EXERCISE SHEET #1

Let $\mathbb{H}^n = \{x \in \mathbb{R}^n : x_n > 0\}$ be the upper half space equipped with hyperbolic metric ρ (induced from Riemannian metric $\frac{|dx|}{x_n}$).

Exercise 0.1 (Nearest projection)**.** (1) *Let x be in $\mathbb{H}^n$. Show that the nearest point to x on the positive n-th axis is $|x|e_n$ and we have*

$$\cosh \rho(x, |x|e_n) = \frac{|x|}{x_n}$$

(2) *Let π be the nearest point projection of a point $x \in \mathbb{H}^n$ onto the positive n-th axis defined by $\pi(x) = |x|e_n$. Prove that for all $x, y \in \mathbb{H}_n$, we have*

$$\rho(\pi(x), \pi(y)) \leq \rho(x, y)$$

with equality if and only if either $x = y$ or x and y lie on the n-th axis.

Let γ be a geodesic in $(\mathbb{H}^n, \rho)$ for $n \geq 2$. Let $\pi_\gamma : \mathbb{H}^n \to \gamma$ be the *nearest point projection map*: for any $x \in \mathbb{H}^n$,

$$\rho(x, \pi_\gamma(x)) = \inf_{y \in \gamma} \rho(x, y).$$

Exercise 0.2 (Strong contraction)**.** *Show that for any hyperbolic ball B in $(\mathbb{H}^n, \rho)$ with $B \cap \gamma = \emptyset$, we have*

$$diam(\pi_\gamma(B)) \leq 2 \log(1 + \sqrt{2})$$

where diam is taken with respect to ρ.

Exercise 0.3 (Poincaré extension)**.** *Suppose that $\phi \in M(\hat{\mathbb{R}}^{n+1})$ preserves $\mathbb{H}^{n+1}$. Then ϕ is the Poincaré extension of some $\psi \in M(\hat{\mathbb{R}}^n)$.*

Let $(\mathbb{B}^n, \rho)$ be the ball model of n-dimensional hyperbolic space, and $\mathbb{S}^{n-1}$ be the $(n-1)$-sphere equipped with Euclidean metric $|x - y|$. As a set, $\mathbb{S}^{n-1}$ can be identified with the set of geodesic rays from the origin o.

For any three points $o, x, y \in \mathbb{B}^n$, we define the Gromov product

$$\langle x, y \rangle_o := \frac{\rho(x, o) + \rho(y, o) - \rho(x, y)}{2}$$

The metric $|\cdot|$ on $\mathbb{S}^{n-1}$ can be recovered from the hyperbolic geometry.

Exercise 0.4 (Visual metric)**.** *Let $\alpha, \beta : [0, \infty) \to (\mathbb{B}^n, \rho)$ be two distinct geodesic rays (i.e.: isometric embedding) from o ending at two points $p, q \in \mathbb{S}^{n-1}$ respectively. Define*

$$\delta(\alpha, \beta) := \lim_{t \to \infty} e^{-\langle \alpha(t), \beta(t) \rangle_o}.$$

Prove that $2\delta(\alpha, \beta) = |p - q|$. The so-defined function δ is called visual metric.

(Tips: use Hyperbolic Cosine Rule (in Beardon's book))

Consequences of thin triangle property. We know that there exists a universal constant $\delta > 0$ (independent of n) such that any geodesic triangle in $(\mathbb{H}^n, \rho)$ is δ-thin: any side is contained in the δ-neighborhood of the other two sides.

Exercise 0.5 (Gromov product)**.** *Prove that*

$$\rho(o, [x, y]) - 2\delta \leq \langle x, y \rangle_o \leq \rho(o, [x, y])$$

where $[x, y]$ *denotes the geodesic with endpoints* x, y*.*

A path $p : [0, 1] \to (\mathbb{H}^n, \rho)$ is called *rectifiable* if its length defined as follows is finite:

$$Len(p) := \sup\{\sum_{i=0}^{n} \rho(p(t_i), p(t_{i+1}))\}$$

over all finite subdivision $t_0 = 0 < t_1 < t_2 \cdots < t_n = 1$ of $[0, 1]$.

Exercise 0.6. *Let* α *be a geodesic segment and* p *be a rectifiable path with length* $s \geq 2$ *with the same endpoints as* α*. Then any point* $z \in \alpha$ *has a distance at most* $(1 + \delta \log_2 s)$ *to* p*:*

$$\rho(z, p) \leq 1 + \delta \log_2 s$$

where $\delta > 0$ *is the thin constant from Exercise 0.1. (Tips: subdivide inductively the path* p *by half and then apply* δ*-thin triangle property.)*

Corollary 0.7 (Exponential perimeter)**.** *If* p *is disjoint with a (hyperbolic) ball centered at the middle point in* α *with radius* $R = \dfrac{Len(\alpha)}{2} - 1$*, then*

$$s \geq 2^{\delta^{-1}(R-2)}.$$