SUBGROUPS OF SURFACE GROUPS ARE ALMOST GEOMETRIC

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Let F be a surface and let α be an element of $\pi_1(F)$. We will say that α is *geometric* if it can be represented by an embedded loop in F. One knows that, in general, not every element of $\pi_1(F)$ is geometric. For example, if F is $S^1 \times I$ so that $\pi_1(F)$ is infinite cyclic, there are only three geometric elements of $\pi_1(F)$ – the identity and the two generating elements. In this example, it is obvious that if we consider a non-geometric element α of $\pi_1(F)$, then α is geometric in a suitable finite covering space of F. We will prove that this is the case for any surface F.

A compact subsurface X of F is said to be *incompressible* if no component of the closure of $F-X$ is a 2-disc whose boundary is contained in ∂X. If X is incompressible, then the natural map $\pi_1(X) \to \pi_1(F)$ is injective, so that we can think of $\pi_1(X)$ as a subgroup of $\pi_1(F)$. Subgroups which arise in this way we call *geometric*. This generalizes the idea of geometric elements of $\pi_1(F)$, because an infinite cyclic subgroup of $\pi_1(F)$ is geometric if and only if one of its generators is geometric. We will prove that any finitely generated (f.g.) subgroup of $\pi_1(F)$ is geometric in some finite covering space of F. The main result of this paper is the following stronger result.

THEOREM 3.3. *Let F be a surface, let S be a* f.g. *subgroup of $\pi_1(F)$ and let $g \in \pi_1(F) - S$. Then there is a finite covering F_1 of F such that $\pi_1(F_1)$ contains S but not g and S is geometric in F_1.*

Our proof uses the geometry of the hyperbolic plane and simple facts about groups generated by reflections. This result also yields new group theoretic results. A group G is called *locally extended residually finite* (LERF) if given a f.g. subgroup S of G and $g \in G - S$, then G has a subgroup G_1 of finite index which contains S but not g. It follows at once from Theorem 3.3 that surface groups are LERF. In fact these two results are equivalent, which we prove in §1. In particular, we have a new proof that surface groups are residually finite. One can go further and deduce that all Fuchsian groups are LERF, because any Fuchsian group is a finite extension of a surface group (**[2]**, **[4]**, **[11]**), and finite extensions of LERF groups are also LERF.

Not all of these residual finiteness results are new. The fundamental group of a non-closed surface is free, and the result that free groups are LERF was first proved by Hall **[5]**. See also **[3]**, **[8]** and Lemma 15.22 of **[6]**. Hall proves a stronger result than this and we also obtain this stronger result. (See Theorem 2.2) In the case of closed surfaces it was not previously known that their fundamental groups were LERF. It was, of course, known that surface groups were residually finite (**[1]**, **[7]**), and it follows from results of Stebe **[12]** that surface groups satisfy a residual finiteness condition called Π_c. This also follows from the fact that they are LERF.

The results in this paper were suggested by the author's work on ends of pairs of groups **[9]**. The fact that surface groups are LERF gives a good source of examples.

When I first set out to prove the results in this paper, I had in mind a proof exactly as presented here, but I lacked one ingredient. Thus the original draft of this paper

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contained a rather crude and complicated proof. I am very grateful to William Thurston for pointing out to me the crucial missing fact—namely that in the hyperbolic plane, one can find a regular pentagon with all angles equal to $\pi/2$.

In §1 of this paper, we give the definitions of various residual finiteness conditions and we give results which connect these conditions with topology. In §2, we prove Theorem 3.3 for the special case of non-closed surfaces. In §3, we deal with the closed case. In §4, we discuss the extension of these results to dimension three.

1. *Preliminaries*

We start by defining the residual finiteness conditions in which we are interested.

A group G is said to be residually finite (RF) if for any non-trivial element g of G, there is a subgroup G_1 of finite index in G which does not contain g. A group G with a subgroup S is *S-residually finite* (S-RF) if for any element g of $G-S$, there is a subgroup G_1 of finite index in G which contains S but not g. A group G is called *extended residually finite* (ERF) if G is $S-$RF for every subgroup S of G, and G is called *locally extended residually finite* (LERF) if G is $S-$RF for every finitely generated subgroup S of G.

Each of these conditions on G can also be defined by considering finite quotient groups of G. A group G is RF if for any non-trivial element g of G, there is a homomorphism ϕ of G to a finite group with $\phi(g)$ non-trivial, and G is $S-$RF if for any element g of $G-S$, there is a homomorphism ϕ of G to a finite group with $\phi(g) \notin \phi(S)$.

