it is discrete in the above-mentioned topology of $PSL(2,\mathbb{C})$.

EXERCISE 3.4. *The group* $G = PSL(2,\mathbb{R})$ *is a* ***topological group*** *endowed with quotient topology: the group multiplication* $(f,g) \in G \times G \to fg \in G$ *is continuous, and the inverse* $g \in G \to g^{-1} \in G$ *is homeomorphism.*

An indirect way to see it is to observe that $SL(2,\mathbb{R})$ covers G so the product $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ covers $G \times G$ as well. The covering map being a local homeomorphism implies that the convergence in $G \times G$ is locally the same as the convergence in $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$.

EXERCISE 3.5. *A group* G *is Fuchsian iff any sequence of elements* $g_n \to 1$ *becomes eventually constant:* $g_n = 1$ *for all but finitely many* n*.*

A Fuchsian group admits a properly discontinuous action on $\mathbb{H}^2$.

THEOREM 3.6. *A subgroup of* $PSL(2,\mathbb{R})$ *is Fuchsian if and only if it acts properly discontinuously on* $\mathbb{H}^2$*.*

PROOF. $\Rightarrow$: Given any compact set K in $\mathbb{H}^2$, let $g \in G$ such that $gK \cap K \neq \emptyset$. Without loss of generality, assume that $i \in K$. Thus, $d_{\mathbb{H}^2}(i, gi) \leq 2R$ where R is the diameter of K. By Exercise 3.2, we have $\| g \| = \| A \|$ is uniformly bounded. This implies that only finitely many g satsifies $gK \cap K \neq \emptyset$. If not, there will be a subsequence of g_n such that $A_n \to A$, where A_n are their matrix represenatives. By local homeomorphism of $SL(2,\mathbb{C}) \to PSL(2,\mathbb{C})$, this subsequence converges in G so giving a contradiction to the discreteness of G.

$\Leftarrow$: If G is not discrete, then there exists a seuqence of elements $g_n \in G$ such that $g_n \to 1$ in G. Recall that $SL(2,\mathbb{C}) \to PSL(2,\mathbb{C})$ is a local isometry, so their matrix represenatives A_n converges to the identity in the norm $\| \cdot \|$. This gives a non-dsicrete orbit $g_n x$ for any $x \in \mathbb{H}^2$. This contradicts to the properly discontinuous action. □

A Fuchsian group is called *elementary* if it admits a finite orbit in $\bar{\mathbb{H}}^2$; otherwise it is *non-elementary*: any orbit is infinite.

THEOREM 3.7. *Let* G *be a subgroup of* $PSL(2,\mathbb{R})$ *acting properly discontinuously on* $\mathbb{H}^2$*. Then*

(1) *a parabolic element cannot have a common fixed point with a hyperbolic element.*

(2) *any two hyperbolic element have either disjoint fixed points or the same fixed points.*

Proof. For (1), we can assume that they have a common fixed point at ∞ so we can write $g(z) = z + a$ and $h(z) = kz$ for $a \in \mathbb{R}, k \neq 1$. Up to taking the inverse, we can assume that $k > 1$. By computation we have $h^{-n}gh^n(z) = z + k^{-n}a$. This contradicts the properly discontinuous action of G.

The statement (2) is similar and left to the reader. □

Theorem 3.8. *If all non-trivial element in a subgroup G of $PSL(2, \mathbb{R})$ is elliptic, then G has a global fixed point in $\mathbb{H}^2$.*

Proof. By Exercise 2.12, all elliptic elements fix the same point. □

Theorem 3.9. *Let G be an elementary Fuchsian group of $PSL(2, \mathbb{R})$. Then G belongs to one of the following cases:*

(1) *G is a finite cyclic group generated by an elliptic element,*
(2) *G is an infinite cyclic group generated by either a parabolic element or a hyperbolic element,*
(3) *G is conjugated to a subgroup $\langle z \to kz, z \to -1/z\rangle$ for some $1 \neq k > 0$.*

Proof. If G admits a finite orbit in $\mathbb{H}^2$, then G contains no parabolic and hyperbolic elements; otherwise some power of them would fix pointwise the finite orbit, giving a contradiction. By Theorem 3.8, all elliptic elements fix the same point. Conjugate the fixed point to the orgin so G is conjugated to a subgroup in S^1. Since the group is discrete, we see that G must be a finite cyclic subgroup.

So assume now that G has a finite orbit in $\bar{\mathbb{H}}^2$ and G is infinite. Since G is infinite, it must contain a hyperbolic or parabolic element (by the first paragraph). And the orbit is finite, some power of an infinite order element must fix pointwise this orbit. Thus the orbit consists of at most two points, since every orientation-preserving isometry fixes at most 2 points in the boundary. If it is just one point, then by Theorem 3.7.1, G consists of only parabolic elements. By conjugating the fixed point to ∞, we see that G must be generated by a parabolic element.

If the orbit conatins exactly two points, by Theorem 3.7, G cannot contain a parabolic element so every nontrivial element in G is either hyperbolic or elliptic. We may conjugate these two points to $0, \infty$ in $\bar{\mathbb{H}}^2$. Note that G must contain hyperbolic elements. If it consists of only hyperbolic elements, then we see that G is cyclic generated by a hyperbolic element.

If G does contain an elliptic element e, then e must switch the two fixed points $0, \infty$ so e can be conjugated to $z \to -1/z$.

Let H be the subgroup of G fixing 0 and ∞. As above, we have that H is generated by a hyperbolic element $z \to kz$ for some $k > 0$. We claim now that $G = \langle z \to kz, e\rangle$. Indeed, it suffices to consider $g \in G \setminus H$ so it switchs 0 and ∞. Then $e \cdot g$ fixes 0 and ∞ and thus belongs to H. Therefore, G is conjugated to $\langle z \to kz, z \to -1/z\rangle$. □

Exercise 3.10. *Prove that if an element in $PSL(2, \mathbb{R})$ switches two points $z, w \in \partial_\infty \mathbb{H}^2$ then it is conjugated to $z \to -1/z$.*

Theorem 3.11. *A non-elementary Fuchsian group contains infinitely many hyperbolic elements, none two of which has the same fixed points.*

PROOF. By Theorem 3.9, there exist at least two hyperbolic elements g, h such that $Fix(g) \cap Fix(h) = \emptyset$. By Lemma 2.15, $g^n h^m$ is hyperbolic for any sufficiently large $n, m > 0$. By Exercise 2.16, the fixed points of $g^n h^m$ are disjoint with those of g, h, but it could arbitrarily close to those of g! Consequently, we could produce infinitely many hyperbolic elements without sharing the same fixed points. □

3.1. Limit sets of Fuchsian groups. Since a Fuchsian group G acts properly discontinuously so any orbit is discrete in $\mathbb{D}^2$, it will be useful to look at their asymptotics at the infinity, $\partial_\infty \mathbb{D}^2$, of $\mathbb{D}^2$. In what follows, we usually consider the ball model, since its compactification by $\partial_\infty \mathbb{D}^2 = S^1$ is obvious and easy to visualize then in upper half space model.

