Let $\mathbb{H}^n = \{ x \in \mathbb{R}^n : x_n > 0 \}$ be the upper half space equipped with hyperbolic metric $\rho$ (induced from Riemannian metric $\frac{|dx|}{x_n}$).

**Exercise 0.1** (Nearest projection). 1. Let $x$ be in $\mathbb{H}^n$. Show that the nearest point to $x$ on the positive $n$-th axis is $|x|e_n$ and we have 
\[
cosh \rho(x, |x|e_n) = \frac{|x|}{x_n}
\]

2. Let $\pi$ be the nearest point projection of a point $x \in \mathbb{H}^n$ onto the positive $n$-th axis defined by $\pi(x) = |x|e_n$. Prove that for all $x, y \in \mathbb{H}_n$, we have 
\[
\rho(\pi(x), \pi(y)) \leq \rho(x, y)
\]
with equality if and only if either $x = y$ or $x$ and $y$ lie on the $n$-th axis.

**Exercise 0.2** (Strong contraction). Show that for any hyperbolic ball $B$ in $(\mathbb{H}^n, \rho)$ with $B \cap \gamma = \emptyset$, we have 
\[
diam(\pi_\gamma(B)) \leq 2 \log(1 + \sqrt{2})
\]
where $\text{diam}$ is taken with respect to $\rho$.

**Exercise 0.3** (Poincaré extension). Suppose that $\phi \in M(\mathbb{R}^{n+1})$ preserves $\mathbb{H}^{n+1}$. Then $\phi$ is the Poincaré extension of some $\psi \in M(\mathbb{R}^n)$.

Let $(\mathbb{B}^n, \rho)$ be the ball model of $n$-dimensional hyperbolic space, and $\mathbb{S}^{n-1}$ be the $(n-1)$-sphere equipped with Euclidean metric $|x - y|$. As a set, $\mathbb{S}^{n-1}$ can be identified with the set of geodesic rays from the origin $o$.

For any three points $o, x, y \in \mathbb{B}^n$, we define the Gromov product 
\[
\langle x, y \rangle_o := \frac{\rho(x, o) + \rho(y, o) - \rho(x, y)}{2}
\]
The metric $|\cdot|$ on $\mathbb{S}^{n-1}$ can be recovered from the hyperbolic geometry.

**Exercise 0.4** (Visual metric). Let $\alpha, \beta : [0, \infty) \rightarrow (\mathbb{B}^n, \rho)$ be two distinct geodesic rays (i.e.: isometric embedding) from $o$ ending at two points $p, q \in \mathbb{S}^{n-1}$ respectively. Define 
\[
\delta(\alpha, \beta) := \lim_{t \rightarrow \infty} e^{-\langle \alpha(t), \beta(t) \rangle_o}.
\]
Prove that $2\delta(\alpha, \beta) = |p - q|$. The so-defined function $\delta$ is called visual metric.

(Tips: use Hyperbolic Cosine Rule (in Beardon’s book))

**Consequences of thin triangle property.** We know that there exists a universal constant $\delta > 0$ (independent of $n$) such that any geodesic triangle in $(\mathbb{H}^n, \rho)$ is $\delta$-thin: any side is contained in the $\delta$-neighborhood of the other two sides.
Exercise 0.5 (Gromov product). Prove that
\[\rho(o, [x,y]) - 2\delta \leq \langle x, y \rangle_o \leq \rho(o, [x,y])\]
where \([x,y]\) denotes the geodesic with endpoints \(x, y\).

A path \(p : [0,1] \to (\mathbb{H}^n, \rho)\) is called rectifiable if its length defined as follows is finite:
\[\text{Len}(p) := \sup \left\{ \sum_{i=0}^{n} \rho(p(t_i), p(t_{i+1})) \right\}\]
over all finite subdivision \(t_0 = 0 < t_1 < t_2 \cdots < t_n = 1\) of \([0,1]\).

Exercise 0.6. Let \(\alpha\) be a geodesic segment and \(p\) be a rectifiable path with length \(s \geq 2\) with the same endpoints as \(\alpha\). Then any point \(z \in \alpha\) has a distance at most \((1 + \delta \log_2 s)\) to \(p\):
\[\rho(z, p) \leq 1 + \delta \log_2 s\]
where \(\delta > 0\) is the thin constant from Exercise 0.1. (Tips: subdivide inductively the path \(p\) by half and then apply \(\delta\)-thin triangle property.)

Corollary 0.7 (Exponential perimeter). If \(p\) is disjoint with a (hyperbolic) ball centered at the middle point in \(\alpha\) with radius \(R = \frac{\text{Len}(\alpha)}{2} - 1\), then
\[s \geq 2^{\delta^{-1}(R-2)}\].