Research Overview

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My research interests lie in the area of *Geometric group theory.* I'm primarily interested in the class of *relatively hyperbolic groups* and their generalizations called *acylindrically hyperbolic groups*, with emphasis on a subclass of the latter with *strongly contracting elements*.

First of all, I give an outline of the results to be discussed in detail:

Boundary of relatively hyperbolic groups (§1):

1. (§§1.1) In [47], we studied a class of quasi-conformal measures called *Patterson-Sullivan measures* on the Bowditch boundary with applications towards the growth of relatively hyperbolic groups. We proved that there exists no uniform gap between the growth rates of the group and all proper quotient groups. This result is motivated by the growth tightness property [48] and resolves a question of Dal'bo-Peigné-Picaud-Sambusetti [14] in Kleinian groups.

2. (§§1.2) In joint work with L. Potyagailo [39], we computed the Hausdorff dimension of Floyd boundary of a relatively hyperbolic group. We are able to identify the Hausdorff dimension with the growth rate of the group, a relation conjectured by M. Bourdon. This suggests Floyd metric as an alternative metric structure on the Bowditch boundary, which deserves to be further explored in greater depth.

3. (§§1.3) In a different but related direction, I made progress with I. Gekhtman, V. Gerasimov and L. Potyagailo in [29] to understand the Martin boundary associated with a random walk on any relatively hyperbolic group by establishing a surjective map from Martin boundary to Floyd boundary. The map fails to be injective only on non-conical points in Floyd boundary, so in a sense, Floyd boundary captures the substantial part of Martin boundary.

Counting in groups with contracting elements (§2): Motivated by counting results in relatively hyperbolic groups through quasi-conformal measures, I developed an elementary approach without using ergodic theory to the following counting results.

1. (§§2.2, 2.4) Coarse asymptotic formula for lattice points [49] and conjugacy classes [22] are obtained in a class of *statistical convex-cocompact actions* (SCC action).

2. (§§2.1, 2.3) The set of contracting elements is exponentially generic in any SCC action [46]. One of the tools in the proof is the study of *growth tight subsets*, which also yields the growth tightness of convex-cocompact subgroups in mapping class groups and cubical convex subgroups in cubical groups, etc.

3. Specializing in the following classes of groups, we obtain the following new results:

   (a) Purely exponential growth and prime conjugacy growth formulae in the class of CAT(0) groups with rank-1 elements;

   (b) Conjugacy growth series for non-elementary relatively hyperbolic groups is transcendental for every finite generating set;

   (c) pseudo-Anosov elements in mapping class groups are exponentially generic in Teichmüller metric. Moreover, the axis of generic pseudo-Anosov elements is contained in the principal stratum of quadratic differentials, answering a question of J. Maher [15].

For brevity, we will not discuss the statistical hyperbolicity of SCC actions [27] and of relatively hyperbolic groups [36].

1 Boundaries of relatively hyperbolic groups

Gromov boundary provides a useful and often efficient way to study hyperbolic groups. Bowditch [8] showed that the existence of local cut-points in the boundary of a one-ended hyperbolic group detects group splittings over two-ended groups, which thus produces a canonical JSJ splitting invariant under quasi-isometry.

In some further applications, finer information such as metric and measurable structures on Gromov boundaries are indispensable. This is in particular featured in the proof of the famous Mostow rigidity theorem, in which quasi-conformal mappings, ergodic theory of conformal measures plays a crucial role. This circle of ideas admits further development in the study of Patterson-Sullivan measures, and Hausdorff dimension, and conformal dimensions, and quasi-isomeric rigidity, just to name a few in our context.

Motivated by these studies, we wish to establish a useful analytic theory on the boundary of relatively hyperbolic groups, which sets the stage for further applications. Before describing our results of this direction in [47] and [39], let us introduce the central objects in our study.

Relatively hyperbolic group. The notion of relative hyperbolicity could be defined in a number of equivalent ways. For our purpose, a group G is called *relatively hyperbolic* if it admits a *cusp-uniform action* on a Gromov-hyperbolic space so that the quotient space is a union of a compact set with finitely many cusps [9, 28]. In terms of boundaries, this is equivalent to say that G acts as a geometrically finite action on the Gromov boundary so that every limit point is either *conical* or *bounded parabolic*. The set of bounded parabolic points is at most countable, and their stabilizers comprises a *peripheral structure* $\mathbb{P}$ of G. Hence, the relative hyperbolicity only makes sense for a pair $(G, \mathbb{P})$, even though the peripheral structure $\mathbb{P}$ is often explicitly mentioned.

Bowditch boundary. Once a peripheral structure is chosen, the Gromov boundary is independent of the choice of the geometrically finite action [9], and thus is a well-defined boundary for a relatively hyperbolic group. We call it the *Bowditch boundary* ΛG of $(G, \mathbb{P})$.

Floyd boundary. By the work of V. Gerasimov [23], the Bowditch boundary could be obtained as quotient of the Floyd boundary of G, which is a sort of absolute boundary defined for any locally finite graph (Γ, d_Γ) without involving peripheral structures.