DEFINITION 3.12. Let G be a Fuchsian group. The *limit set* denoted by $\Lambda(G)$ is the set of accumulation points of an orbit Go where $o \in \mathbb{D}^2$ is a preferred basepoint. Each point in $\Lambda(G)$ will be called a *limit point*

By Exercise 2.6, the limit set does not depend on the choice of basepoints.

EXERCISE 3.13. *The limit set of G is a G-invariant, closed subset in the topology of $\bar{\mathbb{H}}^2$.*

The following result is a consequence of Theorem 3.8.

LEMMA 3.14. *A Fuchsian group is elementary iff its limit set consists of at most two points (it may be 0, 1, 2). A non-elementay Fuchsian group must have infinitely many limit points.*

The limit set can be characterized by the following property.

THEOREM 3.15. *Let G be a non-elementary Fuchsian group. Then the limit set $\Lambda(G)$ is the minimal G-invariant closed set in $\partial_\infty \mathbb{D}^2$. And there is no isolated point in $\Lambda(G)$.*

By definition, a *perfect* set is a subset of a topological space that is closed and has no isolated points. It is known that a perfect set has uncountablely many points. So the limit set of a non-elementay Fuchsian group is a prefect set so contains uncountably many points.

PROOF. Let L be a G-invariant closed set in $\partial_\infty \mathbb{D}^2$. We shall prove that $\Lambda(G) \subset L$. Recall that G contains infinitely many hyperbolic elements g_n without same fixed points. Since L is closed and G-invaraint so $g_n L = L$, by dynamics of hyperbolic elements in Theorem 2.13, the set L contains at least three points.

By definition, $\Lambda(G)$ is the set of accumulation points of Gz. So for any $x \in \Lambda(G)$, there exists a sequence of elements $h_n \in G$ such that $h_n o \to x$ for some $o \in \bar{\mathbb{D}}^2$. Let $z \neq w \in L \setminus \{x\}$ be two points, which exist by the first paragraph. We connect z and w by a geodesic γ. We claim that up to passage of subsequences, one of the two sequences $\{h_n z\}$ and $\{h_n w\}$ converges to x.

Indeed, we choose the basepoint o on the geodesic γ for convenience. Passing to a subsequence, we assume that the endpoints $h_n z$ and $h_n w$ of geodesics $h_n \gamma$ converge to a, b respectively. It is possible that $a = b$.

Since L is closed and $z, w \in L$, we thus obtain $a, b \in L$. Note that $h_n o$ belongs to the geodesics $h_n \gamma$ so it must converge to a point in $\{a, b\}$ (cf. Exercise 3.16). Hence, the claim follows. As a consequence, x belongs to $\{a, b\}$ so $\Lambda(G) \subset L$ is proved.

Now it remains to show that x is not isolated in $\Lambda(G)$. Indeed, since $\Lambda(G)$ contains infinitely many points, we then choose three distinct points $z_1, w_1, w_2 \in \Lambda(G) \setminus \{x\}$. We apply the claim above to there pairs (z_1, w_1), (z_1, w_2) and (w_1, w_2) separately: there must be a pair of points, denoted by (z, w), from z_1, w_1, w_2 such that $g_n z \to x$ and $g_n w \to x$. Since $g_n z \neq g_n w$, we thus obtain a sequence of distinct points tending to x, thereby completing the proof that x is not isolated. □

Exercise 3.16. *Give a proof of the above fact that if a sequence of points z_n on geodesics γ_n converges to a point $z \in \partial_\infty \mathbb{D}^2$, then z must lie in the set of accumulation points of endpoints of γ_n.*

One way to prove this exercise is to use the following fact:

Let γ be a geodesic in $\mathbb{D}^2$ outside the ball $B(0, r)$ of Euclidean radius $r < 1$ centered at the origin. Then the Euclidean diameter of γ tends to 0 as $r \to 1$.

Exercise 3.17. *Consider a Fuchsian group G with a subgroup H.*

(1) *If H is of finite index in G, then $\Lambda(H) = \Lambda(G)$.*
(2) *If H is an infinite normal subgroup in G, then $\Lambda(H) = \Lambda(G)$. In particular, if G is non-elementary, then H is also non-elementary. (Tips: use Theorem 3.15.)*

Corollary 3.18. *Let G be a non-elementay Fuchsian group. Then the following holds:*

(1) *Any orbit is dense in the limit set $\Lambda(G)$.*
(2) *The closure of fixed points of parabolic elements coincides with $\Lambda(G)$, provided that parabolic elements exist.*
(3) *The closure of fixed points of hyperbolic elements coincides with $\Lambda(G)$.*

CHAPTER 3

Geometry of Fuchsian groups

In this chapter, we will always consider a Fuchsian group G acting on $\mathbb{H}^2$ or $\mathbb{D}^2$ if no explicit mention. We shall begin with some examples of non-elementary Fuchisan groups.

1. Schottky groups

Fix a basepoint $o \in \mathbb{H}^2$. If g is a non-elliptic element of $\mathbb{D}^2$, then the set X_g represents the open half-plane in $\mathbb{D}^2$ bounded by the bisector $L_{o,go}$, containing $g(o)$. The sets X_g and $X_{g^{-1}}$ are disjoint (resp. tangent) if and only if g is hyperbolic (resp. parabolic).

Exercise 1.1. *Prove that the sets X_g and $X_{g^{-1}}$ are disjoint (resp. tangent) in $\bar{\mathbb{H}}^2$ if and only if g is hyperbolic (resp. parabolic).*

We have

$$gX_{g^{-1}} = \mathbb{H}^2 \setminus \overline{X_g}.$$

Definition 1.2. Let $g_1, g_2, \cdots, g_n$ be a set of non-elliptic elements such that

$$\overline{\left(X_{g_i} \cup X_{g_i^{-1}}\right)} \cap \overline{\left(X_{g_j} \cup X_{g_j^{-1}}\right)} = \emptyset$$

for any $i \neq j$. The group generated by $\{g_1, g_2, \cdots, g_n\}$ is called *Schottky group.*

Lets repeat the Ping-Pong Lemma 2.20 here.

