EXERCISE SHEET #3

Injectivity radius. A group G is called *residually finite* if the trivial group is the intersection of all finite index subgroups. It is known that any finitely generated linear group is residually finite.

Exercise 0.1. *Suppose $M = \mathbb{H}^n/G$ is compact. Prove that for any $C > 0$, there exists a finite covering N of M such that the injective radius at any point of N is at least C.*

Exercise 0.2. *Let G be a non-elementary discrete group in $I(\mathbb{H}^n)$ so that $\mathbb{H}^n/G$ is compact for $n \geq 2$. Prove that G does not contain $\mathbb{Z} \times \mathbb{Z}$ as a subgroup.*

Dirichlet domain. A *geodesic triangulation* of a hyperbolic surface is a decomposition of the surface as the union of finitely many geodesic triangles such that every two triangles are either disjoint, or share a vertex or only one edge.

Exercise 0.3. *Prove that every closed orientable hyperbolic surface Σ_g admits a geodesic triangulation, and derive that the area of is $4\pi(g-1)$. Tips:*

(1) *Let X be a discrete set of points in $\mathbb{H}^2$. For given $p \in X$, let $D(p)$ be the set of points in $\mathbb{H}^2$ which is closer to p than any point of X. Prove that D_p is a convex geodesic hyperbolic polygon. (Compare with Dirichlet construction)*
(2) *Choose appropriately and then lift finitely many points on the surface to the universal covering and apply the previous construction.*

Exercise 0.4. *Prove that the Dirichlet domain for $PSL(2,\mathbb{Z})$ with center at ti for $t > 1$ is $\{z \in \mathbb{C} : |z| \geq 1, |Re(z)| \leq 1/2, Im(z) > 0\}$.*

Cayley graph. Let G be a group generated by a finite set of elements S. The **Cayley graph** $\mathcal{G}(G,S)$ of G with respect to S is a graph with the vertex set G such that two vertices $g_1, g_2 \in G$ are connected by one edge if and only if $g_1 = g_2 s$ for some $s \in S$. By assigning each edge with unit length, the **word metric** $d_S(g_1, g_2)$ is the minimal length of connected paths from g_1 to g_2.

Exercise 0.5 (Changing generating set)**.** *Prove that the identification $(G, d_S) \to (G, d_T)$ is a quasi-isometry for two finite generating sets S, T: there exists $K > 0$ such that for any $g_1, g_2 \in G$,*

$$\frac{1}{K} d_S(g_1, g_2) \leq d_T(g_1, g_2) \leq K d_S(g_1, g_2)$$

Exercise 0.6 (Milnor-Svarc Lemma)**.** *Let G be a discrete subgroup of $I(\mathbb{H}^n)$ so that $\mathbb{H}^n/G$ is compact with diameter $R > 0$. Fix a basepoint $o \in \mathbb{H}^n$. Consider the orbital map $\phi : G \to \mathbb{H}^n$:*

$$\phi : g \in G \mapsto go \in \mathbb{H}^n$$

Prove that

(1) *the set $S := \{g \in G : \rho(o, go) \leq 2R + 1\}$ generates the group G.*
(2) *the map $\phi : (G, d_S) \to \mathbb{H}^n$ is a quasi-isometric embedding: there exist constants $\lambda > 1, c > 0$ such that*

$$\frac{1}{\lambda} d_S(g_1, g_2) - c \leq d(\phi(g_1), \phi(g_2)) \leq \lambda d_S(g_1, g_2) + c$$

Tips for (3): connect $g_1 o, g_2 o$ *be a geodesic segment and subdivide into segments of length 1, then apply (1).*