Definition. [19, 30] Let Γ be a Cayley graph of G. Upon fixing a summable series $f(n)$ with bounded decay

$$\exists \lambda: \ 1 \geq f(n+1)/f(n) > \lambda > 0,$$

we can then re-scale each edge e in the graph Γ by the length $f(d_\Gamma(e, 1))$. The induced length metric ρ_1^f is called the *Floyd metric*, whose metric completion is a compact space by the summability of $f(n)$. This defines the so-called *Floyd boundary* $\partial_f G$ of the group G (depending on Γ and f).

The Floyd boundary is called *nontrivial* if $\sharp \partial_f G \geq 3$. In [30], A. Karlsson proved that the action of the group G on a nontrivial Floyd boundary is a convergence action.

1.1 Patterson-Sullivan measures on Bowditch and Floyd boundaries

Recall that the *critical exponent* of a discrete group action of G on a metric space (X, d) is defined as follows:

$$\omega(G) := \limsup_{n \to \infty} \frac{\log \sharp N(o, n)}{n} \tag{1}$$

where $N(o, n) := \{g \in G : d(o, go) \leq n\}$ counts the number of elements in a ball of radius n at a basepoint $o \in X$.

If G acts on its Cayley graph with respect to a finite generating set S, the critical exponent $\omega(G)$ is usually called *growth rate* denoted by $\delta_{G,S}$. This setup is our motivating example in this section and main applications we have in mind.

In [47], I developed a theory of Patterson-Sullivan measure on Bowditch boundary and Floyd boundary of a relatively hyperbolic group. Previously, M. Coornaert [10] has generalized much of work of Patterson and Sullivan to any discrete group action on a Gromov-hyperbolic space with finite critical exponent.

The novelty of our Patterson-Sullivan theory lies in the consideration of the action of a relatively hyperbolic group G on the Cayley graph X, which is not Gromov hyperbolic anymore. On one hand, several new ingredients are indeed required to handle the non-hyperbolicity, and on the other hand, the theory in this setup turns out to be particularly tight and useful: the PS-measures have no atoms, and thus, the set of conical points admits full PS-measures. The atom-less property is deduced from the convergence at the growth rate $s = \omega(G)$ of the following *Poincare series* $\Theta_P(s)$ associated with parabolic subgroups $P \in \mathcal{P}$:

$$\forall s \geq 0, \ \ \Theta_P(s) := \sum_{p \in P} \exp(-s \cdot d(o, po)). \tag{2}$$

This property fails for certain geometrically finite actions on negatively pinched Riemannian manifolds by Dal'bo-Otal-Peigné [13].

The next important consequence is the Sullivan shadow lemma which allows to derive the *purely exponential growth functions* of balls

$$\forall n > 1, \ \ \sharp N(o, n) \asymp \exp(n \cdot \omega(G)). \tag{3}$$

The main application of Patterson-Sullivan measures in [47] is to study the growth rate of quotients of a relatively hyperbolic group.

In [26], R. Grigorchuk and P. de la Harpe introduced the *growth tightness* property of a group which characterizes the group by its growth rate among all of its quotients. Namely,

Definition. A group G is called *growth tight* if given any finite generating set S, the strict inequality

$$\delta_{G,S} > \delta_{\bar{G},\bar{S}}$$

holds for all proper quotients $\bar{G}$, with $\bar{S}$ denoting the canonical image of S in $\bar{G}$.

In [48], we obtain the following result using the growth tightness criterion in [14]:

Theorem 1. *[48, Theorem 1.4] Any group with nontrivial Floyd boundary is growth tight for every generating set. In particular, this holds for any non-elementary relatively hyperbolic group.*

Motivated by growth tightness, it is natural to ask whether there exists a uniform gap between the growth rates of the group and all quotient groups. With help of rotating family theory in [12], we give a negative answer to this question using our counting results.

Theorem 2. *Let G be a relatively hyperbolic group with a finite generating set S. Then there exists a sequence of proper quotients $\bar{G}_n$ such that*

$$\delta_{\bar{G}_n,S} \to \delta_{G,S}$$

as $n \to \infty$.

This theorem admits an analogue in the setting of cusp-uniform actions with parabolic gap property, which actually resolves a question posed in [14].

Some idea in proof of Theorem 2 In fact, $\bar{G}_n$ is constructed as small cancellation quotients $G/\langle\langle h^n \rangle\rangle$ for any hyperbolic element h. Using the *partial cones*, a sequence of geodesic trees T_n rooted at $1 \in G$ is constructed in the Cayley graph X so that

1. Each branch of T_n issuing at 1 contains uniformly spaced transitional points;
2. The growth rate of T_n tends $\delta_{G,S}$, as $n \to \infty$.

We call a rooted tree satisfying (1) and (2) by *large transitional tree*. The proof of Theorem is completed by showing that T_n injects into $\bar{G}_n$.

