EXERCISE SHEET #4

Recall that $V(G, r) = \{x \in \mathbb{H}^n : d(x, gx) < r \text{ for some non-elliptic } g \in G\}$.

Exercise 0.1 (Thin part for parabolic type)**.** *Let G be an elementary discrete subgroup of parabolic type in $I^+(\mathbb{H}^n)$ with fixed point p. Assume that G is virtually $\mathbb{Z}^{n-1}$. Prove that for any $r > 0$, there exist G-invariant horoballs $B_1 \subset B_2$ centered at p such that $B_1 \subset V(G, r) \subset B_2$.*

Exercise 0.2. *Show that a complete hyperbolic manifold has finite volume if and only if for any $\epsilon > 0$, the thick part $M^{\geq \epsilon}$ has finite volume. Try to give an example showing that the quantifier "any" cannot be improved to "some". (Tips: prove that $M^{\geq \epsilon}$ is compact.)*

Recall that Mostow rigidity theorem says that the fundamental group of a finite volume hyperbolic n-manifold for $n \geq 3$ determines isometry type.

Exercise 0.3. *For any given $V, \epsilon > 0$ and $n \geq 2$, the set of complete hyperbolic n-manifolds with volume bounded above by V and injectivity radius bounded below by ϵ is finite.*

(Tips: prove that there are only finitely many homeomorphism types via a cellulation of $M^{\geq \epsilon}$ by polytopes by using Dirichlet construction in Ex. 3.)

Exercise 0.4. *For any closed orientable hyperbolic surface Σ_g, there exists a closed simple geodesic with length less than $2 \log(4g - 2)$.*

Exercise 0.5 (Simple closed geodesics)**.** *Let $M = \mathbb{H}^n/G$ for a discrete torsion-free subgroup G of $I^+(\mathbb{H}^n)$. Every loxodromic element g in G gives rise to a closed geodesic $[g]$ in M by projecting the axis $Ax(g)$ to M.*

Let $\epsilon_n > 0$ be the Margulis constant. Prove that

(1) *If the length of $[g]$ is smaller than $\epsilon_n/2$, then $[g]$ is simple.*
(2) *If two distinct closed geodesics $[g]$ and $[h]$ have length less than $\epsilon_n/2$, then $[g]$ and $[h]$ are disjoint.*

Separability. A subgroup H of a group G is called *separable* if H is the intersection of all finite index subgroups of G containing H. (In this language, G is residually finite if $\{1\}$ is separable.)

Let $G < I^+(\mathbb{H}^n)$ be a torsion-free discrete group. Let $A \subset \mathbb{H}^n$ be a subset invariant under a subgroup H of G (i.e.: $\forall h \in H, hA = A$). If for any $g \in G \setminus H$ we have $gA \cap A = \emptyset$, then A is called **strictly invariant** in G (under H).

Exercise 0.6. *Assume that H is separable in G. If A/H is compact with quotient topology, show that there exists a finite index subgroup Γ of G containing H such that A is strictly invariant under H in Γ. Consequently, the map $f : A/H \to \mathbb{H}^n/\Gamma$ is a topological embedding.*