Lemma 1.3 (Ping-Pong Lemma). *Suppose that G is generated by a set S, and G acts on a set X. Assume, in addition, that for each $s \in \tilde{S} = S \sqcup S^{-1}$, there exists a set $X_s \subset X$ with the following properties.*

(1) *$\forall s \in \tilde{S}$, $s \cdot X_t \subset X_s$, where $t \in \tilde{S} \setminus \{s^{-1}\}$.*
(2) *$\exists o \in X \setminus \cup_{s \in \tilde{S}} X_s$, such that $s \cdot o \in X_S$ for any $s \in \tilde{S}$.*

Then $G \cong F(S)$.

Corollary 1.4. *A Schottky group is free.*

1.1. Fundamental domain. We give a general introduction to the notion of a fundamental domain. More details can be found in [**4**, Ch. 6.6] or [**1**, Ch. 9].

Definition 1.5. A closed subset F is called a *fundamental domain* for the action of G on $\mathbb{H}^2$ if the following two conditions hold:

(1) $\cup_{g \in G} gF = \mathbb{H}^2$,
(2) $g\mathring{F} \cap \mathring{F} = \emptyset$ for any $g \neq 1 \in G$.

Exercise 1.6. *If there exists a point $o \in \mathbb{H}^2$ such that Go is discrete and the point-stabilizer G_o is finite, then G acts properly and discontinuously on $\mathbb{H}^2$.*

LEMMA 1.7. *If a group action of G on $\mathbb{H}^2$ admits a fundamental domain then G is a Fuchsian group.*

PROOF. Let F be a fundamental domain for the action of G on $\mathbb{H}^2$. For any interior point $o \in \mathring{F}$, we see that Go is discrete, and G_o is trivial. Hence, G acts properly discontinuously on $\mathbb{H}^2$ so it is a Fuchsian group. □

A fundamental domain F is called *locally finite* if any compact set intersects only finitely many translates gF for $g \in G$. The importance of a locally finite fundamental domain lies in the following fact.

THEOREM 1.8. [**4**, Theorem 6.6.7][**1**, Theorem 9.2.4] *Let F be a locally finite fundamental domain for the action of G on $\mathbb{H}^2$. Then $\mathbb{H}^2/G$ is homeomorphic to the quotient space of F by the restriction of the map $\mathbb{H}^2 \to \mathbb{H}^2/G$.*

Assume that G acts properly discontinuously on $\mathbb{H}^2$. We define a metric on $\mathbb{H}^2/G$ as follows:

$$\bar{d}(Gx, Gy) = \inf\{d(x, Gy)\}$$

for $x, y \in \mathbb{H}^2$.

EXERCISE 1.9.
1. *Prove that $\bar{d}$ is indeed a metric on the set $\mathbb{H}^2/G$ of orbits.*
2. *The map $\pi : \mathbb{H}^2 \to \mathbb{H}^2/G$ sends $B(x, r)$ onto $B(\pi(x), r)$ for each $r > 0$. In particular, π is an open map.*
3. *The quotient topology on $\mathbb{H}^2/G$ coincides with the metric topology by $\bar{d}$.*

THEOREM 1.10 (Covering is local isometry). *Assume that G acts freely and properly discontinuously on $\mathbb{H}^2$. Then the covering map $\pi : \mathbb{H}^2 \to \mathbb{H}^2/G$ is a local isometry: for each point $x \in \mathbb{H}^2$, there exists $r > 0$ (depending on x) such that $\pi : B(x, r) \to B(\pi(x), r)$ is an isometry.*

PROOF. First note that for each $x \in \mathbb{H}^2$ there exists $r > 0$ such that $B(x, r) \cap B(gx, r) = \emptyset$ for all $1 \neq g \in G$. The constant r is thus the desired one. □

EXERCISE 1.11. *Prove that the quotient spaces $\mathbb{H}^2/\langle h \rangle$ and $\mathbb{H}^2/\langle p \rangle$ endowed with the above metrics are not isometric, where h is a hyperbolic element and p is a parabolic element. (Tips: find metric-invariants to distinguish them: for instance, whether they contain closed loops which are locally shortest (i.e.: closed geodesics), or the maximal radius of embeded disks in spaces (i.e. injective radius)...)*

1.2. Dirichlet domain. In this subsection, we are going to construct a fundamental domain for any Fuchsian group. This in particular implies that the converse of Lemma 1.7 is also true.

LEMMA 1.12. *Suppose that G acts properly discontinuously on $\mathbb{H}^2$. Then there exists a point o such that it is not fixed by any non-trivial element $g \in G$.*

PROOF. Fix arbitrary point $z \in \mathbb{H}^2$, and consider the discrete orbit Gz. Then there exists $r > 0$ such that $B(z, r) \cap gB(z, r) = \emptyset$ if $go \neq o$. Thus, any point o in $B(z, r)$ satisfies the conclusion, since the point z is the only fixed point of the stabilizer of G_z. □

A special kind of fundamental domain called *Dirichlet domain* can be constructed as follows. Let o be a point not fixed by any nontrivial element in G.

Denote by $H_o(g)$ be the closed half-plane containing o bounded by the bisector $L_{o,go}$. The *Dirichlet domain* is defined as follows:

$$D_o(G) := \cap_{g\in G} H_o(g).$$

Equivalently, it contains exactly the shortest points from each orbit Gz. This is formulated in the following.

LEMMA 1.13. $D_o(G) = \{z \in \mathbb{H}^2 : d(o,z) = d(Go,z) = d(o,Gz)\}$.

PROOF. Let $z \in D_o(G)$ so $d(o,z) \le d(go,z)$ for any $g \in G$. Hence, $d(z,o) = d(z,Go)$. For the other direction, take $z \in \mathbb{H}^2$ such that $d(o,z) = d(Go,z)$. Since Go is discrete, for any $g \in G$, we have $d(o,z) \le d(go,z)$ so $z \in H_o(g)$. This implies that $z \in D_o(G)$ completing the proof. □

LEMMA 1.14. *For any point $o \in \mathbb{H}^2$ fixed only by the trivial element in G, the Dirichlet domain $D_o(G)$ is a connected convex fundamental domain.*

PROOF. The set $D_o(G)$ is path connected, and convex as the intersection of convex half-planes. Since it consists of points $z \in \mathbb{H}^2$ such that $d(o,z) = d(o,Gz)$, the condition (1) for a fundamental domain holds. So it remains to prove (2).