The Patterson-Sullivan theory here on word metrics is further used in studying the random walk on relatively hyperbolic groups (e.g. see [16][17][21]).

1.2 Hausdorff dimension of Floyd and Bowditch boundary

In joint work with L. Potyagailo [39], we investigate the relation between the Hausdorff dimension of the boundary and growth rate.

In geometrically finite Kleinian groups, the identification of these two quantities began in the work of Patterson [38], Sullivan [45] and after many efforts of different authors, it eventually lead to a very general result proved by Bishop-Jones [7] for any Kleinian groups. In a coarse context of hyperbolic groups, Coornaert [10] showed that there exists a family of visual metric on Gromov boundary such that the Hausdorff dimension is a constant multiple of the critical exponent of the action.

Bowditch boundary of a relatively hyperbolic group can be realized as the Gromov boundary of a proper hyperbolic space. Thus, it might be tempting to calculate the Hausdorff dimension of Bowditch boundary endowed with visual metric.

However, its value may be infinite by an example of Gaboriau-Paulin [20]. Even in case that the Hausdorff dimension is finite, it has little to do with the geometry of Cayley graph. From a group theoretical point of view, we would like to find a boundary with Hausdorff dimension directly related to the intrinsic geometry of the group.

An important observation of Marc Bourdon is that the Hausdorff dimension of Floyd boundary with Floyd metric is always bounded above by the growth rate. He then conjectured that it is the exact value of Hausdorff dimension.

Conjecture. *The Hausdorff dimension of Floyd boundary of a relatively hyperbolic group equals a constant times of the growth rate.*

In [39], the conjecture is confirmed for $f(n) = \lambda^n$. We can in fact prove more. The Floyd metric can be push-forward to get a "shortcut metric" on the Bowditch boundary ΛG, with respect to which the same relation can also be established.

Theorem 3. *[39, Theorem 1.1] Let G be a relatively hyperbolic group with a finite generating set S. There exists a constant $0 < \lambda_0 < 1$ such that*

$$dim_H(\partial_f G) = dim_H(\Lambda G) = -\delta_{G,S}/\log\lambda$$

for any $\lambda \in [\lambda_0, 1)$.

Some idea in proof of Theorem 3 In Bishop-Jones [7], a very general strategy of getting the lower bound of Hausdorff dimension is exhausting the visual boundary by geodesic rays of *bounded type*. However, their argument uses crucially the hyperbolic geometry and is hard to apply in our setup: either the Cayley graph is not Gromov-hyperbolic or the action on hyperbolic coned-off graphs is not proper.

Our proof of Theorem 3 consists in constructing a sequence of *periodic* large transitional trees satisfying 1.1 (1) and (2), where periodicity means some semi-group structure. Moreover, the ends space of those trees consists of *uniformly conical points* and admits a family of quasi-conformal measures thanks to the periodicity of those large trees. We are then able obtain a formula of Floyd metric akin to visual metric on the ends space. Given these ingredients, the proof of theorem is completed by standard arguments.

1.3 Martin boundary covers Floyd boundary

Let μ be a finitely supported *irreducible* probability measure on a group G so that the support of μ generates G as a semi-group. A random walk with step distribution μ on the group G is a *time homogeneous Markov chain* on the states G with transition probability given by $p(x, y) := \mu(x^{-1}y)$. Through the potential theory, one can define a natural compactification of the states G called *Martin boundary* $\partial_\mu G$ which gives an integral representation of **all positive** μ-harmonic functions.

In joint work with I. Gekhtman, V. Gerasimov and L. Potyagailo [29], we aim to get a concrete description of the Martin boundary of a relatively hyperbolic group G by geometric boundaries. We now briefly explain the construction of Martin boundary.

Definition. Define the *Green's function* as follows

$$G(x, y) = \sum_{n\geq 0} p_n(x, y)$$

where, as the n-th convolution of μ, $p_n(x,y)$ is the probability of random trajectories of length n from x to y. It is straightforward that $h(\cdot) := G(\cdot, y)$ is *super μ-harmonic*: $h(g) \geq \sum_{x\in G} h(gx)$ for any $g \in G$. The normalized Green functions called *Martin kernels*

$$K(\cdot, y) := \frac{G(\cdot, y)}{G(1, y)}$$

live in the *compact base* B of the positive cone $\mathcal{H}^+(G, \mu)$ of all super μ-harmonic functions. The boundary of Martin kernels $y \in G \mapsto K(\cdot, y) \in B$ is called *Martin boundary* $\partial_\mu G$.

The Martin boundary is in contrast with the Poisson boundary, which is a measurable space with a family of harmonic measures and represents all **bounded** harmonic functions via Poisson formula. In this sense, Martin boundary is "larger" than Poisson boundary.

The identification of Poisson boundary has obtained significant progress in last 30 years. Only few of Martin boundary is computed, among which we wish to mention hyperbolic groups and virtually Abelian groups of rank n. The former is homeomorphic to Gromov boundary [1] and the latter is either a point or a sphere of rank $n-1$ [35].

