EXERCISE SHEET #11

Quasi-isometry. Recall that a (not-necessarily continuous) map $\phi : (X, d) \to (Y, d)$ between two metric spaces is called (λ, c)-*quasi-isometric embedding* if for any two $x_1, x_2 \in X$, we have

$$\frac{1}{\lambda} d(x_1, x_2) - c \leq d(\phi(x_1), \phi(x_2)) \leq \lambda d(x_1, x_2) + c.$$

If, in addition, there exists $R > 0$ such that $Y \subset N_R(\phi(X))$, then it is called (λ, c)-*quasi-isometry.*

A (λ', c')-quasi-isometric embedding $\psi : Y \to X$ is called a *quasi-inverse* of ϕ if there exists $D = D(\lambda, c, \lambda', c') > 0$ such that for any $x \in X, y \in Y$

$$d(x, \psi(\phi(x)) \leq D, \ d(y, \phi(\psi(y))) \leq D.$$

Exercise 0.1 (Quasi-inverse always exists)**.** *A (λ, c)-quasi-isometry $\phi : X \to Y$ admits a quasi-inverse.*

Quasi-geodesics. Recall that a path p in a metric space (X, d) is called a (λ, c)-*quasi-geodesic* for some $\lambda, c > 0$ if for any rectifiable subpath q, we have

$$Len(q) \leq \lambda \cdot d(q_-, q_+) + c$$

where q_-, q_+ are the endpoints of q. A (λ, c)-quasi-isometric embedding $\phi : (I, |\cdot|) \to (X, d)$ of an interval $I \subset \mathbb{R}$ into X is also called a (λ, c)-quasi-geodesic. In practice, we can assume that ϕ is continuous up to a uniform finite neighborhood.

Exercise 0.2. *Let $\phi : (I, |\cdot|) \to (X, d)$ be a (λ, c)-quasi-isometric embedding. Show that there exist constants $\lambda', c', D > 0$ depending only on λ, c and a (λ', c')-quasi-geodesic p such that $p \subset N_D(\phi(I))$ and $\phi(I) \subset N_D(p)$.*

Application of Milnor-Svarc Lemma.

Exercise 0.3. *Prove that any finite index subgroup of a finitely generated group is finitely generated.*

Exercise 0.4. *Let $1 \to F \to G \to \Gamma \to 1$ be an exact sequence, where F is finite and G is finitely generated. Prove that G is quasi-isometric to Γ (equipped with any finite generating sets).*

Growth function. Let G be a group generated by a finite set S. The *growth function* $\phi : n \in \mathbb{N} \to \mathbb{N}$ of (G, S) defined by $\phi(n) := \sharp\{g \in G : d_S(1, g) \leq n\}$ counts the number of group elements in a ball of radius n with respect to word metric d_S.

Exercise 0.5. *Prove that the growth function of a free abelian group $\mathbb{Z}^n$ of rank n is a polynomial of degree n with respect to any finite generating set. Conclude that if $\mathbb{Z}^n$ acts properly discontinuously and co-compactly on $\mathbb{R}^m$, then $m = n$.*

Remark. A famous theorem of Gromov says that if the growth function of a finitely generated group is polynomial, then it must be virtually nilpotent. (the converse is also true and much easier by Wolf.)

Exercise 0.6. *Prove that there exists no quasi-isometry between a free group $\mathbb{F}_n$ of rank $n \geq 2$ and a free abelian group $\mathbb{Z}^m$ for $m \geq 1$.*