Hyperbolic Boundaries of Groups

Lecture 1

2022.08.15

Hyperbolic Boundaries of Groups:

Lecture 1: End boundary

Lecture 2: Floyd boundary

Lecture 3: Horofunction boundary

Compactification: X locally compact, Hausdorff top. space

$$X \hookrightarrow \overline{X} \quad \text{compact Hausdorff}$$

open dense

$$\partial X := \overline{X} \setminus X \quad \text{boundary}$$

Question:

Given $(X,d) \xrightarrow{\Phi} (Y,d)$, with $\partial X \subseteq \overline{X}$ and $\partial Y \subseteq \overline{Y}$, and a dashed arrow $\partial X \overset{\partial_\infty \Phi}{\dashrightarrow} \partial Y$: when Φ extends continuously to their boundaries as a continuous map?

$$\partial_\infty \Phi : \partial X \to \partial Y$$

Rmk: Application in Mostow Thm

Def: $\Phi : (X,d) \longrightarrow (Y,d)$ is called (λ, c)-quasi-isometry for $\lambda \geq 1$, $c \geq 0$

- Φ is coarsely bilipschitz map, coarsely injective

$$\frac{1}{\lambda} d(x,x') - c \leq d(\Phi x, \Phi x') \leq \lambda\, d(x,x') + c$$

↳ (λ, c)-q. isometric embedding

- coarsely surjective:

$$\exists R \geq 0 : \quad N_R(\Phi X) \supseteq Y$$

I. End boundary (H. Freudenthal)

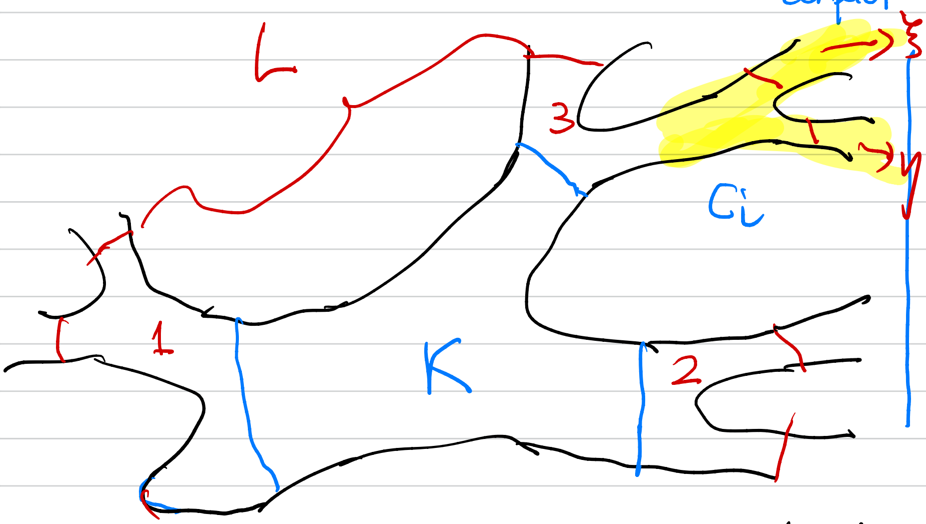
Assume X is metric & proper [locally compact ⊕ completeness]

Ex: X = Graph

$$\partial_E X \triangleq \varprojlim \left\{ \mathrm{Comp}^{\infty}(X \setminus K) \xleftarrow{\text{inclusion}} \mathrm{Comp}^{\infty}(X\setminus L) \right\}$$

$\mathrm{Comp}^{\infty}(X \setminus K)$ = { ∞-components of $X \setminus K$ }

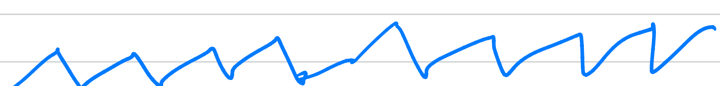
compact $K \subseteq L$



$$\partial_E X = \left\{ (C_1, C_2 \cdots C_n \cdots) \in \prod_K \underbrace{\mathrm{Comp}^{\infty}(X \setminus K)}_{\text{finite set}} : C_1 \supseteq C_2 \supseteq C_3 \supseteq \cdots \right\}$$

Cantor set

- $\partial_E X$ is compact space: $\partial_E X \subseteq \overbrace{\prod_K \underbrace{\mathrm{Comp}^{\infty}(X \setminus K)}_{<\infty}}^{\text{closed}}$ Compact
  - $\xi \in \partial_E X$: $\xi = (C_1 \supseteq C_2 \supseteq C_3 \cdots)$
  - Neighborhood base: $\{C_i\}$
- $\partial_E X$ = { one-sided embedded rays α, } $/ \sim$



$\alpha \sim \beta \iff \alpha, \beta$ is eventually in the component of $X \setminus K$

Proper ⟺ locally finite degree

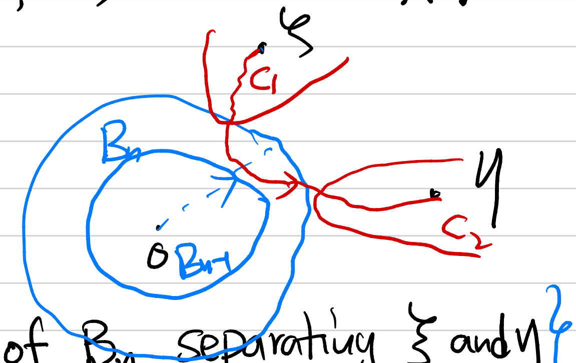
We are interested in graphs X:

$$B_n \triangleq \{ v \in X^0 : d(o, x) \leq n \}$$

X^0 = vertex set

(Cornulier, ...)