In [29], we compare Martin boundary with the geometric boundaries, Floyd and Bowditch boundary, of relatively hyperbolic groups:

Theorem 4. *[29] Fix a Floyd function f and a probability measure μ as above. There exists a continuous surjective G-equivariant map π from $\partial_\mu G$ to $\partial_f G$. Moreover, the pre-image of a conical point under π in $\partial_f G$ is a singleton.*

This statement is analogous to the existence of a continuous and surjective map called *Floyd map* from Floyd boundary to Bowditch boundary, firstly established in Kleinian groups by W. Floyd [19] and then in the general class of relatively hyperbolic groups by V. Gerasimov [23].

The main ingredient in the proof of Theorem 4 is the following *relative version of Ancona inequality*:

Lemma. *For any triple (x, y, z) in G, there exists a constant $C > 0$ depending only on the Floyd distance $\rho_y^f(x, z)$ (and the measure μ) such that the following holds,*

$$G(x, z) \leq C \cdot G(x, y)G(y, z)$$

where ρ_y^f is the Floyd metric with respect to y.

Remark. By Karlsson's lemma in [30], the distance $d(x, [y, z])$ is bounded above by a constant depending only on $\rho_y^f(x, z)$. The relative Ancona inequality gives a quantitative sense of the fact that the Green function is multiplicative along a quasi-geodesic if it is Floyd visible. When the group G is Gromov-hyperbolic, then C is a uniform constant so the corresponding inequality is proved by Ancona [1].

These results are further developed to characterize the full Martin boundary of *toral* relatively hyperbolic groups [32], to obtain the strict fundamental inequality of random walks [16], to investigate stability of Martin boundary up to spectral radius [17].

2 Groups with contracting elements

In a series of work [49, 46, 22, 27], we studied several counting problems in a class of proper group actions called *statistically convex-cocompact actions* for a countable group G on a proper geodesic metric space (X, d) with a contracting element.

Contracting elements Roughly speaking, the notion of a contracting element represents a certain negatively curved direction along periodic geodesics. A subset S is called *C-contracting* for $C > 0$ if the shortest point projection to S of any metric ball has diameter bounded by C. An element $g \in G$ of infinite order is *contracting* if admits a cocompact action on a contracting subset. In literature, this definition is often called *strongly contracting.*

The prototype of a contracting element is a hyperbolic isometry on Gromov hyperbolic spaces. Beyond the negatively curved spaces, the following examples of contracting elements are known:

- hyperbolic elements in relatively hyperbolic groups [25], [24];
- rank-1 elements in CAT(0) groups [6];
- pseudo-Anosov elements in mapping class groups [34].

Although a group with a contracting element is acylindrically hyperbolic in the sense of Osin [37], our study focuses on the asymptotic geometry of the following class of actions.

Statistically convex-cocompact actions This is a generalization of the geometric notion of convex-cocompact actions in a statistical sense. This will be made precise by using the *critical exponent* of a subset $\Gamma \subset G$ as follows:

$$\omega(\Gamma) := \limsup_{n\to\infty} \frac{\log \sharp N(o,n) \cap \Gamma}{n} \tag{4}$$

where $N(o,n) := \{g \in G : d(o,go) \le n\}$.

Fix constants $0 \le M_1 \le M_2$. Denote by $\mathcal{O}_{M_1,M_2}$ the set of elements $g \in G$ such that there exists some geodesic γ between $B(o, M_2)$ and $B(go, M_2)$ with the property that the interior of γ lies outside $N_{M_1}(Go)$.

Definition (SCC action)**.** If there exist two positive constants $M_1, M_2 > 0$ such that $\omega(\mathcal{O}_{M_1,M_2}) < \omega(G)$, then the action of G on a proper geodesic metric space X is called *statistically convex-cocompact.*

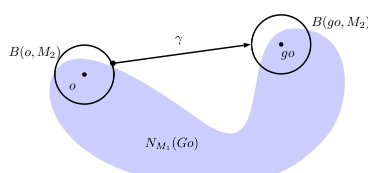

Figure 1: Definition of $\mathcal{O}_{M_1,M_2}$

Among others, the SCC actions with contracting elements include relatively hyperbolic groups, geometrically finite actions with parabolic gap property, and mapping class groups on Teichmüller spaces [18].

Remark. The SCC action (or *complementary growth gap* in [3]) is motivated by a *parabolic gap condition* introduced by Dal'bo-Otal-Peigné [13] for a geometrically finite Hadamard manifold. In any Hadamard manifold this notion is studied under the term *strongly positive recurrent* independently by Shapira-Tapie [44], and further in [41] on the *amenability problem*. Our definition and technics apply to non-Riemannian manifolds and coarse geometric spaces, such as Teichmüller spaces etc.

2.1 Exponential genericity of contracting elements

The statistical properties of random walks on geometric groups have been investigated by many authors. The analogue of the study in counting measure is significantly different and is receiving a great deal of recent interests. Our study is motivated by the following long-standing conjecture.