Suppose not, there exist $z, w \in \mathring{D}_o(G)$ such that they are in the same G-orbit: there exists $1 \ne g \in G$ such that $w = gz$. Hence, we have $d(o,z) = d(o,Gz) = d(o,w)$ thus $d(o,z) = d(g^{-1}o,z)$: $z \in L_{o,g^{-1}o}$ lies in the boundary of $D_o(G)$. This is a contradiction. □

COROLLARY 1.15. *For any $z \in \mathbb{H}^2$, the intersection $Gz \cap D_o(G)$ is a finite nonempty set.*

PROOF. By the proof of Lemma 1.14, any two points w_1, w_2 has the same distance to o. By the properly discontinuous action, there are only finitely many such points in $Gz \cap D_o(G)$. □

In what follows, the set $Gz \cap D_o(G)$ shall be referred to as a *cycle*.

LEMMA 1.16 (Local finiteness). *The Dirichlet domain is locally finite: any compact set K intersects only finitely many tranlsates of $D_o(G)$.*

PROOF. Without loss of generality, assume that K is a closed ball of radius R centered at o. Given $gD_o(G) \cap K \ne \emptyset$, we are going to prove that $d(o,go) \le 2R$ so the conclusion follows by proper actions.

Let $z \in gD_o(G) \cap K$. Then $d(o,z) \le R$ and $g^{-1}z \in D_o(G)$. Since $D_o(G)$ contains closet points in each orbit, we see that $d(g^{-1}z,o) \le d(z,o) \le R$. Hence, $d(o,go) \le 2R$. □

Let F be a convex set in $\mathbb{D}^2$. The *sides* of F correspond to the collection of maximal non-empty convex subsets of the boundary of F in $\mathbb{D}^2$, and two sides intersect at a *vertex*.

LEMMA 1.17 (Sides paired). *For each side S of $D_o(G)$, there exists a unique element $g \in G$ such that the following holds:*

(1) *S is contained in a bisector $L_{o,go}$.*
(2) *$S = D_o(G) \cap gD_o(G)$.*
(3) *$g^{-1}S$ is also a side of $D_o(G)$.*

PROOF. Observe that the collection of bisectors $\{L_{o,go} : g \in G\}$ is locally finite: any compact set K intersects finitely many of them. Indeed, we can assume that K is a closed ball of radius R centered at o. If $K \cap L_{o,go} \neq \emptyset$, then $d(o, go) \leq 2R$. The properly discontinuously action thus implies the local finitenes of bisectors.

As a consequence of local finiteness, each side S contains at least two points so has positive length. Moreover, S must belong to a bisector $L_{o,go}$ for some $g \in G$.

We first prove that $D_o(G) \cap gD_o(G) = S$. If not, then $D_o(G) \cap gD_o(G)$ is a proper subset of S, and there exists $g \neq h \in G$ such that $S \cap hD_o(G)$ contains at least two points so has positive length. Let $z \in D_o(G) \cap gD_o(G)$ so $z, g^{-1}z \in D_o(G)$. Thus, $d(z, o) = d(go, z)$ by Lemma 1.13. This implies that o, go are symmetric with respect to $L_{o,go}$. By the same reasoning, we see that o, ho are symmetric about the same line $L_{o,go}$. Thus, we must have $go = ho$. By the choice of the basepoint o, we have $g = h$. This is a contradiciton, so $S = D_o(G) \cap gD_o(G)$.

By the maximality of sides by definition, we see that $g^{-1}S$ is also an edge of $D_o(G)$.

Let us prove the uniqueness of the above g. If there exists $g \neq h$ such that $S = D_o(G) \cap hD_o(G)$, then S lies on $L_{o,ho}$ so $L_{o,go} = L_{o,ho}$. Hence, we would obtain $go = ho$ and then $g = h$, a contradiction. □

REMARK. When a side of a convex fundamental domain is preserved by an elliptic element, the middle point is fixed by the elliptic element. In this case, we shall divide this side into two sides with a new vertex at the middle point. It is clear that the above statements still hold for these new sides.

Note that the pair (g, g^{-1}) corresponds to the pair of sides $(S, g^{-1}S)$. It is possible that $S = gS$. If this happens, then g must have fixed point inside S and $g^2 = 1$.

The set Φ of elements g determined by sides S shall be called *side pairings* of the Dirichlet domain.

COROLLARY 1.18 (Generating sets). *The set of side pairings Φ generates the group G.*

SKETCH OF PROOF. By the same argument of Theorem 3.5, the set of elements $\{g \in G : gF \cap F\}$ generates G. Thus it remains to show that the elements in vertex stabilizers can be written as products over Φ. □

EXERCISE 1.19. *Give a proof of the above corollary.*

1.3. Schottky groups are Fuchsian.

LEMMA 1.20 (Fundamental domain). *The set $F = \mathbb{D}^2 \setminus \cup_{1 \leq i \leq n}(X_{g_i} \cup X_{g_i^{-1}})$ coincides with the Dirichlet domain $D_o(G)$ based at o.*

PROOF. By definition of $D_o(G)$, we know that $D_o(G)$ is a subset of F. For the other direction, suppose that there exists $x \in F \setminus D_o(G)$. Then there exists $1 \neq g \in G$ such that $gx \in D_o(G)$. Since G is a free group on the generators $S = \{g_1, g_2, \cdots, g_n\}$, we write $g = s_1 s_2 \cdots s_m$ as a reduced word where $s_i \in S$. It thus follows that $gx \in X_{s_1}$. However, $X_{s_1} \cap F = \emptyset$ so this gives a contradiction that $x \in F$. Hence, it is proved that $F = D_o(G)$. □

THEOREM 1.21. *A Schottky group is a free Fuchsian group.*

1.4. Modular groups. The *modular group* $PSL(2,\mathbb{Z})$ is clearly a Fuchsian group, since the entries in matrices are integers so the group is discrete in $PSL(2,\mathbb{R})$.

LEMMA 1.22. *The Dirichlet fundamental domain at $o = ki$ for $k > 1$ is*

$$D_o(G) = \{z \in \mathbb{H}^2 : |z| > 1, |Re(z)| \leq 1/2\}.$$

PROOF. It is clear that $D_o(G) = H_o(g) \cap H_o(h)$ for $g(z) = z + 1$ and $h(z) = -1/z$. So it remains to show that for any $\phi = \frac{az+b}{cz+d}$, $\phi\mathring{F} \cap \mathring{F} = \emptyset$.

