EXERCISE SHEET #4

Let X be a proper δ-hyperbolic space with Gromov boundary ∂X.

Execise 0.1. *There exists a uniform constant C depending only on δ such that the following thin triangle property holds.*

Let $x, y, z \in X \cup \partial X$ be any triple of distinct points. Then any geodesic $[x, y]$ is contained in the C-neighborhood of $[x, z] \cup [y, z]$.

Let X be a metric complete space and A be a closed subset. Let $\pi_A : X \to A$ be the shortest projection (set-valued) map so that $\pi_A(x)$ is the set of points $a \in A$ satisfying $d(x, a) = d(x, A)$.

Execise 0.2. *Let $\phi : X \to Y$ be a (λ, c)-quasi-isometry between two proper geodesic δ-hyperbolic spaces X, Y. Let γ be a geodesic. Prove that there exists a constant $D = D(\lambda, c, \delta)$ such that for any point $x \in X$,*

$$d_H(\phi(\pi_\gamma(x)), \pi_{\phi\gamma}(\phi(x))) \leq D$$

where d_H denotes the Hausdorff distance.

We say that a (not necessarily geodesic) metric space X is δ-hyperbolic if for any four points x, y, z, w, we have

$$\langle x, y\rangle_w \geq \min\{\langle x, z\rangle_w, \langle z, y\rangle_w\} - \delta. \tag{1}$$

If X is a geodesic metric space, this is equivalent to the usual thin triangle property.

Definition 0.3 (Gromov boundary defined by sequences)**.** A sequence (x_n) in X *converges at infinity* if $(x_i, x_j)_o \to \infty$ as $i, j \to \infty$. Two such sequences $(x_n), (y_n)$ are called *equivalent* if $(x_i, y_j)_o \to \infty$ as $i, j \to \infty$. The *Gromov boundary* $\partial_s X$ of X is the set of all equivalent classes of sequences converging at infinity.

Execise 0.4. *If X is a proper geodesic hyperbolic space, there exists a natural bijection from $\partial_s X$ to ∂X.*

By using (1), we can prove the following.

Execise 0.5. *Consider $w, x, y, z \in X$, and $C \geq 0$. Assume $\langle w, y\rangle_x \leq C$ and $\langle x, z\rangle_y \leq C + \delta$ and $d(x, y) \geq 2C + 2\delta + 1$. Then $\langle w, z\rangle_x \leq C + \delta$.*

Definition 0.6. For $C, D \geq 0$, a sequence of points $x_0, \cdots, x_n$ is a (C, D)-*chain* if one has $\langle x_{i-1}, x_{i+1}\rangle_{x_i} \leq C$ for all $0 < i < n$, and $d(x_i, x_{i+1}) \geq D$ for all $0 \leq i < n$.

Using induction via the previous exercise, we can prove the following very useful fact, saying that a "long" local quasi-geodesic is a global quasi-geodesic.

Execise 0.7. *Let $x_0, \cdots, x_n$ be a (C, D)-chain with $D \geq 2C + 2\delta + 1$. Then $\langle x_0, x_n\rangle_{x_1} \leq C + \delta$, and*

$$d(x_0, x_n) \geq \sum_{i=0}^{n-1} (d(x_i, x_{i+1}) - (2C + 2\delta)) \geq n.$$

Corollary 0.8. *In particular, if $D > 2(2C+2\delta)$ then $2d(x_0, x_n) \geq \sum_{i=0}^{n-1} d(x_i, x_{i+1})$. This implies that the path $\cup_{i=0}^{n-1}[x_i, x_{i+1}]$ is a $(2, 4C + 4\delta + 2)$-quasi-geodesic.*