Conjecture (Farb)**.** *The set of pseudo-Anosov elements is generic in counting measure on the mapping class group of an orientable surface with negative Euler characteristic.*

We actually look at a general form of this conjecture. Let G be a group acting properly on a proper geodesic metric space (X, d) with a contracting element.

Question. Whether the set of contracting elements is *generic* in counting measure? i.e.

$$\lim_{n\to\infty} \frac{\sharp\{g \in N(o, n) : g \text{ is contracting}\}}{\sharp N(o, n)} = 1.$$

The set is called *exponentially generic* if the rate of convergence is exponentially rapid.

In [46], we showed that the question admits a clean answer when the action is SCC.

Theorem 5. *[46, Theorem A] Suppose that a non-elementary group G admits a SCC action on a geodesic metric space (X, d) with a contracting element. Then the set of contracting elements is exponentially generic.*

We now explain a consequence of this theorem in mapping class groups. Note that the action on Cayley graph is always cocompact, and thus is a SCC action. The Farb's conjecture would follow from the following conjecture:

Conjecture. *The mapping class groups of orientable surfaces with negative Euler characteristic contain a contracting element.*

The current knowledge of word geodesics in mapping class groups seems insufficient to prove/disprove this conjecture (but note the recent work [40]). On the other hand, the action of mapping class group on Teichmüller space contains contracting elements and satisfies the SCC property. We can thus strengthen an earlier result of J. Maher [33] with exponential convergence speed.

Theorem 6. *[46, Theorem 1.3] The set of pseudo-Anosov elements is exponentially generic with respect to the Teichmüller metric.*

Theorem 5 admits many other consequences, among which we have the exponential genericty of hyperbolic elements in relatively hyperbolic groups, and rank-1 elements in CAT(0) groups. A corollary of the latter is that the set of rank-1 elements is exponentially generic in a right-angled Artin group which are not direct product.

In the next two subsections, we explain the basic idea in the proof of Theorem 5 and some of further applications.

2.2 Lattice point counting: purely exponential growth

For a proper action of G on X and a basepoint $o \in X$, it has been a long history to study the growth of lattice points in Go. The simplest exponential growth function is given by

$$\forall n \geq 1, \ \sharp N(o,n) \asymp \exp(\omega(G)n),$$

for which we shall call *purely exponential growth* (PEG).

There are two approaches in literature to establish purely exponential growth: **ergodic theory** on geodesic flow spaces and **geodesic automatic structures** on groups. The first method has been employed to get a precise asymptotic formulae for discrete groups on negatively curved Riemannian manifolds, CAT(-1) spaces [43] and Teichmüller spaces with Teichmüller metric [4]. The second one uses the Perron-Frobenius theorem [21]. These methods are powerful and efficient in counting. However, it is not easy to implement them in concrete examples, if it is not impossible. In [49], an elementary approach via contracting elements is developed to study counting problems, such as PEG and conjugacy class growth etc.

Theorem 7. *[49, Theorem B] Suppose that a non-elementary group G admits a SCC action on a geodesic metric space (X,d) with a contracting element. Then G has purely exponential growth.*

This is new in the class of CAT(0) groups with rank-1 elements in this generality. For rank-1 compact Riemiannian manifolds, the corresponding result was obtained in [31] through the conformal measure on boundary. This also gives a simplified way to obtain (coarse) Teichmüller lattice counting in [4].

2.3 Growth tight subsets and exponential genericty

The growth tightness for a group (Definition 1.1) first introduced in [26] has raised many interests in recent years [14, 11, 3]. In [49], we found out that it is also very fruitful to look at *growth tight subsets* of a group: a subset $S \subset G$ is called *growth tight* if $\omega(S) < \omega(G)$.

In view of Genericity Question 2.1, one notes the following connection when the group action has purely exponential growth: A set of elements is exponentially generic if and only if its complement is growth tight.

In [49], we prove the growth tightness of a class of barrier-free sets. Its definition is a bit technical, but is line with the situation that a word does not contain a particular sub-word. Roughly speaking, an element g is called *(ϵ, f)-barrier-free* if the ϵ-neighborhood of $[o, go]$ does not contain any entire segment labeled by f.

Lemma. *[49, Theorem C] Suppose G admits a SCC actions and contains a contracting element. Then there exists $\epsilon > 0$ such that for any $f \in G$, the set of (ϵ, f)-barrier-free elements is growth tight.*

The result admits many consequences. Generalizing Theorem 1, the following corollary recovers a theorem of Arzhantseva-Cashen-Tao [3].

Corollary 8. *Suppose that G admits a SCC action on X and contains a contracting element. Then the action of G is growth tight:*

$$\omega(\bar{G}) < \omega(G)$$

where $\omega(\bar{G})$ is the critical exponent of any proper action of a proper quotient $\bar{G} = G/N$ on the quotient space X/N.