For any $z \in \mathring{F}$, we see that $|cz + d|^2 > 1$ so $Im(\phi(z)) = \frac{Im(z)}{|cz+d|^2} < Im(z)$. The conclusion thus follows. □

2. Geometry of Dirichlet domains

2.1. Ford domains. The reference to this subsection is [**1**, Section 9.5], where the notion of a generalized Dirichlet domain is introduced.

We first give an alternative way to interprete the Dirichlet domain. This is best illustrated in the upper plane model $\mathbb{H}^2$. Consider a LFT

$$\phi(z) = \frac{az + b}{cz + d}$$

where $a, b, c, d \in \mathbb{R}$ and $ad - bc = 1$. By computation, we see that

$$\phi'(z) = \frac{1}{(cz + d)^2}.$$

Hence, the Euclidean length $|dz|$ is sent under ϕ to the Euclidean length $|d\phi(z)|$ by a ratio $\frac{1}{|cz+d|^2}$. If $c \neq 0$, then ϕ is a Euclidean isomtery restricting on the points satsifying $|cz + d| = 1$. Since $c, d \in \mathbb{R}$, the set $|cz + d| = 1$ is a circle centered at $z = -d/c \in \mathbb{R}$ with radius $|1/c|$, which is orthgonal to the x-axis.

Equivalently, $c \neq 0$ is amount to saying that ϕ does not fix ∞.

DEFINITION 2.1. If $c \neq 0$, then $|z + d/c| = |1/c|$ is called the *isometric circle* of $\phi(z) = \frac{az+b}{cz+d}$.

Recall that an orientation-preserving isometry is a product of two reflexions about two geodesics whose configuration determines the isometry type (cf. Theorem 2.10). An isometric circle is clearly a geodesic, so giving rise to the following decompostion of an element as a product of an inversion about isometric circle and a Euclidean reflexion.

LEMMA 2.2. *If $g \in PSL(2,\mathbb{R})$ does not fix ∞ in $\mathbb{H}^2$, then $g = \rho_{L_1}\rho_{L_2}$, where L_2 is its isometric circle and L_1 is orthogonal to the real axis so ρ_{L_1} is a Euclidean reflexion. Moreover, $\rho_{L_1}(L_2)$ is the isometric circle of g^{-1}.*

By Theorem 2.10, we see that the isometric circles of g and g^{-1} are parallel (resp. asymptotic / intersecting) iff g is hyperbolic (resp. parabolic / elliptic).

The following theorem is proved in [**1**, Theorem 9.5.2].

THEOREM 2.3. *The intersection of exteriors of the isometric circles of all elements in G is a fundamental domain. In particular, when o is the origin in $\mathbb{D}^2$, it coincides with the Dirichlet domain based at o.*

2.2. Classification of limit points. We shall introduce a class of limit points called *conical points* which generalize the fixed points of hyperbolic elements. They constitute the most frequently occurring points in limit sets.

Definition 2.4 (Conical points). Let G be a Fuchsian group. A limit point $z \in \Lambda(G)$ is called a *conical point* if there exists a sequence of elements $g_n \in G$ such that $g_n o \to z$, and for some basepoint o and some geodesic ray γ ending at z, the points $g_n o$ stay within a finite neighborhood of γ.

The definition is independent of the choice of the basepoints and geodesic rays:

Exercise 2.5. *If z is a conical point given by the above definition, then the last statement holds for any basepoint o and any geodesic ray γ ending at z.*

Exercise 2.6. *In a Fuchsian group, the fixed points of a hyperbolic element are conical points.*

Via the above exercise, the following result generalizes the first statement of Theorem 3.7.

Lemma 2.7. *In a Fuchsian group, a conical point cannot be fixed by a parabolic element.*

Proof. Assume that the conical point is at ∞ and is fixed by a parabolic element p which has the form $p(z) = z + c$ for $c \in \mathbb{R}$. By Exercise 2.5, we fix the basepoint at i, and the geodesic ray γ is put on the y-axis, for instance. By definition, there exists a sequence of elements $g_n \in G$ such that $g_n i \in N_M(\gamma)$ converges to ∞ for a uniform constant $M > 0$. The idea of the proof is similar to that of Theorem 3.7: we shall examine the values of a sequence of parabolic elements $g_n^{-1} p g_n$ at i.

First, after passage to subsequence, we see that $p(g_n(i)) = g_n(i) + c$ has a uniform bounded hyperbolic distance to $g_n(i)$. Indeed, since $g_n i \in N_M(\gamma) \to \infty$, the y-cooridnate of $g_n i$ tends to ∞. By definition of hyperbolic distance $\frac{|dz|}{y}$, there exists a constant K depending on c such that $d(p(g_n(i)), g_n(i)) \le K$. Hence, we see that $d(g_n^{-1} p g_n(i), i) \le K$ for all n. Since G acts properly on $\mathbb{H}^2$, we obtain that the set of elements $g_n^{-1} p g_n$ is finite.

As a consequence, there exist infinitely many distinct n_i such that $g_{n_i}^{-1} p g_{n_i}$ equal to the same element so $g_{n_0} g_{n_i}^{-1} p = p g_{n_0} g_{n_i}^{-1}$. Thus, $g_{n_0} g_{n_i}^{-1}$ is a parabolic element fixing ∞ as well, sending $g_{n_0} g_{n_i}^{-1}$ to $g_{n_i} i$ to $g_0 i$. However, the y-coordinate of $g_{n_i} i$ differs from that of $g_0 i$ as $g_n i \to \infty$. This is a contradiction, because a parabolic element fixing ∞ preserves the y-coordinate. Therefore, the proof is complete. □

In $\mathbb{D}^2$, a *horocycle* based at $q \in S^1$ is a Euclidean circle in $\mathbb{D}^2$ tangent at q with S^1. The Euclidean disk bounded by a horocycle is called *horodisk*.

Exercise 2.8. *In a Fuchsian group G, let $q \in \partial_\infty \mathbb{H}^2$ be a point fixed by a parabolic element p. Denote by G_q the stabilizer of q in G. Prove that there exists a horodisk H based at q such that $gH \cap H = \emptyset$ for any $g \in G \setminus G_q$.*

[Tips: use Lemma 2.7 prove that for any point $o \in \mathbb{H}^2$, there exists a finite number $M > 0$ such that y-coordinates of $go \in Go$ are bounded by M.]

Let H be a subgroup of a Fuchsian group G. A subset K in $\mathbb{H}^2$ is called *strictly H-invariant* if $hK = K$ for any $h \in H$, and $gK \cap K = \emptyset$ for any $g \in G \setminus H$.

