4. Groups acting on trees: a brief introduction to Bass-Serre trees

4.1. **Free products.** In this section, we first introduce a *free product* G of two given groups H, K. The group G is the biggest one among the groups generated by H, K with the property that any group are the quotient of G.

Precisely, a group G is called a *free product* of H and K if there exist a pair of homomorphisms $\iota_H : H \to G$ and $\iota_K : K \to G$ such that they are universal in the following diagram:

$$\begin{array}{ccccc} H & \xrightarrow{\iota_H} & G & \xleftarrow{\iota_K} & K \\ & \searrow^{\phi_H} & \vdots\,\Phi & \swarrow_{\phi_K} & \\ & & \Gamma & & \end{array}$$

By the universal property, it is easy to see that $\iota_H : H \to G$ and $\iota_K : K \to G$ are both injective. Moreover, G is unique up to isomorphism, so G must be generated by H and K.

Suppose that $H = \langle S|D\rangle, K = \langle T|E\rangle$ are given by presentations. Then by Theorem 3.3, the group G given by the presentation $\langle S \cup T|D \cup E\rangle$ is the free product of H, K.

Understanding H and K as disjoint alphabet sets, an *alternating word* w is of form $h_1k_1 \cdots h_mk_m$, where $h_i \in H, k_j \in K$. The length of w is the number of letters in word. It is called *reduced* if $h_i \in H \setminus 1, k_j \in K \setminus 1$.

We consider the set Ω of all reduced alternating words $h_1k_1 \cdots h_mk_m$ in H and K. The following result is fundamental in understanding free products.

Theorem 4.1. *[Normal form theorem]*[4, Thm 3.1] *Every element of $G = H \star K$ is equal to a unique alternating expression of the form $h_1k_1 \cdots h_mk_m \in \Omega$*

Sketch of the proof. Since G is genearted by H and K, any element in G can be written as an alternating expression of the form $h_1k_1 \cdots h_mk_m$. To prove the uniqueness, we shall construct a free action of G on the set Ω of all alternating expressions. To that end, we first construct the homomorphisms of H and K into the symmetry group of Ω and then by the above universal property, the homomorphisms of $G \to Sym(\Omega)$ is defined correspondingly.

For each $h \in H$, the associated bijection $\phi_H(h)$ is given by sending $h_1k_1 \cdots h_mk_m$ to $hh_1k_1 \cdots h_mk_m$ with a neccessary modification so that the image is alternating expression. Note that $\phi_H : H \to Sym(\Omega)$ is an injective homomorphism. Similarly, we can define $\phi_K : K \to Sym(\Omega)$ which is also injective. These define a group homormorphism $G \to Sym(\Omega)$.

Consider the empty word $\emptyset$ in Ω. Any alternating expression $h_1k_1 \cdots h_mk_m$ of element g in G maps $\emptyset$ to the alternating word $h_1k_1 \cdots h_mk_m$. This word is nonempty iff $g \neq 1$ This establishes the uniqueness of the statement. □

Corollary 4.2. *If an alternating word $w = h_1k_1 \cdots h_mk_m$ reprensents the identity in G, then it must be not reduced: there exists some i such that $h_i = 1$ or $k_i = 1$.*

In particular, if two reduced words represent the same group element, then they are equal letter by letter.

4.2. **Free products acting on trees.** Let $G = H \star K$ be a free product. We define a graph Γ as follows.

(1) The vertex set V consists of two types H and K: $V = \{gH, gK : g \in G\}$.
(2) The edge set E consists of all group elements in G.

(3) The edge $g \in E = G$ connects gH and gK.

Then G acts on Γ: each element $g \in G$ sends xH to gxH and xK to gxK. The edge relation is preserved. So G acts on Γ by graph isomorphism.

Theorem 4.3 (Bass-Serre Trees for free products)**.** *The graph Γ is a tree so that the degree of vertex of type H (resp. K) equals $\sharp H$ (resp. $\sharp K$).*

Moreover, the action of G on Γ has trivial edge stablizers and vertices stablizers of type H and K conjugated to H and K respectively so that the quotient is an interval.

Proof. By definition of action of G on the graph, there are two different orbits of vertices: $G \cdot H$ and $G \cdot K$. The vertex H is adjacent to hK for $h \in H$. That is to say, the set of edges adjacent to H has one-to-one correspondence with the set of elements in H. Similarly, the edges adjacent to K correspond to the set K. Since G is generated by H and K, the graph Γ is connected.