We observe that if a group G is ERF, then every quotient of G is RF. Hence the existence of non-residually finite groups shows that free groups cannot be ERF. We will need the following results about these conditions.

LEMMA 1.1. *If G is* RF *or* ERF *or* LERF, *then any subgroup of G has the same property and so does any group K which contains G as a subgroup of finite index.*

Proof. The results for subgroups of G are obvious.

Let K be a group in which G has finite index. If G is not normal in K, we consider G_0, the intersection of the conjugates of G in K, Then G_0 is a subgroup of G, and G_0 is a normal subgroup of K of finite index. Hence, by replacing G by G_0, we can suppose that G is normal in K. Let F denote the finite quotient group and let $p: K \to F$ be the projection map. We now consider the three cases separately.

Suppose that G is RF and that we are given a non-trivial element k of K. If k lies in G, then G has a subgroup G_1 of finite index which does not contain k. Now G_1 is also of finite index in K, so that K has the required subgroup of finite index which does not contain k. If k does not lie in G, then G itself is the required subgroup of K. Hence K is RF.

Now suppose that G is ERF and that we are given a subgroup S of K and an element k of $K-S$. Then $S \cap G$ is a normal subgroup of S with quotient some subgroup F_1 of F. Let K_1 denote $p^{-1}(F_1)$. If k does not lie in K_1, then K_1 is the required subgroup of finite index in K, as K_1 contains S. If k does lie in K_1, we proceed as follows.

We can write $k = gs$, where $g \in G$ and $s \in S$. As k does not lie in S, we know that g does not lie in $S \cap G$. We use the fact that G is ERF to obtain a subgroup G_2 of finite index in G which contains $S \cap G$ but not g. Let G_3 denote the intersection of the conjugates of G_2 by the elements of S. Then G_3 is also a subgroup of G of finite index which contains $S \cap G$ but not g and G_3 is normalized by S. Let K_3 be the subgroup of K_1 generated by G_3 and S. Then G_3 is a normal subgroup of K_3 with quotient F_1. Clearly K_3 is of finite index in K and contains S. Also k cannot lie in K_3 as K_3 contains S but not g. Thus K_3 is the required subgroup of K and it follows that K is ERF.

If G is LERF, then the above proof shows that K must also be LERF. One needs only to observe that if S is a f.g. subgroup of K, then $S \cap G$ is also f.g. as it is of finite index in S. This completes the proof of Lemma 1.1.

In order to make the connection between residual finiteness conditions and topology, we use the following result.

LEMMA 1.2. *If X is a Hausdorff topological space with a regular covering $\tilde{X}$ and covering group G and if C is a compact set of $\tilde{X}$, then $\{g \in G : gC \cap C \neq \varnothing\}$ is finite.*

We will only apply this result and its corollaries to the case when X is a PL manifold and hence a simplicial complex. The result in this case is very easy to prove, so we do not give a proof of Lemma 1.2. We give this general statement in order to emphasize that X need not be a manifold.

We can now prove the following result.

LEMMA 1.3. *Let X be a Hausdorff topological space with a regular covering $\tilde{X}$ and covering group G. Then the following conditions are equivalent:*

(i) *G is* RF,

(ii) *If C is a compact set in $\tilde{X}$ then, G has a subgroup G_1 of finite index such that $gC \cap C$ is empty for every non-trivial element g of G_1,*

(iii) *If C is a compact set in $\tilde{X}$, then the projection map $\tilde{X} \to X$ factors through a finite covering X_1 of X such that C projects by a homeomorphism into X_1.*

Proof. The equivalence of (ii) and (iii) is obvious by taking $X_1 = \tilde{X}/G_1$.

Now suppose that condition (i) holds, and let C be a compact set in $\tilde{X}$. Lemma 1.2 tells us that the set $T = \{g \in G : gC \cap C \neq \varnothing\}$ is finite. For each non-trivial element t of T, we know that G has a subgroup G_t of finite index which does not contain t. The intersection of the groups G_t is a subgroup G_1 of G, which satisfies the conditions in (ii). Hence condition (ii) holds.

Conversely suppose condition (ii) holds and let g be a non-trivial element of G. Let x be a point in $\tilde{X}$ and apply conditions (ii) with $C = x \cup gx$. The subgroup G_1 of G obtained this way cannot contain g. Hence G is RF. We can connect the LERF condition with a topological condition in a similar way. We prove the following result.