- Visual metric: Fix $\lambda \in (0,1)$ and $o \in X^0$.



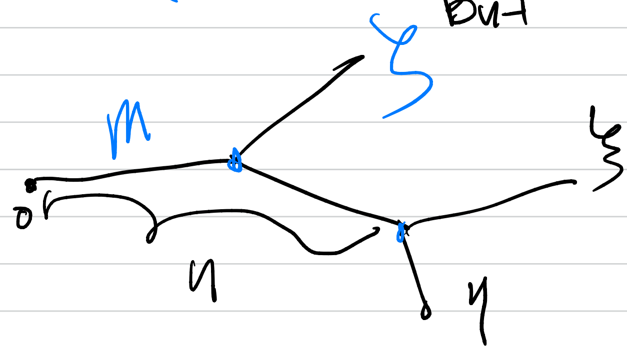
$$\rho_\lambda^o(\xi, \eta) \triangleq \lambda^n$$

where ($\xi \neq \eta$)

$$n \triangleq \min \{ \text{radius } n \text{ of } B_n \text{ separating } \xi \text{ and } \eta \}$$

$$= \min \{ n: \text{any path from } \xi \text{ to } \eta \text{ passes through } B_n \}$$

$$= \max \{ n+1: \text{some path from } \xi \text{ to } \eta \text{ disjoint } B_{n-1} \}$$

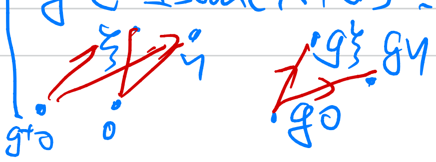


- Ex: 1) ρ_λ^o is a metric. It's ultrametric.

2) $\lambda^{d(o,o')} \rho_\lambda^{o'}(\xi, \eta) \leq \rho_\lambda^o(\xi, \eta) \leq \lambda^{-d(o,o')} \rho_\lambda^{o'}(\xi, \eta)$

3) Quasi-isometries extend to boundaries as bilipshitz homeomorphism.

[$g \in Isom(X,d)$: $\partial_\infty g : \partial_E X \longrightarrow \partial_E X$ is bilip of constant $\lambda^{d(o, go)}$



$\partial_\infty \mathbb{E}^2 = \{1\,pt\}$ $\partial_\infty \mathbb{R} = \{+\infty, -\infty\}$

- Thm (Freudenthal - Hopf)

If X = Cayley graph of finitely generated group G

$G \curvearrowright Cay(G, S)$, $V = G$

$E: \{g_1 \to g_2 \Leftrightarrow g_2 = g_1 s \quad s \in S\}$

then $|\partial_E X| \in \{0, 1, 2, \underset{\text{uncountable}}{\infty}\}$

$0 \Updownarrow$ finite gp; $2 \Updownarrow$ $G \underset{\text{fi}}{\geq} \mathbb{Z}$

- Thm (Stallings) G f.g.

$Cay(G, S)$ has ∞-many ends

$\Updownarrow$

$G = H *_F K$ or $G = H *_{F_1 \sim F_2}$

F: finite $\qquad F_1 \cong F_2$ finite

$\Leftrightarrow$ $G \curvearrowright$ Tree with finite edge stabilizer

Rem: ∞-ends gp is quasi-isom. invariant.

- Thm: (Stallings) $G \overset{\text{homeo}}{\curvearrowright} \underbrace{\partial_E Cay(G, S)}_{\partial_E X}$ is a minimal convergence

group action: $\forall \underset{\text{inf. set}}{\{g_n\}} \subseteq G$ $\exists$ subsequence $\{g_{n_i}\}$ & $a, b \in \partial_E X$

$x \quad g_{n_i} x \quad b \quad g_{n_i}^{-1} y \quad a$

s.t.