We now prove that Γ is a tree. Let γ be an immersed loop: there exists no backtracking. Up to a tranlation, we can assume that γ is based at H; the case at K is similar. According to the adjacency, there are an even number of edges in γ, where edges in γ must be of form $H \leftrightarrow hK$ for $h \in H$ or $K \leftrightarrow kH$ for $k \in K$ up to translation. Thus, tracing out the loop γ, we see that the terminal point is the vertex $h_1k_1 \cdots h_nk_nH$ for $h_i \in H$ and $k_i \in K$.

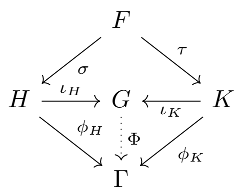
Since γ is a loop, we have the equality $h_1k_1 \cdots h_nk_nH = H$. We obtain that

$$h_1k_1 \cdots h_nk_n = h \tag{4}$$

for some $h \in H$. Since there exists no backtracking, we see that $k_i \neq 1$ for $i < n$ and $h_i \neq 1$ for $i > 1$. It is possible that $k_n = 1$ or $h_1 = 1$, but they cannot happen at the same time. So, up to removing h_1 or k_n from the left side in (4), we obtain a reduced word of length at least 2. But the right side in (4) is a reduced word of length 1. This is a contradiction to the normal form theorem 4.1. The graph Γ is thus a tree. The proof is complete. □

4.3. **Free products with amalgamation.** Now suppose that each of H and K contain a subgroup isomorphic to F: there exists monomorphisms $\sigma : F \to H$ and $\tau : F \to K$. We want to formulate a biggest group G generated by H and K so that $H \cap K = F$ is realized inside G.

Precisely, a group G is called a *free product* of H and K with malgamation over F if there exist a pair of homomorphisms $\iota_H : H \to G$ and $\iota_K : K \to G$ such that they are universal in the following diagram:



By the universal property, it is easy to see that $\iota_H : H \to G$ and $\iota_K : K \to G$ are both injective. Moreover, G is unique up to isomorphism, so G is generated by H and K.

Let $w = h_1k_1 \cdots h_nk_n$ be an alternating word over the alphabet set H and K such that $h_i \in H$ and $k_i \in K$. If it has length strictly bigger than 1 and

$h_i \in H \setminus F, k_i \in K \setminus F$, then w is called a *reduced* alternating word. Let Ω be the set of all such that $h_i \in H \setminus F$ and $k_i \in K \setminus F$, unless the alternating word is of length 1.

The following fact is obvious.

Lemma 4.4. *Every element of $G = H \star_F K$ is equal to a reduced alternating expression of the form $h_1 k_1 \cdots h_m k_m$. It may not be unqiue, but the length of the alternating word is unique.*

However, such an expression will not be unique, due to the nontrivial intersection F. To obtain a unique normal form, we have to choose a *right coset transversal* T_H and T_K of F in H and K respectively: namely, in each right H-coset, choose a right coset representative. We then consider the set Ω of words concatenating F with the alternating words in T_H and T_K.

Given a reduced alternating form $h_1 k_1 \cdots h_m k_m$, we convert the letters from right to left so that they become to be the corresponding right coset representatives. In the final form, we will get a *normal form* $f h_1' k_1' \cdots h_m' k_m'$ for some $f \in F$. So, any element has a normal form which turns out to be unique. Also note that in this process, the length of a normal form is the same as that of the original one.

Theorem 4.5. *[Normal form theorem]*[4, Thm 3.7] *With the choice of the right coset transversal T_H and T_K as above, every element of $G = H \star_F K$ is equal to a unique normal form $f h_1 k_1 \cdots h_m k_m$ with $f \in F, h_i \in T_H$ and $k_i \in T_K$ when present.*

Corollary 4.6. *If an alternating expression $h_1 k_1 \cdots h_m k_m$ gives the identity, then it is not reduced: there exists h_i or k_i such that $h_i, k_i \in F$.*

Let $G = H \star_F K$. We define a graph Γ as follows.

(1) The vertex set V consists of two types H and K: $V = \{gH, gK : g \in G\}$.
(2) The edge set E consists of all left F-cosets in G.
(3) The edge $gF \in E$ connects gH and gK.

Again, G acts on Γ: each element $g \in G$ sends xH to gxH and xK to gxK. The edge relation is preserved. So G acts on Γ by graph isomorphism.

We have the same result for free product with amalgamation. The only difference is that the edge stabilizer is a conjugate of F, instead of a trivial group.