LEMMA 1.4. *Let X be a Hausdorff topological space with a regular covering $\tilde{X}$ and covering group G. Then G is* LERF *if and only if given a* f.g. *subgroup S of G and a compact subset C of $\tilde{X}/S$, there is a finite covering X_1 of X such that the projection $\tilde{X}/S \to X$ factors through X_1 and C projects homeomorphically into X_1.*

Proof. Suppose that the geometric condition holds and that S is a f.g. subgroup of G and g is an element of $G - S$. Pick $x \in \tilde{X}$ and let C in $\tilde{X}/S$ be the image of $x \cup gx$.

The geometric condition provides us with a finite covering X_1 of X. Hence $X_1 = \tilde{X}/G_1$, where G_1 is of finite index in G. Clearly g does not lie in G_1 as $x \cup gx$ projects homeomorphically into X_1. Hence G is LERF.

Now suppose that G is LERF and that we are given a f.g. subgroup S of G and a compact subset C of $\tilde{X}/S$. Let p denote the projection map $\tilde{X} \to \tilde{X}/S$, and let Y denote $p^{-1}(C)$ in $\tilde{X}$. It is easy to show that Y has a compact subset D such that $p(D) = C$. Recall that $\{g \in G : gD \cap D \neq \emptyset\}$ is finite, by Lemma 1.2. As G is LERF we can find a subgroup G_1 of finite index in G such that G_1 contains S and, in addition, if g is an element of G_1 such that gD meets D then g lies in S. Then $X_1 = \tilde{X}/G_1$ is the required finite covering of X. This completes the proof of Lemma 1.4.

Finally, we need the following standard result about surfaces.

LEMMA 1.5. *Let F be a surface such that $\pi_1(F)$ is finitely generated and let C be a compact subset of F. Then there is a compact, connected, incompressible subsurface Y of F which contains C such that the natural map $\pi_1(Y) \to \pi_1(F)$ is an isomorphism.*

Proof. Choose a basepoint $*$ in F and a finite generating set for $\pi_1(F, *)$. For each generator of $\pi_1(F)$, choose a based map $S^1 \to F$. Let N be a regular neighbourhood of C and of the union of the images of these cycles. Then N is a compact subsurface of F. By adding 1-handles to N, we can arrange that N is connected. Now the natural map $\pi_1(N) \to \pi_1(F)$ is surjective by our construction of N. If $F - N$ has any component whose closure is a 2-disc, we enlarge N by adding all such components and let Y be the resulting subsurface of F. As Y contains N, we still have that the natural map $\pi_1(Y) \to \pi_1(F)$ is surjective and now van Kampen's Theorem implies that this map is injective. This completes the proof of Lemma 1.5.

If we put together the results contained in the preceding two lemmas, we immediately obtain the following result.

LEMMA 1.6. *Let F be a surface. Then $\pi_1(F)$ is* LERF *if and only if given a* f.g. *subgroup S of $\pi_1(F)$ and $g \in \pi_1(F) - S$, there is a finite covering F_1 of F such that $\pi_1(F_1)$ contains S but not g and S is geometric in F_1.*

Remark. An analogue of Lemma 1.5 also holds in three dimensions **[10]**. Hence so does an analogue of Lemma 1.6.

Proof of Lemma 1.6. If F satisfies the stated geometric condition it is immediate that $\pi_1(F)$ is LERF.

Now suppose that $\pi_1(F)$ is LERF, and that we are given a f.g. subgroup S of $\pi_1(F)$ and $g \in \pi_1(F) - S$. Let F_S denote the covering of F with $\pi_1(F_S) = S$. Pick a point x in the universal covering space $\tilde{F}$ of F and let C denote the image of $x \cup gx$ in F_S. Apply Lemma 1.5 to obtain a compact connected incompressible subsurface Y of F_S which contains C in its interior and such that $\pi_1(Y) = \pi_1(F_S)$. Now Lemma 1.4 tells us that F has a finite covering F_1 such that the projection $F_S \to F$ factors through F_1 and Y projects homeomorphically into F_1. This completes the proof of Lemma 1.6.

2. *Non-closed surfaces*

In this section, we prove the following result.

THEOREM 2.1. *Let F be a non-closed surface, let S be a* f.g. *subgroup of $\pi_1(F)$ and let $g \in \pi_1(F)-S$. Then there is a finite covering F_1 of F such that $\pi_1(F_1)$ contains S but not g and S is geometric in F_1. Further S is a free factor of $\pi_1(F_1)$.*

The proof of Lemma 1.6 shows that the following group theoretic result is equivalent.

