Limit set. Suppose that $G$ acts properly on $\mathbb{H}^n$ for $n \geq 2$. The limit set $\Lambda G$ of $G$ is the set of accumulation points in $\partial \mathbb{H}^n := S^{n-1}$ of any $G$-orbit in $\mathbb{H}^n$.

Exercise 0.1. (1) If $H$ is a finite index subgroup in $G$, then $\Lambda H = \Lambda G$.
(2) Let $H$ be an infinite normal subgroup in $G$. If $|\Lambda G| \geq 3$, then $\Lambda H = \Lambda G$.

Exercise 0.2. Assume that $|\Lambda G| \geq 3$.
(1) Let $x \neq y \in \Lambda G$ be any two points. For any open neighborhoods $x \in U$ and $y \in V$ there exists a loxodromic element $h \in G$ such that $h_- \in U$ and $h_+ \in V$.
(2) Prove that $G$ contains infinitely many conjugacy classes of loxodromic elements with pairwise disjoint fixed points.
(3) Prove that for any $x \neq y \in \Lambda G$ there exists a sequence of elements $g_n$ such that $g_n o \rightarrow x$ and $g_n^{-1} o \rightarrow y$ for some (or any) $o \in \mathbb{H}^n$.

Tips for (1): choose loxodromic $h,k$ such that $h_- \in U$ and $k_+ \in V$. Then consider $k^n h^n$ for large $n$.

Discontinuity domain. The complement $\Omega G := S^{n-1} \setminus \Lambda G$ of the limit set is called the discontinuity domain. Assume that $\Omega G$ is non-empty.

Exercise 0.3. (1) Prove that for any $p \in \Omega G$, the set of accumulation points for the orbit $G \cdot p$ coincides with $\Lambda G$.
(2) Prove that $\Omega G$ is open and dense in $S^{n-1}$, and $\Lambda G$ is nowhere dense (i.e. the interior of $\Lambda$ is empty).
(3) Prove that $\Omega G$ is the maximal discontinuity domain on $S^{n-1}$: let $U$ be an open subset of $S^{n-1}$ on which $G$ acts properly. Then $U \subset \Omega G$.

The topology of Cantor set. Let $T$ be a tree (i.e. a connected graph without loops) endowed with a metric so that every edge is isometric to the unit interval $[0,1]$. Fix a basepoint $o \in T$. The visual boundary $\partial T$ is the set of all geodesic rays from $o$.

For any $\alpha \neq \beta \in \partial T$, the function $\delta(\alpha, \beta) := 2^{-n}$ is a metric on $\partial T$, where $n$ is the length of the intersection $\alpha \cap \beta$.

Exercise 0.4 (Cantor set). Assume that $T$ is an infinite tree of valence 3 (the valence of any vertex is 3).
(1) Construct a bijective map from $\partial T$ to the Cantor set $C$. (Tips: write the numbers in $C$ in base 3 decimal expansion with only 0 and 2’s)
(2) Prove that this map is a homeomorphism of $\partial T$ to the Cantor set $C \subset [0,1]$ with subspace topology.
(3) Prove that the visual boundary for any tree with valence between 3 and a fixed $M \geq 3$ is homeomorphic to the Cantor set.

Schottky groups. Two elements $a, b$ in $I(\mathbb{H}^2)$ are called ping-pong players if there are disjoint open halfspaces (bounded by bi-infinite geodesics) $H_a, H_b, H_{a^{-1}}, H_{b^{-1}}$ in $\mathbb{H}^2$ so that...
(1) \( s(\mathbb{H}^2 \setminus H_{s^{-1}}) = \tilde{H}_s \) for any \( s \in S := \{a, b, a^{-1}, b^{-1}\} \).
(2) \( \cup_{s \in S} \tilde{H}_s \neq \mathbb{H}^2 \).

The group generated by \( a, b \) is called (classical) Schottky groups.

**Exercise 0.5** (Limit set of Schottky groups). Set \( P := \mathbb{H}^2 \setminus \cup_{s \in S} H_s \). Assume that \( d(H_s, H_t) > 0 \) for any \( s \neq t \in S \). Prove that \( \cup_{g \in G} (g \cdot P) = \mathbb{H}^2 \).

**Exercise 0.6** (Examples of Schottky groups). (1) Find examples of Schottky groups \( G \) such that the limit set \( \Lambda G \) is homeomorphic to the circle \( S^1 \).

(2) Find examples of Schottky groups \( G \) such that \( \cup_{g \in G} (g \cdot P) \nsubseteq \mathbb{H}^2 \).

Tips for (1): find \( a, b \) so that the boundaries of \( H_a, H_b, H_{a^{-1}}, H_{b^{-1}} \) form an ideal quadrilateral.

**Exercise 0.7** (Convex hull of limit set). Let \( CH(\Lambda G) \) be the minimal convex subset of \( \mathbb{H}^n \) whose boundary in \( S^{n-1} \) is equal to \( \Lambda G \).

(1) Prove that \( CH(\Lambda G) \) is contained in a finite neighborhood of the union of all bi-infinite geodesics with their two endpoints in \( \Lambda G \).

(2) Assume that \( d(H_s, H_t) > 0 \) for any \( s \neq t \in S \). Prove that \( G \) acts co-cocompactly on \( CH(\Lambda G) \).

Tips: build an orbital map from the free group \( G \) into \( \mathbb{H}^n \) and use Morse Lemma.