Theorem 4.7 (Bass-Serre Tree for amalgamation)**.** *The graph Γ is a tree so that the degree of vertex of type H (resp. K) equals $\sharp H/F$ (resp. $\sharp K/F$).*

Moreover, the action of G on Γ has edge stablizers conjugated to F and vertices stablizers of type H and K conjugated to H and K respectively so that the quotient is an interval.

Proof. The proof is similar to that of Theorem 4.3. We emphasize the differences below.

We now prove that Γ is a tree. Let γ be an immersed loop: there exists no backtracking. Up to a tranlation, we can assume that γ is based at H; the case at K is similar. According to the adjacency, there are an even number of edges in γ, where edges in γ must be of form $H \leftrightarrow hK$ for $h \in H/F$ or $K \leftrightarrow kH$ for $k \in K/F$ up to translation. Tracing out the loop γ, we see that the terminal point is the vertex $h_1 k_1 \cdots h_n k_n H$ for $h_i \in H$ and $k_i \in K$.

Since γ is a loop, we have the equality $h_1k_1 \cdots h_nk_nH = H$. We obtain that

$$h_1k_1 \cdots h_nk_n = h \tag{5}$$

for some $h \in H$. Since there exists no backtracking, we see that $k_i \notin 1$ for $i < n$ and $h_i \notin F$ for $i > 1$. It is possible that $k_n \in F$ or $h_1 \in F$, but they cannot happen at the same time. So, combinning h_1 with k_1 or k_n with h_n from the left side in (5) if necessary, we obtain a reduced word of length at least 2. But the right side in (5) is a reduced word of length 1. This is a contradiction to the normal form theorem 4.5. The graph Γ is thus a tree. The proof is complete. □

4.4. **HNN extension.** Let G be a group with two isomorphic subgroups H and K. Let $\tau : H \to K$ be an isomorphism. We want to build a new group $\tilde{G}$ such that $G \subset \tilde{G}$ and H, K become conjugate in $\tilde{G}$. If G is given by a presentation

$$\langle S | \mathcal{R} \rangle.$$

As usual, we request $\tilde{G}$ to be the biggest one with this property. Then the desired group $\tilde{G}$ must have presentation as follows

$$\langle S, t | \mathcal{R}, tht^{-1} = \tau(h), \forall h \in H \rangle,$$

which is called *HNN* extension of G over H, K, denote by $G\star_{H\sim K}$. The new generator t is usually called *stable* letter.

By definition, every element in $\tilde{G}$ can be written as a product of form called *t-expression* as follows:

$$g_0 t^{\epsilon_1} g_1 \cdots t^{\epsilon_n} g_n$$

where $g_i \in G$ or $\epsilon_i \in \{1, -1\}$. Any subword tht^{-1} for $h \in H$ and $t^{-1}kt$ for $t \in K$ is called *t-pinch* in the above form. A t-expression form without t-pinches is called *reduced*.

A reduced t-expression of an element may not be unique. In order to get a normal form, we choose right coset transversal T_H and T_K of H and K in G respectively.

Theorem 4.8. *[Normal form theorem]*[4, Thm 3.1] *Every element of $\tilde{G} = G\star_{H\sim K}$ is equal to a unique reduced t-expression of the form $g_0 t^{\epsilon_1} g_1 t^{\epsilon_2} g_2 \cdots t^{\epsilon_n} g_n$ with $g_i \in T_H \cup T_K$ when present. If $\epsilon_i = 1$ for $i > 0$, then $g_i \in T_H$; if $\epsilon_i = -1$ for $i > 0$ then $g_i \in K$.*

Corollary 4.9 (Briton's Lemma)**.** *If a t-expression in $\tilde{G} = G\star_{H\sim K}$ represents the trivial element, then it must contain t-pinches.*

Let $\tilde{G} = G\star_{H\sim K}$. We define a graph Γ as follows.

(1) The vertex set V consists of all left cosets of G: $V = \{xG : x \in \tilde{G}\}$.
(2) The edge set E consists of two types H and K: all left H-cosets and K-cosets in $\tilde{G}$.
(3) The edge $xK \in E$ connects xG and xtG; And $xH \in E$ connects $xt^{-1}G$ and xG.

We define an action of $\tilde{G}$ on Γ: each element $g \in \tilde{G}$ sends xG to gxG. The edge relation is preserved. So G acts on Γ by graph isomorphism.

Though we have two types of edges, they actually belongs to the same orbit: the element t sends the edge $[t^{-1}G, G]$ to $[G, tG]$.

