EXERCISE SHEET #3

Prove that there are only finitely many conjugacy classes of finite subgroups in a hyperbolic group. You may proceed by the following steps:

Execise 0.1. *Assume that a group G acts geometrically on a proper hyperbolic space (X, d).*

(1) *Define a notion of the center for any bounded set B in a metric space X. Define first the radius of B:*

$$r_B := \inf\{r : B \subset B(x, r), r \geq 0, x \in X\}.$$

where $B(x, r)$ is the closed ball of radius r at x. The center of B is then defined to be set of points $o \in X$ such that

$$B \subset B(o, r_B + 1).$$

(2) *Prove that if X is δ-hyperbolic space, the center of any bounded set is bounded by a constant depending only on δ.*

(3) *Apply the assertion (2) to the orbit $B = F \cdot x$ of a finite group F of G, and conclude the proof that there are finitely many conjugacy classes of finite subgroups F.*

Here are two useful facts about quasiconvex subgroups.

Execise 0.2. *If H is a undistorted subgroup in a hyperbolic group G, then it is quasi-convex.*

Execise 0.3. *Prove that any finitely generated subgroup in a free group of finite rank is quasiconvex.*

The following fact allows to solve conjugacy problem for hyperbolic groups.

Execise 0.4. *Let g, h be two conjugate elements in a hyperbolic group G. Prove that there exists a short conjugator $f \in G$ of length at most $D = D(|g|, |h|)$ so that $g = fhf^{-1}$.*