Then Exercise 2.8 implies that every maximal parabolic subgroup P has a strictly invariant horodisk H. By the following exercise, we see that the corresponding quotient space H/P is embedded into $\mathbb{H}^2/G$, which shall be referred to as a *cusp* of $\mathbb{H}^2/G$.

EXERCISE 2.9. *Let K be a strictly H-invariant open subset in $\mathbb{H}^2$. Prove that the quotient space K/H is homeomorphic to $\pi(K)$ in $\mathbb{H}^2/G$ where $\pi : \mathbb{H}^2 \to \mathbb{H}^2/G$.*

Let F be a convex set in $\mathbb{D}^2$. It will be useful to consider the *infinity boundary* of F, denoted by F^∞, which is the intersection with S^1 the closure of F in the compactification $\bar{\mathbb{D}}^2$. A *free side* is a connected component of F^∞ of positive length in S^1.

LEMMA 2.10. *The interior of a free side of the Dirichlet domain is not a limit point.*

PROOF. This is straightforward by definition of a limit point. □

2.3. Parabolic fixed points and proper vertex. Recall that a *vertex* of a convex set F is the intersection of two sides. When considering the infinity boundary of F, it is useful to define vertices there as follows. A *proper vertex* of F is a point on S^1 which is the intersection of two sides; otherwise it is called an *improper vertex* if one of the two sides is a free side.

LEMMA 2.11. *Every parabolic fixed point is sent by an element $g \in G$ into the infinity boundary $D_o^\infty(G)$ of $D_o(G)$. Moreover, it is sent to a proper vertex.*

PROOF. Let q be a point fixed by a parabolic element p. We fix a geodesic ray γ ending at q. For convenience, we consider the upper plane model $\mathbb{H}^2$ and assume $q = \infty$, so γ belongs to the y-axis. Write $F = D_o(G)$ in the proof.

Since interior points of the infinity boundary of F are not limit points, it suffices to prove that γ will eventually stay in a translate gF for some $g \in G$. Equivalently, we need to show there are only finitely many gF intersecting γ.

We argue by contradiction. Assume that there exists infinitely many g_nF such that $g_nF \cap \gamma \neq \emptyset$. Choose $z_n \in g_nF \cap \gamma$. Since the Dirichlet domain is locally finite, we conclude that $z_n \to \infty$. We claim now that $d(g_no, \gamma) < M$ for a uniform constant M.

Indeed, since F is exacty the set of shortest points to the basepoint o in each orbit Gz, it follows that the set g_nF consists of shortest points in orbits to g_no. Since $z_n \in g_nF$, we see that $d(z_n, g_no) \le d(\langle p\rangle z_n, g_no)$ for each fixed n. Since p is of the form $z \to z + c$, it preseves the horocycle H through z_n. Note that the shortest path from g_no to H is orthogonal to H, so we see that the x-coordinate of $g_n(o)$ differs that of z_n at most $c/2$. This implies that there exists a uniform constant M such that $d(g_no, \gamma) < M$ where M depends on c. The claim thus follows.

A consequence of the claim shows that $g_no \to \infty$ and $g_n \in N_M(\gamma)$. This contradicts to Lemma 2.7. The proof is thus complete. □

The claim of the above proof proves the following fact. See [**1**, Thm 9.2.8] for a general statement with ANY LOCALLY FINITE fundamental domain.

COROLLARY 2.12. *Let p be a parabolic element with the fixed point at q. Then any geodesic ray ending at q intersects in only finitely many translates of Dirichlet domains.*

LEMMA 2.13. [**1**, Thm 9.3.8] *Let $q \in S^1$ be any point of $D_o^\infty(G)$ fixed by a nontrivial element p. Then p must be a parabolic element. Moreover, the cycle of q consists of a finite number of proper vertices.*

PROOF. Assume to the contrary that p is hyperbolic. Let γ be the axis of p with one endpoint at q. Let $z_n \in [o,q]$ tending to q where $[o,q] \subset D_o(G)$ by the convexity. Clearly, there exists $w_n \in \gamma$ such that $d(z_n, w_n) \to 0$ as $n \to \infty$. Since $\langle p \rangle$ acts cocompactly on γ, there exists a sequence of distinct elements $h_n \in \langle p \rangle$ sending z_n to a compact set K of γ: $h_n z_n \in K$. Noting that $d(z_n, w_n) \to 0$, there exists a compact set $K \subset K'$ such that $h_n D_o(G)$ intersects K' for infinitely many h_n. This is a contradiciton to the local finiteness of $D_o(G)$. Thus, p must be parabolic.

It remains to show that the cycle of q is finite. If not, there exist infinitely many $q_n \in D_o^\infty(G)$ and $g_n q_n = q$ for $g_n \in G$. As a consequence, each $g_n D_o(G)$ intersects a fixed geodesic ray ending at q so it is impossible by the proof of Lemma 2.11. Thus g_n must be a finite set, contradicting that $q_n \in D_o^\infty(G)$ are distinct. So the proof is finished. □

Recall that an improper vertex is the intersection of a side with a free side.

EXERCISE 2.14. *If a Dirichlet domain has finitely many sides, then every improper vertex is not a limit point.*

2.4. Conjugacy classes of elliptic and parabolic elements. A *cycle* is a maximal subset of vertices in F if they belong to the same G-orbit. If one of point in a cycle is fixed by an elliptic element, then the cycle is called an *elliptic cycle*. A cycle of proper vertices is called a *parabolic cycle*.

LEMMA 2.15 (Elliptic cycle). *Let C be a cycle of vertices in a Dirichlet domain F, and Θ be the sum of the angles at vertices in C. Then there exists some integer $m \ge 1$ such that $\Theta = 2\pi/m$. If $m > 1$, then every vertex in C is fixed by an elliptic element order m, otherwise its stabilizer is trivial.*

PROOF. By Corollary 1.15, C is a finite set. We list $C = \{x_0, x_1, \cdots, x_n\}$ such that $h_i x_i = x_{i-1}$ for some h_i where $1 \le i \mod (n+1)$. Note that $h_0 x_0 = x_n$. Thus the product $h_1 h_2 \cdots h_n h_0$ fixes x_0.

Since the sides of F is paired by Lemma 1.17, the point x_i is the common endpoint of two sides e_i and e_i' such that $h_i e_i = e_{i-1}'$ is the intersection $F \cap h_i F$ and the other side of $h_i F$ is $h_i e_i'$. Note that $h_0 e_0 = e_n'$ and $F \cap h_0 F = e_n'$. Let θ_i be the angle between e_i and e_i'.

