EXERCISE SHEET #9

Conical points. Let G be a discrete subgroup in $I(\mathbb{H}^n)$. A limit point $p \in \Lambda G$ is called *conical* if there exist a sequence of distinct elements $g_n \in G$ and a point $x \in \mathbb{H}^n$ and a geodesic ray γ ending at p such that

$$\sup_{n\geq 1}\{\rho(g_n x, \gamma)\} \leq R < \infty.$$

Exercise 0.1. *Let p be a conical limit point.*

1. *Prove that there exist a sequence of distinct elements $g_n \in G$ such that for any point $x \in \mathbb{H}^n$ and any geodesic ray γ at p we have*

$$\sup_{n\geq 1}\{\rho(g_n x, \gamma)\} < R < \infty$$

   where R depends on the choice of x and γ.

2. *Prove that there exist a point $q \in S^{n-1} \setminus p$ and a sequence of distinct elements $g_n \in G$ such that*

$$g_n^{-1}|_{S^{n-1}\setminus p} \to q \text{ locally uniformly.}$$

   Conclude that g_n are loxodromic elements for sufficiently large n.

Exercise 0.2. *Prove that a conical limit point p can not be fixed by a parabolic element h. You could proceed by applying Exercise 0.1 with the following setup:*

1. *assume that $p = \infty$ and γ is the geodesic ray issuing from $o := x_n e_n$ and ending at ∞.*
2. *and then prove the set $\{g_n^{-1} h g_n(o)\}$ is finite.*

Margulis Lemma. Let M be a metric space (cf. §9.6 in [R]):

1. A *geodesic line* in M is a locally distance preserving map $\gamma : \mathbb{R} \to M$: for any $t \in \mathbb{R}$ there exists $\delta > 0$ such that $\gamma : [t - \delta, t + \delta] \to M$ is distance preserving.
2. It is called *periodic* if there exists a *period* $p > 0$ such that $\gamma(t + p) = \gamma(t)$ for any $t \in \mathbb{R}$.
3. The image of a periodic geodesic line in M is called a *closed geodesic* and the (unique) smallest period is the *length* of the closed geodesic.
4. A closed geodesic is *simple* if $\gamma : [t, t + p) \to M$ is injective for any $t \in \mathbb{R}$.

Exercise 0.3 (Simple closed geodesics). *Let $M = \mathbb{H}^n/G$ for a discrete torsion-free subgroup G of $I(\mathbb{H}^n)$. Every loxodromic element g in G gives rise to a closed geodesic $[g]$ in M by projecting the axis $Ax(g)$ to M.*

Let $\epsilon_n > 0$ be the Margulis constant. Prove that

1. *If the length of $[g]$ is smaller than $\epsilon_n/2$, then $[g]$ is simple.*
2. *If two distinct closed geodesics $[g]$ and $[h]$ have length less than $\epsilon_n/2$, then $[g]$ and $[h]$ are disjoint.*

Exercise 0.4 (Thin part for parabolic type). *Let G be an elementary discrete subgroup of parabolic type in $I(\mathbb{H}^n)$ with fixed point p. Prove that for any $r > 0$, there exists a horoball B_r centered at p such that $V(G, r) \subset B_r$.*