EXERCISE SHEET #4

Profinite topology. We define a topology called *profinite topology* on a group G using finite index subgroups. The relevant facts (exercise) are as follows:

1. the intersection of two finite index subgroups is of finite index.
2. any finite index subgroup contains a normal finite index subgroup.

Exercise 0.1 (Profinite topology)**.** *Let $\mathcal{O}$ be the collection of all finite index subgroups and their left and right cosets.*

1. *Verify that $\mathcal{O} \cup \{\emptyset\}$ is an open base for a topology called profinite topology.*
2. *The group G with the profinite topology is a topological group.*
3. *Show that a subgroup H is a closed subset in the profinite topology if and only if it is the intersection of all finite index subgroups containing H.*
4. *Show that the trivial subgroup $\{1\}$ is closed if and only if for any $1 \neq g \in G$, there exists a homomorphism π of G onto a finite group F such that $\pi(g) \neq 1 \in F$.*

Remark. In literature, a group G is called *residually finite* if $\{1\}$ is closed; equivalently if the profinite topology is Hausdorff. Closed subgroups are also called *separable* which are important in studying 3 dim. topology.

Group actions. A *group action* of G on a set X is a map

$$\cdot : G \times X \to X$$

by

$$(g, x) \mapsto g \cdot x$$

such that if for any $g, h \in G$ and $x \in X$, we have

1. $1 \cdot x = x$ for any $x \in X$
2. $g \cdot (h \cdot x) = (gh) \cdot x$.

For any $g \in G$, define a map $\phi_g : X \to X$ by $\phi_g(x) := g \cdot x$.

Exercise 0.2 (Group action on sets)**.** *Show that $g \mapsto \phi_g$ is a homomorphism of G into the group of all bijections of X.*

Remark. We say that G acts by homeomorphism on a topological space X if the above map $\phi_g : X \to X$ is a homeomorphism for each $g \in G$.

A continuous map $f : X \to Y$ is called *proper* if for every compact subset $K \subset Y$, $f^{-1}(K)$ is compact. The relevant facts (exercise) about product topology:

1. The projection map $X \times Y \to X$ is continuous, so sends compact sets to compact sets.
2. Any compact set in a Hausdorff topological space X is closed, any closed subset in a compact set is compact.
3. Any compact set C in $X \times Y$ is contained in a product of compact sets $K \times L$ where $K \subset X, L \subset Y$ are compact.

Exercise 0.3 (p.d.c action)**.** *Let G act by homeomorphism on a topological Hausdorff space X. We equip G with discrete topology (i.e. every subset is open). Define a continuous map $\Pi : G \times X \to X \times X$ by*

$$(g, x) \mapsto (g \cdot x, x)$$

where the product of spaces are equipped with product topology.

Show that Π is a proper map if and only if G acts properly discontinuously on X: for any compact set $K \subset X$, the set $\sharp\{g \in G : K \cap g \cdot K \neq \emptyset\}$ is finite.

Compact open topology. Denote by $\mathcal{C}(X, Y)$ the set of all continuous maps from a topological space X to a metric space (Y, d). For $K \subset X, U \subset Y$, define

$$V(K, U) := \{f \in \mathcal{C}(X, Y) : f(K) \subset U\}.$$

For $\epsilon > 0$ and $f \in \mathcal{C}(X, Y)$, define

$$D_K(f, \epsilon) := \{g \in C(X, Y) : \sup_{x \in K} d(f(x), g(x)) < \epsilon\}.$$

(1) The *compact-open topology* on $\mathcal{C}(X, Y)$ is generated by the open sub-base $\{V(K, U) : \text{compact } K \subset X, \textit{open } U \subset Y\}$.
(2) The *uniform convergence topology* on $\mathcal{C}(X, Y)$ is generated by the open sub-base $\{D_K(f, \epsilon) : \text{compact } K \subset X, \epsilon > 0, f \in C(X, Y)\}$.

Exercise 0.4. *Prove that the compact-open topology coincides with the uniform convergence topology on $\mathcal{C}(X, Y)$.*

Ascoli-Arzela Lemma. Let $S \subset \mathcal{C}(X, Y)$ be a set of continuous maps from a compact metric space (X, d) to a complete metric space (Y, d). We say that S is

(1) *equicontinuous* if for any $\epsilon > 0$ there exists $\delta > 0$ such that whenever $d(x, x') \leq \delta$ we have $d(f(x), f(x')) \leq \epsilon$ for all $f \in S$.
(2) *bounded* if for any $x \in X$, the set $\{f(x) : f \in S\}$ in Y is *precompact*: its closure in Y is compact.

Exercise 0.5. *Assume that S is equicontinuous and bounded. Prove that the closure of S in $\mathcal{C}(X, Y)$ is compact with the uniform convergence topology.*