Note that $h_1 F \cap F = e_0'$, then $h_1 h_2 F \cap h_1 F = h_1 e_1'$, continuously we get

$$h_1 h_2 \cdots h_i F \cap h_1 h_2 \cdots h_{i-1} F = h_1 h_2 \cdots h_{i-1} e_{i-1}' = h_1 h_2 \cdots h_i e_i$$

for $i \le n$. The other side of $h_1 h_2 \cdots h_n F$ is $h_1 h_2 \cdots h_n e_n'$. Noting that $e_n' = h_0 e_0$, the sides $h_1 h_2 \cdots h_n h_0 e_0$ and e_0 extends a total angle $\theta_0 + \theta_1 + \cdots + \theta_n$.

Since $h_1 h_2 \cdots h_n h_0$ fixes x_0 and sends e_0 to $h_1 h_2 \cdots h_n h_0 e_0$ with angle Θ, it must be an elliptic element of order $2\pi/\Theta$. □

LEMMA 2.16. *If a Dirichlet domain has finitely many sides, then each proper vertex is fixed by a parabolic element.*

PROOF. Let v be a proper vertex so it is the intersection of two sides. Then there exists infinitely many translates of Dirichlet domains $g_n D_o(G)$ in which v is a proper vertex. Hence, $g_n^{-1}v$ are proper vertices in $D_o(G)$. By hypothesis, the cycle of proper vertices is finite. As a result, there are infinitely many distinct elements $g_{n_i}^{-1}$ such that $g_{n_i}^{-1}v$ are the same. Thus, the proper vertex v is fixed by a non-trivial element which must be parabolic by Lemma 2.13. □

By defintion, a subgroup is called a *parabolic (resp. elliptic) subgroup* if every nontrivial element is parabolic (resp. elliptic). It is called *maximal* if it is maximal with respect to the inclusion.

By Lemma 2.18, a parabolic (resp. elliptic) subgroup fixes a unique point v so it is included in a *unique* maximal parabolic (resp. elliptic) subgroup which is the stabilizer of the point v.

Theorem 2.17. *In a Dirichlet domain, there exists a one-one correspondence between elliptic cycles and conjugacy classes of maximal elliptic subgroups. If the Dirichlet domain has finitely many sides, then parabolic cycles correspond to conjugacy classes of maximal parabolic subgroups.*

Proof. The correspondence for elliptic cycles and conjugacy classes of maximal elliptic subgroups is straightforward. We prove the correspondence for parabolic cycles.

Let C be a parabolic cycle which consists of proper vertices in the same G-orbit. Then each $v \in$ is fixed by a parabolic element by Lemma 2.16 so the stabilizer G_v of v is a maximal parabolic subgroup. Hence, C corresponds to the conjugacy class of G_v.

Conversely, a maximal parabolic subgroup fixes a unqiue point $v \in S^1$ so its conjugacy class corresponds to the orbit Gv. By Lemma 2.11, v is sent by an element g to a proper vertex. This clearly establishes the correspondence between parabolic cycles and conjugacy classes of maximal parabolic subgroups. □

3. Geometrically finite Fuchsian groups

3.1. Convex hull and Nielsen kernel. Let K be a closed subset in S^1. The *convex hull* $C(K)$ of K is the minimal closed convex subset of $\mathbb{D}^2$ such that the infinity boundary of $C(K)$ contains K. Equivalently, $C(K)$ is the intersection of closed half planes H whose infinity boundary contains K.

Recall that the infinity boundary of a subset K in $\mathbb{D}^2$ is the intersection of the Euclidean closure of K in $\bar{\mathbb{D}}^2$ with S^1.

Exercise 3.1. *For a closed subset K, the convex hull $C(K)$ is the intersection of closed half planes H whose infinity boundary contains K, and the infinity boundary of $C(K)$ coincides with K.*

Let G be a non-elementary Fuchsian group with limit set $\Lambda(G)$. The *Nielsen kernel* $N(\Lambda(G))$ is defined to be the convex hull of $\Lambda(G)$. Thus, $N(\Lambda(G))$ is G-invariant.

Lemma 3.2 (Retractions). *There exists a G-equivariant retraction map*

$$r : \bar{\mathbb{D}}^2 \setminus \Lambda(G) \to N(\Lambda(G)).$$

Proof. For $x \in \bar{\mathbb{D}}^2$, we define $r(x)$ to be the point at which a hyperbolic ball around x is tangent with $N(\Lambda(G))$. If $x \in S^1 \setminus \Lambda(G)$, then $r(x)$ is the point at which a Euclidean ball around x is tangent with $N(\Lambda(G))$.

Since $N(\Lambda(G))$ is G-invariant, it follows that the retraction map is G-invariant as well. □

The set $S^1 \setminus \Lambda(G)$ is called the *discontinuity domain* of the action. It is the maximal open set in S^1 on which G acts properly discontinuously.

LEMMA 3.3. *A Fuchsian group G acts properly discontinuously on $\bar{\mathbb{D}}^2 \setminus \Lambda(G)$.*

PROOF. Recall that a properly discontinuous action means that for every point $x \in \bar{\mathbb{D}}^2 \setminus \Lambda(G)$, there exists an open neighborhood U such that $\{g : gU \cap U \neq \emptyset\}$ is finite. We know that G acts properly discontinuously on $\mathbb{D}^2$. It remains to show that G acts properly discontinuously on $S^1 \setminus \Lambda(G)$. This is equivalent to prove the following condition: for every compact set K in S^1, the set $S := \{g \in G : gK \cap K \neq \emptyset\}$ is finite. On the other hand, for the compact set $r(K)$ in $\mathbb{D}^2$, the proper action implies that $T := \{g : g \cdot r(K) \cap r(K) \neq \emptyset\}$ is finite. Hence, $r(gK) \cap r(K) \neq \emptyset$ for these $g \in T$. This shows that $S \subset T$ so S is finite. The proof is complete. □

THEOREM 3.4. *If G is a non-elementary torsion-free Fuchsian group, then the quotient space $N(\Lambda(G))/G$ is the minimal convex submanifold which is homotopic to $\mathbb{D}^2/G$.*

PROOF. By Lemma 3.2, the G-equivariant retraction map $r : \mathbb{D}^2 \to N(\Lambda(G))$ decends to the retraction: $\mathbb{D}^2/G \to N(\Lambda(G))/G$. The $N(\Lambda(G))/G$ is minimal because the limit set is the minimal G-invariant closed susbet in S^1. □

3.2. Geometrically finite groups. Consider a Fuchisan group G with a Dircilet domain D. The *area* of $\mathbb{H}^2/G$ is defined to be the area of D. It is easy to verify that the area of $\mathbb{H}^2/G$ does not depend on the choice of a Dirichlet domain. (cf. [**3**, Thm 3.1.1].)

