Conical points. Let $G$ be a discrete subgroup in $I^+(\mathbb{H}^n)$. A limit point $p \in \Lambda G$ is called conical if there exist a sequence of distinct elements $g_n \in G$ and a point $x \in \mathbb{H}^n$ and a geodesic ray $\gamma$ ending at $p$ such that
\[
\sup_{n \geq 1} \{ \rho(g_n x, \gamma) \} \leq R < \infty.
\]

Exercise 0.1. Let $p$ be a conical limit point. Prove that there exist a sequence of distinct elements $g_n \in G$ such that for any point $x \in \mathbb{H}^n$ and any geodesic ray $\gamma$ at $p$ we have
\[
\sup_{n \geq 1} \{ \rho(g_n x, \gamma) \} < R < \infty
\]
where $R$ depends on the choice of $x$ and $\gamma$.

Exercise 0.2. Prove that a conical limit point $p$ can not be fixed by a parabolic element $h$ in a discrete group. (Tips: adapt the proof that hyperbolic elements cannot share fixed points with parabolic ones.)

Exercise 0.3 (Finite sided fundamental polyhedron). Suppose that a discrete group $G$ of $I^+(\mathbb{H}^n)$ admits a fundamental polyhedron with only finitely many sides. Show that $G$ is geometrically finite. (Tips: examine the boundary point which are contained in infinitely many copies of fundamental polyhedron)

Critical exponent. Suppose that $G$ acts properly on $\mathbb{H}^n$. Consider the orbital counting function defined by
\[
N(r, x, y) = \sharp \{ g \in G : d(x, gy) \leq r \}
\]
for $x, y \in \mathbb{H}^n$ and $r \geq 0$. Fix a constant $\Delta > 0$ and consider the annulus counting function
\[
A(r, x, y) = \sharp \{ g \in G : |d(x, gy) - r| \leq \Delta \}.
\]
The critical exponent $\delta_G$ is given by the following
\[
\delta_G = \limsup_{r \to \infty} \frac{\log N(r, x, y)}{r}
\]

Exercise 0.4. Prove that
\begin{enumerate}
\item The Poincaré series $\Theta_s(x, y) = \sum_{g \in G} e^{-s d(x, y)}$ converges for any $s > \delta_G$, and diverges for $s < \delta_G$.
\item For any fixed $\Delta > 0$, we have
\[
\delta_G = \limsup_{r \to \infty} \frac{\log A(r, x, y)}{r}
\]
\end{enumerate}

Exercise 0.5. Let $G$ be a discrete elementary group of parabolic type so that $G$ is virtually $\mathbb{Z}^k$ for some $1 \leq k \leq n - 1$. Prove that $\delta_G = k/2$.

What is the critical exponent of an elementary group of hyperbolic type?

(Tips: verify that the hyperbolic distance is of order $2 \log$ “the induced Euclidean distance” on the horosphere. Count elements in $\mathbb{Z}^k$ using the co-compact action of $\mathbb{Z}^k$ on some $k$-hyperplane in the horosphere.)
Exercise 0.6. Suppose that $G$ acts properly on $\mathbb{H}^n$. Prove that there exist a constant $D > 0$ depending on $x, y$ such that for any $n \geq 1$

$$N(r, x, y) \leq De^{r(n-1)}$$

In particular, we always have $\delta_G \leq n-1$. Prove that $\delta_G > 0$ if $G$ is non-elementary.

If, in addition, $G$ acts co-compactly on $\mathbb{H}^n$, then there exists a constant $C$ independent of $x, y$ such that for any $n \geq 1$

$$Ce^{r(n-1)} \leq N(r, x, y)$$

(Tips: note that the hyperbolic volume of a ball of radius $r$ is of order $e^{(n-1)r}$.)