THEOREM 2.2. *Let G be a free group, let S be a* f.g. *subgroup of G and let $g \in G-S$. Then G has a subgroup G_1 of finite index which contains S but not g. Further S is a free factor of G_1.*

Remark. This result for arbitrary free groups follows from the result for f.g. free groups, as S and g must lie in some f.g. free factor of G.

This result was first proved by Hall [5]. See also [3], [8] and Lemma 15.22 of [6]. My excuse for giving yet another proof is that this proof motivated me to find the proof of the general result in §3. Hempel [6] also gives a geometric proof of Theorem 2.2. His proof which uses graphs is essentially the same as ours which uses surfaces. The connection is that a non-closed surface is a regular neighbourhood of some graph.

Finally, we observe that Theorem 2.2 is equivalent to Hall's Theorem 5.1 in [5]. Hall's result looks stronger for he asserts that one can choose G_1 to miss a given finite subset T of $G-S$, whereas Theorem 2.2 is the special case when T has a single element. However, Hall's result follows from ours by taking the intersection of the subgroups G_1 of G obtained for each element of T.

Proof of Theorem 2.1. We start by observing that it suffices to prove Theorem 2.1 in the case when F is compact. For then Theorem 2.2 will follow for the case of f.g. free groups, and hence will hold for all free groups. But now Theorem 2.1 must hold in the non-compact case also as Theorems 2.1 and 2.2 are equivalent.

Let F be a compact surface with boundary. We know that F can be constructed from a 2-disc D by attaching 1-handles. However, it will be more convenient to think of F as obtained from D by identifying in pairs certain arcs in the boundary of D. Thus D can be thought of as a fundamental region in the universal covering space of F. If $\pi_1(F)$ is free of rank n, we can divide ∂D into $4n$ arcs which, after identification, will lie alternately in ∂F and in the interior of F. We orient those arcs of ∂D which are to be identified so that the identifications are all orientation preserving. We also label these arcs with a non-zero integer between $-n$ and n so that arcs to be identified carry equal but opposite numbers. Now any covering space of F can be expressed as a countable collection of copies of D with all the labelled edges identified in pairs, so that the arcs in a pair carry equal but opposite numbers and are identified preserving their orientations. Conversely, any space constructed in this way out of countably many copies of D has a natural projection to F which is the standard projection on each copy of D, and this projection is a covering map.

Let S be a f.g. subgroup of $\pi_1(F)$ and let $g \in \pi_1(F)-S$. Let F_S denote the based covering space of F with fundamental group S. This means that the covering map $p: F_S \to F$ is a based map and that $p_*(\pi_1(F_S)) = S \subset \pi_1(F)$. Let l be a path in F_S

starting at the basepoint whose projection in F is a loop representing g. As $g \notin S$, we know that l is not a loop. Now Lemma 1.5 tells us that there is a compact incompressible sub-surface Y of F_S such that Y contains l and the natural map $\pi_1(Y) \to \pi_1(F_S)$ is an isomorphism. Let X be the union of all the copies of D in F_S which meet Y. Thus Y lies in the interior of X. As Y is compact, X consists of finitely many copies of D. We have by restriction a projection map $X \to F$, but this is not a covering map as there are some labelled edges in ∂X which have not been glued together. We cure this by choosing some pairing and glueing all these edges allowably. This gives us a finite covering space F_1 of F which contains Y. Hence $\pi_1(F_1)$ contains S and S is geometric in F_1. As l is not a loop we see that $g \notin \pi_1(F_1)$. The only point left to prove is that $\pi_1(Y) = S$ is a free factor of $\pi_1(F_1)$. Recall that $Y \subset X \subset F_S$. As X must be incompressible in F_S, we see that $\pi_1(Y) = \pi_1(X) = \pi_1(F_S)$. Now F_1 is obtained from X by glueing certain pairs of edges in ∂X. Hence $\pi_1(F_1)$ is the free product of $\pi_1(X)$ with a free group of rank equal to the number of pairs of edges glued together. The result follows.

3. *The Main Theorem*

We start this section by discussing one particular Fuchsian group. Let H denote the hyperbolic plane, and consider a regular pentagon in H. If the pentagon is very small it will be approximately Euclidean and so will have vertex angles greater than $\pi/2$. If we let the vertices of the pentagon tend to infinity in H, then the angles will tend to zero. Hence, by the Intermediate Value Theorem, there is a regular pentagon P in H with all its vertex angles equal to $\pi/2$. We let Γ denote the group of isometries of H generated by reflections in the sides of P. The fact that every vertex angle of P is $\pi/2$ means that the translates of P by Γ tesselate H and that P is a fundamental region for the action of Γ. Each edge of P determines a line in H and we let L denote the family of lines in H consisting of the translates of these five lines.

