# PRODUCT SET GROWTH

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This text clarifies some statements in the Introduction of the paper [CJY22], and some comments in its Math Review written by Jack Button [But].

0.1. **Growth tightness.** Let $G$ be a finitely generated group with a finite symmetric generating set $S$. Define the growth rate as follows

$$\omega(G,S) := \lim_{n\to\infty} \frac{\log \sharp(S^{\le n})}{n}$$

Let us define

$$\Theta(G) := \left\{ \frac{\omega(G,S)}{\log(\sharp S - 1)} : \sharp S < \infty,\ \langle S\rangle = G \right\}.$$

There is a typo in [CJY22, Introduction, page 800] where $\log(\sharp S)$ is used in the denominator. In this case, the following claim is definitely false, as pointed out in [But]. Once $\Theta(G)$ is so defined as above, we have the claim in [CJY22, Introduction, page 800].

**Claim 0.1.** $1 \in \Theta(G)$ *if and only if* $G$ *is a free group.*

Indeed, a finitely generated group $G$ is called growth tight if for any finite symmetric generating set $S$ and for any infinite normal subgroup $H$, we have $\omega(G,S) > \omega(\bar{G},\bar{S})$ where $\bar{G} = G/H$ and $\bar{S}$ is the natural image of $S$ in $\bar{G}$. Grigorchuk-de la Harpe [GdlH97] proved that non-abelian free groups $G$ are growth tight. In particular, if $\omega(G,S) = \log(\sharp S - 1)$, then $G$ is a free group. The claim follows.

0.2. As pointed out by [But], the following fact in [CJY22, Introduction, page 800] holds for any finitely generated group $G$.

By taking $T_n := S^n$, we have

$$\frac{\omega(G,T_n)}{\log \sharp T_n} \to 1, \text{ as } n \to \infty$$

This fact was suggested in [CJY22] under the assumption that $G$ has purely exponential growth.

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0.3. **Product set growth for sets with identity.** For any finite subset $U \subset G$ and $n \in \mathbb{N}$, the *n-th product set* of $U$ is defined as follows

$$U^n := \{s_1 \cdots s_i \cdots s_n : s_i \in U\}$$

To prove product set growth for $U$, we may assume that $U$ contains the identity $e$. Indeed, we have the following fact.

**Lemma 0.2.** *Let $U$ be a finite non-empty set in $G$. Denote $U_1 = U \cup \{e\}$ where $e$ is identity in $G$. For some $\alpha, \beta > 0$, assume that $|U_1^n| \ge (\alpha|U_1|)^{n\beta}$ holds for any $n \ge 1$. Then $|U^n| \ge (\frac{\alpha}{2^{1/\beta}}|U|)^{n\beta}$.*

*Proof.* First note that $|U^n| \le |U^{n+1}|$ for any $n \ge 1$. Then $|U_1^n| = |U^n \cup \cdots \cup U \cup \{e\}| \le |U^n| + \cdots + |U| + 1 \le n|U^n| + 1$. This implies for $n \ge 1$:

$$|U^n| \ge \frac{|U_1^n| - 1}{n} \ge \frac{|U_1^n|}{2n} \ge \frac{|U_1^n|}{2^n} \ge \frac{(\alpha|U_1|)^{\beta n}}{2^n} \ge \frac{(\alpha|U|)^{\beta n}}{2^n}$$

where $|U_1^n| \ge 2$ for each $n \ge 1$. The lemma is proved. □

This clarifies some comments appearing in the Math Review [But]:

> Thus the main result ([CJY22, Theorem 1.1]) in this paper is precisely showing that a relatively hyperbolic group has uniform product set growth away from virtually cyclic groups and groups conjugate into maximal parabolic subgroups, but for all finite symmetric subsets containing id rather than for all finite subsets.

The product set growth was not the main concern of the paper [CJY22]. By this lemma, the product set growth could be derived from [CJY22, Theorem 1.1] for finite *symmetric* subsets $S$ **without** the identity. The product set growth of relatively hyperbolic groups for possibly asymmetric subsets was dealt in [WY25] (among other groups).

## References

[But] Jack Button. Math review #MR4472896 to the paper CJY22.

[CJY22] Yu-Miao Cui, Yue-Ping Jiang, and Wen-Yuan Yang. Lower bound on growth of non-elementary subgroups in relatively hyperbolic groups. *J. Group Theory*, 25(5):799–822, 2022.

[GdlH97] R. Grigorchuk and P. de la Harpe. On problems related to growth, entropy, and spectrum in group theory. *J. Dynam. Control Systems*, 3(1):51–89, 1997.

[WY25] Renxing Wan and Wenyuan Yang. Uniform exponential growth for groups with proper product actions on hyperbolic spaces. *J. Algebra*, 676:189–235, 2025.

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