Other consequences include also that convex-cocompact subgroups in mapping class groups, quasi-convex subgroups in cubical groups are growth tight. The latter resolves a question posed by Dahmani-Futer-Wise [11]

Some idea in proof of Theorem 5 By purely exponential growth of SCC actions, the exponential genericity of contracting elements follows from the following

Theorem 9. *Suppose G admits a SCC action and contains a contracting element. Then non-contracting elements is growth tight.*

In order to prove this result, we make a crucial use of the recent work of Bestvina-Bromberg-Fujiwara [5] on the projection complex. The projection complex $\mathcal{P}(\mathbb{X})$ is constructed out of the set $\mathbb{X}$ of all the axis of a contracting element f. Then there exists a natural projection map π from the group action of G on X to the quasi-tree $\mathcal{P}(\mathbb{X})$. This map should be compared with the shadowing map from Teichmüller space to the curve graph in the work of Masur-Minsky.

The projection complex is a quasi-tree so the hyperbolicity via the map π entails to prove that almost every non-contracting element g admits a conjugation $g = tht^{-1}$ so that it labels a quasi-geodesic in the space. This is enough by computation to conclude the proof of Theorem 9.

2.4 Applications: counting conjugacy classes

We fix a basepoint $o \in \mathrm{Y}$. The *algebraic length* of a conjugacy class $[g]$ is defined as $\ell_o[g] := \inf_{g' \in [g]} d(o, g'o)$. In joint work with I. Gekhtman [22], we count the number $\mathcal{C}(o, n)$ of conjugacy classes $[g]$ in a SCC action with algebraic length less than n.

The starting point of our study is Theorem 9 that the non-contracting ones are exponentially negligible. Thus, we only need to count conjugacy classes of contracting elements. We then go ahead to analyze the generic properties of contracting elements, some of which are collected into the following.

Proposition. *There exist constants $M = M(o) > 0$ and $\theta = \theta(o) \in (0, 1)$ and an exponentially generic set of contracting elements $g \in \mathcal{G}$ with the following properties:*

1. *The stable length coincides with algebraic length up to an additive constant M;*
2. *The axis of each element g spends at least θ-percent time in the thick part $N_M(Go)$;*
3. *The cyclic group $\langle g \rangle$ has finite index at most M in its maximal elementary group.*

Remark. One consequence in Teichmüller spaces is that the Teichmüller axis of a generic pseudo-Anosov element is contained in the principal stratum of quadratic differentials. This gives a positive answer to the (first part of) question posed by J. Maher [15, Question 6.4].

With those generic properties at hand, we are able to count conjugacy classes of contracting elements in any SCC action. The growth formulae is the same as that of prime numbers in integers, so we refer to as *prime conjugacy growth formulae* below.

Theorem 10. *[22, Main Theorem] Suppose that a non-elementary group G admits a SCC action on a geodesic metric space (X, d) with a contracting element and $\omega(G) < \infty$. There exists a constant $D = D(o) > 0$ such that the following hold:*

$$\mathcal{C}(o, n) \asymp_D \frac{\exp(\omega(G)n)}{n}.$$

We also count the number of conjugacy classes using *stable length* defined by

$$\tau([g]) := \lim_{n\to\infty} \frac{d(o, g^n o)}{n}$$

which does not depend on the basepoint. Even though the number of conjugacy classes with bounded stable length could be infinite, we can obtain a satisfactory formula with stable length in many interesting examples. One of them is the following.

Theorem 11. *[22, Corollary 1.4] Let G be a relatively hyperbolic group. Then for the action on the Cayley graph, the prime conjugacy growth formulae holds for conjugacy classes of hyperbolic elements with stable length.*

Conjugacy growth series. Consider the counting with respect to the action on Cayley graph. An application of prime conjugacy growth formulae is to the formal *conjugacy growth series*:

$$\mathcal{P}(z) = \sum_{[g]\subset G} z^{\ell_1[g]} \in \mathbb{Z}[[z]],$$

whether it is rational, algebraic, or transcendental over $\mathbb{Q}(z)$. By computation, I. Rivin [42] computed the irrational conjugacy growth series for free group, and then conjectured that this series of a hyperbolic group is rational if and only if it is virtually cyclic. This is in contrast with the well-known result that, if counting $N(1, n)$ instead of $\mathcal{C}(1, n)$, the growth series $\sum_{g\in G} z^{d(1,g)}$ is rational for any hyperbolic group.

The Rivin's conjecture in hyperbolic groups is confirmed by Antolín-Ciobanu [2]. By Theorem 10, the Rivin's conjecture holds for a large subclass of acylindircally hyperbolic groups with a contracting element in word metric. In particular, this includes relatively hyperbolic groups, and non-direct product right-angled Artin groups.

Theorem 12. *[22, Theorem 1.8] The conjugacy growth series of a non-elementary relatively hyperbolic group is transcendental for every finite generating set.*

References

[1] A. Ancona, *Positive harmonic functions and hyperbolicity*, In Potential Theory - Surveys and Problems (Prague, 1987), Lecture Notes in Math. Springer, Berlin., 1988. (Cited on page 6.)