South-North dynamic $\begin{cases} g_{n_i} x \xrightarrow[\text{unif convergence}]{} a & \forall x \in \partial_E X \setminus b \\ g_{n_i}^{-1} y \longrightarrow b & \forall y \in \partial_E X \setminus a \end{cases}$

Cor: $|\partial_E X| = \infty$ then $X \supseteq \mathbb{F}_2$

- Thm [Dussaule – Y. 2020]

Assume $X = \mathrm{Cay}(G, S)$ is ∞-ended

Then: $\forall \lambda \in (0, 1)$

$$\mathrm{Hdim}_{\rho_\lambda}(\partial_E X) = -\frac{1}{\log \lambda} \omega_G$$

where

Growth rate $\omega_G \triangleq \lim_{n\to\infty} \frac{\log \# B_n = \{ v \in X^0 : d(o, v) \leq n \}}{n}$

$$\mathrm{Hdim}(X, d) = \sup \{ r : H^r(X) = \infty \}$$
$$= \inf \{ r \geq 0 : H^r(X) = 0 \}$$

$s \geq 0$:

$$H^s(X) \triangleq \lim_{\varepsilon \to 0} \mathrm{Inf} \left\{ \sum_{i\in I} r_i^s : X \subseteq \bigcup_{i\in I} B(x_i; r_i) \quad r_i \leq \varepsilon \right\}$$

I Countable

↳ ε-covering of X

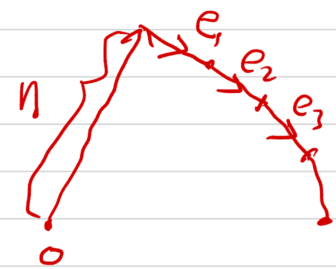
Hyperbolic Boundaries of Groups

Lecture 2

2022.08.17

I. Floyd boundary $X = $ (locally finite graph, combinatorial metric d)

- Fix basepoint $o \in X$
- Fix a summable, λ-slow decay function: $f: \mathbb{N} \to \mathbb{R}_{>0}$
  - $\sum_{n=1}^{\infty} f(n) < \infty$
  - $\exists \lambda \in (0,1)$: $\lambda f(n) \leq f(n+1) \leq f(n)$



Examples: • $f(n) = \frac{1}{n^2}$, $\frac{1}{n^p}$ $p > 1$

• $f(n) = \lambda^n$

- Floyd metric: edge e $\rightsquigarrow$ $\ell_f(e) \triangleq f(d(o,e))$

  path $\gamma = e_1 \cdots e_n$ $\rightsquigarrow$ $\ell_f(\gamma) \triangleq \sum_{i=1}^{n} \ell_f(e_i)$

  $\rightsquigarrow$ $\delta_f^o(x,y) \triangleq \inf \{ \ell_f(\gamma) : \gamma : x \to y \text{ path} \}$

  $x, y \in X^0$

- Floyd Compactification: $(\overline{X}_f, \delta_f^o) \triangleq$ Cauchy completion of (X, δ_f^o)

$$\partial_f X \triangleq \overline{X}_f \setminus X$$

- Basic Facts:

1) $\overline{X}_f$ is compact. $\Leftrightarrow$ complete + totally bounded. $\forall \varepsilon > 0$ $\exists F \subseteq \overline{X}_f$ $|F| < \infty$

   $\forall \varepsilon, \exists n$. $F \triangleq B(o,n) \cap X^0$ $N_\varepsilon(F) \supseteq \overline{X}_f$

2) Geodesic rays from o • converges to a boundary pt

   • is Floyd geodesic

3) change basepoints: $\lambda^{d(o,o')}\delta_f^{o}(x,y) \leq \delta_f^{o'}(x,y) \leq \lambda^{-d(o,o')}\delta_f^{o}(x,y)$

$\Uparrow$ $f(d(o',e)) \geq f(d(o,e)+d(o,o'))$

$\Rightarrow G \curvearrowright \partial_f Cay(G,S)$ by bilipschitz homeomorphisms

4) change scaling function: 1) If $f \geq g$, then $\partial_f X \to \partial_g X$. 1-Lip.

2) If $\forall c>0$, $\exists M, N$ s.t. $M f(n) \leq f(cn) \leq N f(n)$

Ex: $f(n) = \frac{1}{n^2}$

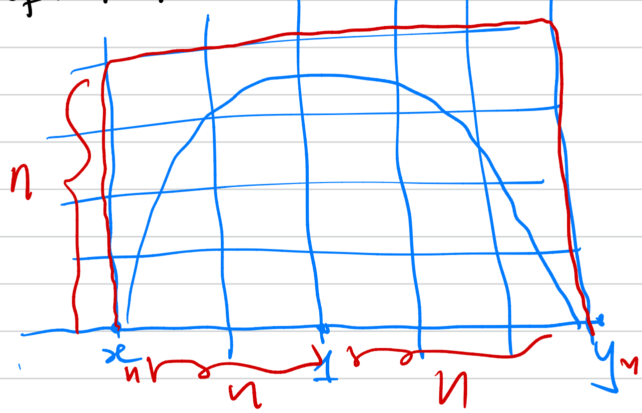
then $\partial_f Cay(G,S) \underset{\text{bi-Lip}}{\overset{\text{homeo}}{\cong}} \partial_f Cay(G,T)$ $\forall S, T$