It is not hard to see that Γ has a subgroup G of index four which is isomorphic to the fundamental group of the non-orientable closed surface F with Euler number -1. Again I am grateful to Thurston for pointing this out to me. Figure 1 shows a disc with two holes divided into four pentagons. If we double this surface along its boundary, we obtain the closed orientable surface of genus two divided into eight pentagons. This surface double covers F and the covering involution preserves pentagons. Hence we see how to divide F into four pentagons. This picture tells

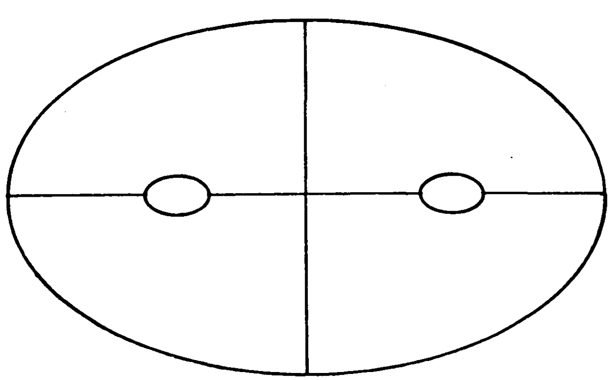
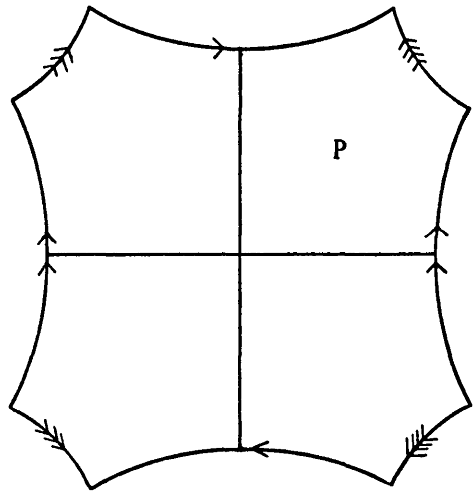


Fig. 1 A disc with 2 holes divided into four pentagons



Four pentagons = an octagon
Fig. 2a. Glue edges of octagon as shown to obtain F
$\chi(F) = -1$

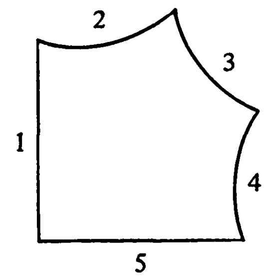


Fig. 2b

us how to find $G = \pi_1(F)$ as a subgroup of Γ of index four. We can obtain F from four pentagons by glueing edges as shown in Fig. 2a. If we number the edges of P from 1 to 5 as shown in Fig. 2b, then G can be generated by $x_1 x_2 x_5$, $x_1 x_4$, $x_3 x_5$ and $x_1 x_3 x_1 x_5$ and the four pentagons shown in Fig. 2a will be a fundamental region for G. This is why G has index four in Γ.

We will shortly prove the following result about G by using the Fuchsian group Γ.

THEOREM 3.1. *This group G is* LERF.

We will then have

THEOREM 3.2. *Every surface group and every Fuchsian group is* LERF.

Proof. The result is trivial for the sphere and projective plane. It is also easy to see that $\mathbb{Z} \times \mathbb{Z}$ is LERF. Hence the fundamental groups of the torus and klein bottle are LERF. Any other closed surface covers F, and so its fundamental group is a subgroup of G and is automatically LERF. Also G contains a free group of rank two, so that we have yet another proof that free groups are LERF. Now any Fuchsian group has a subgroup of finite index which is a surface group **[2]**, **[4]**, **[11]**. Hence all Fuchsian groups are LERF, by Lemma 1.1.

We also have the following geometrical result, which is equivalent to Theorem 3.2, by Lemma 1.6.

THEOREM 3.3. *Let F be a surface, let S be a* f.g. *subgroup of $\pi_1(F)$ and let $g \in \pi_1(F) - S$. Then there is a finite covering F_1 of F such that $\pi_1(F_1)$ contains S but not g and S is geometric in F_1.*

Before proving Theorem 3.1 we give a geometrical proof that Γ is RF. This explains the key idea of the proof that G is LERF.