[2] Y. Antolín and L. Ciobanu, *Formal conjugacy growth in acylindrically hyperbolic groups*, Int Math Res Notices **1** (2017), 121–157, arXiv:1508.06229. (Cited on page 11.)

[3] G. Arzhantseva, C. Cashen, and J. Tao, *Growth tight actions*, Pacific Journal of Mathematics **278** (2015), 1–49. (Cited on pages 7 and 9.)

[4] J. Athreya, A. Bufetov, A. Eskin, and M. Mirzakhani, *Lattice point asymptotics and volume growth on Teichmüller space*, Duke Math. J. **161** (2012), no. 6, 1055–1111. (Cited on page 9.)

[5] M. Bestvina, K. Bromberg, and K. Fujiwara, *Constructing group actions on quasi-trees and applications to mapping class groups*, Publications mathématiques de l'IHÉS **122** (2015), no. 1, 1–64, arXiv:1006.1939. (Cited on page 10.)

[6] M. Bestvina and K. Fujiwara, *A characterization of higher rank symmetric spaces via bounded cohomology*, Geometric and Functional Analysis **19** (2009), no. 1, 11–40, arXiv:math/0702274. (Cited on page 7.)

[7] C. Bishop and P. Jones, *Hausdorff dimension and Kleinian groups*, Acta Math. **3** (1995), no. 179, 1–39. (Cited on pages 4 and 5.)

[8] B. Bowditch, *Cut points and canonical splittings of hyperbolic groups*, Acta Math. **180** (1998), 145–186. (Cited on page 2.)

[9] ———, *Relatively hyperbolic groups*, Int. J. Algebra Comput. (2012), no. 22, p1250016. (Cited on page 2.)

[10] M. Coornaert, *Mesures de Patterson-Sullivan sure le bord d'un espace hyperbolique au sens de Gromov*, Pac. J. Math. (1993), no. 2, 241–270. (Cited on pages 3 and 4.)

[11] F. Dahmani, D. Futer, and D. Wise, *Growth of quasiconvex subgroups*, Math. Proc. Cambridge Philos. Soc **167** (2019), no. 3, 505–530, arXiv:1602.08085. (Cited on pages 9 and 10.)

[12] F. Dahmani, V. Guirardel, and D. Osin, *Hyperbolically embedded subgroups and rotating families in groups acting on hyperbolic spaces*, Memoirs of the American Mathematical Society **245** (2014), arXiv:1111.7048. (Cited on page 4.)

[13] F. Dal'bo, P. Otal, and M. Peigné, *Séries de Poincaré des groupes géométriquement finis*, Israel Journal of Math. **118** (2000), no. 3, 109–124. (Cited on pages 3 and 7.)

[14] F. Dal'bo, M. Peigné, J.C. Picaud, and A. Sambusetti, *On the growth of quotients of Kleinian groups*, Ergodic Theory and Dynamical Systems **31** (2011), no. 3, 835–851. (Cited on pages 1, 4, and 9.)

[15] K. Delp, D. Hoffoss, and J. Manning, *Problems in groups, geometry and 3-manifolds*, arXiv:1512.04620, 2015. (Cited on pages 2 and 10.)

[16] M. Dussaule and I. Gekhtman, *Entropy and drift for word metrics in relatively hyperbolic groups*, to appear, Groups, Geometry and Dynamics, arXiv:1811.10849. (Cited on pages 4 and 6.)

[17] ———, *Stability phenomena for martin boundaries of relatively hyperbolic groups*, arXiv:1909.01577. (Cited on pages 4 and 6.)

[18] A. Eskin, M. Mirzakhani, and K. Rafi, *Counting geodesics in a stratum*, Inventiones Mathematicae **215** (2019), 535–607, Arxiv: 1206.5574. (Cited on page 7.)

[19] W. Floyd, *Group completions and limit sets of Kleinian groups*, Inventiones Math. **57** (1980), 205–218. (Cited on pages 3 and 6.)

[20] D. Gaboriau and F. Paulin, *Sur les immeubles hyperboliques*, Geom. Dedi. (2001), 153–197. (Cited on page 5.)

[21] I. Gekhtman, S. Taylor, and G. Tiozzo, *Counting loxodromics in graph products and relatively hyperbolic groups*, to appear, Israel Journal of Mathematics, arXiv:1711.04177. (Cited on pages 4 and 9.)

[22] I. Gekhtman and W.Y. Yang, *Counting conjugacy classes in groups with contracting elements*, arXiv: 1812.0625, 2018. (Cited on pages 1, 6, 10, and 11.)

[23] V. Gerasimov, *Floyd maps for relatively hyperbolic groups*, Geom. Funct. Anal. (2012), no. 22, 1361 – 1399. (Cited on pages 2 and 6.)