Def: Trivial Floyd boundary $\iff$ $|\partial_f X| \leq 2$ for some $f>0$

- Examples: $X = Cay(G,S)$

1) $\mathbb{Z}^2$: $|\partial_f X| = 1$

$\rho_f^o(x_n, y_n) \leq 4n f(n) \to 0$



2) $\mathbb{Z}$: $\partial_f X \cong \{\pm\infty\}$

3) Any subexponential growth group: $|\partial_f X| \leq 2$

4) Mapping class groups: $\partial_f X = 1$

Right-Angled Artin gps: $|\partial_f X| \leq 2$ unless graph is NOT connected

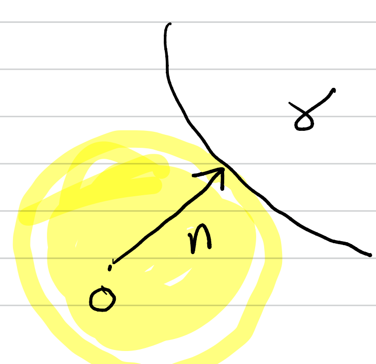
- Visibility Lemma (Karlsson)

Lemma: $\exists k: \mathbb{N}_{>0} \to \mathbb{R}_{>0}$ $k(n) \to 0$ as $n \to +\infty$ s.t.

for any geodesic γ,

$\ell_f(\gamma) \leq k(n)$

where $n \triangleq d(o,\gamma)$



Proof:

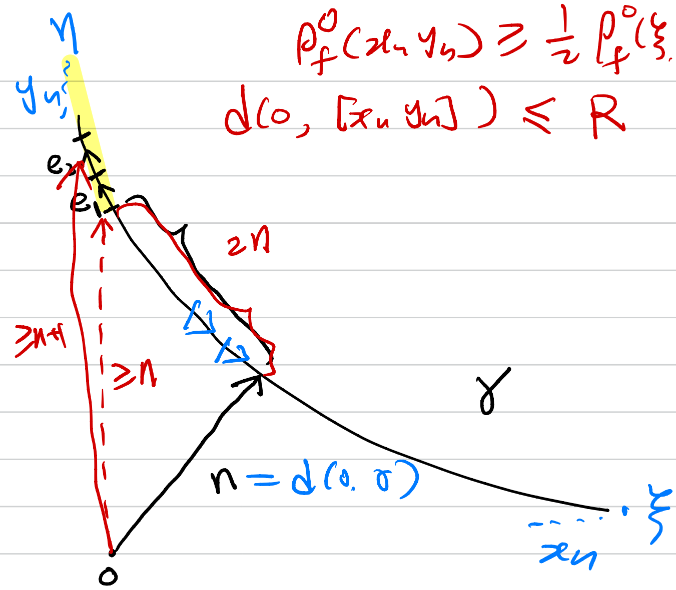
$$\ell_f(\gamma) \leq \sum_{i=1}^{\infty} f(n+i) + 2n \cdot f(n)$$

$$\triangleq k(n) \qquad\qquad 2n\cdot f(n) \to 0$$

$$\lim_{n\to\infty} k(n) = 0$$

$\rho_f^0(x_n, y_n) \geq \frac{1}{2} \rho_f^0(\xi, \eta)$

$d(o, [x_n, y_n]) \leq R$



Cor: $\exists R: \mathbb{R}_{>0} \longrightarrow \mathbb{N}_{>0}$ s.t.

If $\delta_f^o(\xi, \eta) \geq k > 0$ for $\xi, \eta \in \partial_f G$

then $d(o, [\xi, \eta]) \leq R(k)$

Any two distinct points $\xi, \eta \in \partial_f X$ can be connected by a geodesic.

- elliptic: $g^n = 1 \quad \exists n$
- parabolic: $|Fix(g)| = 1$ $\{g^n\}$
- hyperbolic: $|Fix(g)| = 2$

homeo $\triangleq \partial_f Cay(G, S)$ $\quad \{g^n\}$

Thm (Karlsson) $G \curvearrowright \partial_f G$ is a convergence group action:

$\forall g_n \subseteq G \quad \exists \{g_{n_i}\}, \ \exists a, b \in \partial_f G$ s.t.

$g_{n_i}|_{\partial_f G \setminus a} \xrightarrow{\text{unif. conv}} b$ & $g_{n_i}^{-1}|_{\partial_f G \setminus b} \xrightarrow{\text{unif. conv}} a$ $\qquad U \supseteq g_{n_i} K$

Proof:

- Fix compact $K \subseteq \partial_f G \setminus \{a\}$.

