EXERCISE SHEET #3

Exercise 0.1 (Thin triangle property)**.** *There exists a universal constant $\delta > 0$ (independent of n) such that any geodesic triangle in $(\mathbb{H}^n, \rho)$ is δ-thin: any side is contained in the δ-neighborhood of the other two sides. In particular, each side contains a point which has a distance at most δ to each other side.*

For any three points $o, x, y \in (\mathbb{H}^n, \rho)$, we define the Gromov product

$$\langle x, y \rangle_o := \frac{\rho(x, o) + \rho(y, o) - \rho(x, y)}{2}$$

Exercise 0.2 (Gromov product)**.** *Prove that*

$$\rho(o, [x, y]) - 2\delta \leq \langle x, y \rangle_o \leq \rho(o, [x, y])$$

where $[x, y]$ denotes the geodesic with endpoints x, y.

A path $p : [0, 1] \to (\mathbb{H}^n, \rho)$ is called *rectifiable* if its length defined as follows is finite:

$$Len(p) := \sup\{\sum_{i=0}^{n} \rho(p(t_i), p(t_{i+1})\}$$

over all finite subdivision $t_0 = 0 < t_1 < t_2 \cdots < t_n = 1$ of $[0, 1]$.

Exercise 0.3. *Let α be a geodesic segment and p be a rectifiable path with length $s \geq 2$ with the same endpoints as α. Then any point $z \in \alpha$ has a distance at most $(1 + \delta \log_2 s)$ to p:*

$$\rho(z, p) \leq 1 + \delta \log_2 s$$

where $\delta > 0$ is the thin constant from Exercise 0.1. (Tips: subdivide inductively the path p by half and then apply δ-thin triangle property.)

Corollary 0.4 (Exponential perimeter)**.** *If p is disjoint with a (hyperbolic) ball centered at the middle point in α with radius $R = \dfrac{Len(\alpha)}{2} - 1$, then*

$$s \geq 2^{\delta^{-1}(R-2)}.$$

A rectifiable path p in $(\mathbb{H}^n, \rho)$ is called (λ, c)-*quasi-geodesic* for some $\lambda, c > 0$ if for any subpath q of p, we have

$$Len(q) \leq \lambda \rho(q_-, q_+) + c$$

where q_-, q_+ are endpoints of q.

Exercise 0.5 (Morse Lemma)**.** *For any $\lambda, c > 0$, there exists a constant $D = D(\lambda, c) > 0$ such that the following property holds.*

Let p be a (λ, c)-quasi-geodesic and α be the geodesic with the same endpoints as p. Then

(1) *Use Corollary 0.4 to deduce that $\alpha \subset N_D(p)$.*
(2) *Furthermore, for any point $w \in p$, we have $d(w, \alpha) \leq 2D\lambda + c$. (Tips: use the connectivity of $\alpha \subset N_D(p)$).*