Theorem 4.10 (Bass-Serre Tree for HNN extension)**.** *The graph Γ is a tree so that the degree of vertex equals $\sharp G/H + \sharp G/K$. Moreover, the action of G on Γ has edge stablizers conjugated to H (or equivalently K) and vertices stablizers G so that the quotient is a loop.*

Proof. The proof is similar to that of Theorem 4.3. In this case, there are only one orbit of vertices and edges.

We now prove that Γ is a tree. Let γ be an immersed loop: there exists no backtracking. Up to a tranlation, we can assume that γ is based at G. According to the adjacency, edges in γ must be of form as follows, up to translation,

$$G \leftrightarrow gtG$$

for $g \in G/K$ or

$$gt^{-1}G \leftrightarrow G$$

for $g \in G/H$. Thus, tracing the loop γ, we see that the terminal point is the vertex $g_1 t^{\epsilon_1} g_2 t^{\epsilon_2} \cdots g_n t^{\epsilon_n} G$ for $g_i \in G$ and $\epsilon_i \in \{1, -1\}$.

Since γ is a loop, we have the equality $g_1 t^{\epsilon_1} g_2 t^{\epsilon_2} \cdots g_n t^{\epsilon_n} G = G$. We obtain that

$$g_1 t^{\epsilon_1} g_2 t^{\epsilon_2} \cdots g_n t^{\epsilon_n} = g \tag{6}$$

for some $g \in G$.

Since there exists no backtracking, we see that if $\epsilon_i = 1$ and $\epsilon_{i+1} = -1$, then $g_{i+1} \in G \setminus H$. Indeed, if not, we have $g_{i+1} \in H$. Recalling that G acts transtively on Γ, we can assume that $i = 1$. Thus, $g_1 t g_2 t^{-1} G = G$. So the subpath

$$G \leftrightarrow g_1 tG \leftrightarrow g_1 t g_2 t^{-1} G = G$$

gives a backtracking in γ.

By the same reasoning, if $\epsilon_i = -1$ and $\epsilon_{i+1} = 1$, then $g_{i+1} \in G \setminus K$. Again, it is possible that $g_1, g_n \in H \cup K$, but it cannot happen at the same time that $g_1, g_n \in H$ and $g_1, g_n K$. So, combinning h_1 with k_1 or k_n with h_n from the left side in (6) if necessary, we obtain a reduced word of length at least 2. But the right side in (6) is a reduced word of length 1. This is a contradiction to the normal form theorem 4.8. The graph Γ is thus a tree. The proof is complete. ☐

Exercise 4.11. *Draw a portion of the Cayley graph of the group $\langle a, t : ta^2t^{-1} = a^3 \rangle$.*

4.5. **Abelianization of free products.** Let $G = H \star K$ be a free product. By the universal property, there exists a natural morphism $G \to H \times K$. We denote by N the kernel of this morphism.

We start with a classification of isometries on trees.

Lemma 4.12. *Let g be an isometry of a tree T. Then either g fixes a point or g preserves a unique geodesic by translation.*

Proof. Suppose that g does not fix any point. Fix a basepoint $o \in T$. Consider the geodesic segments $[o, go]$ and $[o, g^{-1}]$ both originating from o. Since T is a tree, let b be the branching point of these two geodesics. We then claim that $d(o, m) < d(o, go)/2$. Otherwise, the middle point m of $[o, go]$ coincides with that of $[o, g^{-1}o]$, which is gm. So m is fixed by g: a contradiction. Hence, $d(o, b) < d(o, go)/2$. We now form a geodesic γ preserved by g. Let $\gamma = \cup_{i \in \mathbb{Z}} [g^{-1}m, m]$. It is clear by the the above Claim that γ is a geodesic. The proof is then complete. ☐

Exercise 4.13. *Prove the uniqueness statement in Lemma 4.12.*

Exercise 4.14. *Any finite group acts on a tree with a global fixed point.*

Corollary 4.15. *All finite subgroups in a free product must be conjuated into H or K.*

Exercise 4.16. *Let $G = H \star_F K$ be a free product of non-trivial groups H, K over F. Using Bass-Serre tree to prove that the center of G is contained in F.*

Lemma 4.17. *N is a free group generated by $S = \{[h,k] : h \in H \setminus 1, k \in K \setminus 1\}$.*

Proof. By Theorem 2.20, we only need to show that N acts freely on the Bass-Serre tree Γ. To prove the freeness of N, any vertex stabilizer of the Bass-Serre Γ is sent to a non-trivial subgroup under the morphism $H \star K \to H \times K$. This implies that the kernel N of this morphism acts freely Γ. Thus the conclusion follows. □