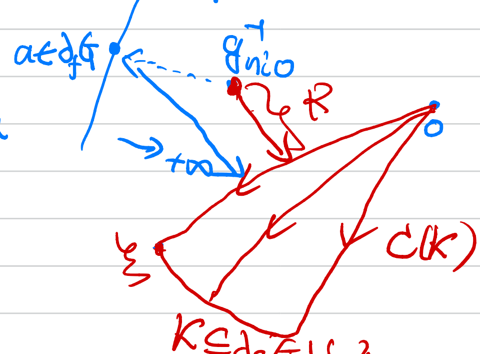
  Take g_{n_i} s.t. $g_{n_i} o \to b$, $g_{n_i}^{-1} o \to a$

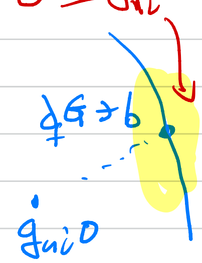
- Consider cone over K from o:

  $C(K) = \bigcup_{\xi \in K} [o, \xi]$

- Key Fact: $a \cap K = \phi$

  $d(g_{n_i}^{-1} o, C(K)) \longrightarrow +\infty$





- $\delta_f^o(g_{n_i} o, g_{n_i} K) = \delta_f^{g_{n_i}^{-1} o}(o, K) \longrightarrow 0$ as $n_i \to +\infty$

  $g_{n_i} o \to b$ $\qquad$ Visibility Lemma

▨

- **Relation with end boundary:**

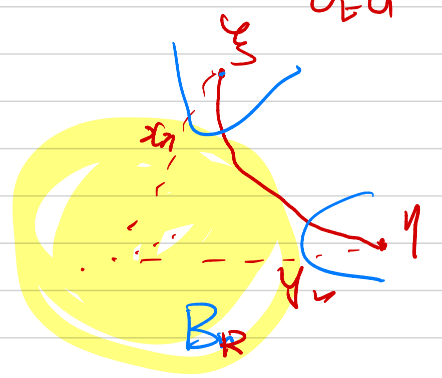
Lem: The identification on G extends to a continuous map ($x_n \in G$)

$$\Phi : \partial_f G \twoheadrightarrow \partial_E G$$

($\forall (x_n) \to p \longmapsto$, $(y_n) \to$)

If $f = \lambda^n$ then it is 1-Lipshitz map

$$\rho_\lambda^0(\Phi\xi, \Phi\eta) \leq \delta_f^0(\xi, \eta)$$



If $\xi \neq \eta \in \partial_E G$

then $\delta_f^0(x_n, y_n) > f(R)$

- **Floyd's thm in Kleinian groups:**

Suppose $G < PSL(2, \mathbb{C}) \curvearrowright \mathbb{H}^3$ is a discrete Geom. finite gp.

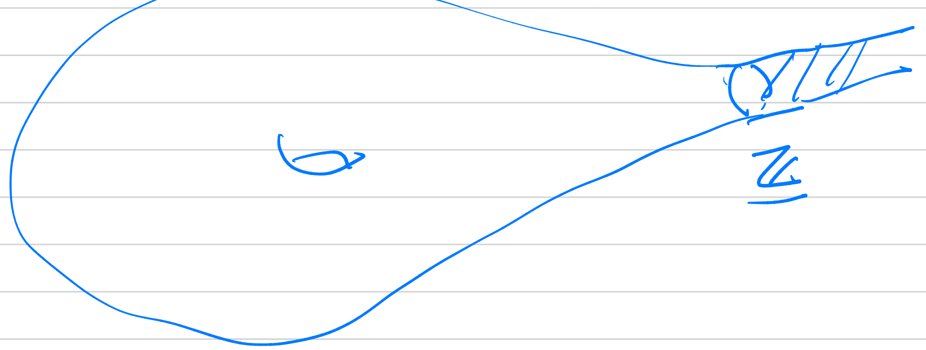
Then for $f(n) := \frac{1}{n^2}$ $\exists$ a continuous surjection

$$\Phi : \partial_f G \twoheadrightarrow \Lambda G \triangleq \{\text{Accumulation pts of } Go \subseteq \overline{\mathbb{H}^3}\}$$

s.t.

Φ fails to injective only on the parbolic fixed pt p of rank-1.

$\# \Phi^{-1}(p) = 2$



Hyperbolic Boundaries of Groups

Lecture 3

2022.08.19

Hyperbolic Boundaries of Groups

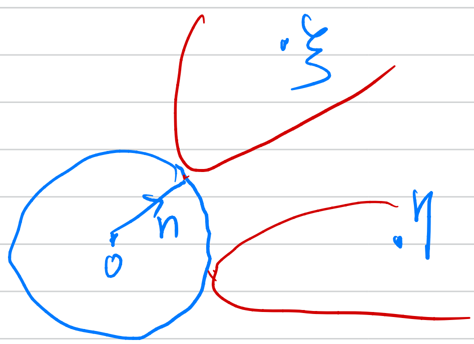
Review: $X = (\text{loc. finite graph}, \text{combinatorial metric})$

1) End boundary:

$$\partial_E X \triangleq \varprojlim_{K:\text{ compact}} \mathrm{Comp}^\infty (X \setminus K)$$

$(\partial_E X, \rho_\lambda^o)$ $\quad o \in X$, $\forall \lambda \in (0,1)$

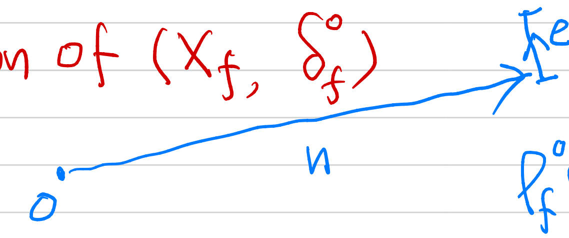
$$\rho_\lambda^o(\xi, \eta) \triangleq \lambda^n \quad n = \text{minimal separating radius}$$



