Inyectivity radius. A group $G$ is called residually finite if the trivial group is the intersection of all finite index subgroups. It is known that any finitely generated linear group is residually finite.

**Exercise 0.1.** Suppose $M = \mathbb{H}^n/G$ is compact. Prove that for any $C > 0$, there exists a finite covering $N$ of $M$ such that the injective radius at any point of $N$ is at least $C$.

**Exercise 0.2.** Let $G$ be a non-elementary discrete group in $I(\mathbb{H}^n)$ so that $\mathbb{H}^n/G$ is compact for $n \geq 2$. Prove that $G$ does not contain $\mathbb{Z} \times \mathbb{Z}$ as a subgroup.

**Dirichlet domain.** A geodesic triangulation of a hyperbolic surface is a decomposition of the surface as the union of finitely many geodesic triangles such that every two triangles are either disjoint, or share a vertex or only one edge.

**Exercise 0.3.** Prove that every closed orientable hyperbolic surface $\Sigma_g$ admits a geodesic triangulation, and derive that the area of is $4\pi (g - 1)$. Tips:

1. Let $X$ be a discrete set of points in $\mathbb{H}^2$. For given $p \in X$, let $D_p$ be the set of points in $\mathbb{H}^2$ which is closer to $p$ than any point of $X$. Prove that $D_p$ is a convex geodesic hyperbolic polygon. (Compare with Dirichlet construction)

2. Choose appropriately and then lift finitely many points on the surface to the universal covering and apply the previous construction.

**Exercise 0.4.** Prove that the Dirichlet domain for $\text{PSL}(2, \mathbb{Z})$ with center at $t_i$ for $t > 1$ is $\{ z \in \mathbb{C} : |z| \geq 1, |\text{Re}(z)| \leq 1/2, \text{Im}(z) > 0 \}$.

**Cayley graph.** Let $G$ be a group generated by a finite set of elements $S$. The Cayley graph $G(G,S)$ of $G$ with respect to $S$ is a graph with the vertex set $G$ such that two vertices $g_1, g_2 \in G$ are connected by one edge if and only if $g_1 = g_2s$ for some $s \in S$. By assigning each edge with unit length, the word metric $d_S(g_1, g_2)$ is the minimal length of connected paths from $g_1$ to $g_2$.

**Exercise 0.5** (Changing generating set). Prove that the identification $(G, d_S) \rightarrow (G, d_T)$ is a quasi-isometry for two finite generating sets $S,T$: there exists $K > 0$ such that for any $g_1, g_2 \in G$,

$$\frac{1}{K} d_S(g_1, g_2) \leq d_T(g_1, g_2) \leq K d_S(g_1, g_2)$$

**Exercise 0.6** (Milnor-Svarc Lemma). Let $G$ be a discrete subgroup of $I(\mathbb{H}^n)$ so that $\mathbb{H}^n/G$ is compact with diameter $R > 0$. Fix a basepoint $o \in \mathbb{H}^n$. Consider the orbital map $\phi : G \rightarrow \mathbb{H}^n$:

$$\phi : g \in G \mapsto go \in \mathbb{H}^n$$

Prove that

1. the set $S := \{ g \in G : \rho(o, go) \leq 2R + 1 \}$ generates the group $G$.
2. the map $\phi : (G, d_S) \rightarrow \mathbb{H}^n$ is a quasi-isometric embedding: there exist constants $\lambda > 1, c > 0$ such that

$$\frac{1}{\lambda} d_S(g_1, g_2) - c \leq d(\phi(g_1), \phi(g_2)) \leq \lambda d_S(g_1, g_2) + c$$
Tips for (3): connect $g_1 o, g_2 o$ be a geodesic segment and subdivide into segments of length 1, then apply (1).