EXERCISE SHEET #2

Exercise 0.1 (B. Ex. 4.1.1)**.** *Let* $\phi(z) = \frac{az+b}{cz+d} \in PSL(2,\mathbb{C})$ *where* $a, b, c, d \in \mathbb{C}, ad - bc = 1$. *Define* $\phi(-\dfrac{d}{c}) := \infty, \phi(\infty) := \dfrac{a}{c}$ *if* $c \neq 0$*, otherwise* $\phi(\infty) := \infty$. *Equip* $\hat{\mathbb{C}}$ *with chordal metric. Prove* $\phi : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ *is a homeomorphism.*

Exercise 0.2. *Let* $\{u_0, u_1, \cdots, u_n\}$ *be an affinely independent set of* $(n+1)$ *unit vectors in* $\mathbb{R}^n$*. Prove that if a Mobius transformation* $\phi \in M(\mathbb{B}^n)$ *fixes each* u_i *for* $0 \leq i \leq n$ *then* ϕ *is the identity.*

Let γ be a geodesic in $(\mathbb{H}^n, \rho)$ for $n \geq 2$. Let $\pi_\gamma : \mathbb{H}^n \to \gamma$ be the *nearest point projection map*: for any $x \in \mathbb{H}^n$,

$$\rho(x, \pi_\gamma(x)) = \inf_{y \in \gamma} \rho(x, y).$$

Exercise 0.3 (Strong contraction)**.** *Show that for any hyperbolic ball* B *in* $(\mathbb{H}^n, \rho)$ *with* $B \cap \gamma = \emptyset$*, we have*

$$diam(\pi_\gamma(B)) \leq 2 \log(1 + \sqrt{2})$$

where diam is taken with respect to ρ*.*

Let $(\mathbb{B}^n, \rho)$ be the ball model of n-dimensional hyperbolic space, and $\mathbb{S}^{n-1}$ be the $(n-1)$-sphere equipped with Euclidean metric $|x - y|$. As a set, $\mathbb{S}^{n-1}$ can be identified with the set of geodesic rays from the origin o.

For any three points $o, x, y \in \mathbb{B}^n$, we define the Gromov product

$$\langle x, y \rangle_o := \frac{\rho(x, o) + \rho(y, o) - \rho(x, y)}{2}$$

The metric $| \cdot |$ on $\mathbb{S}^{n-1}$ can be recovered from the hyperbolic geometry.

Exercise 0.4 (Visual metric)**.** *Let* $\alpha, \beta : [0, \infty) \to (\mathbb{B}^n, \rho)$ *be two distinct geodesic rays (i.e.: isometric embedding) from* o *ending at two points* $p, q \in \mathbb{S}^{n-1}$ *respectively. Define*

$$\delta(\alpha, \beta) := \lim_{t \to \infty} e^{-\langle \alpha(t), \beta(t) \rangle_o}.$$

Prove that $2\delta(\alpha, \beta) = |p - q|$*. The so-defined function* δ *is called visual metric.*

(Tips: use Hyperbolic Cosine Rule (in Beardon's book))