2) Floyd boundary: $f(n) = \frac{1}{n^2}$ $\quad f(n) = \lambda^n$ $\quad \lambda \in (0,1)$

$$\partial_f X \triangleq (\overline{X}_f \setminus X, \delta_f^o)$$

Cauchy completion of (X_f, δ_f^o)



$\rho_f^o(e) = f(n)$

Fact:

1) $\exists\, \tilde{\Phi}: (\partial_f X, \delta_f^o) \xrightarrow{\text{continuous}} (\partial_E X, \rho_\lambda^o)$

If $f(n) = \lambda^n$, it is 1-Lipschitz.

(Potyagailo.-Y.)

Thm: Let G be Relatively hyperbolic groups { • G: ∞-ends • hyp groups • $vol(M) < \infty$ }

Then for $f(n) = \lambda^n$, for some $\lambda \in (0,1)$

$$\mathrm{Hdim}(\partial_f G, \delta_f^o) = -\frac{1}{\log \lambda}\, \omega_G \qquad \omega_G = \lim_{n\to\infty} \frac{\log |B_n|}{n}$$

2) $G \curvearrowright \partial_f G / \partial_E G$ is a convergence action

elliptic, para. hyp

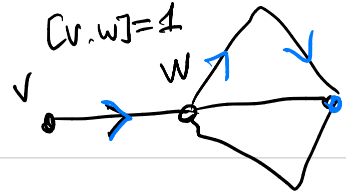
Thm (Floyd 80', Gerasimov 12')

• G Relatively hyperbolic group: $G \curvearrowright$ proper hyp space [geom. finite]

Then for some $\lambda \in (0,1)$ Bowditch

$\exists\, \phi: \partial_f G \twoheadrightarrow \partial_B G \overset{\Delta}{=}$ Gromov bdry of some

s.t. Φ fails to be injective only at countable parabolic pts



Examples:

MCG

1) RAAGs have trivial Floyd boundary (≤ 2 pts) unless the defining graph is disconnected.

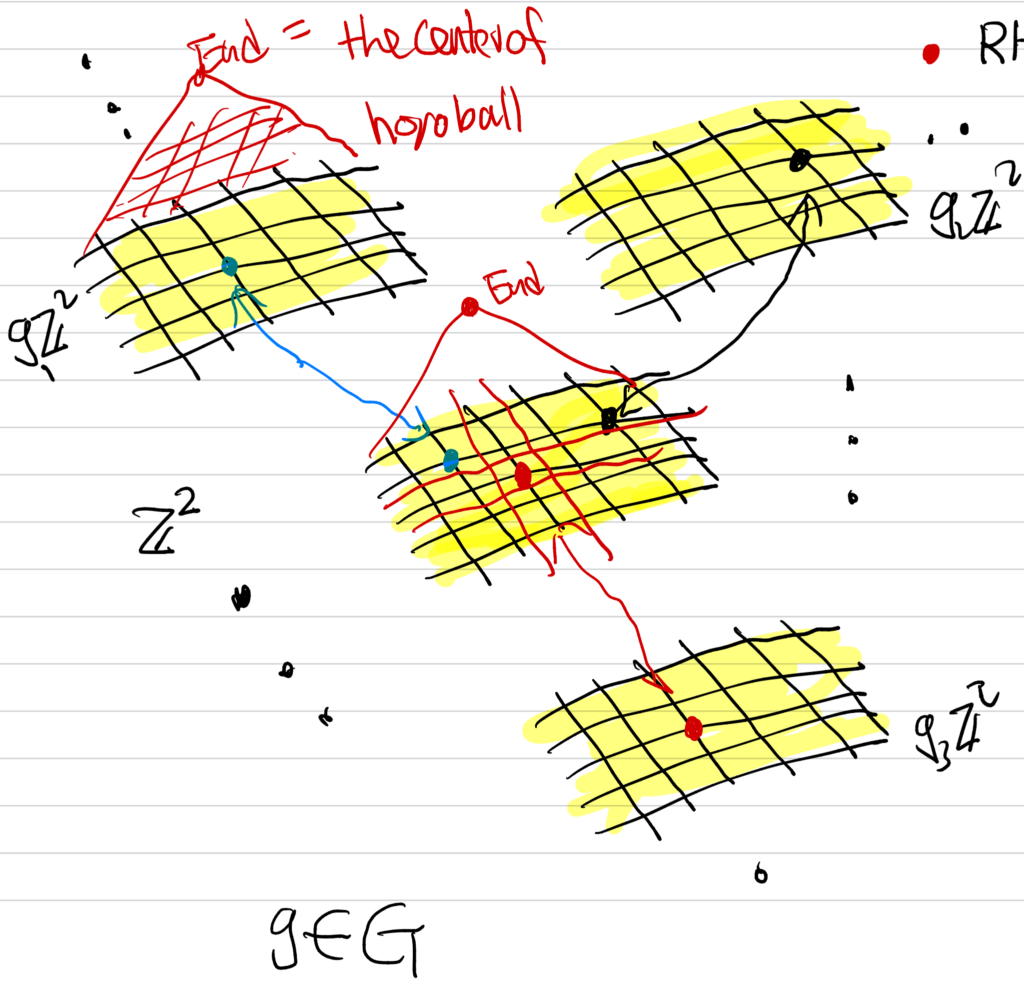
proof: By contradiction, $G \curvearrowright \partial_f G$ $|\partial_f G| \geq 3$

