EXERCISE SHEET #10

Surface topology. A closed curve is called *essential* if it is non-identity in the fundamental group. An essential simple closed curve (SCC) α on Σ_g for $g \ge 1$ is called *non-separating* if $\Sigma_g \setminus \alpha$ consists of only one component. Otherwise, it is called *separating*.

Exercise 0.1. *Let α and β be any two non-separating essential SCCs on Σ_g for $g \ge 1$. Prove that up to homeomorphism there exists only one topological type of non-separating SCCs: there exists a homeomorphism ϕ of Σ_g such that $\phi(\alpha) = \beta$.*

Prove that there are finitely many topological types of separating essential SCCs on Σ_g up to homeomorphism. Classify all topological types of separating essential SCCs.

Hyperbolic Surface. A *geodesic triangulation* of a hyperbolic surface is a decomposition of the surface as the union of finitely many geodesic triangles such that every two triangles are either disjoint, or share a vertex or only one edge.

Exercise 0.2. *Prove that every closed orientable hyperbolic surface admits a geodesic triangulation. Tips:*

1. *Let X be any finite set of points in $\mathbb{H}^2$. For given $p \in X$, let $D(p)$ be the set of points in $\mathbb{H}^2$ which is closer to p than any point of X. Prove that D_p is a convex geodesic hyperbolic polygon. (Compare with Dirichlet construction)*
2. *Cover the hyperbolic surface by finitely many metric disks which are isometric to disks in $\mathbb{H}^2$. First partition into polygons and then subdivide into triangles.*

Exercise 0.3. *The area of any closed orientable hyperbolic surface Σ_g is $-2\pi\chi(\Sigma_g)$, where $\chi(\Sigma_g)$ is the Euler characteristic of Σ_g.*

Exercise 0.4. *Prove that there are uncountably many closed orientable hyperbolic surfaces so that no two of them are isometric.*

Cayley graph. Let G be a group generated by a finite set of elements S. The **Cayley graph** $\mathcal{G}(G,S)$ of G with respect to S is a graph with the vertex set G such that two vertices $g_1, g_2 \in G$ are connected by one edge if and only if $g_1 = g_2 s$ for some $s \in S$.

By assigning each edge with unit length, we obtain a **word metric** d_S between $g_1, g_2 \in G$ on $\mathcal{G}(G,S)$ so that $d_S(g_1, g_2)$ is the minimal length of connected paths from g_1 to g_2.

Exercise 0.5 (Changing generating set)**.** *Prove that*

1. *the Cayley graph $\mathcal{G}(G,S)$ is a connect graph.*
2. *the identification $(G, d_S) \to (G, d_T)$ is a quasi-isometry for two generating sets S, T: there exists $K > 0$ such that for any $g_1, g_2 \in G$,*

$$\frac{1}{K} d_S(g_1, g_2) \le d_T(g_1, g_2) \le K d_S(g_1, g_2)$$

Remark. Let G be discrete subgroup in $I(\mathbb{H}^n)$ with Dirichlet domain $D(o)$. Let S be the set of side pairing for D. Then the Cayley graph of G with respect to S can be identified with the **dual graph** for the tessellation $\mathbb{H}^n = \cup_{g\in G} g\bar{D}$ with vertex set Go where two points $go, g'o$ are connected by an edge iff $gD, g'D$ shares a common side.

Exercise 0.6 (Milnor-Svarc Lemma)**.** *Let G be a discrete subgroup of $I(\mathbb{H}^n)$ so that $\mathbb{H}^n/G$ is compact. Fix a basepoint $o \in \mathbb{H}^n$. Consider the orbital map $\phi : G \to \mathbb{H}^n$:*

$$\phi : g \in G \mapsto go \in \mathbb{H}^n$$

Prove that

(1) *there exists a constant $R > 0$ such that $N_R(\phi(G)) = \mathbb{H}^n$.*
(2) *the set $S := \{g \in G : \rho(o, go) \le 2R + 1\}$ generates the group G.*
(3) *the map $\phi : (G, d_S) \to \mathbb{H}^n$ is a quasi-isometric embedding: there exist constants $\lambda > 1, c > 0$ such that*

$$\frac{1}{\lambda} d_S(g_1, g_2) - c \le \rho(\phi(g_1), \phi(g_2)) \le \lambda d_S(g_1, g_2) + c$$

Tips for (3): connect g_1o, g_2o be a geodesic segment and subdivide into segments of length 1, then apply (1).