LEMMA 3.4. *Γ is residually finite.*

Proof. Let C be a compact set in H^2. Lemma 1.3 tells us that we must show that Γ has a subgroup Γ_1 of finite index such that if g is a non-trivial element of Γ_1, then gC does not meet C. We do this by exhibiting a subgroup Γ_1 of Γ with a fundamental region X which contains C in its interior and consists of a finite number of pentagons from our tesselation of H. The index of Γ_1 in Γ is equal to the number of pentagons in X and so must be finite.

As C is compact, C can meet only finitely many of the pentagons in our tesselation of H. Hence we can replace C by the union of a finite number of pentagons. Further by enlarging this union if necessary, we may suppose that C is the union of a finite number of pentagons and that C is connected. As C is compact, only finitely many lines in L meet C. Each line l in H separates H into two half planes l_+, l_-. If l is a line in L which does not meet C, then C lies in the interior of one of these half planes, say l_+, because C is connected. We let X be the intersection of all such half planes l_+, for all lines l in L which do not meet C. Clearly X is convex, is a union of pentagons and contains C in its interior. If a line l in L meets the interior of X, then l must meet C by our construction of X. Hence only finitely many lines in L can meet the interior of X, and X must be compact. We now define Γ_1 to be the group of isometries of H generated by reflections in the sides of X. The sides of X are lines in L and a reflection in a line of L is an element of Γ. Hence Γ_1 is a subgroup of Γ. Also, as X is a convex union of pentagons every vertex angle of X must be $\pi/2$. For we can disregard angles equal to π, and an angle of $3\pi/2$ is excluded by convexity. Hence X is a fundamental region for Γ_1, for the same reason that P is a fundamental region for Γ. The result follows.

Comments on the proof. The fact that our original fundamental region P has all angles equal to $\pi/2$ is crucial for this proof. If, for example, P has an angle of $\pi/3$, then X might have an angle $2\pi/3$ and would not be a fundamental region for Γ_1. We remark that this proof can also be used to show $\mathbb{Z}\times\mathbb{Z}$ is RF by considering the usual action of $\mathbb{Z}\times\mathbb{Z}$ on the Euclidean plane $\mathbb{R}^2$, which has a square fundamental region. For any compact set in $\mathbb{R}^2$ lies in the interior of a rectangular union X of squares, and there is an obvious subgroup of $\mathbb{Z}\times\mathbb{Z}$ with X as its fundamental region. This is how the idea for the proof of Lemma 3.4 arose.

We now use essentially the same idea to show that Γ is LERF. However, it seems simpler to work with G rather than Γ.

THEOREM 3.1. *The group G is* LERF.

Proof. Suppose we are given a f.g. subgroup S of G and a compact subset C of H/S. Let $p: H \to H/S$ denote the projection, which is a covering map. Let D be a compact set in H such that $p(D) = C$. We must produce a subgroup G_1 of finite index in G such that G_1 contains S and, in addition, if an element g of G_1 has gD meeting D then g lies in S. This will prove that G is LERF from Lemma 1.4, for C will project homeomorphically into H/G_1.

As C is a compact subset of H/S and as $\pi_1(H/S) = S$ is f.g., Lemma 1.5 tells us that there is a compact connected subsurface C_1 of H/S, such that C_1 contains C and the natural map $\pi_1(C_1) \to \pi_1(H/S)$ is an isomorphism. Let Y denote $p^{-1}(C_1)$, a connected surface in H. As before, we define $\bar{Y}$ to be the intersection of all the closed half spaces in H which contain Y in their interior and are bounded by a line in

L. Thus $\overline{Y}$ is a convex union of pentagons in H. Now Y is invariant under the action of S on H, and so it follows that $\overline{Y}$ is S-invariant. Hence $\overline{Y}$ covers its image $p(\overline{Y})$ in H/S, so that $p(\overline{Y})$ is a union of pentagons in H/S. If a line l of $p(L)$ in H/S meets the interior of $p(\overline{Y})$, it follows from the definitions of Y and $\overline{Y}$ that l meets C_1. Hence only finitely many lines in $p(L)$ can meet $p(\overline{Y})$, and $p(\overline{Y})$ must be compact. Let X be a (compact) fundamental region in $\overline{Y}$ for the action of S.