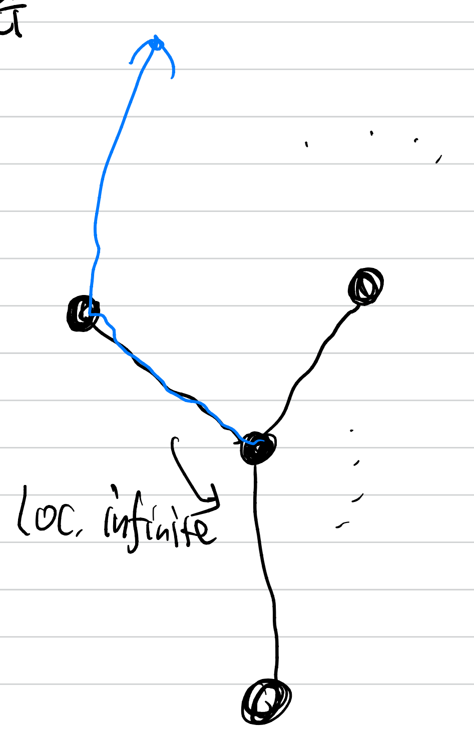
- ∞-order elements are either parabolic or hyperbolic
- Two commuting elements have the same fixed pts ($hk = kh$)

$\Rightarrow$ All generators share the same fixed pts $\Rightarrow$ RAAG fixes 1-pts or 2pts

2). $G = \mathbb{Z}^2 * \mathbb{Z}^2$ / $\mathbb{Z}^2 * \mathbb{Z}$

- RAAG
- ∞-ends
- RHG





T

Bass-Serre

Cantor set $=$ $\partial_E G = \partial T \sqcup \{ v(g\mathbb{Z}^2) : g \in G \}$

where ∂T = Ends of T = Gromov bdry of T, and the points $v(g\mathbb{Z}^2)$ are parabolic pts.

homeo $\Vert$

$\partial_f G \overset{\text{homeo}}{=} \partial_B G \triangleq$ Gromov bdry of hyp space Y

$$Y \triangleq Cay(G,S) \bigsqcup_{\{g\mathbb{Z}^2\}} \{\text{combinatorial horoballs } H(g\mathbb{Z}^2)\}$$