DEFINITION 3.5. A non-elementary Fuchsian group G is called *geometrically finite* if there exists a Dirichlet domain D such that $N(\Lambda(G)) \cap D$ has finite area.

By convention, any elementary Fuchsian group is geometrically finite.

THEOREM 3.6. *The following statements are equivalent:*

(1) *G is geometrically finite;*
(2) *G is finitely generated;*
(3) *Any Dirichlet domain of G has finitely many sides;*
(4) *G has a Dirichlet domain with finitely many sides;*
(5) *The limit set consists of conical points and parabolic fixed points.*

The direction "(4) ⇒ (2)" follows by Corollary 1.18.

LEMMA 3.7 ((4) ⇒ (1)). *If G admits a Dirichlet domain with finitely many sides, then G is geometrically finite.*

PROOF. Let D be a Dirichlert domain. Replace each free side by a geodesic with the same endpoints to form a convex set K with finite area. It is clear that $N(\Lambda(G) \subset G \cdot K$ so G is geometrically finite. □

We wont present the proof here that (1) ⇒ (3) and (1) ⇒ (2)which can be found in [**1**, Theorem 10.1.2]. The remaining part is aiming to establish the equivalence between (3), (4) and (5).

The following exercise implies that we can separate two cusps.

EXERCISE 3.8. *Let z, w be two parabolic fixed points of a Fuchsian group G which are not in the same G-orbit. Denote by G_z and G_w the stabilizers of z and w respectively. Prove that there exist a G_z-strictly invariant horodisk H_z based at z and a G_w-strictly invariant horodisk H_w based at w such that*

$$gH_z \cap H_w = \emptyset$$

for any $g \in G$.

If a Dirichlet domain has finitely many sides, then it has finitely many parabolic cycles. For each parabolic cycle C, we choose equivariantly a horodisk H_v at each $v \in C$ such that $H_{gv} = gH_v$ for $gv \in C$.

LEMMA 3.9 (Cusp decomposition). *If G admits a Dirichlet domain with finitely many sides, then there exists finitely many horodisks H_i centered at proper vertices for each parabolic cycle and a compact subset $K \subset D_o(G)$ such that*

$$C(\Lambda(G)) \setminus GH_i = GK.$$

PROOF. Let F be the Dirichlet domain so $F \cap N(\Lambda(G))$ is a fundamental domain for the action of G on $N(\Lambda(G))$. Define $K = F \cap N(\Lambda(G)) \setminus \cup_i H_i$. To finish the proof, it suffices to prove that K is a compact set.

Suppose to the contrary that K is not compact. Then there exists a sequence of points $z_n \in K$ tending to a point z at the infinity S^1. Since $z_n \in N(\Lambda(G))$ and the Euclidean boundary of $N(\Lambda(G))$ at S^1 coincides with $\Lambda(G)$), we see that $z \in \Lambda$ is a limit point. Meanwhile, z belongs to the infinity boundary F^∞. The interior of a free side of F cannot contain a limit point. Hence, z has to be a proper vertex or improper vertex. By Exercise 2.14, an improper vertex is not a limit point as well. So z must be a proper vertex. However, by assumption, for each proper vertex z, a horodisk H_i based at z is removed from F: $K = F \cap N(\Lambda(G)) \setminus \cup_i H_i$. This thus gives a contradiction to that z belongs to the infinity boundary of K. This concludes the proof of the lemma. □

LEMMA 3.10 ((4) ⇒ (5)). *If G admits a Dirichlet domain with finitely many sides, then every limit point is either conical or parabolic.*

PROOF. We fix a basepoint $o \in K$ where $K \subset N(\Lambda(G))$ is the compact set given by Lemma 3.9. By Exercise 3.8, we can assume that those horodisks at parabolic fixed point are pairwise-disjoint.

Consider a limit point $z \in \Lambda(G)$. We connect o and z by a geodesic ray γ so $\gamma \subset N(\Lambda(G))$. By the cusp decomposition, we have two possibilities:

Case 1. The geodesic ray γ eventually enters into a horodisk at z. In this case, z is a fixed point by a parabolic element.

Case 2. The geodesic ray γ returns $G \cdot K$ infinitely often. Each return produces an intersection point x_n with $G \cdot K$. Since K is compact, there exists $g_n \in G$ such that $d(g_n o, x_n) \le R$ where $R := \mathrm{Diam}(K) < \infty$. This implies that z is a conical point. □

LEMMA 3.11 ((5) ⇒ (3)). *If a Dirichlet domain has infinitely many sides, then there exists a limit point which is neither conical nor a parabolic point.*

PROOF. Let $z \in S^1$ be an accumulation point of the set of endpoints of the sides. We shall prove that z is our desired point. If z is a parabolic point, then by Lemma 2.11, z should be a proper vertex of a translate of the Dirichlet domain

F so z cannot be an accumulation point of sides. Thus, z is not a parabolic fixed point.

We next prove that z is not conical. This follows from the convexity of the Dirichlet domain F. Let γ be a geodesic ray ending at z which lies entirely in F. By definition of conical points, there exists $g_n o$ tending to z in a finite M-neighborhood of γ. Thus, $B(o, M) \cap g_n^{-1}\gamma \neq \emptyset$ so $g_n^{-1}F \cap B(o, M) \neq \emptyset$. This contradicts to the local finiteness of F. Hence, z cannot be a conical point. The proof is complete. □

3.3. Convex-cocompact subgroups and lattices. A Fuchsian group is called *convex-cocompact* if $N(\Lambda(G))/G$ is compact.

4. Hyperbolic surfaces

DEFINITION 4.1. A metric space Σ is called a hyperbolic surface if every point $p \in \Sigma$ has an open neighborhood which is isometric to an open disk in $\mathbb{H}^2$.

4.1. Glueing polygons.

THEOREM 4.2. *Every closed orientable surfaces of genus* ≥ 2 *admits a hyperbolic structure.*

4.2. Developping hyperbolic surfaces.

Bibliography

1. A. Beardon, *The geometry of discrete groups*, Graduate Texts in Mathematics, 1982.
2. J.Meier, *Groups, graphs and trees: An introduction to the geometry of infinite groups*, Cambridge University Press, Cambridge, 2008.
3. S. Katok, *Fuchsian groups*, Chicago Lectures in Mathematics.
4. J. Ratcliffe, *Foundations of hyperbolic manifolds*, Cambridge University Press, Cambridge, 2008.