We define Γ_2 to be the group of isometries of H generated by reflections in the sides of $\overline{Y}$. As usual, $\overline{Y}$ is a (non-compact) fundamental region for Γ_2, because all the vertex angles of $\overline{Y}$ are equal to $\pi/2$. We define Γ_1 to be the group of isometrics of H generated by Γ_2 and S. Observe that Γ_2 is a normal subgroup of Γ_1, because $\overline{Y}$ is S-invariant, and the quotient of Γ_1 by Γ_2 is isomorphic to S. It follows that X is a fundamental region in H for the action of Γ_1, for $H/\Gamma_1 = (H/\Gamma_2)/S = \overline{Y}/S$. Hence Γ_1 is of finite index in Γ. Let D be a compact set in the interior of $\overline{Y}$ such that $p(D) = C_1$. If we have $g \in \Gamma_1$ such that gD meets D, then $g(\text{int } \overline{Y})$ meets int $\overline{Y}$ and so g must lie in S. Hence if we define $G_1 = \Gamma_1 \cap G$, then G_1 will have all the required properties. This completes the proof of Theorem 3.1.

The proof of Theorem 2.1 can be thought of in much the same way as that of Theorem 3.1. The difference is that the universal covering space of a surface with boundary is not the hyperbolic plane. We can remove the boundary of the surface to cure this and then a fundamental region in H will have vertices at infinity. This means that any connected union of fundamental regions is automatically convex and that one does not need to worry about the angles at the vertices. This is why the proof of Theorem 2.1 is so much simpler than that of Theorem 3.1.

4. *The* 3-*dimensional case*

As we have just shown that surface groups are LERF, it seems reasonable to ask whether 3-manifold groups are LERF. One must restrict this question to compact 3-manifolds because, for example, the additive group of the rationals is the fundamental group of a 3-manifold and is not RF. Our previous results allow us to prove the following.

THEOREM 4.1. *The fundamental group of a compact Seifert fibre space is* LERF.

Proof. Let G be the fundamental group of a compact Seifert fibre space M. If G is finite, the result is trivial. Otherwise G has a normal infinite cyclic subgroup J with quotient a group Γ which contains a surface group Γ_1 as a subgroup of finite index **[6]**. The inverse image of Γ_1 in G is a subgroup G_1 of finite index in G. As J admits only two automorphisms, G_1 has a subgroup G_2 of finite index such that J is central in G_2 and has quotient a surface group Γ_2. The covering space M_2 of M with fundamental group G_2 is a Seifert fibre space with no singular fibres and so is a S^1-bundle over a surface F. If F is non-orientable, let F_3 denote the orientable double covering of F and let M_3 be the corresponding double covering of M_2. Thus M_3 is a S^1-bundle over the orientable surface F_3.

Now $\pi_1(M)$ is LERF if and only if $\pi_1(M_3)$ is LERF. Hence it suffices to prove Theorem 4.1 in the special case when M is a S^1-bundle over an orientable surface F and the fundamental group J of the fibre is central in $\pi_1(M)$. Let G denote the fundamental group of such a 3-manifold M, let Γ denote $\pi_1(F)$ and let $p : G \to \Gamma$ denote the projection map. Let S be a f.g. subgroup of G and let $g \in G - S$. If $p(g) \notin p(S)$, then Γ must have a subgroup Γ_1 of finite index which contains $p(S)$ but not $p(g)$. This is because Γ is LERF. In this case, $p^{-1}(\Gamma_1)$ is a subgroup of G of finite index which contains S but not g. If $p(g) \in p(S)$, we can write $g = hs$ for some elements $h \in J$, $s \in S$. As $g \notin S$, we must have $h \notin S \cap J$. Let J_1 be a subgroup of J of finite index which contains $S \cap J$ but not h. (Take J_1 equal to $S \cap J$ unless $S \cap J$ is trivial.) Then Lemma 4.2 below tells us that G has a subgroup G_1 of finite index such that G_1 contains S and $G_1 \cap J = J_1$. Thus $h \notin G_1$ and so $g \notin G_1$. It follows that G is LERF, so that Lemma 4.2 will complete the proof of Theorem 4.1.

LEMMA 4.2. *Let Γ denote the fundamental group of a compact orientable surface F and let G be a central extension of an infinite cyclic group J by Γ, with projection $p : G \to \Gamma$. Let S be a f.g. subgroup of G and let J_1 be a subgroup of finite index k in J such that J_1 contains $S \cap J$. Then G has a subgroup G_1 of finite index such that G_1 contains S and $G_1 \cap J = J_1$.*

Remark. Any such extension G is the fundamental group of a S^1-bundle over F.

