EXERCISE SHEET #5

Conical points. Let G be a discrete subgroup in $I^+(\mathbb{H}^n)$. A limit point $p \in \Lambda G$ is called *conical* if there exist a sequence of distinct elements $g_n \in G$ and a point $x \in \mathbb{H}^n$ and a geodesic ray γ ending at p such that

$$\sup_{n\geq 1}\{\rho(g_n x, \gamma)\} \leq R < \infty.$$

Exercise 0.1. *Let p be a conical limit point. Prove that there exist a sequence of distinct elements $g_n \in G$ such that for any point $x \in \mathbb{H}^n$ and any geodesic ray γ at p we have*

$$\sup_{n\geq 1}\{\rho(g_n x, \gamma)\} < R < \infty$$

where R depends on the choice of x and γ.

Exercise 0.2. *Prove that a conical limit point p can not be fixed by a parabolic element h in a discrete group. (Tips: adapt the proof that hyperbolic elements cannot share fixed points with parabolic ones.)*

Exercise 0.3 (Finite sided fundamental polyhedron)**.** *Suppose that a discrete group G of $I^+(\mathbb{H}^n)$ admits a fundamental polyhedron with only finitely many sides. Show that G is geometrically finite. (Tips: examine the boundary point which are contained in infinitely many copies of fundamental polyhedron)*

Critical exponent. Suppose that G acts properly on $\mathbb{H}^n$. Consider the orbital counting function defined by

$$N(r, x, y) = \sharp\{g \in G : d(x, gy) \leq r\}$$

for $x, y \in \mathbb{H}^n$ and $r \geq 0$. Fix a constant $\Delta > 0$ and consider the annulus counting function

$$A(r, x, y) = \sharp\{g \in G : |d(x, gy) - r| \leq \Delta\}.$$

The critical exponent δ_G is given by the following

$$\delta_G = \limsup_{r\to\infty} \frac{\log N(r, x, y)}{r}$$

Exercise 0.4. *Prove that*

1. *The Poincaré series $\Theta_s(x, y) = \sum_{g\in G} e^{-sd(x,y)}$ converges for any $s > \delta_G$, and diverges for $s < \delta_G$.*
2. *For any fixed $\Delta > 0$, we have*

$$\delta_G = \limsup_{r\to\infty} \frac{\log A(r, x, y)}{r}$$

Exercise 0.5. *Let G be a discrete elementary group of parabolic type so that G is virtually $\mathbb{Z}^k$ for some $1 \leq k \leq n-1$. Prove that $\delta_G = k/2$.*

What is the critical exponent of an elementary group of hyperbolic type?

(Tips: verify that the hyperbolic distance is of order $2\log$ "the induced Euclidean distance" on the horosphere. Count elements in $\mathbb{Z}^k$ using the co-compact action of $\mathbb{Z}^k$ on some k-hyperplane in the horosphere.)

Exercise 0.6. *Suppose that G acts properly on $\mathbb{H}^n$. Prove that there exist a constant $D > 0$ depending on x, y such that for any $n \geq 1$*

$$N(r, x, y) \leq De^{r(n-1)}$$

In particular, we always have $\delta_G \leq n-1$. Prove that $\delta_G > 0$ if G is non-elementary.

If, in addition, G acts co-compactly on $\mathbb{H}^n$, then there exists a constant C independent of x, y such that for any $n \geq 1$

$$Ce^{r(n-1)} \leq N(r, x, y)$$

(Tips: note that the hyperbolic volume of a ball of radius r is of order $e^{(n-1)r}$.)