EXERCISE SHEET #2

Limit set. Suppose that G acts properly on $\mathbb{H}^n$ for $n \ge 2$. The limit set ΛG of G is the set of accumulation points in $\partial \mathbb{H}^n := S^{n-1}$ of any G-orbit in $\mathbb{H}^n$.

Exercise 0.1.
(1) *If H is a finite index subgroup in G, then $\Lambda H = \Lambda G$.*
(2) *Let H be an infinite normal subgroup in G. If $|\Lambda G| \ge 3$, then $\Lambda H = \Lambda G$.*

Exercise 0.2. *Assume that $|\Lambda G| \ge 3$.*
(1) *Let $x \neq y \in \Lambda G$ be any two points. For any open neighborhoods $x \in U$ and $y \in V$ there exists a loxodromic element $h \in G$ such that $h_- \in U$ and $h_+ \in V$.*
(2) *Prove that G contains infinitely many conjugacy classes of loxodromic elements with pairwise disjoint fixed points.*
(3) *Prove that for any $x \neq y \in \Lambda G$ there exists a sequence of elements g_n such that $g_n o \to x$ and $g_n^{-1} o \to y$ for some (or any) $o \in \mathbb{H}^n$.*

Tips for (1): choose loxodromic h, k such that $h_- \in U$ and $k_+ \in V$. Then consider $k^n h^n$ for large n.

Discontinuity domain. The complement $\Omega G := S^{n-1} \setminus \Lambda G$ of the limit set is called the discontinuity domain. Assume that ΩG is non-empty.

Exercise 0.3.
(1) *Prove that for any $p \in \Omega G$, the set of accumulation points for the orbit $G \cdot p$ coincides with ΛG.*
(2) *Prove that ΩG is open and dense in S^{n-1}, and ΛG is nowhere dense (i.e. the interior of Λ is empty).*
(3) *Prove that ΩG is the maximal discontinuity domain on S^{n-1}: let U be an open subset of S^{n-1} on which G acts properly. Then $U \subset \Omega G$.*

The topology of Cantor set. Let T be a *tree* (i.e. a connected graph without loops) endowed with a metric so that every edge is isometric to the unit interval $[0, 1]$. Fix a basepoint $o \in T$. The *visual boundary* ∂T is the set of all geodesic rays from o.

For any $\alpha \neq \beta \in \partial T$, the function $\delta(\alpha, \beta) := 2^{-n}$ is a metric on ∂T, where n is the length of the intersection $\alpha \cap \beta$.

Exercise 0.4 (Cantor set)**.** *Assume that T is an infinite tree of valence 3 (the valence of any vertex is 3).*
(1) *Construct a bijective map from ∂T to the Cantor set C. (Tips: write the numbers in C in base 3 decimal expansion with only 0 and 2's)*
(2) *Prove that this map is a homeomorphism of ∂T to the Cantor set $C \subset [0, 1]$ with subspace topology.*
(3) *Prove that the visual boundary for any tree with valence between 3 and a fixed $M \ge 3$ is homeomorphic to the Cantor set.*

Schottky groups. Two elements a, b in $I(\mathbb{H}^2)$ are called *ping-pong players* if there are disjoint open halfspaces (bounded by bi-infinite geodesics) $H_a, H_b, H_{a^{-1}}, H_{b^{-1}}$ in $\mathbb{H}^2$ so that

(1) $s(\mathbb{H}^2 \setminus H_{s^{-1}}) = \bar{H}_s$ for any $s \in S := \{a, b, a^{-1}, b^{-1}\}$.
(2) $\cup_{s\in S}\bar{H}_s \neq \mathbb{H}^2$.

The group generated by a, b is called *(classical) Schottky groups.*

Exercise 0.5 (Limit set of Schottky groups)**.** *Set* $P := \mathbb{H}^2 \setminus \cup_{s\in S} H_s$. *Assume that* $d(H_s, H_t) > 0$ *for any* $s \neq t \in S$. *Prove that* $\cup_{g\in G}(g \cdot P) = \mathbb{H}^2$.

Exercise 0.6 (Examples of Schottky groups)**.**
(1) *Find examples of Schottky groups* G *such that the limit set* ΛG *is homeomorphic to the circle* S^1.
(2) *Find examples of Schottky groups* G *such that* $\cup_{g\in G}(g \cdot P) \subsetneq \mathbb{H}^2$.

Tips for (1): find a, b *so that the boundaries of* $H_a, H_b, H_{a^{-1}}, H_{b^{-1}}$ *form an ideal quadrilateral.*

Exercise 0.7 (Convex hull of limit set)**.** *Let* $CH(\Lambda G)$ *be the minimal convex subset of* $\mathbb{H}^n$ *whose boundary in* S^{n-1} *is equal to* ΛG.

(1) *Prove that* $CH(\Lambda G)$ *is contained in a finite neighborhood of the union of all bi-infinite geodesics with their two endpoints in* ΛG.
(2) *Assume that* $d(H_s, H_t) > 0$ *for any* $s \neq t \in S$. *Prove that* G *acts co-cocompactly on* $CH(\Lambda G)$.

Tips: build an orbital map from the free group G *into* $\mathbb{H}^n$ *and use Morse Lemma.*