Proof. If F has boundary, then Γ is free. Hence the extension splits and $G \cong J \times \Gamma$. The required result is now obvious for one can take $G_1 = J_1 \times \Gamma$.

If F is closed, then Γ has presentation $\{\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g : \prod[\alpha_i, \beta_i] = 1\}$. Let $a_1, \ldots, a_g, b_1, \ldots, b_g$ be elements of G such that $p(a_i) = \alpha_i$, $p(b_i) = \beta_i$, and let t be a generator of J. Then G has presentation

$$\{t, a_1, \ldots, a_g, b_1, \ldots, b_g : a_i^{-1} t a_i = t,\ b_i^{-1} t b_i = t,\ \prod[a_i, b_i] = t^n\}$$

for some integer n. Central extensions of J by Γ correspond to elements of $H^2(\Gamma, J) \cong \mathbb{Z}$, and the integer n in the presentation can be identified with the element of $H^2(\Gamma, J)$ which corresponds to G.

If $n = 0$, we again have $G = J \times \Gamma$ and the result is obvious. So we suppose that $n \neq 0$. From the presentation above for G, one sees that $H_1(G) \cong \mathbb{Z}_n \oplus H_1(F)$. Let $\pi : H_1(G) \to \mathbb{Z}_n$ denote projection onto the first factor. If k divides n, there is an epimorphism $\mathbb{Z}_n \to \mathbb{Z}_k$. Hence by composing with π and the abelianisation homomorphism we have an epimorphism $G \to \mathbb{Z}_k$. The kernel G_1 of this map is of index k in G_1 and is generated by $t^k, a_1, \ldots, a_g, b_1, \ldots, b_g$. Thus $G_1 \cap J = J_1$ and $p(G_1) = \Gamma$. Hence G_1 contains S and $G_1 \cap J = J_1$ as required.

If k does not divide n, we will show that Γ has a subgroup Γ_2 of finite index l, a multiple of k, such that Γ_2 contains $p(S)$. Let G_2 denote $p^{-1}(\Gamma_2)$, so that G_2 contains S. Then G_2 has a presentation of the same type as G but the genus of Γ_2 will not equal that of Γ. (Unless F is a torus.) I claim that the integer n_2 which determines the extension G_2 satisfies $n_2 = ln$ and so is divisible by k. Now we can apply the argument of the previous paragraph to show that G_2 has the required subgroup G_1.

In order to obtain the subgroup Γ_2 of Γ, we first recall from Theorem 3.3 that as S is a f.g. subgroup of $\Gamma = \pi_1(F)$, there is a finite covering F_3 of F in which S is geometric. Hence $H_1(S)$ has rank less than $H_1(F_3)$. Hence there is an epimorphism $H_1(F_3) \to \mathbb{Z}_k$ whose kernel contains $H_1(S)$. The corresponding finite covering F_2 of F_3 is a finite covering of F of index l, a multiple of k, and $\pi_1(F_2) = \Gamma_2$ contains S. The reason for the equation $n_2 = ln$ is simply that the map

$$\mathbb{Z} \cong H^2(\Gamma, J) \to H^2(\Gamma_2, J) \cong \mathbb{Z}$$

induced by the inclusion of Γ_2 in Γ, is multiplication by l. This equation can also be seen directly and geometrically by considering the covering space determined by G_2 of the S^1-bundle over F with fundamental group G.

Theorem 4.1 tells us, in particular, that any S^1-bundle over a surface has LERF fundamental group. However, I am unable to decide whether the same holds for bundles over S^1 with fibre a surface. It seems quite possible that this is false.

One could try to apply the methods of §3 rather than the results. Thus one considers hyperbolic 3-manifolds i.e. 3-manifolds which are the quotient of hyperbolic 3-space H^3 by a group of isometries which act as a group of covering translations. If one has a polyhedron P in H^3 which is a fundamental region for the group Γ generated by reflections in the faces of P, and if all the dihedral angles of P are equal to $\pi/2$, then the methods of §3 will apply to show that Γ is LERF. However, this is not a common situation in dimension three. Thurston has pointed out that one can have a regular octahedron P in H^3 all of whose dihedral angles are $\pi/2$, by having the vertices of P at infinity. He has also shown that the group of the Borromean rings is a subgroup of index two in the reflection group Γ determined by P. Hence the group of the Borromean rings is LERF. However, not all hyperbolic 3-manifold groups are commensurable with groups generated by reflections, so this leaves a large field for further study.

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