EXERCISE SHEET #5

Exercise 0.1 (Indiscrete groups)**.** *Let ϕ, ψ be two isometries of $\mathbb{H}^n$ one of which is loxodromic. If ϕ, ψ share one and only one fixed point, then the group $\langle \phi, \psi \rangle$ is not discrete in $Isom(\mathbb{H}^n)$.*

Limit set. Suppose that G acts properly discontinuously on $\mathbb{H}^n$ for $n \geq 2$. The limit set ΛG of G is the set of accumulation points in $\partial_\infty \mathbb{H}^n := S^{n-1}$ of any G-orbit in $\mathbb{H}^n$.

Exercise 0.2.

(1) *If H is a finite index subgroup in G, then $\Lambda H = \Lambda G$.*

(2) *Let H be an infinite normal subgroup in G. Assume that $|\Lambda G| \geq 3$. Then $\Lambda H = \Lambda G$. Give an example to show that the infinity of H is necessary.*

(3) *Assume that $|\Lambda G| \geq 3$. Prove that G contains infinitely many loxodromic elements with pairwise disjoint fixed points.*

Discontinuity domain. The complement $\Omega G := S^{n-1} \setminus \Lambda G$ of the limit set is called the discontinuity domain. Assume that ΩG is non-empty.

Exercise 0.3.

(1) *Prove that for any $p \in \Omega G$, the set of accumulation points of the orbit $G \cdot p$ coincides with ΛG.*

(2) *Prove that ΩG is open and dense in S^{n-1}, and ΛG is nowhere dense (i.e. the interior of Λ is empty).*

(3) *Prove that ΩG is the maximal discontinuity domain on S^{n-1}: let U be an open subset of S^{n-1} on which G acts properly discontinuously. Then $U \subset \Omega G$.*