[24] V. Gerasimov and L. Potyagailo, *Quasi-isometries and Floyd boundaries of relatively hyperbolic groups*, J. Eur. Math. Soc. **15** (2013), 2115 – 2137. (Cited on page 7.)

[25] ______, *Quasiconvexity in the relatively hyperbolic groups*, Journal für die reine und angewandte Mathematik (Crelle Journal) (2015), arXiv:1103.1211. (Cited on page 7.)

[26] R. Grigorchuk and P. de la Harpe., *On problems related to growth, entropy and spectrum in group theory*, J. Dyn. Control Syst. **3** (1997), no. 1, 51 – 89. (Cited on pages 3 and 9.)

[27] S.Z. Han and W.Y. Yang, *Generic free subgroups and statistical hyperbolicity*, Preprint, Arxiv: 1812.06265, 2018. (Cited on pages 2 and 6.)

[28] G. Hruska, *Relative hyperbolicity and relative quasiconvexity for countable groups*, Algebr. Geom. Topol. **10** (2010), 1807–1856. (Cited on page 2.)

[29] L. Potyagailo I. Gekhtman, V. Gerasimov and W.Y. Yang, *Martin boundary covers floyd boundary*, Preprint, Arxiv: 1708.02133, 2017. (Cited on pages 1, 5, and 6.)

[30] A. Karlsson, *Free subgroups of groups with non-trivial Floyd boundary*, Comm. Algebra. **31** (2003), 5361–5376. (Cited on pages 3 and 6.)

[31] G. Knieper, *On the asymptotic geometry of nonpositively curved manifolds*, GAFA **7** (1997), 755–782. (Cited on page 9.)

[32] V. Gerasimov M. Dussaule, I. Gekhtman and L. Potyagailo, *Martin boundary of relatively hyperbolic groups with virtually abelian parabolic subgroups*, arXiv:1602.03767. (Cited on page 6.)

[33] J. Maher, *Asymptotics for pseudo-Anosov elements in Teichmüller lattices*, Geom. Funct. Anal. **20** (2010), 527–544. (Cited on page 8.)

[34] Y. Minsky, *Quasi-projections in Teichmüller space*, J. Reine Angew. Math. **473** (1996), 121–136. (Cited on page 7.)

[35] P. Ney and F. Spitzer, *The martin boundary for random walks*, Trans. Amer. Math. Soc. **349** (1997), no. 5, 1851–1875. (Cited on page 6.)

[36] J. Osborne and W. Yang, *Statistical hyperbolicity of relatively hyperbolic groups*, Algebr. Geom. Topol. **16** (2016), 2143–2158, arXiv:1504.01011. (Cited on page 2.)

[37] D. Osin, *Acylindrically hyperbolic groups*, Trans. Amer. Math. Soc. **368** (2016), no. 2, 851–888. (Cited on page 7.)

[38] S. Patterson, *The limit set of a Fuchsian group*, Acta Mathematica (1976), no. 1, 241–273. (Cited on page 4.)

[39] L. Potyagailo and W. Yang, *Hausdorff dimension of boundaries of relatively hyperbolic groups*, Geometry & Topology **23** (2019), 1779–1840. (Cited on pages 1, 2, 4, and 5.)

[40] Kasra Rafi and Yvon Verberne, *Geodesics in the mapping class group*, arXiv:1810.12489, 2018. (Cited on page 8.)

[41] R.Coulon, R. Dougall, S. Tapie, and B. Schapira, *Twisted Patterson-Sullivan measures and applications to amenability and coverings*, arXiv:1809.10881. (Cited on page 8.)

[42] I. Rivin, *Some properties of the conjugacy class growth function*, Contemp. Math. **360** (2004), Amer. Math. Soc., Providence, RI,. (Cited on page 11.)

[43] T. Roblin, *Ergodicité et équidistribution en courbure négative*, no. 95, Mémoires de la SMF, 2003. (Cited on page 9.)

[44] B. Schapira and S. Tapie, *Regularity of entropy, geodesic currents and entropy at infinity*, To appear in Annales scientifiques de l'ENS, arXiv:1802.04991. (Cited on page 8.)

[45] D. Sullivan, *Entropy, hausdorff measures old and new, and limit sets of geometrically finite Kleinian groups*, Acta Mathematica (1984), 259–277. (Cited on page 4.)

[46] W. Yang, *Genericity of contracting elements in groups*, to appear in Mathematische Annalen, arXiv:1707.06006. (Cited on pages 1, 6, and 8.)

[47] ———, *Patterson-Sullivan measures and growth of relatively hyperbolic groups*, Preprint, arXiv:1308.6326, 2013. (Cited on pages 1, 2, and 3.)

[48] ———, *Growth tightness for groups with contracting elements*, Math. Proc. Cambridge Philos. Soc **157** (2014), 297 – 319. (Cited on pages 1 and 4.)

[49] ———, *Statistically convex-cocompact actions of groups with contracting elements*, Int Math Res Notices **23** (2019), 7259–7323, arXiv:1612.03648. (Cited on pages 1, 6, and 9.)