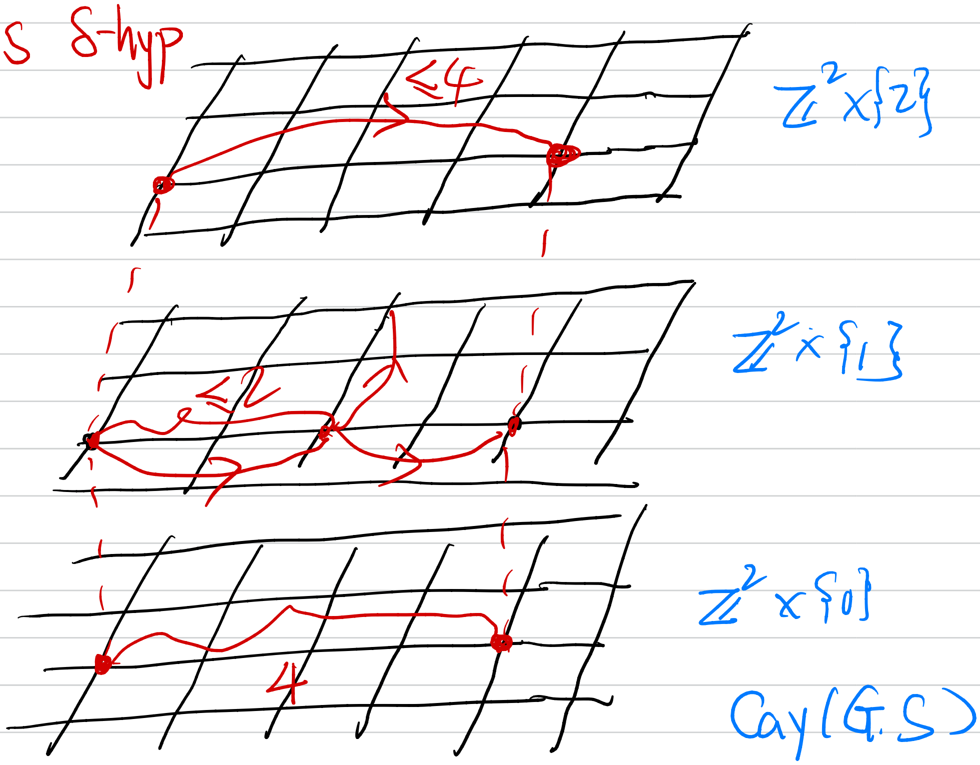
Y is δ-hyp

$\Uparrow$

$H(\mathbb{Z}^2)$ is δ-hyp

$\Vert$

$\mathbb{Z}^2 \times \mathbb{N} \sqcup \{0\}$



horizontal edge at level n (v,w) $\Longleftrightarrow$ $d(v,w) \leq 2^n$

Buseman

Horofunction Boundary: X = proper metric space

Fix $o \in X$

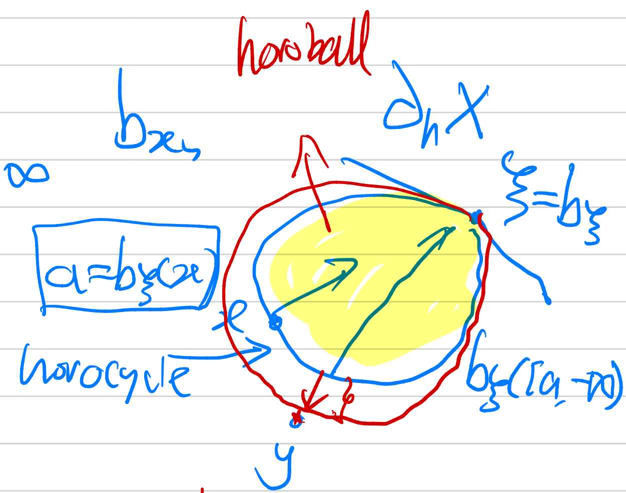
$$X = \left\{ b_x : \begin{array}{l} y \in X \to \mathbb{R} \\ b_x(y) = \boxed{d(x,y)} - d(x,o) \end{array} \right\}_{x \in X} \subseteq \mathrm{Lip}_1^o(X, \mathbb{R})$$

"uniform convergence top"

$$\mathrm{Lip}_1^o(X,\mathbb{R}) \triangleq \{\varphi : \varphi \text{ is Lip. } \& \ \varphi(o) = 0\}$$

$$\partial_h X \triangleq \overline{X} \setminus X$$

- $\partial_h X = \{\text{horofunction } b_\xi : \triangleq \lim_{x_n \to \xi} b_{x_n},\ \{x_n\} \text{ unbounded}\}$
- Geodesic ray $\longrightarrow b_\gamma = \lim_{\gamma \ni x_n \to \infty} b_{x_n}$



- $\mathrm{Isom}(X,d) \overset{\text{homeo}}{\curvearrowright} \partial_h X$

  Quasi-isometries may not!

- Buseman cocycles: $B_\xi(x,y) = b_\xi(x) - b_\xi(y)$

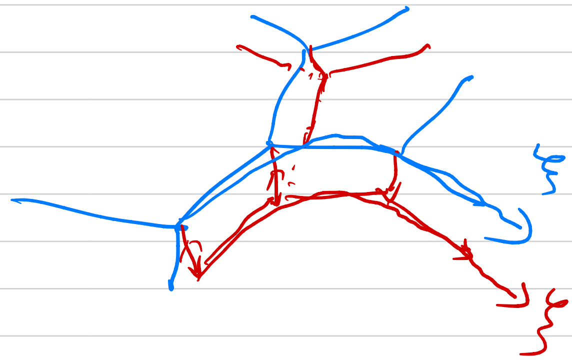
$$B_\xi(x,y) = B_\xi(x,z) + B_\xi(z,y)$$

Examples:

1) $X = CAT(0)$ space.

$\partial_h X = \partial_{vis} X \triangleq \{\text{geodesic rays}\} / \text{Hausd. dist}$

2) $X = \delta$-hyperbolic spaces



$\partial_h X \longrightarrow \partial_G X$

in general, it is NOT injective.

3) $\partial_h X \longrightarrow \partial_f X$

4) $X = (CAT(0)$ Cube complex, combinatorial metric$)$

[$\partial_h X =$ Roller boundary]

$\partial_h(X, CAT(0)) = \partial_{vis} X$

5). X Teichmüller space: Eskin–Mirzakhani

Thurston boundary $=$ $\partial_h(X,$ Thurston metric$)$

Gardiner–Masur bdry $=$ $\partial_h(X,$ Teich metric$)$

6). Martin boundary of random walks: Buseman

Miyachi

Furture Directions

- Patterson – Sullivan Theory on Horofunction boundary:

  ∃ Good theory [Y, 2022, Coulon]

  when $G \curvearrowright X$ contain contracting element

- Mostow Rigidity Type Applications:

  Step 1: Find boundary map

  Step 2: Use ergodicity of geodesic flow w.r.t B.M.S. measure