EXERCISE SHEET #1

Let (X, d) be a geodesic metric space. We denote by $[x, y]$ a choice of a geodesic between x and y. Here we collect a few elementary facts in general metric spaces.

Exercise 0.1. *Let γ be a geodesic in X. Let $x \in X$ and $y \in \pi_\gamma(x)$. Then for any point $z \in \gamma$, we have the path $[x, y][y, z]$ is a $(3, 0)$-quasi-geodesic.*

Could you propose a version of this statement if γ is a (λ, c)-quasi-geodesic.

Exercise 0.2. *Let p be a rectifiable path in X so that $Len(p) \le d(p_-, p_+) + c$ for some $c > 0$. Then any subpath q of p satisfies $Len(q) \le d(q_-, q_+) + c$.*

Exercise 0.3. *Let x, y, z be any points in X. Then $\langle x, y\rangle_z \le d(z, [x, y])$.*

Exercise 0.4. *Let α, β be two (λ, c)-quasi-geodesics for $\lambda, c > 0$. If $\alpha \subset N_D(\beta)$ for some $D > 0$, then $\beta \subset N_{2\lambda D + c}(\alpha)$.*

A geodesic α is *C-contracting* for some $C \ge 0$ if for any metric ball B with $B \cap \alpha = \emptyset$, $diam(\pi_\alpha(B)) \le C$.

Exercise 0.5 (Alternative proof of Morse Lemma)**.** *Let α be a C-contracting geodesic. Then for any $\lambda, c > 0$, there exists $D = D(\lambda, c, C) > 0$ with the following property. Let p be any (λ, c)-quasi-geodesic with two endpoints on α. Then $p \subset N_D(\alpha)$. (Tips: find an appropriate cover of p by balls and then project them to α.)*

We assume now that (X, d) is a δ-hyperbolic space.

Exercise 0.6 (Strengthened version of Morse Lemma)**.** *Let p be a path in (X, d). Given a non-decreasing function $f : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$, let p be a path such that $Len(q) \le f(d(q_-, q_+))$ for any subpath q of p. Assume that f is sub-exponential, i.e.:*

$$\lim_{n\to\infty} \log f(n)/n = 0$$

Then p is a quasi-geodesic. (Tips: prove that p is contained in a uniform neighborhood of $[p_-, p_+]$.)